Quasistatic inflation processes within rigid tubes

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Abstract

In this paper the authors consider mechanical devices that can be seen as segments of an artificial worm or as a balloon for angioplasty. Continuing former work [St 2003] the segment is now placed within a cylindrical or constricted rigid tube that will be touched or pressed during inflation of the segment. Both the segment's shape and the forces of contact are investigated. The main mathematical tool is the Principle of Minimal Potential Energy - handled as an optimal control problem with state constraint. The necessary optimality conditions are carefully analyzed and simulation results for characteristic examples are presented.

The treatment of the problem is primarily mathematical but aiming at application.

Keywords: Optimal control, state constraint, biomechanics.

MSC: 49J15, 49S05, 74F10, 92C10.

1 Introduction

In the following we continue investigations published in [St 2003]. There, the author considered the statical behavior of compliant mechanical elements called "segments", that could be seen as dilatable parts of a worm or of an artificial muscle.

A *segment* has a hull that consists of two parallel rigid circular discs connected by a deformable membrane of circular cylindrical original shape. When this (stress-free) cylinder will be filled with some (incompressible) fluid of fixed volume greater than that of the cylinder the segment deforms into some body of revolution. The membrane enters a state of stress, the discs longitudinally displace, and a hydrostatic pressure arises within the fluid.

Under some working hypotheses concerning the kind of compliance of the membrane (particularly meridional inextensibility) local equilibrium conditions (membrane equations from shell theory) were applied whence an ordinary boundary value problem containing the internal pressure as a parameter was obtained. Then, numerical evaluation of this bvp yielded in particular the shape, volume, expansion, etc. of the segment in dependence of the internal pressure, i.e., a description of the quasistatic inflation process. As a side effect

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the problem of a body of revolution with maximal volume under fixed length of meridian was solved (limit case of infinite pressure). This result showed up to be both interesting and useful.
Among the Conclusions of the above mentioned paper the author emphasized the following *open problem*: Think of the segment centered within a surrounding cylindrical rigid or compliant tube. Under certain geometrical opportunities, the membrane could contact and more and more press against the tube during the inflation process. Then one should determine the shapes of both the segment and the tube, the common surface of contact and the contact force (per unit of area) acting there, all depending on the prescribed segment's internal pressure. This mechanical problem is intimately connected with practical tasks like creeping of an artificial worm along a pipe or dilating a vessel in medical surgery.

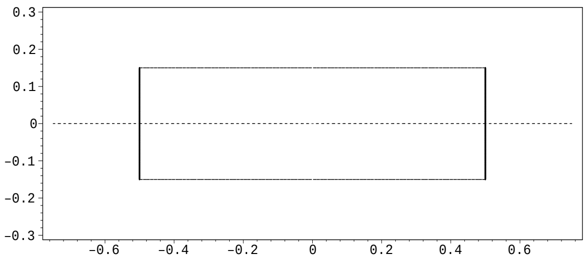
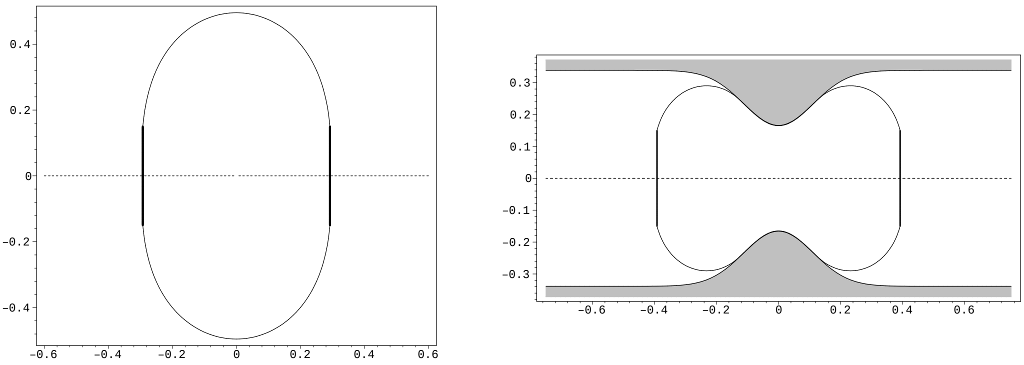


Figure 1: Original segment (longitudinal cut, zero pressure)

Principally, the problem could be tackled in a synthetic way: consider the system segment-tube in a state where contact is achieved and write down the equilibrium equations separately for segment and tube. These equations are both entered by the unknown force of contact. Find this force so that at every point of the membrane the condition 'radius of membrane not greater than inner radius of tube' is fulfilled.
This procedure is apparently rather unwieldy. For not only the force of contact is unknown but so is also the interval along a meridian where there is contact and the unknown force acts.
It seems better to prefer an analytical way, and this will determine the philosophy of the problem treatment in the sequel: Start by formulating the *Principle of Minimal Potential Energy* for the system segment-tube (in what state ever). This principle shows up as a variational problem or, equivalently, as an optimal control problem under state constraint. The crucial point is that in this formulation the unknown forces of contact,



(a) Segment under high internal pressure (b) Segment within constricted tube (same pressure)

Figure 2: a,b

being constraint forces, do not occur. Therefore the corresponding optimality conditions immediately serve for to determine the shape of the overall system, again in dependence of the internal pressure within the subsystem 'segment'. After this has been managed then, finally, the equilibrium equations of the membrane, whose geometry is now well-known, can be established. There, the only unknown is the force of contact (acting on an interval that is known as a result of the foregoing control problem) which now can be found from the equilibrium equations.
In order to open this way of investigation the authors considered in [AS 2005] optimal control problems under state constraint and, following classical patterns, developed necessary optimality conditions in a form that essentially allows for an immediate application to the present problems. As a relevant type of problems again maximum volume problems were solved in the above mentioned paper. Especially, since the respective solutions can be presented in analytical form, they will be used with profit in coming iterative numerical evaluations.
The paper will be organized in the following way. In section 2 first the inevitable facts from geometry and membrane theory will be outlined, some type of control problems from [AS 2005] essential in the sequel together with the corresponding optimality conditions will be presented. In section 3 the inflation of a segment within a rigid tube will be considered. A thorough analysis of the necessary optimality conditions both yields interesting theoretical insights and allows numerical investigation of some choice examples. Simulation results are presented in section 4. The conclusions sketch some open problems.
This paper presents investigations of inflated segments surrounded by *rigid* tubes. Forthcoming investigations will aim at segments within *compliant* tubes. Corresponding tools from optimal control are already prepared in [AS 2005].
There exists plenty of literature on ballooning of segment-like elements, see in [St 2003], [A 1992], [V 2005], mostly based on practical interests in connection with worms, artificial muscles, angioplasty. In many cases the investigations exhibit a specific insufficiency since

the authors start from more or less arbitrary presuppositions about the shape of the deformed element. In the following such inappropriate foundation will be avoided. Instead, the starting-point is an ensemble of stringent assumptions concerning the rheology of the segment. In particular in view of the complicated if not obscure structure of the plaque forming a stenosis of a vein such assumptions seem to be inevitable.
It is not the aim of this primarily mathematical paper to end up in presenting an apparatus of formulas that immediately fits for application by (e.g., medical) practitioners. Besides useful theoretical insights our results are to give stimulation for further investigations straightforwardly aiming at applications.

2 Tools from geometry, membrane theory, optimal control

The mechanical problems to be solved in this paper concern elements of rotational symmetry, and partly of a compliance coming from a skin-like hull. Therefore it makes sense to put together the main things from the geometry of surfaces of revolution and from the statics of membranes (as known from the theory of elastic shells). For optimal control theory those parts from [AS 2005] are sketched that serve as the starting points of the investigations.

2.1 Surfaces of revolution

Using the surface coordinates $u^1 = \phi$ (latitude) and $u^2 = s$ (arc-length of meridian) a general surface point has the radius vector

$$\mathbf{r}(\phi, s) = x(s)\mathbf{e}_x + y(s)\{\cos\phi\mathbf{e}_y + \sin\phi\mathbf{e}_z\}, \;\; \phi \in [0, 2\pi), \; s \in [s_1, s_2]. \tag{1}$$

The moving frame then is

$$\begin{aligned} \mathbf{g}_1 &= y(s)\{-\sin\phi\mathbf{e}_y + \cos\phi\mathbf{e}_z\}, \\ \mathbf{g}_2 &= x'(s)\mathbf{e}_x + y'(s)\{\cos\phi\mathbf{e}_y + \sin\phi\mathbf{e}_z\}, \\ \mathbf{n} &= -y'(s)\mathbf{e}_x + x'(s)\{\cos\phi\mathbf{e}_y + \sin\phi\mathbf{e}_z\}. \end{aligned} \tag{2}$$

It entails
the metric tensor: $g_{11} = y^2, \;\; g_{12} = 0, \;\; g_{22} = 1,$
the Christoffel symbols: $\Gamma^2_{11} = -yy', \;\; \Gamma^1_{12} = y'/y,$ rest zero,
the 2nd fundamental tensor: $b_{11} = -x'y, \;\; b_{12} = 0, \;\; b_{22} = x'y'' - x''y'$.

The standard meridian $\phi = 0$ is given by $(x(\cdot), y(\cdot))$. Its natural equations are

$$x' = \cos u, \;\; y' = \sin u, \;\; u' = \kappa,$$

where $u(s)$ is the angle from $\mathbf{e}_x$ to the tangent vector $\mathbf{g}_2(0, s)$ of the meridian, and $u'(s)$ is the curvature of the meridian at that point ($\kappa = b_{22}$). Most geometry books suppose $u \in C^\infty$, the present context demands to relax this strong smoothness. To start with we shall accept the minimal assumption

u piecewise continuously differentiable.

Thus u is piecewise continuous (class D^0) and, up to finitely many points, everywhere differentiable with continuous derivative. Then x and y describe a piecewise smooth meridian which has a piecewise well defined continuous curvature, the surface of revolution may have edges (along circles of latitude).

2.2 Membranes

The compliant segment to be investigated will in particular be characterized by the hypothesis that its deformable latitudinal hull statically behaves like a membrane shell, i.e., like a solid shell with no stress couples, see [GZ 1954]. The middle surface of this shell will be called *membrane* for short. In any actual state of the segment the membrane is a surface of revolution, and the formulas from above apply.

The local equilibrium of the membrane under the action of the *external force per unit area*

$$\mathbf{P} = P^\alpha \mathbf{g}_\alpha + P_n \mathbf{n}$$

is governed by the *membrane equations*

$$\nabla_\beta N^{\alpha\beta} + P^\alpha = 0, \quad N^{\alpha\beta} = N^{\beta\alpha}, \quad b_{\alpha\beta} N^{\alpha\beta} + P_n = 0. \tag{3}$$

Here, $N^{\alpha\beta}$ are the stress resultants per unit of length, they determine the stress vector $d\mathbf{T} = dT^\rho \mathbf{g}_\rho$ acting at the one-dimensional cut element $d\mathbf{f} = df^\alpha \mathbf{g}_\alpha$: $dT^\rho = N^{\rho\sigma} df_\sigma$, $df_\sigma = g_{\sigma\alpha} df^\alpha$. ∇ is covariant derivation, $\nabla_\beta N^{\alpha\beta} = N^{\alpha\beta}{}_{,\beta} + \Gamma^\alpha_{\rho\beta} N^{\rho\beta} + \Gamma^\beta_{\rho\beta} N^{\alpha\rho}$.

Remark. Since equilibrium takes place in the actual state of the segment, area and length are those in the *deformed* membrane!

The membrane equations will be solved under the following assumptions:
- surface of revolution,
- only normal forces acting ($P^\alpha = 0$),
- rotationally symmetric state of stress ($N^{\alpha\beta}{}_{,1} = 0$).

Then the equations appear as

$$\begin{aligned} &\tfrac{d}{ds} N^{12} + 2\tfrac{y'}{y} N^{12} = 0, \\ &\tfrac{d}{ds} N^{22} + \tfrac{y'}{y} N^{22} - yy' N^{11} = 0, \\ &-yx' N^{11} + (x'y'' - x''y') N^{22} + P_n = 0. \end{aligned} \tag{4}$$

As to the rheological behavior of the membrane the following *working hypotheses* will be introduced again [St 2003]:
- The membrane shell has original constant thickness h and it is incompressible;
- the membrane is *skin-like*, i.e., any state of the segment is stable only if the stress resultants are tensile, $N^{11} \geq 0$, $N^{22} \geq 0$, else a breakdown occurs (total flexibility);
- the membrane is meridionally inextensible;
- latitudinally, the membrane is homogeneously hyperelastic.

