# Precoloring extension for $K_4$-minor-free graphs

Anja Pruchnewski

Institut für Mathematik, TU Ilmenau, PF 10 05 65, D-98684 Ilmenau

Margit Voigt

Fachbereich Informatik/Mathematik, Hochschule für Technik und Wirtschaft Dresden (FH), Friedrich-List-Platz 1, D-01069 Dresden

November 5, 2007

**Abstract**

Let $G = (V, E)$ be a graph where every vertex $v \in V$ is assigned a list of available colors $L(v)$. We say that $G$ is list colorable for a given list assignment if we can color every vertex using its list such that adjacent vertices get different colors. If $L(v) = \{1, \ldots, k\}$ for all $v \in V$ then a corresponding list coloring is nothing other than an ordinary $k$-coloring of $G$. Assume that $W \subseteq V$ is a subset of $V$ such that $G[W]$ is bipartite and each component of $G[W]$ is precolored with two colors. The minimum distance between the components of $G[W]$ is denoted by $d(W)$. We will show that if $G$ is $K_4$-minor-free and $d(W) \geq 7$, then such a precoloring of $W$ can be extended to a 4-coloring of all of $V$. This result clarifies a question posed in [10]. Moreover, we will show that such a precoloring is extendable to a list coloring of $G$ for outerplanar graphs, provided that $|L(v)| = 4$ for all $v \in V \setminus W$ and $d(W) \geq 7$. In both cases the bound for $d(W)$ is best possible.

**Keywords:** precoloring extension, list coloring, minor-free graphs

# 1 Introduction

Precoloring problems for special graph classes and different types of precolored subsets are studied by many authors, including [1]-[8], [10],[11],[13],[14]. Let $G = (V, E)$ be a simple graph and $W \subseteq V$ a precolored subset of the vertex set. The shortest distance between components of $W$ in $G$ is denoted by $d(W)$. Obviously, $d(W)$ influences the extendability of the precoloring of $W$ to all of $G$. Thus we ask for bounds for $d(W)$ such that a precoloring of $W$ can be extended to a proper coloring of $V$ using a given number of colors. An analogous problem for list colorings, introduced independently by Vizing [12] and Erdős, Rubin and Taylor [9] at the end of the 1970s, can also be investigated. In a graph with list coloring, every vertex is assigned a list of available colors $L(v)$ and the graph's vertices are colored in such a way that each vertex is colored with a color from its list and adjacent vertices are colored differently.
Assume that $G$ is a simple graph with chromatic number $\chi(G) = k$. The first result on precoloring extension that takes $d(W)$ into consideration was publicized by Albertson [1]. If $W \subseteq V$ is an independent set and $d(W) \geq 4$, then every $(k+1)$-coloring of $W$ can be extended to a proper $(k+1)$-coloring of $V$. This result can be generalized. If $G[W]$ is an $s$-colorable graph where every component is $s$-colored and $d(W) \geq 4$, then every $(k + s)$-coloring of $W$ can be extended to a proper $(k + s)$-coloring of $V$ [3]. Clearly, the bound for $d(W)$ will grow if we use fewer colors. If $W$ is the union of complete graphs $K_s$ and $d(W) \geq 4s$, then any $(k + 1)$-coloring of $G[W]$ is extendable to a $(k + 1)$-coloring of $V$ by a result of Kostochka proved in [6].
Consider the bounds for $d(W)$ if we use $k + s - 1$ colors. Hutchinson and Moore [10] showed that no distance can insure a color extension with $k+s-1$ colors without topological constraints. They found several results for special graph classes in [10]. Let $G$ be a simple graph with $\chi(G) = k$ and without a minor $K_{k+1}$ and let $G[W]$ be $s$-colorable with every component of $G[W]$ precolored with $s$ colors.

| $k \backslash s$ | 2 | 3 | 4 | 5 |
|---|---|---|---|---|
| 2 | 3 | – | – | – |
| 3 | 7, 8 | 5 | – | – |
| 4 | 7, 8 | 7 | 6 | – |
| 5 | 7, 8 | 7, 8 | 7 | 6 |

