

Edge colouring by total labellings

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Abstract

We introduce the concept of an edge-colouring total k -labelling. This is a labelling of the vertices and the edges of a graph G with labels $1, 2, \dots, k$ such that the weights of the edges define a proper edge colouring of G . Here the weight of an edge is the sum of its label and the labels of its two endvertices. We define $\chi'_t(G)$ to be the smallest integer k for which G has an edge-colouring total k -labelling. This parameter has natural upper and lower bounds in terms of the maximum degree Δ of G : $\lceil(\Delta + 1)/2\rceil \leq \chi'_t(G) \leq \Delta + 1$. We improve the upper bound by 1 for every graph and prove a general upper bound of $\chi'_t(G) \leq \Delta/2 + \mathcal{O}(\sqrt{\Delta \log \Delta})$. Moreover, we investigate some special classes of graphs.

Keywords Edge colouring; total labelling; irregularity strength; discrepancy

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1 Introduction

For a graph $G = (V(G), E(G))$ an *edge-colouring total k -labelling* is a function $f : V(G) \cup E(G) \rightarrow \{1, 2, \dots, k\}$ such that the weights of the edges defined by

$$w(uv) := f(u) + f(uv) + f(v)$$

form a proper edge colouring. The smallest integer k for which there exists an edge-colouring total k -labelling is denoted by $\chi'_t(G)$.

A related concept which has recently received a lot of attention was proposed by Karoński, Łuczak and Thomason [16]. They conjectured that the edges of every graph G with no K_2 component can be labeled with labels $1, 2, 3$ such that the sums of the edge labels incident to the vertices of G define a proper vertex colouring. Addario-Berry, Dalal and Reed [2] recently proved that the labels $1, 2, \dots, 16$ are always sufficient, i.e. every

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graph with no K_2 component has a *vertex-colouring edge 16-labelling* (cf. also [1, 3]). A total version of vertex-colouring labellings was discussed by Przybyło and Woźniak who proved [19] by similar methods as in [2] that every graph has a *vertex-colouring total 11-labelling* and conjecture that 2 labels are enough.

The vertex-colouring edge labellings can be considered a relaxation of the well-known *irregularity strength* of graphs [10, 4, 18, 14] where the label sums for all vertices are required to be different. Similarly, the edge-colouring total labellings which we study here can be considered a relaxation of *edge-irregular total labellings* introduced by Bača, Jendrol', Miller, and Ryan [6], where the weights of all edges are required to be different. The *total edge irregularity strength* $\text{tes}(G)$ is defined as the smallest integer k for which a graph G has an edge-irregular total k -labelling. A simple lower bound is

$$\text{tes}(G) \geq \max \left\{ \left\lceil \frac{|E(G)| + 2}{3} \right\rceil, \left\lceil \frac{\Delta(G) + 1}{2} \right\rceil \right\}.$$

and Ivančo and Jendrol' [15] conjectured that this bound is attained for all graphs except K_5 . Brandt, Miškuf, and Rautenbach [8, 9] recently proved that this is true for graphs whose size is at least 111000 times their maximum degree.

Let us return to the edge-colouring total k -labellings and the corresponding graph parameter $\chi'_t(G)$ which has natural upper and lower bounds in terms of the maximum degree Δ of G . Obviously,

$$\chi'_t(G) \leq \Delta + 1$$

by Vizing's Theorem [20], since a proper edge colouring together with a constant labelling of the vertices defines an edge-colouring total labelling of G . Furthermore, since the possible weights of the edges incident with a vertex v of maximum degree Δ in an edge-colouring k -labelling f are $f(v) + \{2, 3, \dots, 2k\}$, we get a lower bound of

$$\chi'_t(G) \geq \left\lceil \frac{\Delta + 1}{2} \right\rceil.$$

The following is our main result whose proof we postpone to Section 3.

Theorem 1.1 *If G is a graph of maximum degree Δ , then*

$$\chi'_t(G) \leq \left\lceil \frac{1}{2} \left(\Delta + \left\lfloor \sqrt{2\Delta(1 + \ln(2\Delta^2 - 2\Delta + 2))} \right\rfloor \right) \right\rceil + 1 = \Delta/2 + \mathcal{O}(\sqrt{\Delta \log \Delta}).$$

Before we proceed to Section 2 where we study $\chi'_t(G)$ for some special graphs, we show how to reduce the upper bound by one for every graph and relate $\chi'_t(G)$ to the chromatic index. The next result already illustrates our general approach which is to combine edge colouring methods with suitable partitions of the vertex set.

