# The Independence Number in Graphs of Maximum Degree Three

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**Abstract.** We prove that a $K_4$-free graph $G$ of order $n$, size $m$ and maximum degree at most three has an independent set of cardinality at least $\frac{1}{7}(4n - m - \lambda - tr)$ where $\lambda$ counts the number of components of $G$ whose blocks are each either isomorphic to one of four specific graphs or edges between two of these four specific graphs and $tr$ is the maximum number of vertex-disjoint triangles in $G$. Our result generalizes a bound due to Heckman and Thomas (A New Proof of the Independence Ratio of Triangle-Free Cubic Graphs, *Discrete Math.* **233** (2001), 233-237).

**Keywords.** independence; triangle; cubic graph

We consider finite simple and undirected graphs $G = (V, E)$ of order $n(G) = |V|$ and size $m(G) = |E|$. The independence number $\alpha(G)$ of $G$ is defined as the maximum cardinality of a set of pairwise non-adjacent vertices which is called an independent set.

Our aim in the present note is to extend a result of Heckman and Thomas [6] (cf. Theorem 1 below) about the independence number of triangle-free graphs of maximum degree at most three to the case of graphs which may contain triangles. With their very insightful and elegant proof, Heckman and Thomas also provide a short proof for the result conjectured by Albertson, Bollobás and Tucker [1] and originally proved by Staton [9] that every triangle-free graph $G$ of maximum degree at most three has an independent set of cardinality at least $\frac{5}{14}n(G)$ (cf. also [7]). (Note that there are exactly two connected graphs for which this bound is best-possible [2, 3, 5, 8] and that Fraughnaugh and Locke [4] proved that every cubic triangle-free graph $G$ has an independent set of cardinality at least $\frac{11}{30}n(G) - \frac{2}{15}$ which implies that, asymptotically, $\frac{5}{14}$ is not the correct fraction.)

In order to formulate the result of Heckman and Thomas and our extension of it we need some definitions.

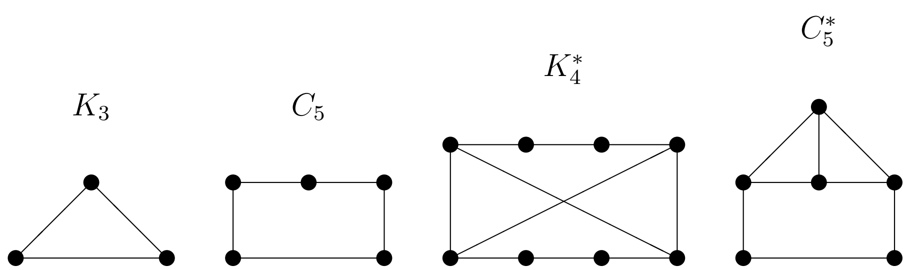


**Figure 1.** Difficult blocks.

A block of a graph is called *difficult* if it is isomorphic to one of the four graphs $K_3$, $C_5$, $K_4^*$ or $C_5^*$ in Figure 1, i.e., it is either a triangle, or a cycle of length five, or arises by subdividing two independent edges in a $K_4$ twice, or arises by adding a vertex to a $C_5$ and joining it to three consecutive vertices of the $C_5$. A connected graph is called *bad* if its blocks are either difficult or are edges between difficult blocks.

For a graph $G$ we denote by $\lambda(G)$ the number of components of $G$ which are bad and by $tr(G)$ the maximum number of vertex-disjoint triangles in $G$. Note that for triangle-free graphs $G$ our definition of $\lambda(G)$ coincides with the one given by Heckman and Thomas [6]. Furthermore, note that $tr(G)$ can be computed efficiently for a graph $G$ of maximum degree at most three as it equals exactly the number of non-trivial components of the graph formed by the edges of $G$ which lie in a triangle of $G$.

**Theorem 1 (Heckman and Thomas [6])** *Every triangle-free graph $G$ of maximum degree at most three has an independent set of cardinality at least $\frac{1}{7}(4n(G) - m(G) - \lambda(G))$.*

Since every $K_4$ in a graph of maximum degree at most three must form a component and contributes exactly one to the independence number of the graph, we can restrict our attention to graphs that do not contain $K_4$'s.