The above table shows bounds and exact values, respectively, for a smallest

$d$ such that $d(W) \geq d$ ensures the required coloring extension. Albertson and Moore [7] found several results for $s = 1$.
For the list coloring version of this problem, some results in the style of Brook's theorem are known. Axenovich [8] and Albertson, Kostochka and West [5] independently showed that for $|L(v)| = \Delta \geq 3\ \forall v \in V$ and $d(W) \geq 8$ every list coloring of an independent set $W$ extends to a list coloring of $V$. If $G$ is 2-connected, then $d(W) \geq 4$ for $\Delta(G) \geq 4$ [13] and $d(W) \geq 6$ for $\Delta(G) = 3$ [14] ensure the extendability of such colorings.
In this paper, we will take a closer look at the case with $k = 3$ and $s = 2$ from the above table. This means that $G$ is a $K_4$-minor-free graph which is always 3-colorable and every component of $G[W]$ is bipartite and precolored with two colors. We can see in the table that $d(W) \geq 8$ ensures that the precoloring is extendable to a $3+2-1 = 4$-coloring of all of $G$. On the other hand, $d(W) \geq 6$ is not sufficient for such an extension, which is shown by an example of Kostochka [11] (see Fig. 1).

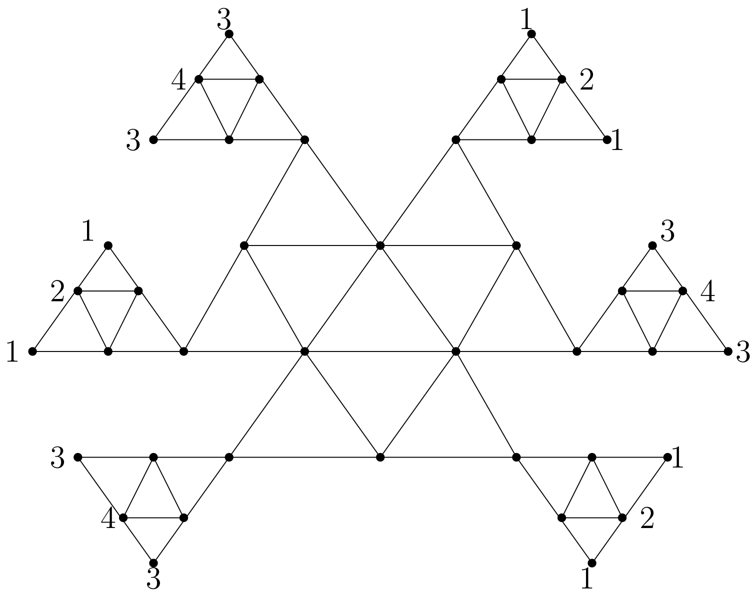


Figure 1: $d(W) = 6$; an extension to a 4-coloring is not possible

To see that the coloring is not extendable, color the central triangle arbitrarily and try to extend the coloring in a proper way.
We will prove in this paper that $d(W) \geq 7$ is sufficient to ensure the extension of the coloring in our case.

Moreover, we investigate the list coloring version of this problem for outerplanar graphs ($K_4$- and $K_{23}$-minor-free graphs). For technical reasons, we start with the results of the list coloring version of our problem.

**Theorem 1** *Let $G$ be an outerplanar graph, $W \subseteq V(G)$ and $G[W]$ bipartite. Furthermore, assume that the shortest distance $d(W)$ between components of $G[W]$ is at least 7. If every vertex of $G \setminus W$ has a list of at least 4 colors, then any precoloring of $G[W]$ which colors every component of $G[W]$ using two colors can be extended to a proper list coloring of all of $G$.*

**Theorem 2** *Let $G$ be a $K_4$ minor-free graph, $W \subseteq V(G)$, and $G[W]$ bipartite. Furthermore, assume that the shortest distance $d(W)$ between components of $G[W]$ is at least 7. Then any proper 4-coloring of $G[W]$ which colors every component of $G[W]$ using two colors can be extended to a 4-coloring of all of $G$.*

Given the example above, it is clear that the bound for $d(W)$ is best possible in both theorems.
The idea behind the proofs of these theorems is taken from the previous example, where the second neighborhood of every precolored component is a single vertex. We start by coloring the first neighborhood of the precolored components and examine the remaining graph $G^*$ after removing these vertices along with the precolored vertices.
In the second section, we will prove a lemma containing a coloring algorithm which will be used for the proofs of both theorems. In both cases, this algorithm realizes the coloring of $G^*$. Theorem 1 is proved in the third section, showing that for this case the remaining graph $G^*$ fulfils the assumption of Lemma 3. For the proof of Theorem 2 in Section 4, we have to construct a special coloring of the first neighborhood to apply Lemma 3 again.