So the principal strain in s-direction vanishes,

$$\varepsilon_2 = 0,$$

and N^{22} appears as reaction to this constraint. The meridians keep their length , the arc-length s is an invariant during deformation.

The principal strain in φ-direction is constant along any circle of latitude, it is given by the original (r) and actual radius (y),

$$\varepsilon_1(s) = (y(s) - r)/r. \tag{5}$$

If σ_1 denotes the principal stress in latitudinal direction then hyperelasticity means

$$\sigma_1 = E\chi(\varepsilon_1) \tag{6}$$

($\chi(\varepsilon) = \varepsilon$ for Hooke material). Generally, E denotes some constant Young's modulus that is given in case of Hooke material and else fictitious and to be suitably chosen. $\chi(\cdot)$ is a smooth function from $\mathbb{R}^+$ to $\mathbb{R}^+$, $\chi(0) = 0$, monotonically increasing in most cases. It is given by experiments or suitably chosen in theory.
Note that σ_1 means force (at a cut $\phi = const$) in ϕ-direction divided by the *original* area of the cut element. Thus (recall ds invariant under deformation) with $\mathbf{g}_1^0 := \mathbf{g}_1/ \parallel \mathbf{g}_1 \parallel$ there holds $\sigma_1 h ds \mathbf{g}_1^0 = N^{11} df_1 \mathbf{g}_1,\ d\mathbf{f} = df^1 \mathbf{g}_1 = ds \mathbf{g}_1^0,\ \ df_1 = g_{11} df^1 = yds$, and it follows

$$y^2 N^{11} = hE\chi(\frac{y}{r} - 1).$$

With regard to later calculations it seems promising to skip to quantities of physical dimension 1. For this end we fix some suitable L_0 as unit of length (i.e., put $x = L_0\widetilde{x}$, etc., and drop the tilda after introduction); later on for L_0 the segment's meridional length will be chosen. Moreover let

$$N^{11} = (\frac{hE}{L_0^2})n^{11},\ \ N^{12} = (\frac{hE}{L_0})n^{12},\ \ N^{22} = (hE)n^{22},\ \ P_n = (\frac{2hE}{L_0})p_n \tag{7}$$

(the parenthesized quantities now are the respective units of measurement, $n^{\alpha\beta}$, p_n to be used in calculations take real numbers as their values).
The constitutive law now takes the normalized form

$$n^{11} = \psi(y)/y^2, \ \text{ where } \ \psi(y) := \chi(\frac{y}{r} - 1), \tag{8}$$

and the membrane equations write, using a dot for $\frac{d}{ds}$,

$$\begin{aligned} &(y^2 n^{12})^{\cdot} = 0, \\ &(yn^{22})^{\cdot} - y^2 \sin u\, n^{11} = 0, \\ &\dot{u} n^{22} - y \cos u\, n^{11} + 2p_n = 0. \end{aligned} \tag{9}$$

Mind that in general the normal load p_n is

$$p_n = p + z, \tag{10}$$

where p is a hydrostatic (constant) pressure while z, the *reaction* to a contact segment-tube, is a priori totally unknown.

If the constitutive law is taken into account, the second membrane equation solves by

$$yn^{22} = \Psi(y) + c^{22}, \text{ where } \Psi(y) := \int_r^y \psi(\eta) d\eta. \tag{11}$$

c^{22} is some integration constant which will be specified by boundary conditions.

The third membrane equation then yields

$$2yz = -2yp - \dot{u}[\Psi(y) + c^{22}] + \psi(y) \cos u. \tag{12}$$

Reminding what had been said in the Introduction, it becomes now clear how to determine the force of contact as soon as the actual shape $(u(\cdot), y(\cdot))$ including the meridional interval of contact is known.

We conclude this section by giving an expression for the potential energy stored in the deformed membrane. Per original unit volume this energy is (no normalization yet)

$$\int_0^{\varepsilon_1} \sigma_1(\varepsilon) d\varepsilon = \int_0^{\varepsilon_1} E\chi(\varepsilon) d\varepsilon = \frac{E}{r} \int_r^y \psi(\eta) d\eta = \frac{E}{r} \Psi(y),$$

an original volume $h\, ds\, r d\phi$ then contains $hE\, ds\, d\phi\, \Psi(y)$, and the total energy follows by integration about *meridian* $\times (0, 2\pi)$. With normalization there results the total potential energy of the deformed membrane

$$W[y] = \int \Psi(y(s)) ds, \quad \text{measured in units } 2\pi h E L_0^2. \tag{13}$$

Integration is along the full (normalized) length of a meridian.

2.3 Optimal control problems

In this section we quote some results from [AS 2005]. There, the authors considered Bolza-type optimal control problems with state $x \in \mathbb{R}^n$ and control $u \in \mathbb{R}^m$ of the following form (roughly):

$$\begin{array}{ll} I[x,u] := g_0(t_0, x(t_0), T, x(T)) + \int_{t_0}^T f_0(t,x,u) dt \to \min, & \\ \quad \dot{x} = f(t,x,u), & (\textit{dynamics}) \\ \quad Q(t,x,u) \geq 0, & (\textit{control restriction}) \\ \quad S(t,x) \geq 0, & (\textit{state restriction}) \\ \quad g(t_0, x(t_0), T, x(T)) = 0, & (\textit{boundary restriction}). \end{array}$$

In order to establish a tool for quick application to the problems under investigation in the following sections we introduce as
(i) *simplifications*: no control restrictions, no finite term of the cost functional (drop Q and g_0), some fixed boundary values (some coordinates of $x(t_0),\ x(T)$ given), simplest dynamics, autonomous cost, scleronomic state restriction;
(ii) *supplement*: internal point condition.

Optimal control problem P1:

Find $(x,u) \in D^1_n[t_0,T] \times D^0_m[t_0,T]$, t_0, T fixed, such that

$$\begin{array}{l} I[x,u] := \int_{t_0}^{T} f_0(x,u)dt \to \min, \quad f_0 \in C^2(\mathbb{R}^{n+m},\mathbb{R}), \\ \quad \dot{x} = f(u), \;\; f \in C^2(\mathbb{R}^m,\mathbb{R}^n), \quad (dynamics) \\ \quad x_j(t_0) = x_{j0},\ j \in J,\;\; x_k(T) = x_{k1},\ k \in K, \quad (boundary\ condition) \\ \quad x_l(\tau) = \xi_l,\ l \in L,\ for\ given\ \tau \in (t_0,T), \quad (internal\ point\ condition) \\ \quad S(x) \geq 0, \;\; S \in C^2(\mathbb{R}^n,\mathbb{R}), \quad (state\ restriction). \end{array} \tag{14}$$

$J,\ K,\ L$ are given subsets of $\{1,..,n\}$.
Regarding the state constraint the following assumption holds (*relative degree* $h = 1$):

$$\begin{array}{l} S(x(T)) > 0, \\ R_0(x,u) := S_{,x}(x)f(u), \;\; R_{0,u}(x,u) \neq 0 \text{ if } S(x) = 0. \end{array}$$

Points $t_\beta \in [t_0,T]$ where the solution curve $t \mapsto x(t)$ enters (joins) or leaves (disjoins) the manifold $S(x) = 0$ are called *junction points*.
Let $B := \{discontinuity\ points\ of\ u(\cdot),\ junction\ points\}$.

Using the *indirect method* (i.e., the Lie-derivative R_0 instead of S enters the Hamiltonian, see [AS 2005]) we have the following

Optimality conditions for P1:

Let (x,u) solve ***P1****.*
Then there exists a multiplier $(l_0, \lambda(\cdot), \rho(\cdot)),\ l_0 \in R^+,\ \lambda \in D^1[t_0,T]$, but $\lambda_l,\ l \in L$, possibly discontinuous at τ, $\rho \in D^0[t_0,T]$ and piecewise differentiable on intervals where $S(x(t)) > 0$, such that with the Hamiltonian

$$H(x,u,l_0,\lambda,\rho) := l_0 f_0(x,u) + \lambda f(u) + \rho R_0(x,u)$$

there holds

$$\begin{array}{l} (a)\;\; \forall t \in [t_0,T] \;\; (l_0,\lambda(t),\rho(t)) \neq 0, \\ (b)\;\; \dot{\lambda}_i = -H_{,x_i}\;,\;\; i \notin L,\ t \in [t_0,T] \setminus B, \\ \qquad \dot{\lambda}_l = -H_{,x_l}\;,\;\; l \in L,\ t \in [t_0,T] \setminus (B \cup \{\tau\}), \\ (c)\;\; \lambda_i(t_0) = 0,\ i \notin J,\;\; (\lambda_i + \rho S_{,x_i})_{t=T} = 0,\ i \notin K, \\ \qquad \text{possibly } \lambda_l(\tau - 0) \neq \lambda_l(\tau + 0),\ l \in L, \\ (d)\;\; H_{,u} = 0,\;\; t \in [t_0,T], \\ (e)\;\; H = const,\;\; t \in [t_0,T] \setminus \{\tau\}, \\ (f)\;\; H_{,uu} \leq 0, \\ (g)\;\; 0 = \dot{\rho} S(x)\ p.w.,\;\; \rho \text{ non-decreasing},\;\; (\rho S)_{t=t_0} = 0. \end{array} \tag{15}$$

Remark. The fact that discontinuities of λ_l and H at τ have to be allowed emerges during the derivation of the optimality conditions by means of variational arguments (see [AS 2005]). Observing the internal point condition $x_l(\tau) = \xi_l$ the variations δt and δx_l must be zero at τ whence certain indeterminations (jumps of λ_l and H) may result.

3 Segment within a rigid tube

We consider a compliant *segment* of original radius r and length L_0 showing the rheological constitution described in section 2.2. Following the lines given there this length L_0 will be taken as the unit of lengths, so, formally $L_0 = 1$. The segment is inserted in a long rigid tube that has an inner radius $R(x)$, where, as usual, x is the longitudinal coordinate. Under certain symmetry assumptions the quasistatic inflation process of the segment will be investigated in the following. Before starting we note that due to the presupposed meridional inextensibility of the segment there exists a maximal volume the free segment (without restriction by the surrounding tube) can approach during inflation, see the corresponding investigations in [AS 2005]. Let $\bar{R}(r)$ be the equatorial radius of the free segment in this maximal volume shape.

Concerning the *internal profile of the tube* we assume D^2-smoothness and adopt the
Assumption:

$$\begin{array}{ll} (a) \ \forall x \in (-\infty, \infty) \ \ R(-x) = R(x), \ \ r < R(x) \leq \bar{R}(r), \\ (b) \ \forall x \in (-\infty, 0) \ \ -\infty < R'(x) < 0 \text{ or } R(\cdot) = const. \end{array} \tag{16}$$

By (a) a symmetry w.r.t. $x = 0$ is ensured, the tube is neither too narrow as to crumple the membrane nor too wide as to avoid any contact under inflation. With (b) the profile either represents a constriction of the tube or the inner radius is constant.
To be concise the arc $\{(x, R(x)) : x \in \mathbb{R}\}$ will be called *profile* in the following.

3.1 Inflation within a rigid tube

Using the tools, notations, and the normalization introduced in section 2 we start with the *Principle of Minimal Potential Energy* for the segment,

$$W[y] - pV[y, u] \to \min\,, \tag{17}$$

where p is the actual internal hydrostatic pressure and $V = \int y^2 \cos u ds$ is the actual volume of the segment, $W = \int \Psi(y) ds$ is the deformation energy of the membrane. Remind $\Psi(y) := \int_r^y \psi(\eta) d\eta$, where ψ governs the latitudinal stress-strain relation of the membrane material (see section 2.2). It makes sense to introduce

$$q := 1/p \in \mathbb{R}_+ \tag{18}$$

as a new parameter; then $q = 0$ exhibits the maximum volume problem (which is purely geometrical though appearing mechanically as the limit case $p \to \infty$).

If we take the above variational principle literally then the task is to find the shape of the segment (at given pressure p) *and* its position within the tube such that the potential energy achieves its minimum. That leads to the following optimal control problem with a parameter ξ (in this first formulation we disregard smoothness classes).