**Theorem 2** *Every $K_4$-free graph $G$ of maximum degree at most three has an independent set of cardinality at least $\frac{1}{7}(4n(G) - m(G) - \lambda(G) - tr(G))$.*

*Proof:* For a graph $G$ we denote the quantity $4n(G) - m(G) - \lambda(G) - tr(G)$ by $\psi(G)$. We wish to show that $7\alpha(G) \geq \psi(G)$. For contradiction, we assume that $G = (V, E)$ is a counterexample to the statement such that $tr(G)$ is smallest possible and subject to this condition the order $n(G)$ of $G$ is smallest possible. If $tr(G) = 0$, then the result follows immediately from Theorem 1. Therefore, we may assume $tr(G) \geq 1$. Since $\alpha(G)$ and $\psi(G)$ are additive with respect to the components of $G$, we may assume that $G$ is connected. Furthermore, we may clearly assume that $n(G) \geq 4$.

**Claim 1.** Every vertex in a triangle has degree three.

*Proof of Claim 1:* Let $x$, $y$ and $z$ be the vertices of a triangle. We assume that $d_G(x) = 2$. Clearly, the graph $G' = G[V \setminus \{x, y, z\}]$ is no counterexample, i.e., $7\alpha(G') \geq \psi(G')$. Since for every independent set $I'$ of $G'$, the set $I' \cup \{x\}$ is an independent set of $G$, we have $\alpha(G) \geq \alpha(G') + 1$. The triangle $xyz$ is vertex-disjoint from all triangles in $G'$, and so $tr(G) \geq tr(G') + 1$.

Suppose $\min\{d_G(y), d_G(z)\} = 2$. Then $\max\{d_G(y), d_G(z)\} = 3$, since $G$ is not just a triangle. Furthermore, by the definition of a bad graph, we have $\lambda(G') = \lambda(G)$ and obtain

$$\begin{aligned} 7\alpha(G) &\geq 7\alpha(G') + 7 \\ &\geq \psi(G') + 7 \\ &= 4n(G') - m(G') - \lambda(G') - tr(G') + 7 \\ &\geq 4(n(G) - 3) - (m(G) - 4) - \lambda(G) - (tr(G) - 1) + 7 \\ &\geq \psi(G) - 12 + 4 + 1 + 7 \\ &= \psi(G), \end{aligned}$$

which implies a contradiction. Therefore, we may assume $d_G(y) = d_G(z) = 3$. Let $N_G(y) = \{x, y', z\}$ and $N_G(z) = \{x, y, z'\}$. Regardless of whether $y' = z'$ or not, we have $tr(G) \geq tr(G') + 1$.

If $y' = z'$, then $G'$ is connected, $y'$ is a vertex of degree one in $G'$ and thus $\lambda(G') = \lambda(G) = 0$. If $y' \neq z'$ and $\lambda(G') \geq 2$, then $\lambda(G') = 2$ and $G$ is a bad graph itself, i.e., $\lambda(G) = 1$. Therefore, in both cases $\lambda(G') \leq \lambda(G) + 1$ and we obtain

$$\begin{aligned} 7\alpha(G) &\geq 7\alpha(G') + 7 \\ &\geq \psi(G') + 7 \\ &= 4n(G') - m(G') - \lambda(G') - tr(G') + 7 \\ &\geq 4(n(G) - 3) - (m(G) - 5) - (\lambda(G) + 1) - (tr(G) - 1) + 7 \\ &\geq \psi(G) - 12 + 5 - 1 + 1 + 7 \\ &= \psi(G), \end{aligned}$$

which implies a contradiction and the proof of the claim is complete. $\square$

**Claim 2.** No two triangles of $G$ share an edge, i.e., $G$ does not contain $K_4 - e$.

*Proof of Claim 2:* Let $x$, $y$, $y'$ and $z$ be such that $xyy'$ and $yy'z$ are triangles. Let $G' = G[V \setminus \{y'\}]$. Clearly, $\alpha(G) \geq \alpha(G')$, $tr(G) \geq tr(G') + 1$ and $G'$ is connected. Note that, by Claim 1, both $x$ and $z$ have degree 3 in $G$ and thus $x$, $y$ and $z$ are all of degree 2 in $G'$.