# 2 A Coloring Algorithm

Let $G$ be a simple graph where every vertex $v$ has a list $L(v)$ of available colors. A vertex with a list of cardinality $i$ is called an $Li$ vertex. We consider the following types of vertex subsets:

- Type 1: an $L1$ vertex $z_1$ or a pair $z_1, z_2$ of adjacent $L1$ vertices with different colors,

- Type 2: a single $L2$ vertex,
- Type 3: a pair $x, y$ of adjacent vertices, where $x$ is an $L2$ vertex and $y$ is an $L3$ vertex,
- Type 4: a tree $T = (V_T, E_T)$, where exactly one of the vertices is an $L2$ vertex and the other vertices are $L3$ vertices and there is no path between two nonadjacent vertices of $T$ in $G$ outside of $T$.

The union of all subsets of Type 1, 2, 3, and 4 is denoted by $S$. A vertex belonging to a subset of Type $i$ is called a $t_i$-vertex.

**Lemma 3** *Let $G = (V, E)$ with $|V| = n$ be a connected $K_4$-minor-free graph where every vertex is assigned a list $L(v)$ of available colors. Let $S$ be a subset of $V$ containing subsets of Type 1, 2, 3 and 4 and $|L(v)| \in \{3, 4\}$ for all $v \in V \setminus S$. Furthermore we assume that*

1. *there is at most one subset of Type 1 in $S$,*
2. *the distance between two different subsets of $S$ in $G$ is at least 3, and*
3. *$|L(w)| = 4$ for all $w \in V \setminus S$ adjacent to a vertex of $S$.*

*Then $G$ is list colorable.*

**Proof.**
We will prove this lemma by induction on the number of subsets of Type 4. If there is no subset of Type 4 then do the following
**Algorithm:**
Since $G$ and all of its subgraphs are $K_4$-minor-free, each of them contains at least two non-adjacent vertices of degree at most 2 [15].

1. Determine an order $v_1, \ldots, v_n$ for the vertices of $V$ such that $v_i$ has degree at most 2 in $G[v_{i+1}, \ldots, v_n]$.

   If there is a $t_1$-vertex $z_1$ in $G$, then choose $v_n = z_1$. If there is a second $t_1$-vertex $z_2$ in $G$ (which is adjacent to $z_1$ by definition), then choose $v_{n-1} = z_2$.

2. If $v_i = x$ is a $t_2$-vertex and there are two neighbors, say $v_j, v_s \in \{v_{i+1}, \dots, v_n\}$ with $i < j < s$, then reorder the vertices by moving $x$ behind $v_j$: $v_k = v_{k+1}$ for $k = i, \dots, j-1$ and $v_j = x$.

   This reordering is shown in Figure 2. Note that only two edges of the graph ($xv_j$ and $xv_s$ in the original order) are given explicitly. A dashed line means that we do not know whether the vertices are adjacent or not.