Find (ξ, x, y, u), $\xi \in \mathbb{R}$, such that

$$\begin{aligned}
&\int_{-\frac{1}{2}}^{\frac{1}{2}} \{q\Psi(y) - y^2 \cos u\} ds \to \min, \\
&\quad \dot{x} = \cos u, \quad x(0) = \xi, \\
&\quad \dot{y} = \sin u, \quad y(\pm\tfrac{1}{2}) = r, \\
&\quad S(x, y) := R(x) - y \geq 0.
\end{aligned}$$

(Equivalently, we could use any $x(s_0)$ instead of $x(0)$). In view of the applications we have in mind our primary interest is *not* directed to the absolute minimum of the potential energy and the corresponding position, ξ, if it exists. Rather, we *prescribe* some ξ (instantaneously, the segment of a worm *is* at some position during the worm's motion, the balloon for angioplasty *is placed* to some position within the tube) and then focus on the shape of the segment that minimizes the energy in this position.

Using Assumption (16,a) it is simple matter to prove the following proposition for the *feasible functions* (satisfying the restrictions within the problem).

Proposition 3.1 *At given ξ let $s \mapsto (\xi, x(s), y(s), u(s))$ be feasible.*

Then also $s \mapsto (-\xi, -x(-s), y(-s), -u(-s))$ is feasible with the same value of the cost functional.

This means that every feasibly shaped segment at position ξ has a mirror twin w.r.t. $x = 0$ (forced by the eveness of $R(\cdot)$). Both segments have the same value of the potential energy, Proposition 3.1 describes their isomorphism. For shapes with symmetry w.r.t. $x = 0$ the twins coincide.

In this paper on radially constrained segments we shall restrict our considerations to segments placed at $\xi = 0$ - though, for applications as in angioplasty, eccentrically (w.r.t. the symmetric profile) positioned inflating segments could be of interest as well.

Now we have to deal with the following **optimal control problem**. ($\widetilde{D}_1^1$ denotes the class of p.w.continuous and p.w.continuously differentiable functions.)

Find $(x, y, u) \in D_2^1[-\frac{1}{2}, \frac{1}{2}] \times \widetilde{D}_1^1[-\frac{1}{2}, \frac{1}{2}]$ such that

$$\begin{aligned}
&\int_{-\frac{1}{2}}^{\frac{1}{2}} \{q\Psi(y) - y^2 \cos u\} ds \to \min, \\
&\quad \dot{x} = \cos u, \quad x(0) = 0, \\
&\quad \dot{y} = \sin u, \quad y(\pm\tfrac{1}{2}) = r, \\
&\quad S(x, y) := R(x) - y \geq 0.
\end{aligned} \tag{19}$$

This is an optimal control problem of type **P1** (see section 2.3) with state (x, y), control u and a state constraint $S(x, y) \geq 0$. The *relative degree* of the problem is $h = 1$, i.e., $\dot{S} = R'(x)\cos u - \sin u$ explicitly depends on u and the rank condition $\dot{S}_{,u} \neq 0$ is generically fulfilled (violated at some s iff at s the extremal hits the profile orthogonally). The relatively strong smoothness of u is inherited from the theory of surfaces in section 2.1. As a peculiarity there occurs the restriction to zero of the state coordinate x at the *inner* point $s = 0$ of $(-\frac{1}{2}, \frac{1}{2})$. This internal point condition entails some discontinuities to be allowed in the optimality conditions (see section 2.3) but, fortunately, it simplifies the search for the optimal solutions by means of the following proposition.

Proposition 3.2 *Any solution of the optimal control problem (19) is symmetric in the sense*

$$\forall s \in [-\frac{1}{2}, \frac{1}{2}] \quad x(-s) = -x(s), \;\; y(-s) = y(s), \;\; u(-s) = -u(s).$$

Proof. For the sake of brevity let

$$(x, y, u) =: X =: \begin{cases} X_1 & \textit{on } [-\frac{1}{2}, 0) \\ X_2 & \textit{on } (0, \frac{1}{2}] \end{cases}$$

and, with feasible X, write the potential energy

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} \{q\Psi(y) - y^2 \cos u\} ds =: I[X] = I_1[X_1] + I_2[X_2]$$

where $I_1 = \int_{-\frac{1}{2}}^{0} \dots$, and $I_2 = \int_{0}^{\frac{1}{2}} \dots$. Now, following Proposition 3.1, let $\widetilde{X}$ be the mirror twin of X, i.e., $\widetilde{X}(s) = (-x(-s), y(-s), -u(-s))$ and introduce $\widetilde{X}_1$, $\widetilde{X}_2$ as done above. Compare the integrals $I_1[X_1]$ and $I_1[\widetilde{X}_1]$, put $X_1^* := X_1$ if $I_1[X_1] \leq I_1[\widetilde{X}_1]$ else $X_1^* := \widetilde{X}_1$ then proceed analogously with I_2 and define X_2^*. It is easily seen that

$$X^* =: \begin{cases} X_1^* & \textit{on } [-\frac{1}{2}, 0) \\ X_2^* & \textit{on } (0, \frac{1}{2}] \end{cases}$$

is feasible and symmetric and there holds $I[X^*] \leq I[X] = I[\widetilde{X}]$. So, if there exists a (minimizing) solution then it is symmetric. ■

This proposition has a nice consequence: *computations of solutions and, partly, the analysis of optimality conditions may be confined to* $s \in [-\frac{1}{2}, 0)$.

In order to formulate the necessary optimality conditions (see [AS 2005] and section 2.3) we define the *Hamiltonian*

$$\begin{aligned} &H(x, y, u, l_0, \lambda_1, \lambda_2, \rho) := \\ &\quad l_0[-y^2 \cos u + q\Psi(y)] + \lambda_1 \cos u + \lambda_2 \sin u + \rho[R'(x)\cos u - \sin u]. \end{aligned} \tag{20}$$

The **optimality conditions** then are the following:

Let (x, y, u) *be a solution of the optimal control problem* (19).
Then there exists a multiplier $(l_0, \lambda_1(\cdot), \lambda_2(\cdot), \rho(\cdot))$, $l_0 \in \mathbb{R}^+, \lambda_1 \in D^1[-\frac{1}{2}, \frac{1}{2}]$ *but possibly*

discontinuous at $s = 0$, $\lambda_2 \in D^1[-\frac{1}{2}, \frac{1}{2}]$, $\rho \in D^0[-\frac{1}{2}, \frac{1}{2}]$ *and piecewise differentiable on intervals where* $S(x(s), y(s)) > 0$, *such that*

$$\begin{array}{ll} (a) & \forall s \in [-\frac{1}{2}, \frac{1}{2}] \ \ (l_0, \lambda_1(s), \lambda_2(s), \rho(s)) \neq 0, \\ (b) & \dot{\lambda}_1 = -H_{,x} = -\rho R''(x) \cos u, \ s \in [-\frac{1}{2}, \frac{1}{2}] \setminus (B \cup \{0\}) \\ (b_1) & \lambda_1(-\frac{1}{2}) = 0, \ \lambda_1(\frac{1}{2}) = -(\rho R'(x))_{s=1/2}, \\ (c) & \dot{\lambda}_2 = -H_{,y} = -l_0[q\psi(y) - 2y \cos u], \ s \in [-\frac{1}{2}, \frac{1}{2}] \setminus B \\ (d) & 0 = H_{,u} = [l_0 y^2 - \lambda_1 - \rho R'(x)] \sin u + [\lambda_2 - \rho] \cos u, \\ (e) & H = const, \ s \in [-\frac{1}{2}, 0) \cup (0, \frac{1}{2}] \\ (f) & 0 \leq H_{,uu} = -H + l_0 q \Psi(y), \\ (g) & 0 = \dot{\rho} S = \dot{\rho}[R(x) - y], \ \ \rho \text{ non-decreasing, } (\rho S)_{s=-1/2} = 0. \end{array} \tag{21}$$

The set B consists of all the discontinuity points of u and junction points meridian-tube.

Remark: If (x, y, u, λ, ρ) solves the necessary conditions (21) then for the sake of brevity (x, y, u) will be called an *extremal* of the problem.

The optimal control task (19) aims at a particular class of objects, the *segments* as introduced in section 2.2. Therefore some *further side conditions* issuing from physical assumptions given in section 2.2 which are not immanent to the Principle as formulated above must be observed as well.
First, the membrane has been assumed to be skin-like, i.e., the stress resultants are non-negative (tensile stresses only). And *second*, it had been tacitly assumed in section 2.2 that active constraint be physically ideal, i.e., the reaction force z acts in normal direction and is governed by (12), while inactive constraint yields $z = 0$. z, applied from outside (from the tube) to the membrane, must be a pressure, therefore $z \leq 0$ (no adhesion).

Now
(i) $0 \leq n^{11}(s) = \psi(y(s))/y^2(s)$ implies

$$y(s) \geq r.$$

(ii) $0 \leq n^{22}(s)$ can be exploited in the following way. At a cut $s = const \leq 0$ the equilibrium of longitudinal forces reads

$$n^{22}(s)y(s)\cos u(s) - py^2(s) - 2\int_{-\frac{1}{2}}^{s} z(t)y(t)\sin u(t)dt = 0, \ s \in [-\frac{1}{2}, 0].$$

Now $z(t) = 0$ if at t there is no contact segment-tube , $z(t) \leq 0$ and (owing to (16)(b)) $\sin u(t) < 0$ if there is contact. So the integral is non-negative thus $n^{22}(s)\cos u(s) > 0$ if $p > 0$ and therefore $n^{22}(s) > 0$ and $\cos u(s) > 0$. Hence there holds

$$p > 0 \Rightarrow \forall s \in [-\frac{1}{2}, 0] \ \ u(s) \in (-\frac{\pi}{2}, \frac{\pi}{2}). \tag{22}$$

Moreover, it is necessary that (if $p > 0$)

$$u(-\frac{1}{2} + 0) =: u(\frac{1}{2}) =: \alpha > 0. \tag{23}$$

Clearly, $\alpha < 0$ would lead to $\dot{y}(-\frac{1}{2}+0) = \sin\alpha < 0$ and $y(s) < r$ in some right neighborhood of $-\frac{1}{2}$.
So assume $\alpha = 0$. Then for the same reason $\dot{u}(-\frac{1}{2}+0) \geq 0$ must hold. Owing to the assumptions (16) there is no active constraint ($z = 0$) in a right neighborhood of $-\frac{1}{2}$. The formula (12) yields with $s \to -\frac{1}{2}$

$$0 < 2rp = -\dot{u}(-\frac{1}{2}+0)rn^{22}(-\frac{1}{2}),$$

and this entails the contradicting boundedness $\dot{u}(-\frac{1}{2}+0) < 0$.

Finally the stronger restriction (for $p > 0$)

$$u \textit{ continuous at } s \in (-\frac{1}{2}, 0) \Rightarrow y(s) > r. \tag{24}$$

will be shown.
Assume $y(s_0) = r$. For $y(\cdot)$ not to go below r it is necessary that $u(s_0) = 0$. Owing to the assumptions (16) there is no active constraint in a neighborhood of s_0, there holds $z(s) = 0$ and thus, according to (12)

$$0 < 2py(s) = -\dot{u}(s)y(s)n^{22}(s) + \psi(y(s))\cos u(s)$$

for all s in this neighborhood (possibly up to finitely many points where $u(\cdot)$ or $\dot{u}(\cdot)$ has a jump). So $s \to s_0 \pm 0$ yields ($\psi(y(s_0)) = \psi(r) = 0$)

$$0 < 2pr = -\dot{u}(s_0 \pm 0)rn^{22}(s_0).$$

Since $n^{22} > 0$ everywhere we obtain $\dot{u}(s_0 \pm 0) < 0$. Hence $u(s) < 0$ and as a consequence $y(s) < r$ in a right neighborhood of s_0: contradiction!

Remark: Later on we shall show that u is continuous everywhere, then $y(s) > r$ will become true on all of $(-\frac{1}{2}, 0)$.

Due to the symmetry of the extremals the inequalities (22) and (24) hold true for the $s > 0$ part of the extremals.

Being strong inequalities, these side conditions (22) and (24) do not influence the optimality conditions via additional multipliers but they will prove important in the sequel.

The analysis of the optimality conditions will be done through several steps.

(1) *Normality*: Assume $l_0 = 0$. At $s = -\frac{1}{2}$ the constraint is inactive ($S > 0$) hence from the optimality conditions we obtain in turn (g): $\rho(-\frac{1}{2}) = 0$, (b$_1$): $\lambda_1(-\frac{1}{2}) = 0$, (d): $\lambda_2(-\frac{1}{2}) = 0$. Thus the total multiplier vanishes at $s = -\frac{1}{2}$: contradiction!
So let $l_0 = 1$ for all that follows.