Theorem 1.2 *If G is a graph of maximum degree Δ , then $\chi'_t(G) \leq \Delta$*

Proof: Let $c : E(G) \rightarrow \{1, 2, \dots, \Delta + 1\}$ be a proper edge colouring of G which exists by Vizing's Theorem [20]. Since the subgraph containing the edges coloured Δ and $\Delta + 1$ consists of paths and even cycles, it is bipartite. Fix a bipartition $A \cup B$ of $V(G)$ such that all edges with colours Δ and $\Delta + 1$ have one endvertex in A and the other endvertex in B .

Assign to all vertices of A the label 1 and to all vertices of B the label Δ . Assign label $c(e)$ to all edges between vertices of A and label $c(e) + 1$ to all edges between vertices of B . Finally, determine the labels of the edges in the bipartite graph spanned by the edges between A and B by a proper Δ -edge colouring c' .

The edges joining vertices of A receive weights between 3 and $(\Delta - 1) + 1 + 1 = \Delta + 1$, the edges joining A to B receive weights between $\Delta + 2$ and $2\Delta + 1$, and the edges joining vertices of B receive weights between $2\Delta + 2$ and 3Δ . Since these weights form proper edge colourings inside and between the sets, they form a proper edge colouring of the entire graph. \square

The upper bound $\chi'_t(G) \leq \Delta$ can only be tight for small values of Δ . From Theorem 1.1 follows that for $\Delta \geq 19$ we have $\chi'_t(G) < \Delta$, and, in fact, with a more refined reasoning along the same lines the threshold can be reduced to $\Delta \geq 14$. We are not aware of any graph with $\Delta > 3$ and $\chi'_t(G) = \Delta$.

Next we show that an edge-colouring total k -labelling gives rise to a proper edge colouring with $2k - 1$ colours. Conversely, this means that for every type II graph (i.e. $\chi'(G) = \Delta(G) + 1$) we have $\chi'_t(G) > \frac{\Delta(G)+1}{2}$.

Lemma 1.3 *If $\chi'_t(G) = k$ for a graph G , then $\chi'(G) \leq 2k - 1$.*

Proof: Consider an edge-colouring total k -labelling f of G . Note that for $l \leq k + 1$ the edges of weights l and $l + 2k - 1$ cannot have a common endvertex and therefore form a matching. Thus we can decompose the edge set into $2k - 1$ matchings: $k - 1$ matchings with the edges of weight l and $l + 2k - 1$ for $3 \leq l \leq k + 1$, and k matchings with the edges of weight l for $k + 2 \leq l \leq 2k + 1$. \square

2 Special classes of graphs

If G is a graph of maximum degree $\Delta = 1$, then $\chi'_t(G) = 1$. If $\Delta = 2$, then $\chi'_t(G) = 2$ by Theorem 1.2. Similarly, if $\Delta(G) = 3$, then $2 \leq \chi'_t(G) \leq 3$. In our first result we characterize cubic graphs with $\chi'_t(G) = 2$.

Theorem 2.1 *A cubic graph G satisfies $\chi'_t(G) = 2$ if and only if its vertex set can be partitioned into two parts A and B that induce perfect matchings.*

Proof: Let f be an edge-colouring total 2-labelling of a cubic graph G . For every vertex $v \in V(G)$ the three edges incident with v must receive the weights 3, 4, 5, if $f(v) = 1$, and the weights 4, 5, 6, if $f(v) = 2$. The edges of weight 3 and weight 6 join two vertices with the same label.

If $f(v) = 1$, then the other endvertex of the edge of weight 5 incident with v has label 2. So there are at least as many vertices with label 2 as with label 1. Conversely, for $f(v) = 2$ the edge of weight 4 incident to v has its other endvertex labelled 1. So there are at least as many vertices labelled 1 as with label 2. Together, there are equally many vertices labelled 1 and 2 and the edges of weights 4 and 5 form a 2-regular graph joining vertices of label 1 to vertices of label 2. Therefore, G has the indicated structure.