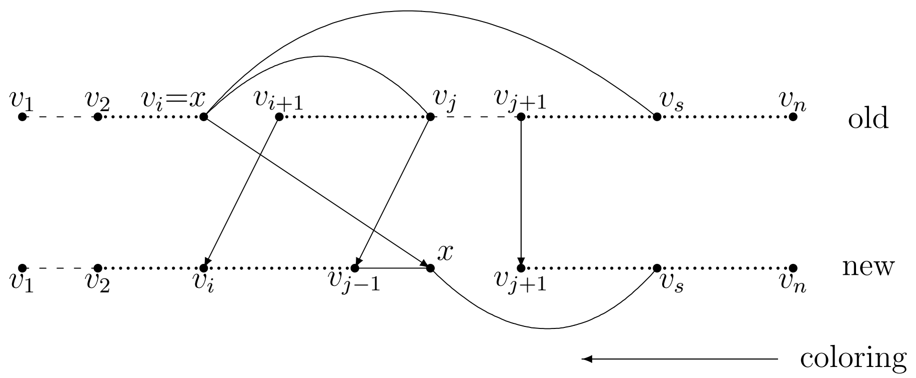


Figure 2: Reordering if $|L(x)| = 2$ and $|L(v_j)| = 4$

   The new vertex $v_{j-1}$ is an $L4$ vertex and has at most three neighbors in $v_j, \dots, v_n$ regarding the new order. For all other vertices in the ordering, the number of neighbors in the graph induced by the vertices of higher index in the ordering does not change.

3. Let $v_i = x$ be a $t_3$-$L2$-vertex and $v_j, v_s \in \{v_{i+1}, \dots, v_n\}$ two neighbors with $i < j < s$.

   If $v_j$ is an $L4$ vertex, reorder analogously to 2.

   If $v_j = y$ is the corresponding $t_3$-$L3$-vertex, then move $y$ to the end of the ordering: $v_k = v_{k+1}$ for $k = j, \dots, n-1$ and $v_n = y$. This reordering is shown in Figure 3.

   If $v_\ell \neq x$ is a neighbor of $y$, then $v_\ell$ is an $L4$-vertex. If $\ell > j$ then the number of $v_\ell$'s neighbors of higher index increases by one to at most three after moving $y$.

Observation after the reordering:

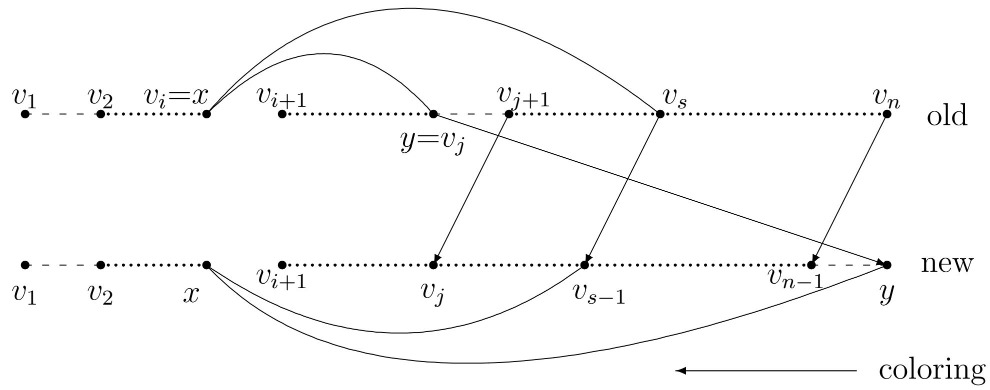


Figure 3: Reordering if $|L(x)| = 2$ and $|L(v_j)| = 3$

- Every $L4$ vertex has at most three neighbors in the graph induced by the vertices of higher index since it has at most one ”reordered neighbor” according to the lemma's second assumption.
- Every $L3$ vertex has at most two neighbors in the graph induced by the vertices of higher index.
- Every $L2$ vertex $x$ has either one neighbor in the graph induced by the vertices of higher index, or two neighbors such that one of them is the $L3$ vertex $y$ belonging to $x$. In the last case, $y$ is one of the last vertices of the ordering and may be followed by some other reordered $L3$ vertices.
- $z_1$ has no neighbors in the graph induced by the vertices of higher index. If the set of vertices of higher index is not empty, then its elements are reordered $L3$ vertices which are not adjacent to $z_1$, due to the lemma's assumption.
  $z_2$ has only one neighbor in the graph induced by the vertices of higher index, namely $z_1$.

4. Color the vertices in the reverse order $v_n, \ldots, v_1$:
   - Let $v_n, v_{n-1}, \ldots, v_i$ be reordered $t_3$-$L3$-vertices. Note that these vertices build an independent set in $G$ because of the lemma's assumption. Color each of these vertices with a color that does not belong to the list of its corresponding $t_3$-$L2$-vertex.

- Color the remaining vertices with an arbitrary available color which has not been used to color its previously colored neighbors.