If $G'$ is bad, then $x$, $y$ and $z$ are three consecutive vertices in a block of $G'$ isomorphic to $C_5$. Since the corresponding block in $G$ is isomorphic to $C_5^*$, the graph $G$ is also bad. Conversely, if $G$ is bad, then $x$, $y$, $y'$ and $z$ belong to a block of $G$ isomorphic to $C_5^*$. Since the corresponding block in $G'$ is isomorphic to $C_5$, the graph $G'$ is also bad.

Therefore, $\lambda(G') = \lambda(G)$ and we obtain

$$\begin{aligned}
7\alpha(G) &\geq 7\alpha(G') \\
&\geq \psi(G') \\
&= 4n(G') - m(G') - \lambda(G') - tr(G') \\
&\geq 4(n(G) - 1) - (m(G) - 3) - \lambda(G) - (tr(G) - 1) \\
&\geq \psi(G) - 4 + 3 + 1 \\
&= \psi(G),
\end{aligned}$$

which implies a contradiction and the proof of the claim is complete. □

Note that, by Claim 2, adding an edge to a subgraph of $G$ cannot create a $K_4$.

Let $xyz$ be a triangle in $G$. By Claim 1, we have $N_G(x) = \{x', y, z\}$, $N_G(y) = \{x, y', z\}$ and $N_G(z) = \{x, y, z'\}$ and, by Claim 2, $x'$, $y'$ and $z'$ are all distinct. Let $G' = G[V \setminus \{x, y, z\}]$.

**Claim 3.** The set $\{x', y', z'\}$ is independent.

*Proof of Claim 3:* For contradiction, we assume that $x'y' \in E$. For every independent set $I'$ of $G'$ either $I' \cup \{x\}$ or $I' \cup \{y\}$ is an independent set of $G$ which implies $\alpha(G) \geq \alpha(G') + 1$. Since $G'$ has at most two components, we have $\lambda(G') \leq \lambda(G) + 2$. Furthermore, $n(G') = n(G) - 3$, $m(G') = m(G) - 6$, $tr(G) \geq tr(G') + 1$ and we obtain a similar contradiction as before which completes the proof of the claim. □

**Claim 4.** There are two edges $e$ and $f$ in $\{x'y', y'z', x'z'\}$ such that $\lambda(G' + e) \leq \lambda(G) + 1$ and $\lambda(G' + f) \leq \lambda(G) + 1$.

*Proof of Claim 4:* For contradiction, we assume that $\lambda(G' + x'y') \geq \lambda(G) + 2$. This implies that $G'$ consists exactly of two bad components and that $G$ itself is not a bad graph. Hence $x'y'$ can not be an edge between two difficult blocks, since otherwise $G$ would be a bad graph. Thus both $G' + x'z'$ and $G' + y'z'$ are connected and the claim follows for $\{e, f\} = \{x'z', y'z'\}$. □

**Claim 5.** If $\lambda(G'+e) = \lambda(G'+f) = \lambda(G)+1$, then either $tr(G'+e) \leq tr(G)-1$ or $tr(G'+f) \leq tr(G)-1$.

*Proof of Claim 5:* We may assume that $e = x'z'$ and $f = y'z'$. For contradiction, we assume that $tr(G'+e), tr(G'+f) \geq tr(G)$. This implies that $x'$ and $z'$ have a common neighbour $x''$ in $G'$ and that $y'$ and $z'$ have a common neighbour $y''$ in $G'$. If possible, we choose $x'' = y''$. Clearly, this implies that $G'$ is connected. Furthermore, since the vertices $x, y, z, x', y', z', x'', y''$ all lie in one block of $G$ which cannot be a bad block, the graph $G$ can not be a bad graph. Since $\lambda(G'+e) = \lambda(G'+f) = \lambda(G)+1$, both $G'+e$ and $G'+f$ must be bad graphs.

If the triangle $x'z'x''$ forms a difficult block in $G'+e$, the edge $x'x''$ forms a block in $G'+f$ which does not connect two difficult blocks. This implies that $G'+f$ can not be bad which is a contradiction. Therefore, by symmetry, we may assume that the triangle $x'z'x''$ is contained in a difficult block $B_e$ in $G'+e$ which is isomorphic to $C_5^*$ and that also the triangle $y'z'y''$ is contained in a difficult block $B_f$ in $G'+f$ which is isomorphic to $C_5^*$.