Conversely, if G has the indicated structure, then $|A| = |B|$. We assign label 1 to the vertices and edges in A and label 2 to the vertices and edges in B . Labelling the edges of the 2-regular bipartite graph between A and B by 1 and 2 according to a proper 2-edge colouring results in an edge-colouring total 2-labelling. \square

It is an easy observation that the lower bound is tight for forests.

Theorem 2.2 *If F is a forest of maximum degree Δ , then $\chi'_t(F) = \lceil \frac{\Delta+1}{2} \rceil$.*

Proof: We prove the stronger statement that an edge-colouring total labelling exists using only two vertex labels 1 and $k = \lceil \frac{\Delta+1}{2} \rceil$. Obviously, it suffices to prove the statement for the tree components.

We proceed by induction on the number of vertices n . The statement is true for $n \leq 2$ so assume $n \geq 3$. Let vw be an edge such that v has degree at least 2 and all neighbours of v except possibly w are leaves. Note that such an edge vw exists. Delete all neighbours of v except w to obtain a tree T' , which by induction has the required total labelling. Now label the deleted vertices with 1 and k such that at most $\frac{d(v)+1}{2}$ of the neighbours of v (including the already labelled vertex w) have the same label. Now the remaining edges can be easily labelled such that all edges incident with v have different weights. \square

Next, we consider edge-colouring total labellings of complete graphs. In a graph G with a given edge colouring a *rainbow (perfect) matching* is a (perfect) matching, where all edges are of different colour. We need a lemma on rainbow matchings in the proof of our next result.

Lemma 2.3 (a) *Every complete bipartite graph $K_{k,k}$ has a proper k -edge colouring with a rainbow perfect matching if k is odd, and a rainbow matching of cardinality $k - 1$ if k is even.*

(b) *Every complete graph K_{2k} of even order has a proper $(2k - 1)$ -edge colouring with a rainbow perfect matching unless $k = 2$.*

Proof:

(a) Let u_1, \dots, u_k and w_1, \dots, w_k be the vertices on both sides of the bipartition. Define a proper edge colouring of G by assigning the colour $\ell \in \{1, \dots, k\}$ to the edge $u_i w_j$, if $j - i \equiv \ell \pmod{k}$. Now let a and b be the largest even and odd integer $< \frac{k}{2} + 1$, respectively. Choose a matching M consisting of the edges $u_i w_{a+1-i}$ for $1 \leq i \leq a$ and $u_{a+i} w_{a+b+1-i}$ for $1 \leq i \leq b$. This is a rainbow matching of cardinality $a + b = k - 1$, if k is even and a rainbow perfect matching, if k is odd.

(b) Let $u_0, u_1, \dots, u_{2k-1}$ be the vertices of K_{2k} . First assume that k is odd. Take as the first colour class of edges the perfect matching M_0 consisting of the edges $u_i u_{2k-i-1}$ for $0 \leq i \leq k-1$. The remaining colour classes are obtained as follows: Embed the vertices of K_{2k} in the plane such that $u_1, u_2, \dots, u_{2k-1}$ form the vertices of a regular $(2k-1)$ -gon with center u_0 . Rotating M_0 by an angle of $\frac{2\pi}{2k-1}$ a total number of $2k-2$ times defines $2k-2$ further perfect matchings (cf. Figure 1). Since the geometric lengths of all edges in one matching are different, this defines a proper edge-colouring of K_{2k} for which the matching $u_0 u_{2k-1}, u_1 u_2, \dots, u_{2k-3} u_{2k-2}$ is a rainbow perfect matching.

Next, assume that k is even. Here we choose as the first colour class of edges the perfect matching M_0 consisting of the edges $u_i u_{2k-i-1}$ for $0 \leq i < \frac{k}{4}$, $u_i u_{2k-i-2}$ for $\frac{k}{4} \leq i < \frac{3k}{4} - 1$, $u_i u_{2k-i-3}$ for $\frac{3k}{4} - 1 \leq i < k-1$, and the additional edge $u_i u_{i+\frac{k}{2}}$ for $i = \lfloor \frac{5k}{4} \rfloor - 1$. Again, the remaining $2k-2$ colour classes are obtained by embedding the vertices of K_{2k} and rotating M_0 as before (cf. Figure 1). Again the matching $u_0 u_{2k-1}, u_1 u_2, \dots, u_{2k-3} u_{2k-2}$ is a rainbow perfect matching.