Now assume that there are subsets of Type 4.
Let $T = (V_T, E_T)$ be a Type 4 subset. If $V(G) = V_T$, then we are done coloring the tree.
Otherwise, note that every component of $G - T$ is adjacent to at most two vertices of $T$ since there are no paths between two non-adjacent vertices of $T$ outside of the tree. If such a component is adjacent to two vertices of $T$, then the vertices are adjacent in $T$ for the same reason. At most one of these components has a $t_1$ vertex or a $t_1$ pair. This component, together with the adjacent vertex/vertices of $T$, is denoted by $G'$. If there is no component with $t_1$-vertex/vertices, choose an arbitrary component and add the corresponding vertices of $T$ to obtain $G'$.
Color $G'$ first. This is possible because $G'$ has fewer Type 4 subsets than $G$ and fulfils the lemma's assumption. Next, color the remaining vertices of $T$. Note that each of the remaining components, together with the corresponding adjacent (and already colored) vertex/vertices of $T$, fulfils the lemma's assumption (having one $t_1$ vertex or one $t_1$ pair belonging to T), and again has fewer subsets of Type 4 than $G$. Therefore, we can color these components as well. □

# 3 List Coloring for Outerplanar Graphs

We will use the following notations throughout the rest of the paper.

- The components of $G[W]$ are denoted by $C_1, \ldots, C_r$, the first neighborhoods by $F_1, \ldots, F_r$, and the corresponding second neighborhoods by $S_1, \ldots, S_r$.

- Let $G^*$ be the graph that remains if we delete $C_1, \ldots, C_r$ and $F_1, \ldots, F_r$ from $G$.

- Let $G_j$, $j = 1, \ldots, s$ be the components of $G^*$.

- $S_{ij} = S_i \cap V(G_j)$.

- If $x \in S_{ij}$ has at least k neighbors in $F_i$ ,then $x$ is called a $k^+$-vertex. If $x \in S_{ij}$ has exactly k neighbors in $F_i$, then $x$ is called a $k$-vertex.

**Proof of Theorem 1.**

Note that an outerplanar graph is $K_4$- and $K_{23}$-minor-free. Thus, such a graph has no subdivision of $K_4$ or $K_{23}$.

First we shall prove the following property:

If $G$ is an outerplanar graph, then every $S_{ij}$ is a collection of vertices where either all vertices are 1-vertices or there is exactly one 2-vertex $x$ (the others being 1-vertices). In this case, the component of $G[S_{ij}]$ containing $x$ is a tree $T = (V_T, E_T)$ and there is no path in $G$ between nonadjacent $v_i, v_j \in V_T$ outside of $T$.

- If $S_{ij}$ has a $3^+$ vertex, then there is a subdivision of $K_{23}$ which is forbidden in an outerplanar graph.
- If there are at least two 2 vertices, then we can always find a subdivision of $K_4$ or $K_{23}$, which contradicts the graph's outerplanarity. Three of these configurations are given in Figure 4.

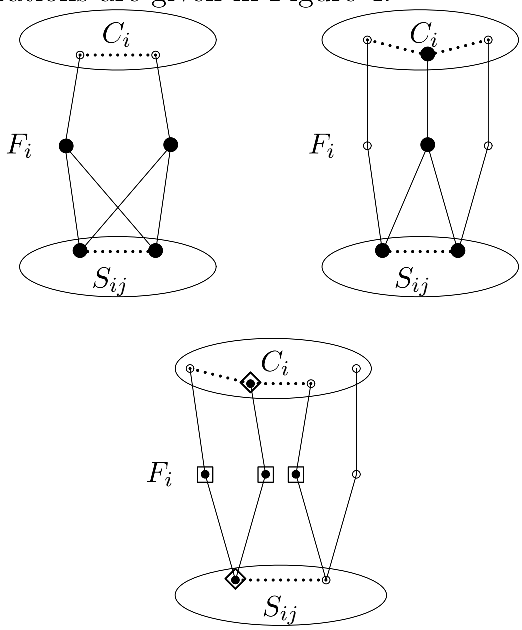


Figure 4: configurations with subdivisions of $K_4$ or $K_{2,3}$

- If there is no 2-vertex we are done.