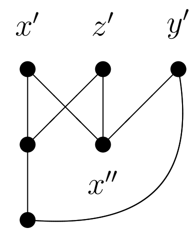


**Figure 2**

First, we assume $x'' = y''$. If $e = x'z'$ is not the edge shared by the two triangles of $B_e$, then either $x'$ and $x''$ or $z'$ and $x''$ have a common neighbour in $G'$. This implies that $y'$ is adjacent to either $x'$ or $z'$ which contradicts Claim 3. Hence the edge $e = x'z'$ must be the edge shared by the two triangles of $B_e$. Now, $G'$ contains the configuration shown in Figure 2. Clearly, all six vertices in Figure 2 belong to one block of $G'+f$ which can not be a difficult block. Therefore, $G'+f$ can not be a bad graph which is a contradiction.

Next, we assume that $x'' \neq y''$. By the choice of $x''$ and $y''$, this implies that no vertex in $G'$ is adjacent to all of $x'$, $y'$ and $z'$. If $e = x'z'$ is the edge shared by the two triangles of $B_e$, then $x'$ and $z'$ must have a common neighbour in $G'$ different from $x''$. This implies that $y''$ is adjacent to all of $x'$, $y'$ and $z'$ which is a contradiction. Hence $x'z'$ is not the edge shared by the two triangles of $B_e$. If $x'x''$ is the edge shared by the two triangles of $B_e$, then the block of $G'+f$ which contains $x'$ contains two vertex-disjoint triangles. Therefore, $G'+f$ can not be a bad graph which is a contradiction. We obtain that $z'x''$ is the edge shared by the two triangles of $B_e$ which implies the existence of a vertex $z''$ such that $G$ contains the configuration shown in Figure 3.

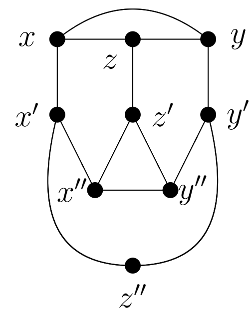


**Figure 3**

Since $G[\{x, y, z, x', y', z', x'', y'', z''\}]$ is not a counterexample, the vertex $z''$ has degree three. Now the graph $G'' = G[V \setminus \{x, y, z, x', y', z', x'', y'', z''\}]$ satisfies $\alpha(G) \geq \alpha(G'') + 3$, $n(G) = n(G'') + 9$, $m(G) = m(G'') + 14$, $\lambda(G'') \leq \lambda(G) + 1$ and $tr(G) \geq tr(G'') + 2$ which implies a similar contradiction as before and completes the proof of the claim. $\Box$

Note that $tr(G' + x'z') \leq tr(G') + 1 = tr(G)$. Therefore, by Claims 4 and 5, we can assume that either $\lambda(G' + x'z') \leq \lambda(G)$ and $tr(G' + x'z') \leq tr(G)$ or $\lambda(G' + x'z') = \lambda(G) + 1$ and $tr(G' + x'z') \leq tr(G) - 1$ both of which imply that $\lambda(G' + x'z') + tr(G' + x'z') \leq \lambda(G) + tr(G)$. Similarly as above, for every independent set $I'$ of $G' + x'z'$ either $I' \cup \{x\}$ or $I' \cup \{z\}$ is an independent set of $G$ which implies $\alpha(G) \geq \alpha(G') + 1$. Since $n(G' + e) = n(G) - 3$ and $m(G' + e) = m(G) - 5$, we obtain a similar contradiction as above which completes the proof. $\Box$

Note that Theorem 2 is best-possible for all bad graphs, all graphs which arise by adding an edge to a bad graph and further graphs such as for instance the graph in Figure 4.

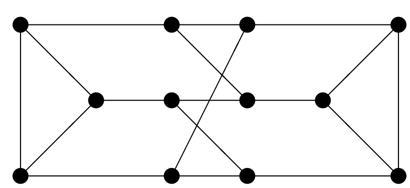

**Figure 4**

In [5] Heckman characterized the extremal graphs for Theorem 1. Similarly, it might be an interesting yet challenging task to characterize the extremal graphs for Theorem 2.

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