(2) *Multipliers and 'energy' constant*: Following (21)(e) let

$$H = c_1 \text{ and } H = c_2 \text{ for } s \in [-\frac{1}{2}, 0) \text{ and } s \in (0, \frac{1}{2}], \text{ respectively.}$$

In corresponding intervals solve the equations $H = c_i$ and $H,_u = 0$ for λ,

$$\begin{array}{ll} \lambda_1 = y^2 - q\Psi(y)\cos u + c_i \cos u - \rho R'(x), & \\ \lambda_2 = \quad -q\Psi(y)\sin u + c_i \sin u + \rho, & \end{array}, \ i = 1, 2.$$

Observing (b_1) and $u(-\frac{1}{2}) = \alpha = -u(\frac{1}{2})$ (symmetry), $s \to \pm\frac{1}{2}$ yields

$$0 = \lambda_1(-\frac{1}{2}) = r^2 + c_1 \cos\alpha, \ \ 0 = \lambda_1(\frac{1}{2}) + (\rho R'(x))_{s=1/2} = r^2 + c_2 \cos\alpha,$$

so that with (22) $c_1 = c_2 =: c$,

$$c = -\frac{r^2}{\cos\alpha} < 0. \tag{25}$$

The multipliers λ now become

$$\begin{array}{l} \lambda_1 = y^2 - [\frac{r^2}{\cos\alpha} + q\Psi(y)]\cos u - \rho R'(x), \\ \lambda_2 = \quad -[\frac{r^2}{\cos\alpha} + q\Psi(y)]\sin u + \rho. \end{array} \tag{26}$$

As a consequence we obtain the smoothness properties of ρ on the full interval $[-1/2, 1/2]$.

(3) *Continuity of the control u*:
a) At $s \neq 0$. In any open interval with active constraint there holds $y = R(x)$, $u(s) = \arctan(R'(x(s)))$, $x(\cdot)$ continuous, thus u is continuous in this interval. Let $s_0 \in (-\frac{1}{2}, 0)$ be any point outside of such intervals, let $y_0 := y(s_0)$, $u_0^{\pm} := u(s_0 \pm 0)$, etc. Continuity of λ_2 at s_0 implies

$$-[q\Psi(y_0) - c]\sin u_0^- + \rho_0^- = -[q\Psi(y_0) - c]\sin u_0^+ + \rho_0^+,$$

and, since ρ is non-decreasing,

$$0 \leq \rho_0^+ - \rho_0^- = [q\Psi(y_0) - c](\sin u_0^+ - \sin u_0^-).$$

Analogously, continuity of λ_1 at s_0 implies

$$0 \leq -R'(x_0)(\rho_0^+ - \rho_0^-) = [q\Psi(y_0) - c](\cos u_0^+ - \cos u_0^-).$$

Eliminating the $\rho_0^{\pm}$ and dropping the positive bracket we obtain

$$-R'(x_0)(\sin u_0^+ - \sin u_0^-) = (\cos u_0^+ - \cos u_0^-).$$

If s_0 *is not* a junction point ($S > 0$, $\rho = const$ around s_0) then $\rho_0^+ - \rho_0^- = 0$, the parenthesized expressions both must vanish whence $u_0^+ = u_0^-$ which means continuity.
If s_0 *is* a junction point where the extremal enters the profile then $R'(x_0) = \tan u_0^+$, the last equation yields $\cos(u_0^+ - u_0^-) = 1$ hence $u_0^- = u_0^+$ (continuity) follows. In a junction point where the extremal leaves the profile (disjunction) the same reasoning yields the same result. If s_0 is a touch point (junction $\wedge$ disjunction) then disjunction demands $4u_0^+ \leq U_0 = \arctan R'(x_0) < 0$ while $\rho_0^+ - \rho_0^- \geq 0$ entails $u_0^- \leq u_0^+$. But then $u_0^- < U_0$

would contradict junction hence $u_0^- = u_0^+ = U_0$ must hold.

Owing to the symmetry, $u(-s) = -u(s)$, continuity has now been ensured on $[-\frac{1}{2}, \frac{1}{2}] \setminus \{0\}$.
b) At $s = 0$. If the constraint is active in a neighborhood of $s = 0$, then the continuity proves as in a).
Again, let $y_0 := y(0)$, etc. Since λ_2 is continuous at 0 (whereas λ_1 may have a jump) we proceed as above and, using the symmetry $u_0^- = -u_0^+$, obtain

$$2[q\Psi(y_0) - c] \sin u_0^+ = \rho_0^+ - \rho_0^- .$$

If in a full neighborhood of 0 the constraint is inactive then $\rho = const$ on this neighborhood thus $u_0^+ = 0$ which implies continuity.
If, finally, $s = 0$ is both junction and disjunction point (one-point touch) then ρ may have a jump and therefore $\sin u_0^+ \geq 0$ holds. This implies $u_0^+ \geq 0$, and in a right neighborhood of $s = 0$ we have $y(s) = R(0) + s \sin u_0^+ + o(s)$. This violates the constraint (note $R'(0) = 0$) if not $u_0^+ = 0$. Then u is continuous with $u(0) = 0$ and this also entails continuity of ρ and λ_1 at $s = 0$.
Summarizing, we have found $u \in C^0[-\frac{1}{2}, \frac{1}{2}]$, $u(0) = 0$. This implies

$$(x, y) \in C_2^1[-\frac{1}{2}, \frac{1}{2}], \quad \lambda \in D_2^1[-\frac{1}{2}, \frac{1}{2}], \quad \rho \in D^1[-\frac{1}{2}, \frac{1}{2}].$$

(4) *Differentiability of the control*: Consider

$$\lambda_1 = \{y^2 - [\frac{r^2}{\cos\alpha} + q\Psi(y)] \cos u\} - \rho R'(x)$$

and condition (b), $\dot{\lambda}_1 = -\rho R''(x) \cos u$, which imply

$$\frac{d}{ds}\{y^2 - [\frac{r^2}{\cos\alpha} + q\Psi(y)] \cos u\} = \dot{\rho} R'(x), \text{ piecewise.}$$

We introduce the *state-control function* Φ,

$$\Phi(y, u; \alpha, q) := y^2 - [\frac{r^2}{\cos\alpha} + q\Psi(y)] \cos u. \tag{27}$$

Then $\lambda_1 = \Phi - \rho R'$ entails

$$\varphi(\cdot; \alpha, q) := \Phi(y(\cdot), u(\cdot); \alpha, q) \in D^1[-\frac{1}{2}, \frac{1}{2}],$$

and there holds, owing to the optimality condition (b),

$$\frac{d}{ds}\varphi = \dot{\rho} R'(x), \quad \text{piecewise.} \tag{28}$$

Using condition (g) we deduce the following alternative: if the constraint is *inactive* then $\dot{\rho} = 0$, thus $\varphi = C = const$, whereas in intervals of *active* constraint $\dot{\rho} \geq 0$ and $R'(x) < 0$ exhibit φ on $[-\frac{1}{2}, 0]$ as a non-increasing function.

The initial values $y(-\frac{1}{2}) = r,\ u(-\frac{1}{2}) = \alpha$ yield $\varphi(-\frac{1}{2};\alpha,q) = 0$, $S_{s=-1/2} > 0$ entails $S > 0$ in some right neighborhood of $s = -\frac{1}{2}$. Therefore $C = 0$ in this utmost left interval of inactive constraint and $C \leq 0$ in any further such interval whithin $[-\frac{1}{2}, 0]$.
In intervals of active constraint, $y(s) = R(x(s))$, there holds $u(s) = \arctan R'(x(s)) \in (-\frac{\pi}{2}, \frac{\pi}{2})$. If $x \to R(x)$ was assumed to be of class $C^k,\ k \geq 2$, then u is C^{k-1} in these intervals.
In intervals of inactive constraint we have $y^2 - [\frac{r^2}{\cos\alpha} + q\Psi(y)]\cos u = C = const \leq 0$ (the value of C depends on the interval). Now $u = \arccos\{(y^2 - C)/[\frac{r^2}{\cos\alpha} + q\Psi(y)]\}$ yields $u \in C^1$ if $u \neq 0$ and, using $\dot{x} = \cos u,\ \dot{y} = \sin u$, iteratively, $u \in C^\infty$ if $u(s) \neq 0$ and $\Psi \in C^\infty$.

(5) *Description of the extremals*: Again, we restrict our considerations to the $s-$interval $[-\frac{1}{2}, 0]$. Now at least for points where $u(s) \neq 0$ we get

$$\frac{d}{ds}\varphi = \sin u\{2y - q\psi(y)\cos u + [\frac{r^2}{\cos\alpha} + q\Psi(y)]\dot{u}\}.$$

So we obtain for the curvature of the extremal *in any interval of inactive constraint*

$$\dot{u} = \{-2y + q\psi(y)\cos u\}/[\frac{r^2}{\cos\alpha} + q\Psi(y)]. \tag{29}$$

The previous condition $u \neq 0$ can now be dropped because of the rhs continuity, $\dot{u}$ is continuous in these intervals. In combination with $\dot{x} = \cos u,\ \ \dot{y} = \sin u$ the last equation (29) forms a differential equation the unconstrained parts of the extremals must satisfy.
In an interval of active constraint, $y(s) = R(x(s))$, $\tan u(s) = R'(x(s))$, the curvature is $K = R''(1 + R'^2)^{-\frac{3}{2}}$, and we get

$$\frac{d}{ds}\varphi = R'(1 + R'^2)^{-\frac{1}{2}}\{2R - q\psi(R)(1 + R'^2)^{-\frac{1}{2}} + [\frac{r^2}{\cos\alpha} + q\Psi(R)]K\}$$

as a function of $x(s)$. Since $R' < 0$, $\dot{\varphi} \leq 0$ then means

$$0 \leq 2R(x) - q\psi(R(x))(1 + R'^2(x))^{-\frac{1}{2}} + [\frac{r^2}{\cos\alpha} + q\Psi(R(x))]K(x) \tag{30}$$

in any interval with active constraint.

Now either the extremal never hits the profile (i.e., $y(s) < R(x(s))$ for all $s \in [-\frac{1}{2}, 0]$) or it is at some smallest $s = t_1 \in (-\frac{1}{2}, 0]$ that the extremal *tangentially* (continuity of u !) enters the profile. If (x, y, u) is the (unconstrained part of the) extremal on the interval $[-\frac{1}{2}, t_1)$ then this means

$$x(t_1 - 0) = x_1,\ y(t_1 - 0) = R(x_1),\ u(t_1 - 0) = \arctan R'(x_1), \tag{31}$$

with some $x_1 \in (-\frac{1}{2}, 0]$ not known in advance but rather, *together with* α, to be determined by means of the last equations (for details see following sections).

Remark: Mechanically, the skin-like membrane would allow edges. So it is by far not self-evident that the extremal has contunous u and, thus, joins the profile tangentially while at most showing a jump of its curvature $\dot{u}$.

If $t_1 < 0$ then an $s-$interval (t_1, t_2) with active constraint could follow. There the extremal is governed by $y(s) = R(x(s))$, $u(s) = \arctan R'(x(s))$, and (30) must be fulfilled. This interval terminates at some $t_2 \in [t_1, t_\star]$ where at $t_\star$ the rhs in (30) turns to negative values. If $t_2 < 0$ the extremal *tangentially* leaves the profile, and again an interval with inactive constraint follows. Along this interval then $\varphi(s; \alpha, q) = C$ holds with $C = \varphi(t_2; \alpha, q) \leq 0$. Whether this case occurs or not seemingly depends both on the profile function R and the function ψ which characterizes the kind of hyperelasticity of the membrane and also on q. There is no practicable and general rule in sight. Merely for convex profiles ($R'' \geq 0$) disjunctions can be excluded in a fairly simple way using phase-space considerations. We come back to this item in section 3.4.