We may assume that there is exactly one 2-vertex $x$ and the rest are 1-vertices. Moreover, the component of $G^*[S_{ij}]$ containing $x$ cannot contain a

cycle since there would otherwise be a subdivision of $K_4$ in $G$. Therefore, such a component is a tree.
If such a tree $T = (V_T, E_T)$ has more than two vertices, then there are no paths between nonadjacent $v_i, v_j \in V_T$ in $G^*$ because there would otherwise be a subdivision of $K_4$ in $G$, since every vertex of $V_T$ has a path to $C_i$.
We are now ready to extend the coloring. Note that for all $i$ the subgraph $G[F_i]$ does not contain a cycle since there would otherwise be a subdivision of $K_4$ in $G$. Thus, every $G[F_i]$ is a forest and every vertex has at least two available colors if we delete the colors used for its neighbors in $C_i$. We are therefore able to color the vertices of $F_i$ properly with respect to $C_i$ for all $i$ and delete the used colors from the lists of the corresponding neighbors in $S_i$. It follows that a 2-vertex of $S_i$ now has at least two available colors and a 1-vertex of $S_i$ now has at least three available colors.
Every component $G_j$ of $G^*$ now fulfills the assumption of Lemma 3. The distance constraints of the assumption are also satisfied because of $d(W) \geq 7$. Thus, according to Lemma 3, we can extend the coloring. □

# 4 Ordinary Coloring for $K_4$-minor-free Graphs

**Proof of Theorem 2.**
We will first find a coloring for the vertices of $F_i$ for all $i$ and forbid the used colors for the coloring of the neighbors. Then every vertex in $S_{ij}$ has a set (list) of some available colors.
We will prove the following claim:
**Claim:** *We can color the $F_i$'s in such a way that every $S_{ij}$ contains at most one vertex with two available colors and all other vertices have at least three available colors.*
**Proof.**

- *$F_i$ is a forest for every $i$.*

  Assume the contrary. Then there is a cycle in $F_i$ which implies, together with a suitable vertex in $C_i$, a subdivision of $K_4$ (and therefore a $K_4$-minor), which leads to a contradiction.

- *Every vertex in $S_{ij}$ has at least two available colors.*

  Assume w.l.o.g. that $C_i$ is colored by $1, 2$. We use the colors 3 and 4 for the coloring of $F_i$. Thus, every vertex in $S_i$ still has at least two available colors, namely 1 and 2.

If $S_{ij}$ contains only one vertex with at least two neighbors in $F_i$, we are done. Thus, we may assume that there is an $S_{ij}$ with at least two $2^+-$vertices, say $v_1, v_2$.

- *$v_1$ and $v_2$ do not have a common neighbor in $F_i$.*

  Assume the contrary. Let $x_1, x_2$ be two neighbors of $v_1$ and $x_3, x_4$ be two neighbors of $v_2$ in $F_i$. If $v_1$ and $v_2$ have two common neighbors in $F_i$, that is $x_1 = x_3$ and $x_2 = x_4$, then we immediately find a subdivision of $K_4$. If $v_1$ and $v_2$ have exactly one common neighbor in $F_i$, say $x_2 = x_3$, we can also find a subdivision of $K_4$ ($v_1, v_2, x_2$ and a suitable vertex of $C_i$).

Let $x_1, x_2$ be two neighbors of $v_1$ and $x_3, x_4$ be two neighbors of $v_2$ in $F_i$.

- *Every path from $x_1$ to $x_2$ in $G$ contains vertices of $G_j$ or $C_i$.*

  If there is a path without vertices of $G_j$ and $C_i$, then we can find a subdivision of $K_4$.

- *Every path from $x_2$ to $x_3$ contains vertices of $G_j$ or $C_i$.*

  If there is a path without vertices of $G_j$ and $C_i$, then we can find a subdivision of $K_4$.

If an $S_{ij}$ has at least two $2^+-$vertices, then the set of all neighbors of $2^+-$vertices in $F_i$ is denoted by $X_{ij}$. Note that every $X_{ij}$ is an independent set due to the above remarks.
Now, for every $i$ consider the subgraph $H_i := G[F_i]$ of $G$ induced by $F_i$. Construct a new graph $T_i$ from $H_i$ identifying the vertices of each set $X_{ij}$ to a vertex $y_j$.