□

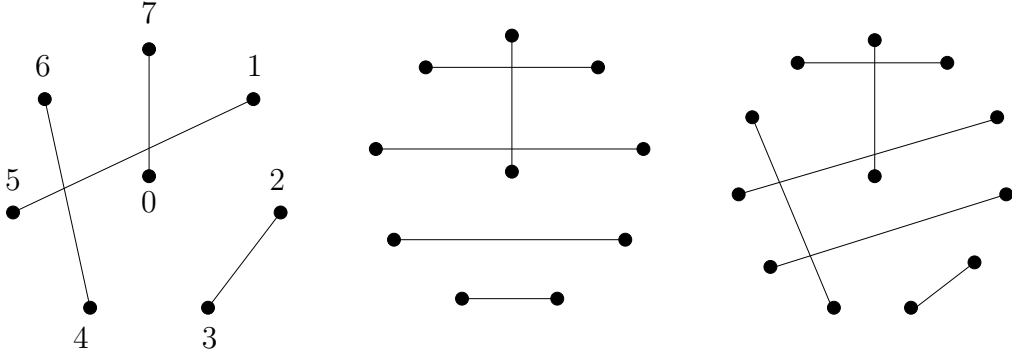


Figure 1

In a graph with a total labelling we denote the set of vertices with label i by V_i .

Theorem 2.4 *If $n \not\equiv 2 \pmod{4}$, then $\chi'_t(K_n) = \lceil \frac{n}{2} \rceil$ and if $n \equiv 2 \pmod{4}$, then $\chi'_t(K_n) \leq \frac{n}{2} + 1$.*

Proof: In view of the lower bound it suffices to describe suitable labellings of the complete graph.

First assume that $n \equiv 0 \pmod{4}$. Label half the vertices by 1 and the other half by $k = \frac{n}{2}$. Determine a proper edge colouring of the edges in V_1 with labels $1, \dots, k-1$, a proper edge colouring of the edges in V_k with labels $2, \dots, k$, and a proper edge colouring of the edges joining V_1 to V_k with labels $1, \dots, k$. It is now easy to verify, that this is an edge-colouring total k -labelling. Note that this also implies the result for $n \equiv 3 \pmod{4}$ and $n \equiv 2 \pmod{4}$ by considering edge-colouring total labellings of complete graphs of order $n+1$ and $n+2$, respectively.

Therefore, only the case $n \equiv 1 \pmod{4}$ remains.

Label $\frac{n-1}{2} = k-1$ vertices by label 1 and k , respectively, and the remaining vertex v by $\frac{k+1}{2}$. Let $u_{\frac{k+1}{2}+2}, \dots, u_{\frac{k+1}{2}+k}$ denote the vertices of V_1 and $w_{\frac{k+1}{2}+k+2}, \dots, w_{\frac{k+1}{2}+2k}$ be the vertices of V_k . Label the edges from v to u_i with $i - \frac{k+1}{2} - 1$ and the edges v to w_j with $j - \frac{k+1}{2} - k$. Note that each vertex u_i and w_j is joined to v by an edge of weight i and j , respectively.

It remains to show that we can find an edge labelling of the edges not incident with v , such that the labels form a proper edge colouring of the remaining graph and the weight of each edge is different from the indices of its endvertices. The edges inside V_1 will obtain the weights $3, \dots, k+1$, inside V_k the weights $2k+2, \dots, 3k$, and the edges between V_1 and V_k will obtain weights $k+2, \dots, 2k+1$.

By Lemma 2.3 (b) we know that the complete graph K_{k-1} induced by V_1 has a proper $(k-2)$ -edge colouring c which has a rainbow perfect matching. Let $\{2, \dots, k-1\}$ be the colours of the colouring and let $\{\frac{k-1}{2}+1, \dots, k-1\}$ be the colours occurring in the rainbow perfect matching M . Assign the indices in such a way that the vertex u_i of index i is incident with the edge of colour $i-2$ in the rainbow matching for $\frac{k+1}{2}+2 \leq i \leq k+1$. Finally, recolour the edges of the rainbow perfect matching M with colour 1 and take the colours of this new colouring c' as the labels of the edges inside V_1 . Note that this edge labelling has the desired property that u_i is not joined to a vertex in V_1 by an edge of weight i . Along the same line of argument we obtain a labelling of the edges inside V_k with labels $\{2, \dots, k\}$ such that each vertex w_j is not joined to a vertex in V_k by an edge of weight j .