Summarizing, *the extremals* $s \mapsto (x, y, u)(s)$, $s \in [-\frac{1}{2}, 0]$, *are characterized in the following way* (the characterization of the right parts, $s \in [0, \frac{1}{2}]$, is obvious because of the symmetry of the extremals).
(a) In intervals of inactive constraint, $y < R(x)$, the extremal with boundary values

$$y(-\frac{1}{2}) = r, \;\; u(-\frac{1}{2}) = \alpha$$

solves the differential equation

$$\begin{aligned} \dot{x} &= \cos u, \\ \dot{y} &= \sin u, \\ \dot{u} &= \{-2y + q\psi(y)\cos u\}/[\tfrac{r^2}{\cos\alpha} + q\Psi(y)], \end{aligned} \tag{32}$$

(*unconstrained extremal*) for which

$$\Phi(y, u; \alpha, q) := y^2 - [\frac{r^2}{\cos\alpha} + q\Psi(y)]\cos u = C = const \leq 0 \tag{33}$$

is a first integral. C depends on the interval. Note that both the differential equation and the first integral explicitly depend on the initial value α which is not known in advance but rather has to be found from the supplementing boundary condition $u(0) = 0$ or in connection with junction conditions (31).
(b) In intervals of active constraint, $y = R(x)$, the extremal is governed by

$$\begin{aligned} y &= R(x(s)), \\ u &= \arctan R'(x(s)), \\ s &= \int \sqrt{1 + R'(x)^2} dx \end{aligned} \tag{34}$$

no longer than (30) is fulfilled, which means nothing else but φ non-increasing along these intervals.
Appropriate junction conditions (31) make (x, y) a C^1-curve which is C^∞ or C^k in intervals of inactive or active constraint, respectively, and may have a jump of its curvature in

junction/disjunction points.

Amazingly, the function $-\Phi$ behaves like an entropy along any extremal ($s \in [-1/2, 0]$). Its derivative along an extremal gets a clear mechanical meaning in the following. (This is indeed a nice fact, since the strongly related multiplier ρ usually remains a bit obscure in literature.)

(6) *The constraint forces*: The constraint force per unit of area, z, is given in section 2.2 by (12). The integration constant there, c^{22}, follows from the equilibrium of longitudinal forces at the left disc: $0 = 2\pi r N^{22}(-\frac{1}{2})\cos\alpha - \pi r^2 P$ as $c^{22} = pr^2/\cos\alpha$. Then (12) yields

$$2qyz = -2y - [\frac{r^2}{\cos\alpha} + q\Psi(y)]\dot{u} + q\psi(y)\cos u. \tag{35}$$

Indeed, the differential equations (32) make z vanish along intervals of inactive constraint whereas along intervals of active constraint z follows from (35) by means of (34).

Consider first the simple but practically important case $R(x) = R_1 = const$ (cylindrical tube). Then there follows at once

$$z = -p + \frac{1}{2R_1}\psi(R_1) \tag{36}$$

for $s \in (t_1, 0]$, where t_1 has to be found via the junction conditions (31). We refer to respective simulations in section 4.

In the general case it is easily seen that

$$2qy\sin u \cdot z = \frac{d}{ds}\{-y^2 + [\frac{r^2}{\cos\alpha} + q\Psi(y)]\cos u\} = -\dot{\varphi} \geq 0. \tag{37}$$

Observing $\sin u = R'(1 - R'^2)^{-1/2} < 0$ the last inequality means $z \leq 0$. This expresses the fact that the reaction z, which the tube exerts on the membrane, cannot be tensile (physically, this excludes adhesion), it must be pressing if not zero. $\dot{\varphi} = 0$ is feasible for both inactive and active constraint. To make things a bit clearer we introduce the

Definition 3.3 *The constraint $R(x) - y \geq 0$ is called* strongly active *if $y = R(x) \wedge z < 0$,* weakly active *if $y = R(x) \wedge z = 0$.*

The following proposition is then obvious. (Remind that we are with $s \leq 0,\ x \leq 0$.)

Proposition 3.4 *If $R(x) \neq const$ then the following alternative holds: the constraint is strongly active iff $\dot{\varphi} < 0$, the constraint is inactive or weakly active iff $\dot{\varphi} = 0$.*

It is obvious that weak activity just means a chance coincidence of the unconstrained extremal and the profile.
Along active constraint intervals there holds $\dot{\varphi} = \frac{\partial}{\partial x}\Phi(R(x), ...)\cos u$ and hence a rule for to calculate z is

$$\begin{aligned} z &= -p\tfrac{1}{2RR'(x)}\tfrac{\partial}{\partial x}\Phi(R(x), \arctan R'(x); \alpha, q) \\ &= -p\{1 + \tfrac{r^2}{cos\,\alpha}\tfrac{1}{2R(x)}K(x)\} + \tfrac{1}{2R(x)}\{\psi(R(x))[1 + R'^2(x)]^{-1/2} - \Psi(R(x))K(x)\}, \end{aligned} \tag{38}$$

where $K = R''(1+R'^2)^{-3/2}$ is the curvature of the profile. Note the structure inherited from $p\Phi = p(y^2 - \frac{r^2}{\cos\alpha}\cos u) - \Psi(y)\cos u$:

$$z = z_1[p, R] + z_2[R, \psi],$$

the first term depends on p (also via α !) and the profile whereas the second, p-free term is determined by the profile and the material hyperelasticity.

From (37) it is easily seen that (up to the measuring unit $2\pi L_0 hE$)

$$-pd\varphi = \sin u \cdot z \cdot 2yds \tag{39}$$

is just the *longitudinal component* of the total constraint force acting at the inner surface element $2\pi yds$ of the tube. This is another interpretation of Φ.

3.2 Maximum volume segment within a rigid tube

The purely geometric problem of maximal volume of a segment within a rigid tube,

$$\text{Find } (x, y, u): \quad V[y, u] \to \max, \quad R(x) - y \geq 0,$$

is embedded in our current considerations as the limit case $q = 0$. This case deserves special attention not only because of its interesting geometric content but in particular for the following fact: it shows up to be solvable by quadratures and, thus, it proves useful as a starting element for iterative calculations in treating problems $q > 0$.

With $q = 0$ we obtain from (32) the curvature $\dot{u} = (-2y\cos\alpha)/r^2$ of the unconstrained parts of the extremals. Thus these parts have negative curvature (right-handed curves). This implies a fact that already had been noted above: *If the profile is convex ($R'' \geq 0$) then at most one junction point $t_1 < 0$ can occur.* Clearly, if there were a disjunction at $t_2 < 0$, then a right-handed part of the extremal would follow which never can hit the left-handed profile again.

Due to the negative curvature u is strictly monotonic and can be used as a parameter for representing the unconstrained part of the extremal. For the sake of brevity let

$$\sqrt{\cos\alpha} =: \gamma > 0,$$

and $x_1 := x(t_1),\ y_1 := y(t_1),\ u_1 := u(t_1)$. Using (32) and (33) with $C = 0$ the initial value problem then reads

$$\begin{array}{l} \frac{dx}{du} = -\frac{r}{2\gamma}\sqrt{\cos u}, \quad \frac{ds}{du} = -\frac{r}{2\gamma}\frac{1}{\sqrt{\cos u}}, \quad u \in (u_1, \alpha], \\ x(u_1) = x_1, \quad s(u_1) = t_1, \quad y_1 = \frac{r}{\gamma}\sqrt{\cos u_1}. \end{array} \tag{40}$$

By means of the elliptic integrals

$$\begin{array}{l} F(z,k) := \int_0^z \frac{1}{\sqrt{1-v^2}\sqrt{1-k^2v^2}}dv, \quad E(z,k) := \int_0^z \frac{\sqrt{1-k^2v^2}}{\sqrt{1-v^2}}dv, \\ \mathcal{F}(z) := F(z,i), \quad \mathcal{E}(z) := E(z,i) \end{array}$$

we have

$$\begin{aligned}
&\int_0^u \sqrt{\cos v}dv = \sqrt{2}E(\sqrt{2}\sin\tfrac{u}{2},\tfrac{1}{\sqrt{2}}) = sign(\sin\tfrac{u}{2})\{\mathcal{E}(1)-\mathcal{E}(\sqrt{\cos u})\},\\
&\int_0^u \tfrac{1}{\sqrt{\cos v}}dv = \sqrt{2}F(\sqrt{2}\sin\tfrac{u}{2},\tfrac{1}{\sqrt{2}}) = 2sign(\sin\tfrac{u}{2})\{\mathcal{F}(1)-\mathcal{F}(\sqrt{\cos u})\},
\end{aligned}$$

and the extremal can be given the representation

$$\begin{aligned}
&x = x_1 - \tfrac{1}{\sqrt{2}}\tfrac{r}{\gamma}\{E(\sqrt{2}\sin\tfrac{u}{2},\tfrac{1}{\sqrt{2}}) - E(\sqrt{2}\sin\tfrac{u_1}{2},\tfrac{1}{\sqrt{2}})\},\\
&y = \tfrac{r}{\gamma}\sqrt{\cos u}, \qquad\qquad u \in (u_1,\alpha].\\
&s = -\tfrac{1}{2} - \tfrac{1}{\sqrt{2}}\tfrac{r}{\gamma}\{F(\sqrt{2}\sin\tfrac{u}{2},\tfrac{1}{\sqrt{2}}) - F(\sqrt{2}\sin\tfrac{\alpha}{2},\tfrac{1}{\sqrt{2}})\},
\end{aligned} \tag{41}$$

Now at the junction point $(s,x,y,u) = (t_1,x_1,y_1,u_1)$ there holds

$$\begin{aligned}
&\tfrac{r}{\gamma}\sqrt{\cos u_1} = R(x_1),\\
&u_1 = \arctan R'(x_1),\\
&t_1 = -\tfrac{1}{2} - \tfrac{1}{\sqrt{2}}\tfrac{r}{\gamma}\{F(\sqrt{2}\sin\tfrac{u_1}{2},\tfrac{1}{\sqrt{2}}) - F(\sqrt{2}\sin\tfrac{\alpha}{2},\tfrac{1}{\sqrt{2}})\},
\end{aligned}$$

while the $x-$equation in (41) becomes an identity. $x_1 \le 0$ clearly implies $u_1 \le 0$. The first two equations lead to

$$\gamma^2 = \cos\alpha = \frac{r^2}{R(x_1)^2}\frac{1}{\sqrt{1+R'(x_1)^2}}. \tag{42}$$

So α is known as soon as x_1 has been found.

If there is no disjunction point $t_2 < 0$, i.e., *if there is contact along the full interval* $(t_1, 0]$, the coincidence of arclengths,

$$t_1 = \int_0^{x_1} \sqrt{1+R'(x)^2}dx,$$

together with the second and third equation above yields another equation for the determination of x_1.

Observing $\sqrt{2}\sin\frac{u_1}{2} < 0$ we obtain

$$\begin{aligned}
&\tfrac{1}{2} + \int_0^{x_1}\sqrt{1+R'(x)^2}dx =\\
&\quad = -R(x_1)[1+R'(x_1)^2]^{\frac{1}{4}}\{\mathcal{F}([1+R'(x_1)^2]^{-\frac{1}{4}}) - 2\mathcal{F}(1)+\\
&\qquad\qquad + \mathcal{F}(\tfrac{r}{R(x_1)}[1+R'(x_1)^2]^{-\frac{1}{4}})\}
\end{aligned} \tag{43}$$

as a single equation for the unknown $x_1 \in (-\frac{1}{2}, 0)$.

For the case $R(x) = R_1 = const$ the relations from [AS 2005] are met again.

Let us finally recall the equatorial radius, $\bar{R}(r)$, of the *unconstrained* maximal volume segment. $\bar{R}(r)$ is the general upper bound of y (for all $q \le 0$). It can easily be found as y_1 from above if junction (and disjunction) happens at $x_1 = t_1 = 0$, $u_1 = 0$. It follows

$$\bar{R}(r) = \frac{r}{\gamma_0(r)}, \quad \text{where } \gamma_0 \text{ solves } \gamma_0 = 2r\{\mathcal{F}(1) - \mathcal{F}(\gamma_0)\}. \tag{44}$$

In [St 2003]

$$r \mapsto .382210 + .745230r + .060065r^2 + .005035r^3 \tag{45}$$

was shown to be a fairly good L^2-approximation of $\bar{R}(r)$ if $r \in [0, 1.5]$.

3.3 Phase-plane considerations

The *unconstrained* (parts of) extremals are described by the autonomous differential equations (32), which admit $\Phi(y, u; \alpha, q)$, given by (33), as a first integral that is zero-valued along any trajectory (x, y, u) with initial values $(*, r, \alpha)$.