- *$T_i$ is a forest for every $i$ and does not contain double edges.*

  Assume the contrary. Then there is a cycle (possibly of length 2) in $T_i$. Thus, there is a cycle in $G[V(G^*) \cup F_i]$ in the original graph which contains at least three vertices of $F_i$, since any two vertices of a fixed $X_{ij}$ are joined by a path which uses vertices from the corresponding $G_j$. But these three vertices, together with a suitable vertex of $C_i$, imply a subdivision of $K_4$ in $G$.

**Coloring:** We are now ready for the coloring of $F_i$. First, color the vertices of $T_i$ properly with 3 and 4, assuming that $C_i$ is colored with 1 and 2. Color each of vertices of $X_{ij}$ with the color which is used for the corresponding vertex $y_j \in V(T_i)$. Such a coloring is possible since every $X_{ij}$ is independent. Thus, all vertices of $X_{ij}$ have the same color for a fixed $i$ and $j$. Consequently, if there are multiple $2^+-$vertices in $S_{ij}$, then all of their neighbors in $F_i$ belong to $X_{ij}$ and have the same color. Thus, the coloring of $F_i$ forbids only one color for each of these vertices. This completes the proof of the claim.
Note that every component of $G^*[S_{ij}]$ is a tree $T = (V_T, E_T)$, because we can otherwise find a subdivision of $K_4$ in $G$, which is forbidden. For the same reason, there is no path between nonadjacent vertices of $T$ outside of $T$ in $G^*$. Now every component $G_j$ of $G^*$ fulfills the assumption of Lemma 3. The distance constraints of the assumption are also satisfied because of $d(W) \geq 7$. Thus, by Lemma 3, we can extend the coloring. □

# References


[1] M.O.Albertson, You can't paint yourself into a corner, J. Combinatorial Theory Ser. B, 73 (1998), 189-194

[2] M.O.Albertson, J.P. Hutchinson, Extending colorings of locally planar graphs, J. Graph Theory 36 (2001), 105-116

[3] M.O.Albertson, J.P. Hutchinson, Graph color extensions: when embedding helps, Elect. J. Comb. 9 (2002), # R37

[4] M.O.Albertson, J.P. Hutchinson, Extending precolorings of subgraphs of locally planar graphs, European J. Comb., 25 (2004), 863-871

[5] M.O. Albertson, A.V.Kostochka, D.B.West, Precoloring extension of Brooks' Theorem, SIAM J. Discr. Math. 18 (2004), 542-553

[6] M.O.Albertson, E.H.Moore, Extending graph colorings, J. Combinatorial Theory Ser. B, 77(1999), 83-95

[7] M.O.Albertson, E.H.Moore, Extending graph colorings using no extra colors, Discrete Math. 234 (2001), 125-132

[8] M.Axenovich, A note on graph coloring extensions and list colorings, Electronic J. Comb. 10 (2003) #N1

[9] P.Erdős, A.L.Rubin, H.Taylor, Choosability in graphs, Proc. West Coast Conference on Combinatorics, Graph Theory and Computing (Humboldt State Univ., Arcata, Calif. 1979), Comgress. Numer. XXVI (Utilitas Math., Winnipeg, 1980), 125-157

[10] J.P. Hutchinson, E.H. Moore, Distance Constraints in Graph Color Extensions, manuscript, 2005

[11] J.P. Hutchinson: personal communication, 2005

[12] V.G.Vizing, Coloring the vertices of a graph in presribed colors (Russian), Diskret. Analiz. No. 29 Metody Diskret. Anal. v Teorii Kodov i Shem (1976), 3-10, 101

[13] M. Voigt: Precoloring extension for 2-connected graphs, SIAM J. Discrete Math. 21, 258 (2007), 258-263

[14] M. Voigt: Precoloring extension for 2-connected graphs with maximum degree 3, Discrete Mathematics, submitted

[15] D.B. West: Introduction to Graph Theory, 1996, Prentice Hall