Finally, we need to label the edges in the bipartite graph spanned by the (V_1, V_k) -edges. This graph is isomorphic to $K_{k-1, k-1}$, where $k-1 \equiv 0 \pmod{2}$. By Lemma 2.3 (a) this graph has a proper $(k-1)$ -edge colouring using the colours $\{1, \dots, k-1\}$ with a rainbow matching M of cardinality $k-2$ that avoids the colour $\frac{k-1}{2}$. Assign the indices in such a way that u_i is incident with the edge of M of weight $i-k-1$ for $k \leq i \leq \frac{k+1}{2}+k$, and w_j is incident with the edge of M of weight $j-k-1$ for $\frac{k+1}{2}+k+2 \leq j \leq 2k$. Moreover, let w_{2k+1} be the vertex in V_k that is not incident with an edge of M . Now recolour the edges of M with colour k to obtain a new colouring, which we use as the labelling of the (V_1, V_k) edges. By the construction it is easy to verify that the result is an edge-colouring total k -edge labelling. \square

We conclude this section with some further results concerning the case $n \equiv 2 \pmod{4}$ which might eventually allow to determine for which $n \equiv 2 \pmod{4}$, $\chi'_t(K_n) = \frac{\Delta+1}{2}$ holds, and for which $\chi'_t(K_n) = \frac{\Delta+1}{2} + 1$. We can show that the second equality holds for $6 \leq n \leq 22$. At the same time our result describes the distribution of the labels in some detail if the first equality holds.

Lemma 2.5 *Let K_n be a complete graph with $k = \chi'_t(K_n) = \frac{n}{2}$. If V_i denotes the set of vertices labelled i in an edge-colouring total k -labelling of K_n , then the cardinality of each set V_i is even, $|V_i| = |V_{k-i+1}|$ and $|V_i| \leq |V_1| = |V_k|$ for $i = 1, \dots, k$. The edges of weight*

$k + 2$ have label 1 and the edges of weight $2k + 1$ have label k . Moreover, if $n \equiv 2 \pmod{4}$ then $6 \leq |V_{\frac{k+1}{2}}| \equiv 2 \pmod{4}$.

Proof: Since $k = \frac{\Delta+1}{2}$, each vertex $v \in V_i$ is incident with an edge of weight $i + \ell$ for $2 \leq \ell \leq 2k$. For $2 \leq i \leq k$ the edges of weight $i + 2$ form a matching between V_i and V_1 and hence $|V_i| \leq |V_1|$. Similarly, each vertex $v \in V_j$ is incident with an edge of weight $j + 2k$ and for $1 \leq j \leq k - 1$ these edges form a matching between V_j and V_k , implying $|V_j| \leq |V_k|$. Since the inequalities hold for $i = k$ and $j = 1$, we obtain $|V_1| = |V_k|$.

Next we show that each of the sets V_i has even cardinality. This is true for V_1 , since the edges of weight 3 form a perfect matching between the vertices in V_1 . Now consider the vertex set $U_i = V_1 \cup V_2 \cup \dots \cup V_i$. Since the edges of weight $i + 2$ form a perfect matching of U_i , and, by induction, U_{i-1} has even cardinality, the set V_i has even cardinality as well.

For $i \leq \frac{k+1}{2}$ we prove by induction over i that the edges of weight $2k + 1$ incident to a vertex in V_i have their other endvertex in V_{k-i+1} , and the edges of weight $k + 2$ incident to a vertex in V_{k-i+1} have their other endvertex in V_i . In particular, $|V_i| = |V_{k-i+1}|$ and the edges of weight $k + 2$ and $2k + 1$ have weight 1 and k , respectively.