Since the x-differential equation decouples from the others it is apparently convenient to look at the core of the problem in a (u, y)-phase-plane. There are several opportunities to do this. Generally, we fix $r > 0$. Then *first*, fixing $q \geq 0$, we could investigate the family of phase-portraits $\mathcal{P}_\alpha$ parameterized by $\alpha \in \mathbb{R}$. Among the trajectories of $\mathcal{P}_\alpha$, described by $\Phi = C = const \in \mathbb{R}$, there is the unique one passing the point (α, r), described by $\Phi = 0$. *Second*, we could gather exactly these exceptional trajectories, one from each $\mathcal{P}_\alpha$. So we get a distinguished phase-portrait $\mathcal{P}$, whose elements are given by $y^2 - [\frac{r^2}{\cos\alpha} + q\Psi(y)]\cos u = 0,\ \alpha \in \mathbb{R}$. Therefore the generating α-free differential equation to $\mathcal{P}$ now is

$$\begin{aligned} \dot{u} &= \{-2y + q\psi(y)\cos u\}\tfrac{\cos u}{y^2}, \\ \dot{y} &= \sin u. \end{aligned} \tag{46}$$

Clearly, in the proper context of extremals all the objects under consideration are of importance only under the restrictions

$$\begin{aligned} &0 < r \leq y \leq \bar{R}(r), \\ &-\tfrac{\pi}{2} < -\alpha \leq u \leq \alpha < \tfrac{\pi}{2}, \\ &C \leq 0. \end{aligned}$$

Remark: One should keep in mind that $x(\cdot\,)$ is disregarded in phase-plane considerations. So phase-plane events are *only necessary* for what happens with extremals in (x, y)-space. Nevertheless the (u, y)-trajectories are in any case parameterized by s, the arc-length of the curve $(x(\cdot\,), y(\cdot\,))$!

In order to have a concrete start we suppose a linear ψ (Hooke constitution of the membrane), i.e.,

$$\psi(y) := \frac{1}{r}(y - r), \quad \Psi(y) := \frac{1}{2r}(y - r)^2.$$

1. Phase portrait $\mathcal{P}_\alpha$ for fixed α

The phase-space is, of course, a cylinder ($u = -\pi,\ u = \pi$ identified).
Differential equations (r, q, α fixed):

$$\begin{aligned} \dot{u} &= \{-2y + q\tfrac{1}{r}(y - r)\cos u\}/[\tfrac{r^2}{\cos\alpha} + q\tfrac{1}{2r}(y - r)^2], \\ \dot{y} &= \sin u. \end{aligned} \tag{47}$$

Geometric location of horizontal line elements: ($\dot{y} = 0$)

$$h = \{(0, y),\ (\pi, y) \mid y \in \mathbb{R}\}.$$

Geometric location of vertical line elements: ($\dot{u} = 0$)

$$v := \{(u, y) \mid (q \cos u - 2r)y - rq \cos u = 0\}.$$

Fixed points: ($h \cap v$)

$$\begin{aligned} FP_1 &: \quad u = 0,\ (q - 2r)y - rq = 0, \\ FP_2 &: \quad u = \pi,\ (q + 2r)y - rq = 0. \end{aligned}$$

Note that the fixed points do not depend on α and that $q = 2r$ is a distinguished value strongly influencing the shape of the portrait. For $q \to 2r \pm 0$ we observe $FP_1 \to (0, \pm\infty)$, $FP_2 \to (\pi, \frac{r}{2})$. Inspecting the linearized differential equations we obtain

$$\begin{aligned} if\ q &< 2r\ then\ FP_1\ is\ a\ focus\ and\ FP_2\ is\ a\ saddle, \\ if\ q &= 2r\ then\ FP_1\ is\ absent\ and\ FP_2\ is\ a\ saddle, \\ if\ q &> 2r\ then\ FP_1\ and\ FP_2\ are\ saddles. \end{aligned}$$

The slope of the eigenvectors to the saddle points depends monotonically increasing on α.

With $r = .1$ and for characteristic values of q (small, big) the following figures sketch the essentials of phase portraits $\mathcal{P}_\alpha$ with arbitrarily chosen $\alpha = \pi/3$. The parameters r and α are indicated by the horizontal lines $y = r$, $y = \bar{R}(r)$, and by the initial point (α, r). Within the direction field the geometric location v reflects the respective q, and a choice collection of some trajectories (each a level curve $\Phi = const$) gives an impression of the portrait. For $q = 0$ the whole problem reduces to a pendulum dynamics.

Remark: Note that it is just the orbit $\Phi = 0$ through (α, r) which passes the point $(\frac{\pi}{2}, 0)$ vertically. Any orbit $\Phi = C < 0$ is located *below* the former one. All these orbits are bounded from above by a heteroclinic if $q \le 2r$ or by the stable and unstable manifold of FP_1 if $q > 2r$. Then in the domain $y > r$ these *bounded* trajectories are, for any q, oriented from right to left, this implies $\dot{u}(s) < 0$, and that means negative curvature of the (x, y)-curves (meridian) corresponding to free (parts of) extremals.
By the same argument as used at the beginning of section 3.2 we deduce the following proposition which holds for any $q \ge 0$ and almost linear $\psi(\cdot)$ (i.e., hypoelasticity close to Hooke one).

Proposition 3.5 *Let $s_* < 0$ be a junction point. If the profile is convex ($R''(x) \ge 0$) for $x \in (x(s_*), 0)$ then disjunction is impossible on the s-interval $(s_*, 0)$. If the profile is everywhere convex then junction-disjunction-rejunction scenarios are excluded once for all.*

We guess that the negative statements of the proposition are the top of what can be gained by the tools used so far. Therefore the general question for circumstances allowing

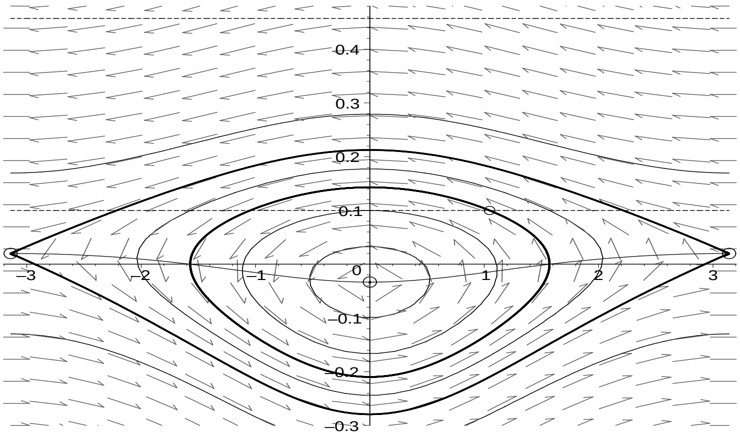


Figure 3: Phase portrait $\mathcal{P}_\alpha$, $\alpha = \pi/3$, with $r = .1$, $q = r/2$

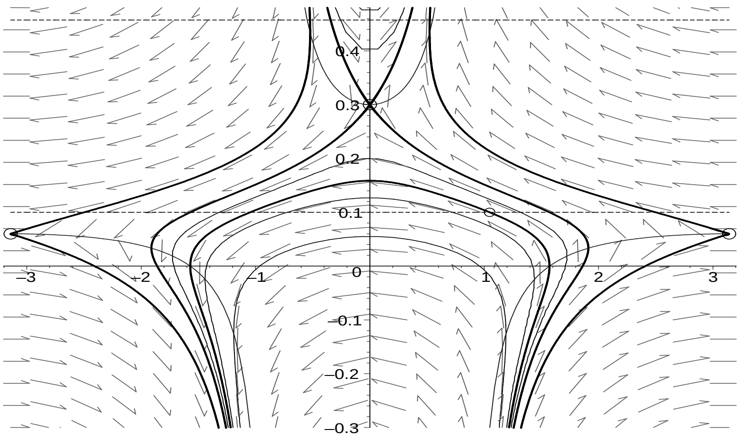


Figure 4: Phase portrait $\mathcal{P}_\alpha$, $\alpha = \pi/3$, with $r = .1$, $q = 3r$

disjunction and rejunction to happen is left as an open problem.

2. Phase portrait $\mathcal{P}$ of orbits through (α, r), $\alpha \in (0, \frac{\pi}{2})$

As phase-space serves now $(-\frac{\pi}{2}, \frac{\pi}{2}) \times (0, \infty)$.
Differential equation:

$$\dot{u} = \{-2y + q\tfrac{1}{r}(y - r)\cos u\}\tfrac{\cos u}{y^2},$$
$$\dot{y} = \sin u.$$

Fixed points:

$$\begin{array}{ll} \textit{saddle } FP_1: \ u=0,\ y=\frac{rq}{q-2r} & \textit{if } q>2r, \\ \textit{none} & \textit{if } q\le 2r. \end{array}$$

Again for $r = .1$ and the same values of q the following figures sketch the orbits belonging to various α, each given as the level curve $\Phi(y, u; \alpha, q) = 0$. On every orbit, starting at (α, r) with $s = -\frac{1}{2}$, the point $s = 0$ is marked. The orbit representing the unrestricted extremal is the one with $u(0) = 0$ (and, then, $(u, y)(\frac{1}{2}) = (-\alpha, r)$), i.e., the corresponding curve (x, y) has length 1.

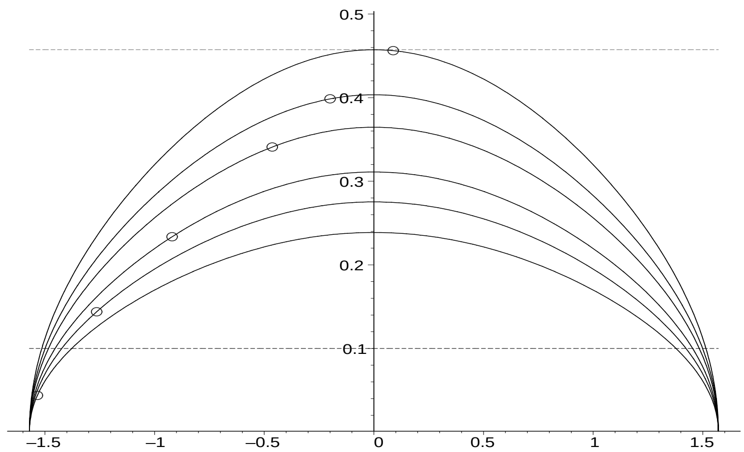


Figure 5: Phase portrait $\mathcal{P}$ with $r = .1$, $q = r/2$. Trajectories starting at $(s, u, y) = (-1/2, \alpha, r)$, phase-points $s = 0$ marked.

Concluding this excursion to phase-spaces let us discuss the junction problem for small $q = \frac{1}{2}r$ (high pressure) and some simple constraints $R(x) - y \geq 0$.

1. Cylindrical tube, $R(x) = R_1 = const$

The phase-curve of the profile $y = R_1$ is the *point* $\mathbf{P} = (0, R_1)$ of the (u, y)-plane. Then, because of the continuity of $u(\cdot)$ and $y(\cdot)$, the correct trajectory in $\mathcal{P}$ is the one meeting the point $\mathbf{P}$ (whereby α gets uniquely fixed). To ensure the end condition $(u, y)(\frac{1}{2}) = (-\alpha, r)$, the phase-point has to 'wait' at $\mathbf{P}$ for twice the 'time' it would need (without constraint) to go from $\mathbf{P}$ to A (phase-point $s = 0$ on orbit) and then proceed its ride until $s = \frac{1}{2}$. This scenario is sketched in Figure 7 and should be recovered in Figure 11.

2. Convex constriction, $R(x) = a + bx^2$

With $a = \frac{3}{2}r$ and $b = \frac{1}{2}$ the phase-curve of the profile $y = R(x)$ is the *convex curve* $\mathbf{P}$: $y = .15 + .5\tan^2 u$ (fat curve) with orientation given by increasing u (corresponding to increasing x in the (x, y)-configuration space) and parameterized by the arc-length of the

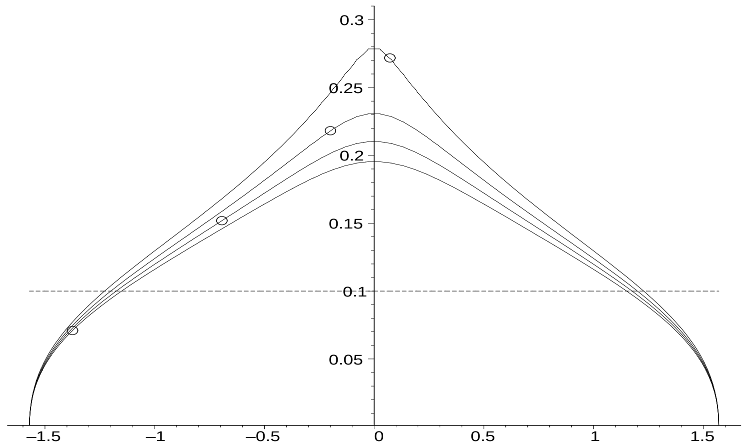


Figure 6: Phase portrait $\mathcal{P}$ with $r = .1$, $q = 3r$. Trajectories starting at $(s, u, y) = (-1/2, \alpha, r)$, phase-points $s = 0$ marked.

