

DAEs - Control and Numerics  
 Exercise Sheet 4 - Stability and optimal control  
*Solutions*

**Exercise 12 (Re-interpretation of variables continued)**

Calculation the ranks and coranks of  $M_i$  we obtain

$$\begin{aligned} \text{rank } M_0 = 2 &\quad \Rightarrow \quad \text{corank } M_0 = 8, \\ \text{rank } M_1 = 11 &\quad \Rightarrow \quad \text{corank } M_1 = 9, \\ \text{rank } M_2 = 20 &\quad \Rightarrow \quad \text{corank } M_2 = 10, \\ \text{rank } M_3 = 30 &\quad \Rightarrow \quad \text{corank } M_3 = 10, \end{aligned}$$

hence  $v_0 = 1, v_1 = 1, v_2 = 0$  and  $a_0 = 7, a_1 = 8, a_2 = 10$ . Let  $Z_{23i}$  be the kernel of  $M_i^\top$  with  $i = 0, 1, 2$ . Then

$$\begin{aligned} \text{rank } Z_{230}^\top N_0 = 8 &\quad \neq \quad a_0 + v_0 = 9, \\ \text{rank } Z_{231}^\top N_1 = 9 &\quad \neq \quad a_1 + v_1 = 10, \\ \text{rank } Z_{232}^\top N_2 = 10 &\quad = \quad a_2 + v_2 = 10, \end{aligned}$$

there  $\mu = 0$  and  $\mu = 1$  is not possible. Let

$$\hat{A}_2 = Z_{232}^\top N_2 = \begin{bmatrix} R & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & R_L & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 1 & 0 & -1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{R_L}{R} & \dot{R}_L & 0 & 0 & 0 \\ 0 & 0 & \frac{R_L}{CL} & 0 & 0 & \frac{L\dot{R}_L - R_L^2}{L^2} & \frac{L\dot{R}_L - R_L\dot{R}_L}{L} & 0 & 0 & 0 \end{bmatrix},$$

then  $\text{rank } \hat{A}_2 = 10$ . Let  $\hat{Z} \in \mathbb{R}^{2 \times 10}$  be the left kernel of the first eight columns of  $\hat{A}_2$ , e.g.

$$\hat{Z} = \begin{bmatrix} R_L^2 & CR_L(R_L^2 - L\dot{R}_L) \\ L\dot{R}_L & L(R_L - CLR\dot{R}_L + CR_L\dot{R}_L) \\ -RR_L^2 & CLR\dot{R}_LR_L - LR_L^2 - CRR_L^3 \\ 0 & LR_L^2 \\ 0 & LR_L^2 \\ R_L^2 & CR_L(R_L^2 - L\dot{R}_L) \\ R_L^2 & CR_L(R_L^2 - L\dot{R}_L) \\ L\dot{R}_L - R_L^2 & LR_L - CR_L^3 - CL^2\dot{R}_L + 2CLR_L\dot{R}_L \\ -LR_L & 0 \\ 0 & CL^2R_L \end{bmatrix}^\top,$$

then  $\begin{bmatrix} I & 0 \\ \hat{Z} & \end{bmatrix} \hat{A}_2 = \begin{bmatrix} A_{22} & B_2 \\ 0 & B_3 \end{bmatrix}$  with invertible  $A_{22} \in \mathbb{R}^{8 \times 8}$  and invertible  $B_3 \in \mathbb{R}^{2 \times 2}$  where

$$B_3 = \begin{bmatrix} RR_L^2 & R_L^2 \\ CRR_L^3 + LR_L^2 - CLR\dot{R}_LR_L & CR_L(R_L^2 - L\dot{R}_L) \end{bmatrix},$$

hence the two input variables  $u_1$  and  $u_2$  must be reinterpreted as a state-variable which have to fulfill the algebraic equation

$$0 = B_3 u + \hat{Z}^\top Z_{232}^\top \begin{pmatrix} f \\ f' \\ f'' \end{pmatrix}.$$

In fact,

$$\begin{aligned} u(t) &= B_3^{-1} \hat{Z} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ f_{10}(t) \\ f'_{10}(t) \\ f''_{10}(t) \end{pmatrix} \\ &= \begin{pmatrix} \frac{R_L(t)^2 - CLR\dot{R}_L(t)R_L(t) + CLR\dot{R}_L(t)^2}{R_L(t)^3} \\ -\frac{RR_L(t)^2 - R_L(t)(L\dot{R}_L(t) + CLR\dot{R}_L(t)) + CLR\dot{R}_L(t)^2}{R_L(t)^3} - 1 \end{pmatrix} f_{10}(t) \\ &\quad + \begin{pmatrix} C - \frac{CLR\dot{R}_L(t)}{R_L(t)^2} \\ -CR - \frac{L(R_L(t) - CRR_L(t))}{R_L(t)^2} \end{pmatrix} f'_{10}(t) + \begin{pmatrix} \frac{CL}{R_L(t)} \\ -\frac{CLR}{R_L(t)} \end{pmatrix} f''_{10}(t) \end{aligned}$$