The statement is true for $i = 1$, so assume that it is true for all indices $< i$. Let vw be the edge of weight $2k + 1$ that is incident to $v \in V_i$. Since the label of vw is at most k , the vertex w has label $s \geq k - i + 1$. If $s > k - i + 1$, then by induction the other endvertex v of the edge of weight $2k + 1$ incident to w has label $t = k - s + 1 < i$, contradicting $v \in V_i$. Analogously, for the vertices of V_{k-i+1} the other endvertex of the incident edge of weight $k + 2$ lies in V_i . This completes the induction. If $n \equiv 2 \pmod{4}$, then $6 \leq |V_{\frac{k+1}{2}}| \equiv 2 \pmod{4}$, because of the parity conditions and since $V_{\frac{k+1}{2}}$ has two disjoint perfect matchings consisting of the edges of weight $k + 2$ and $2k + 1$. \square

Lemma 2.6 *Every edge-colouring total $(2p + 1)$ -labelling of K_{4p+2} for $p \geq 1$ uses at least 5 different vertex labels.*

Proof: For contradiction, we assume the existence of an edge-colouring total $(2p + 1)$ -labelling using less than 5 different vertex labels. By Lemma 2.5, this implies that it has exactly 3 label classes V_1 , V_{p+1} , and V_{2p+1} . Moreover $|V_{2p+1}| = |V_1| \geq |V_{p+1}| \geq 6$. We know that all edges with weights $3, \dots, p + 2$ have both endvertices in V_1 and for each such weight value these edges form a perfect matching in V_1 . Furthermore, all edges of weight $p + 3, \dots, 2p + 2$ incident with a vertex in V_{p+1} have the other endvertex in V_1 , and, finally, there is a perfect matching between V_{2p+1} and V_1 of edges of weight $2p + 3$.

Let n_1 be the number of vertices in V_1 and n_{p+1} the number of vertices in V_{p+1} . Since we have $n_1 = \frac{n - n_{p+1}}{2}$ for $n = 4p + 2$, there are exactly $n_1/2$ edges in V_1 of weight w for each $3 \leq w \leq p + 2$ and $\frac{n_1 - n_{p+1}}{2}$ edges in V_1 of weight w for each $p + 3 \leq w \leq 2p + 2$ and no edges of weight $\geq 2p + 3$. Altogether, there are at most

$$p \binom{n_1}{2} + p \binom{n_1 - n_{p+1}}{2} = p \left(n_1 - \frac{n_{p+1}}{2} \right)$$

edges in V_1 . Since $2n_1 + n_{p+1} = n = 4p + 2$ we get $p = \frac{1}{2}(n_1 + \frac{n_{p+1}}{2} - 1)$ and

$$\binom{n_1}{2} \leq \binom{n_1}{2} - \frac{1}{2}n_{p+1}^2 + \frac{1}{2}(n_1 - n_1 + 1)n_{p+1},$$

which is a contradiction since $n_{p+1} \geq 6 > 1$. \square

3 The general upper bound

Our goal in this section is to prove Theorem 1.1. In order to clarify our approach, we present a number of intermediate results, some of which we think to be interesting on their own right. The first is a consequence of Vizing's Adjacency Lemma [21] (see also [13]). A graph $G = (V, E)$ of maximum degree Δ is called *critical* if $\chi'(G) = \Delta + 1$ but $\chi'(G - e) = \Delta$ for all $e \in E$.

Lemma 3.1 (Vizing's Adjacency Lemma [21]) *Let $G = (V, E)$ be a critical graph with maximum degree Δ and $\chi'(G) = \Delta + 1$. If $uv \in E$ then u is adjacent to at least $\max\{2, \Delta - d_G(v) + 1\}$ many vertices of maximum degree.*

Proposition 3.2 *Every graph G with maximum degree Δ has a proper $(\Delta + 1)$ -edge colouring such that no edge of colour $\Delta + 1$ is incident with a vertex of degree less than Δ .*

Proof: We apply induction on $m := |E(G)|$. If G has a proper Δ -edge colouring, then the statement is vacuously true. Note that this already implies the result for $m \leq 2$. Therefore, we assume now that $m \geq 3$ and that $\chi'(G) = \Delta + 1$.

It follows from Lemma 3.1 applied to a critical subgraph of G and a vertex u of maximum degree, that a neighbour w of u has maximum degree as well. By induction, $G - uw$ has a proper $(\Delta + 1)$ -edge colouring such that no edge of colour $\Delta + 1$ is incident to a vertex of degree less than Δ . Therefore, assigning the colour $\Delta + 1$ to the edge uw yields the desired colouring. \square

The construction in the next result relies on a suitable partition of the vertex set.