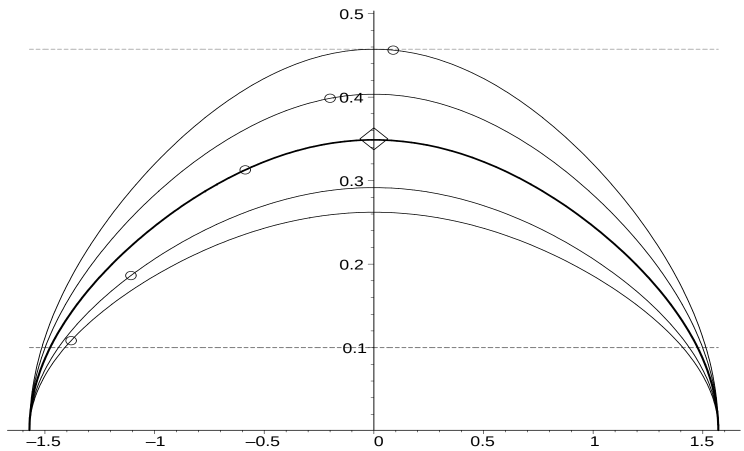


Figure 7: Junction scenario in $\mathcal{P}$ for $r = .1$, $q = r/2$, $R(x) = .35$ (no constriction).

profile (put to zero at $u = 0$). The correct trajectory in $\mathcal{P}$ starting at $s = -\frac{1}{2}$ with some positive $\overline{\alpha}$ (emphasized orbit) now hits $\mathbf{P}$ at $s = t_1 < 0$ in that point P_1 with $u_1 < 0$ whose parameter value (on the profile) equals t_1. Then the phase-point runs down $\mathbf{P}$ arriving the lowest point $(0, a)$ with $s = 0$. So the left part $s \in [-\frac{1}{2}, 0]$ of the extremal is described, rest follows by symmetry. Practically, this construction suffers from the lack of an effective way to find the correct $\overline{\alpha}$.

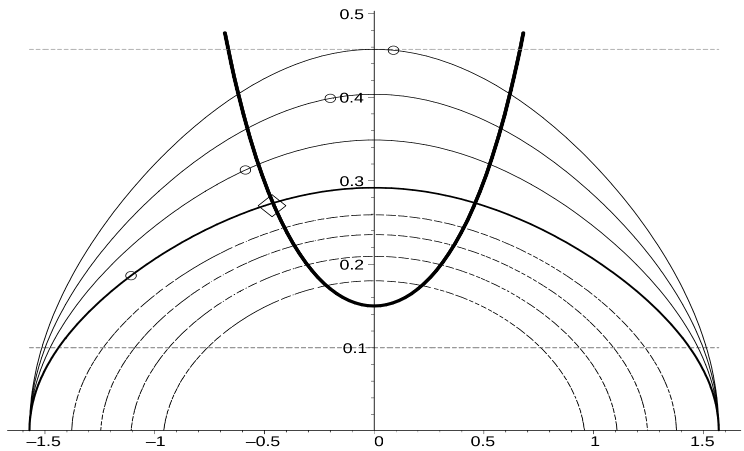


Figure 8: Junction scenario in $\mathcal{P}$ for $r = .1$, $q = r/2$, convex constriction.

Clearly, a disjunction at some $t_2 \in (t_1, 0)$ cannot occur because a following unconstrained (part of the) phase-curve (dashed curves) would be governed by $\Phi(y, u; r, \overline{\alpha}, q) = C < 0$, $\dot{u}(s) < 0$, and never hits $\mathbf{P}$ again (see Proposition 3.5). This scenario is sketched in Figure 8 and should be checked in Figure 14.

3.4 On the join-disjoin-rejoin problem

According to Proposition 3.5 it is clear that the occurence of disjunction-rejunction events essentially depends on a variation of sign of the profile curvature K. Experimental simulations, aimed at the construction of appropriate profiles $R(\cdot)$, hint at the necessity of big negative part of K. Unfortunately, neither such a property of the profile nor the 'supervizing' condition "parameter interval of trajectory has length 1" can be recognized in the (u, y)-phase-plane. Analytical methods (emerging from necessary optimality conditions) do not, of course, provide any sufficient statement about disjunction. The problem addressed here exhibits as a global one. It is not startling that the experimental and analytical attempts to prove or disprove disjunction-rejunction phenomena failed.

In the following we give another qualitative attempt to show (by *gedankenexperiment*) that disjunction-rejunction eventually may happen. These considerations are based on the

Conjecture: Let $R(\cdot)$ be a feasible profile (class C^2 under Assumption (16)), and let $(x, y)_R \in C_2^2$ be the corresponding solution of the optimal control problem. Then the map $R \mapsto (x, y)_R$ is in $C^0[C^2, C_2^0]$.

The strategy is to start from a well understood configuration and then to deform the profile (at fixed q) in appropriate way. We confine to $q = 0$ and use a very narrow parabolic (thus convex) profile in order to have the explicit representations of section 3.2 at our disposal. The starting configuration is shown in Figure 9 (4th quadrant of (x,y)-plane only).

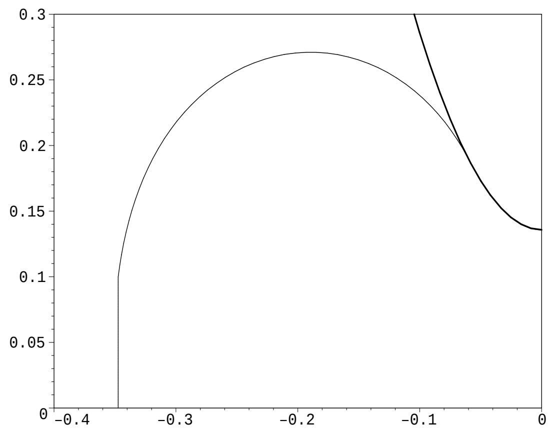


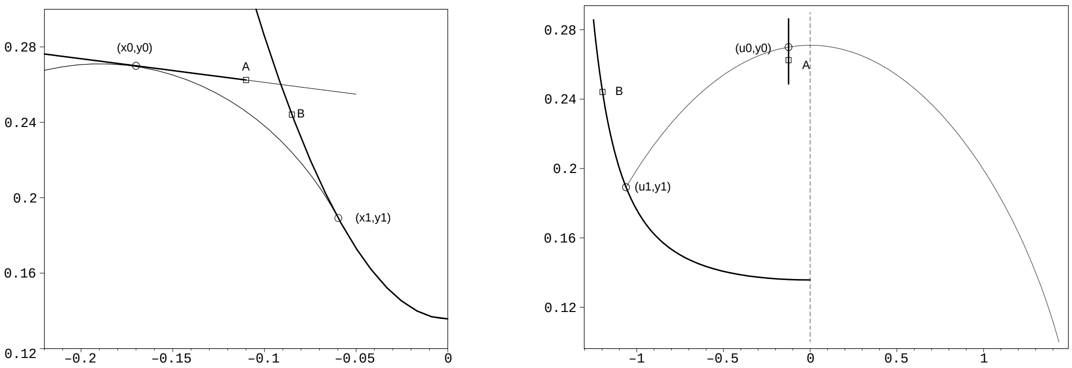
Figure 9: Parabolic profile and extremal ($q = 0$)

Now there is space enough to to perform a C^2-deformation of the profile such that (a) a bit more than the contact region (from $(0, R(0))$ until point B) is kept fixed, and (b) the upper part gets a straight line (beginning at point A) of negative slope u_0 that *touches* the extremal in (x_0, y_0). Mind that the deformation effectively takes place only within a non-contact domain and, thus, does not influence the extremal. Figures 10 sketch the new configuration and the corresponding phase-portrait (the respective C^2-splines $\widehat{AB}$ are not drawn in the figures).

This configuration will be called *configuration* c_0, and the characteristic data are α_0 and $\gamma_0 = \sqrt{\cos \alpha_0}$. Due to junction and touch the points $(u_1, R(x_1))$ and (u_0, y_0) are on the orbit $\Phi(y, u; \alpha_0, 0) = y^2 - \frac{r^2}{\gamma_0^2} \cos u = 0$, so it holds

$$\gamma_0^2 = r^2 \cos u_1 / R^2(x_1) = r^2 \cos u_0 / y_0^2, \quad u_1 = \arctan R'(x_1).$$

Now we perform another C^2-deformation of the profile such that the right part from $(0, R(0))$ until point B is kept fixed, the part left of point A achieves a downward translation ($y \to y - \delta,\ 0 < \delta$, small) while the splines $\widehat{AB}$ deform. Thereby, surely, the extremal is deformed. According to the continuity conjecture the deformation is small, therefore neither the contact part (below (x_1, y_1)) nor the free part (formerly between (x_0, y_0) and (x_1, y_1)) will disappear. For this *configuration* c_δ there are three topological options:
1^o: (x_0, y_0), displaced, is again a *single touch point*;
2^o: (x_0, y_0) has split into *two neighbored touch points*;
3^o: (x_0, y_0) has turned into a small *contact interval.*



(a) Deformed profile and extremal

(b) Corresponding (u, y)-phase portrait

Figure 10: a, b

Claim: *Options 1 and 2 are impossible.*
Proof. If 2^o were true then, between the two touch points, the extremal would need a piece of positive curvature, which is impossible (see section 3.2).
If 1^o happened then $(u_0, y_0 - \delta)$ belongs to the deformed orbit $\Phi(y, u; \alpha_\delta, 0) = 0$ and, with $\cos \alpha_\delta = \gamma_\delta^2$ there holds

$$\gamma_\delta^2 = r^2 \cos u_0/(y_0 - \delta)^2 > \gamma_0^2, \;\; \textit{thus } \alpha_\delta < \alpha_0.$$

The slightly displaced junction point $(x_{1\delta}, y_{1\delta}),\ y_{1\delta} = R(x_{1\delta})$ can be found from

$$\gamma_\delta^2 = r^2 \cos u_{1\delta}/R^2(x_{1\delta}), \;\; \textit{with} \;\; u_{1\delta} = \arctan R'(x_{1\delta}),$$

since $(u_{1\delta}, y_{1\delta})$ is on the same orbit $\Phi(y, u; \alpha_\delta, 0) = 0$. It is simple matter to show $\gamma_\delta > \gamma_0 \Rightarrow x_{10} < x_{1\delta} < 0 \Rightarrow u_1 < u_{1\delta} < 0$. Then we can compare the parameter intervals of the free parts of orbits using (40) (mind signs of integrals):

$$\begin{aligned} & s_{1\delta} + \tfrac{1}{2} = -\tfrac{r}{2\gamma_\delta} \int_{\alpha_\delta}^{u_{1\delta}} \tfrac{1}{\sqrt{\cos u}} du \\ & \ldots\ldots < -\tfrac{r}{2\gamma_0} \int_{\alpha_\delta}^{u_{1\delta}} \tfrac{1}{\sqrt{\cos u}} du = -\tfrac{r}{2\gamma_0} \big(\int_{\alpha_\delta}^{\alpha_0} + \int_{\alpha_0}^{u_1} + \int_{u_1}^{u_{1\delta}} \big) \tfrac{1}{\sqrt{\cos u}} du \\ & \ldots\ldots\ldots < -\tfrac{r}{2\gamma_0} \int_{\alpha 0}^{u_1} \tfrac{1}{\sqrt{\cos u}} du = s_1 + \tfrac{1}{2}. \end{aligned}$$

$T_{1\delta} = \int_{x_{1\delta}}^0 \sqrt{1 + R'(x)^2} dx$ is the length of the contact interval, therefore $T_{1\delta} < T_1$, where in configuration c_0 there holds $s_1 + T_1 = \frac{1}{2}$ (half of parameter interval, $T_1 = \int_{x_1}^0 \sqrt{1 + R'(x)^2} dx$). So we obtain $s_{1\delta} + T_{1\delta} < s_1 + T_1 = \frac{1}{2}$: c_δ with $\delta > 0$ is not a feasible configuration.■

Finally we see that only option 3^o describes a feasible topology, and this proves that *junction-disjunction-rejunction scenarios must not principally be excluded from investigations.*

4 Simulation results

The results presented in the following three sections are the outcome of calculations done by means of Maple7 straight on the basis of the theory developed above. The simulations focus on segments within tubes of three different types: cylindrical tube, tube with convex constriction, tube with non-convex constriction. The strategy of tackling the problems is always the same: (i) start with the maximum volume problem $q = 0$ (evaluation of formulas and one finite equation containing elliptic integrals), important outcome is the junction point t_1^0; (ii) subdivide the interval $(t_1^0, 0]$ into (equal) parts by $t_{1\nu},\ \nu = 1, .., n$, and determine for positive q the unconstrained parts of the extremals on $[-\frac{1}{2}, t_{1\nu})$ by solving the respective boundary value problem, i.e., initial values $(u, y)(-1/2) = (\alpha, r)$, and junction condition (31) with $t_1 = t_{1\nu}$ using a shooting procedure to find the corresponding $q_\nu > 0$ and α_ν; (iii) for $q > q_n$ there is no contact anymore, the extremals can be found by solving boundary value problems with some decreasing $y_0 = y(0)$ using again a shooting procedure to obtain the corresponding q and α.