### Exercise 13 (Stability of DAEs and higher index)

The solution of the homogeneous DAE is given by

$$x(t) = \begin{pmatrix} e^{-tc} \\ 0 \\ e^{-tc} \end{pmatrix}, \quad c \in \mathbb{R},$$

which converges to zero for all  $c \in \mathbb{R}$ . The solution to the inhomogeneous DAE is given by

$$x(t) = \begin{pmatrix} e^{-t}c + \sin(t^2) \\ 2t \cos(t) + \sin(t^2) \\ e^{-t}c + 2t \cos(t) + \sin(t^2) \end{pmatrix}, \quad c \in \mathbb{R},$$

which is unbounded for all  $c \in \mathbb{R}$ .

**Exercise 14 (Optimal control)**

In the notation of the lecture,  $Q = S = R = 1$ , the necessary optimality condition reads as

$$\begin{bmatrix} 0 & 0 & \varepsilon & 0 & 0 \\ 0 & 0 & -1 & \varepsilon & 0 \\ -\varepsilon & 1 & 0 & 0 & 0 \\ 0 & -\varepsilon & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{pmatrix} \dot{x} \\ \dot{\mu} \\ \dot{u} \end{pmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 \end{bmatrix} \begin{pmatrix} x \\ \mu \\ u \end{pmatrix}$$

with boundary condition  $x(0) = x_0$  and  $\begin{bmatrix} \varepsilon & 0 \\ -1 & \varepsilon \end{bmatrix} \mu(1) = 0$ , i.e.  $\mu(1) = 0$ . The last row of the DAE yields

$$u(t) = -x_2(t) - \mu_2(t).$$

Plugging this into the rest of the equations yields

$$\begin{bmatrix} \varepsilon & 0 \\ -1 & \varepsilon \end{bmatrix} \dot{\mu} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix},$$

which together with  $\mu(1) = 0$  yields  $\mu(t) = 0$  for all  $t \in \mathbb{R}$ . Hence the optimal control is given by  $u(t) = -x_2(t) = -y(t)$  independently of  $\varepsilon$ .

For  $\varepsilon = 0$  we get the same control law

$$u(t) = -x_2 + \mu_2(t),$$

but now  $\mu$  is a solution of

$$\begin{bmatrix} 0 & 0 \\ -1 & 0 \end{bmatrix} \dot{\mu} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 \\ -1 & 0 \end{bmatrix} \mu(1) = 0,$$

which implies  $\mu_1(t) = 0$  and  $\mu_2(t)$  can be arbitrary. Plugging this into the necessary condition for  $x$  yields

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \dot{x} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} x + \begin{pmatrix} 0 \\ \mu_2 \end{pmatrix},$$

hence  $\mu_2(t) = 0$  and  $\dot{x}_2 = x_1$ . Therefore the optimal control is given by  $u(t) = -\int_0^t x_1 - x_{2,0}$ , where  $x_1(t)$  can be arbitrary with  $x_1(0) = x_{1,0}$ .

**Exercise 15 (Lyapunov regularity)**

The underlying ODE for the first DAE is  $\dot{x}_1 = -x_1$  which is Lyapunov-regular because its solutions are given by  $e^{-t}c$ ,  $c \in \mathbb{R}$ . However the DAE has the solution  $x_1(t) = e^{-t}c$  and  $x_2(t) = x_1 e^{t \sin(t)} = e^{-t \sin(t)}c$  for  $c \in \mathbb{R}$ . Since

$$\lambda^u = \limsup_{t \rightarrow \infty} \frac{1}{t} \ln(|e^{-t}| + |e^{-t \sin(t)}|) \geq \limsup_{t \rightarrow \infty} \frac{1}{t} \ln |e^{-t \sin(t)}| = 1$$

and

$$\lambda^\ell = \liminf \frac{1}{t} \ln(|e^{-t}| + |e^{-t \sin(t)}|) \leq \liminf \frac{1}{t} \ln(2|e^{-t \sin(t)}|) = -1,$$

hence the DAE is not Lyapunov regular.

The adjoint DAE is given by

$$\begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix} \dot{z} = \begin{bmatrix} 1 & -1 \\ 0 & e^{-t+t \sin(t)} \end{bmatrix},$$

which has the solution  $z_2(t) = 0$  and  $z_1(t) = e^{-t}c$ ,  $c > 0$ . Hence the adjoint DAE is Lyapunov regular.