Theorem 3.3 *If G is a graph of maximum degree Δ whose vertex set has a partition $V(G) = A \cup B$ such that every vertex has at most $k - 1$ neighbours in A and at most $k - 1$ neighbours in B for some k with $k - 1 > \frac{\Delta}{2}$, then $\chi'_t(G) \leq k$.*

Proof: Let $V(G) = A \cup B$ be a partition as in the statement. Label the vertices of A with 1 and the vertices of B with k .

By Proposition 3.2, $G[A]$ has a proper k -edge colouring that avoids colour k at the vertices of degree $d_{G[A]}(v) < k - 1$. Similarly, $G[B]$ has a proper k -edge colouring that avoids colour 1 at the vertices of degree $d_{G[B]}(v) < k - 1$. We choose these edge colourings as the labellings of the edges in A and B , respectively. Let A' denote the set of vertices in

A incident with an edge labelled k and let B' denote the set of vertices in B incident with an edge labelled 1.

It remains to label the edges between A and B . Let $G(A, B)$ denote the bipartite spanning subgraph of G of maximum degree at most $k - 1$ containing all edges between A and B . Considering a perfect matching in a bipartite $(k - 1)$ -regular supergraph of $G(A, B)$, it follows that $G(A, B)$ has a minimal matching M that saturates all vertices v with $d_{G(A, B)}(v) = k - 1$. Note that by the minimality requirement, M does not contain an (A', B') -edge, since for each vertex in $u \in A' \cup B'$ we have $d_{G(A, B)}(u) \leq \Delta(G) - (k - 1) < k - 1$. We label the edges of M with one endvertex in A' with k and the remaining edges with 1. Now $G(A, B) - M$ has maximum degree $\leq k - 2$ and hence has a proper $(k - 2)$ -edge colouring with colours $2, 3, \dots, k - 1$ which we use as the labelling for the edges. It is easy to verify that the edge weights defined by this total k -labelling form a proper edge colouring of G . \square

Our next goal is to find a partition as in Theorem 3.3 for some k close to $\Delta/2$. We do this using the probabilistic method via a discrepancy argument: For a graph G we consider the *discrepancy* $\text{disc}(G)$ defined as follows:

$$\text{disc}(G) := \min_{g: V(G) \rightarrow \{-1, 1\}} \max_{u \in V(G)} \left| \sum_{v \in N_G(u)} g(v) \right|.$$

Note that $\text{disc}(G)$ corresponds to the ordinary discrepancy of the hypergraph on the ground set $V(G)$ whose hyperedges are the neighbourhoods of vertices in G .

Together with Theorem 3.3 we obtain.

Corollary 3.4 *If G is a graph of maximum degree Δ , then*

$$\chi_t(G) \leq \frac{\Delta + \text{disc}(G)}{2} + 1.$$

Proof: Let $g : V(G) \rightarrow \{-1, 1\}$ be such that $\text{disc}(G) = \max_{u \in V(G)} \left| \sum_{v \in N_G(u)} g(v) \right|$. Let $A = g^{-1}(1)$ and $B = g^{-1}(-1)$. For $u \in V(G)$ let $d_A(u) = \{v \in N_G(u) \mid g(v) = 1\}$ and $d_B(u) = \{v \in N_G(u) \mid g(v) = -1\}$. Since $|d_A(u) - d_B(u)| \leq \text{disc}(G)$ and $d_A(u) + d_B(u) \leq \Delta$, we have $\max\{d_A(u), d_B(u)\} \leq \frac{\Delta + \text{disc}(G)}{2}$ for every $u \in V(G)$ and Theorem 3.3 implies the desired result. \square

In order to bound the discrepancy we combine Chernoff's inequality with the Lovász Local Lemma.

Lemma 3.5 (Chernoff's inequality [11], see also [5]) *Let X_1, \dots, X_n be mutually independent random variables with $\mathbf{P}(X_i = 1) = \mathbf{P}(X_i = -1) = \frac{1}{2}$. Then for $S = X_1 + \dots + X_n$ and $\delta > 0$ we get $\mathbf{P}(|S| > \delta) < 2 \exp\left(\frac{-\delta^2}{2n}\right)$.*

Lemma 3.6 (Lovász Local Lemma [12], see also [5]) *Let A_1, A_2, \dots, A_n be events in an arbitrary probability space. Let $\mathbf{P}(A_i) \leq p$ and let A_i be mutually independent of all but at most $d \geq 2$ of the events A_j with $j \neq i$ for each $1 \leq i \leq n$. If $ep(d+1) \leq 1$, then $\mathbf{P}(\bigwedge_{i=1}^n \overline{A_i}) > 0$, i.e. with positive probability none of the events A_i occurs.*

Proposition 3.7 *If G is a graph of maximum degree Δ , then*

$$\text{disc}(G) \leq \left\lceil \sqrt{2\Delta(1 + \ln(2\Delta^2 - 2\Delta + 2))} \right\rceil.$$

Proof: We consider a random function $g : V(G) \rightarrow \{-1, 1\}$ where all values $g(v)$ are 1 independently at random with probability $1/2$.

For some $\delta > 0$ and $u \in V(G)$ consider the event $A_u: \left| \sum_{v \in N_G(u)} g(v) \right| > \delta$. By Chernoff's inequality,

$$\mathbf{P}(A_u) \leq 2 \exp\left(\frac{-\delta^2}{2d_G(u)}\right) \leq 2 \exp\left(\frac{-\delta^2}{2\Delta}\right).$$

The events A_u and A_v are dependent only if there is a path of length exactly two between u and v . Therefore, A_u is independent of all but at most $\Delta(\Delta - 1)$ many events A_v with $v \neq u$. For $\delta := \sqrt{2\Delta(1 + \ln(2\Delta^2 - 2\Delta + 2))}$ we obtain

$$2 \exp\left(1 - \frac{\delta^2}{2\Delta}\right) (\Delta(\Delta - 1) + 1) = 1$$

and the Lovász Local Lemma implies the existence of a function $g : V(G) \rightarrow \{-1, 1\}$ with $\left| \sum_{v \in N_G(u)} g(v) \right| \leq \delta$ for all $u \in V(G)$. \square

Proof of Theorem 1.1: The result follows immediately from Corollary 3.4 and Proposition 3.7. \square

4 Concluding remarks

The upper bound $\mathcal{O}(\sqrt{\Delta \log \Delta})$ for the discrepancy of a Δ -regular graph is not far from being best possible. This is due to the fact, that there are graphs with discrepancy $\Omega(\sqrt{\Delta})$. The Paley graphs for example form an infinite sequence of graphs with $\Delta = \frac{n-1}{2}$ and discrepancy $\Omega(\sqrt{\Delta})$ by a result of Lovász and Sós (see [17, Theorem 4.5]). Conversely, the Beck-Fiala Conjecture (see [17]) says that the vertices of every hypergraph where each vertex belongs to at most Δ hyperedges has discrepancy $\mathcal{O}(\sqrt{\Delta})$. If the Beck-Fiala Conjecture — or its restriction to Δ -regular, Δ -uniform hypergraphs — is true then we can improve the upper bound in Theorem 1.1 with the same reasoning to

$$\chi'_t(G) \leq \frac{\Delta+1}{2} + \mathcal{O}(\sqrt{\Delta}). \tag{1}$$

On the other hand, if, like in most of our explicit labellings, the typical total k -labellings use on the vertices almost only the labels 1 and k , then the reduced upper bound in the formula above is tight in view of the Paley graphs.

So the main open question in this context might be the following:

Problem 4.1 *Is there a constant K with*

$$\chi'_t(G) \leq \frac{\Delta + 1}{2} + K \quad (2)$$

for all graphs G of maximum degree Δ ?

Surely there are further options except (1) and (2). Indications could be obtained from an answer to the question whether all graphs G have an edge-colouring total $\chi'_t(G)$ -labelling with only few vertex labels.

In view of the graphs where we know the exact value of $\chi'_t(G)$, the constant K must be at least 1. With $K = 1$ the bound (2) is attained with equality e.g. for cubic snarks and K_{4k+2} for $1 \leq k \leq 5$. For even Δ we are not aware of any graph with $\chi'_t(G) > \lceil \frac{\Delta+1}{2} \rceil$. One first question in this direction is whether $\chi'_t(G) = 3$ for all graphs with $\Delta = 4$. As a potential candidate for the general problem we checked the unique Paley graph on 17 vertices ($\Delta = 8$), which is at the same time the (4, 4)-Ramsey graph, with a computer program, that came up with an edge-colouring total 5-labelling.

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