Remark: For non-convex constriction the occurence of disjunctions has not been excluded! These cases will be tackled under the *assumption* that there is contact, if any, on $[t_1, 0]$.

Remark: In [St 2003] it had been argued that for segments with $r > .15$ the latitudinal strain ε_1 will never be greater than 2.5. This guarantees that a Hooke law, $\psi(y) = (y-r)/r$, is a good approximation of the hyperelastic (Mooney-Rivlin) behavior of incompressible rubber or latex materials the membrane is made of. We shall adopt this approximation in all following simulations.

4.1 Cylindrical tube (no constriction)

Let $R(x) = R_1 = const,\ r < R_1 < \bar{R}(r)$.
Because of various specific peculiarities this case deserves some extra considerations before going to calculations.

With $R'(x) = 0$, (28) yields $\dot{\varphi} = 0$, and this means that the state-control function

$$\Phi(y, u; \alpha, q) := y^2 - [\frac{r^2}{\cos\alpha} + q\Psi(y)] \cos u$$

now vanishes on the *full* $s-$interval $[-\frac{1}{2}, 0]$ along *any* extremal (x, y, u) which has initial values $y(-\frac{1}{2}) = r,\ u(-\frac{1}{2}) = \alpha$. This corresponds to the fact that the constraint force has zero longitudinal component in this case. Therefore the bracket within Φ, wherever it appears, can be replaced by $y^2/\cos u$.

So we obtain from (26) the multiplier λ_2 in the equivalent form

$$\lambda_2 = -y^2 \tan u + \rho,$$

and differentiation observing the optimality condition (21)(c) yields

$$(2y - q\psi(y)\cos u)\cos u + y^2\dot{u} = \dot{\rho}\cos^2 u.$$

Thus the *unconstrained part* of the extremal, $s \in [-\frac{1}{2}, t_1),\ \dot\rho = 0$, is governed by

$$\begin{array}{lll} \dot u = (-2y + q\psi(y)\cos u)\frac{\cos u}{y^2}, & u(-\frac{1}{2}) = \alpha, & u(t_1) = 0, \\ \dot y = \sin u, & y(-\frac{1}{2}) = r, & y(t_1) = R_1 \text{ (if } t_1 < 0), \\ \dot x = \cos u, & & x(t_1) = t_1, \end{array}$$

whereas for *active constraint* $(s \in (t_1, 0],\ y = R_1,\ u = 0)$

$$\dot\rho = 2R_1 - q\psi(R_1)$$

and

$$\cos\alpha = r^2/[R_1^2 - q\Psi(R_1)] \tag{48}$$

follow. Now $\dot\rho \geq 0$ and $0 < \cos\alpha < 1$ demand bounds for q,

$$q \leq 2R_1/\psi(R_1) \text{ and } q < (R_1^2 - r^2)/\Psi(R_1).$$

The latter inequalities express the evident fact (remind pressure $p = 1/q$) that sufficiently high pressure is needed to achieve contact segment/tube. Comparing with (36) the first inequality also means $z \leq 0$. Nevertheless the minimal value p_0 of p that ensures contact remains still unknown, since possibly

$$p \mapsto z(p) = \begin{cases} 0, & p < p_0, \\ -p + \psi(R_1)/2R_1, & p > p_0. \end{cases}$$

is discontinuous at p_0 (z, the constraint force per unit of area (!) "starts" during inflation with a positive value).
For $p > p_0$ we find

$$z(p) = -p\frac{1}{2R_1}\{2R_1 - q\psi(R_1)\} = -\frac{p}{2R_1}\dot\rho.$$

This gives another interpretation of the multiplier ρ, valid in this case of constant $R(x)$.

And here are some simulation results for

$$r = .15,\ \ R_1 = r + (\bar R(r) + r)/2.$$

Figure 11 shows the inflated segment under different pressures, $p = \infty$ gives maximal volume, 1-point touch occurs at $p = 2.081$.
The next two figures (Figure 12(a) and Figure 12(b)) sketch the constraint force z per unit area, divided by p to avoid values to tend to infinity. We clearly observe that z is discontinuous at p_0.
Finally, the junction point t_1 is depicted as function of p (Figure 13).

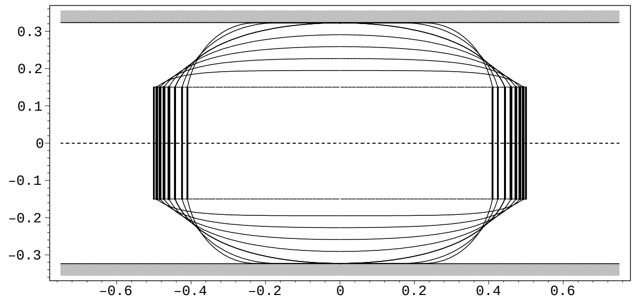
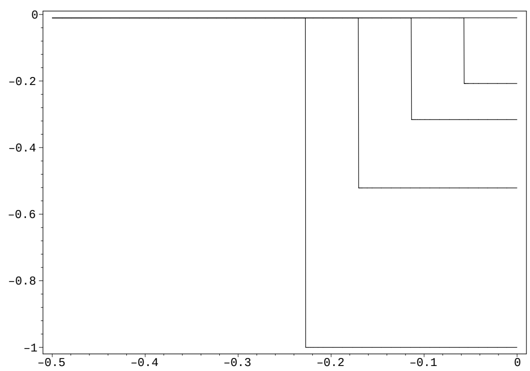


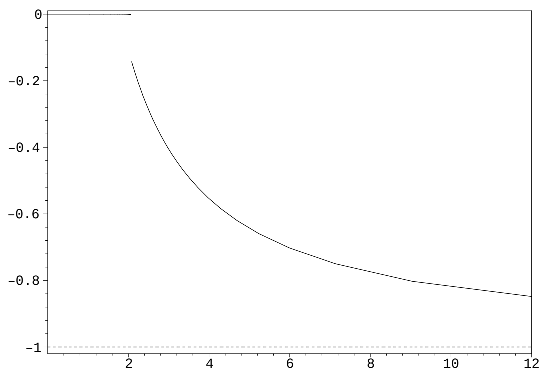
Figure 11: Inflated segment under pressure $p = \infty$, 3.72, 2.08, 1.76, 1.46, .77, 0 (longitudinal cut). 1-point touch at $p_0 = 2.081$.

4.2 Tubes with convex constriction

(A) We consider first a tube having a *quadratic constriction.* And the results are given in Figures 14 and 15.
(B) Since all calculations are done on the interval $(-.5, 0)$, it is evident that there also a *wedge-shaped constriction* 'fulfills' the assumption (16) and can thus be successfully tackled. And the results are given in Figures 16 and Figure 17.



(a) $z(s, p)/p$ vs. s for $p = \infty$, 3.72, 2.61, 2.25, 2.08.



(b) $z(0, p)/p$ vs. p.

Figure 12: a,b

4.3 Tubes with non-convex constriction

Certainly, violation of the assumed convexity of the tube profile is of no relevance if the intervals of active constraint do not contain flat points. In the following a *Gauss-curve profile* is considered whose flat points are inside the interval $(t_1, -t_1)$ in the maximum volume shape. The results for this example are in Figures 18 and 19.

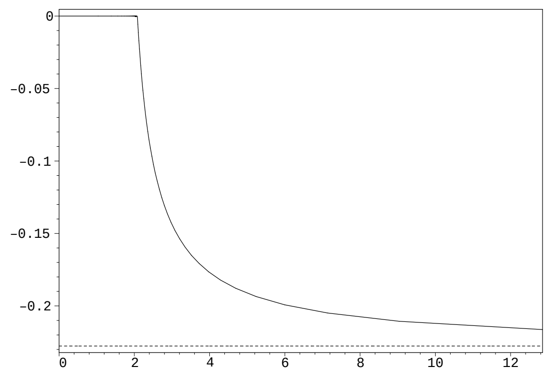


Figure 13: Junction point t_1 vs. p.

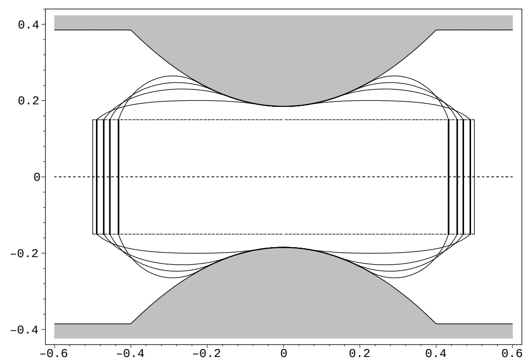
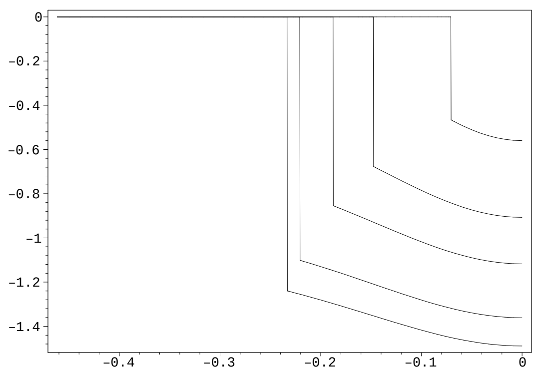
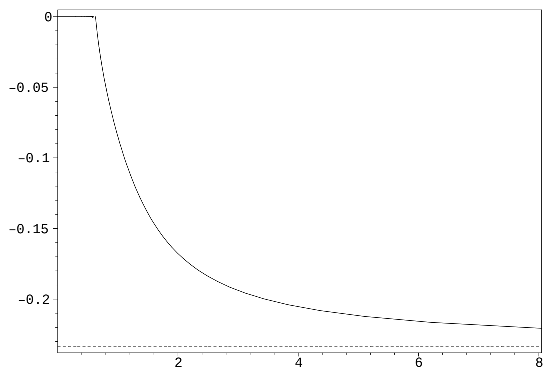


Figure 14: Inflated segment under pressure $p = \infty$, 2.74, 1.61, .89, 0 (longitudinal cut). 1-point touch at $p_0 = .58$.



(a) $z(s,p)/p$ vs. s for $p = \infty$, 8.05, 2.66, 1.62, .91 .



(b) Junction point t_1 vs. p.

Figure 15: a,b

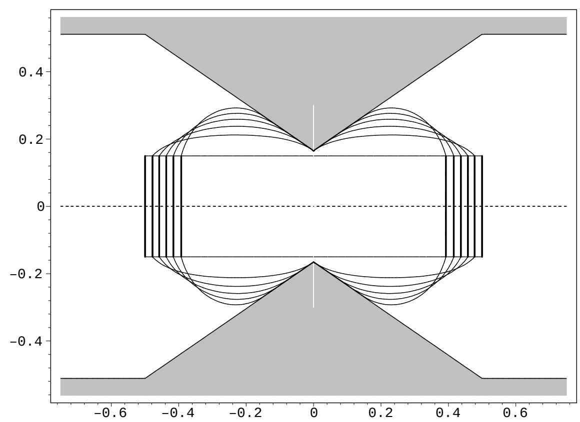
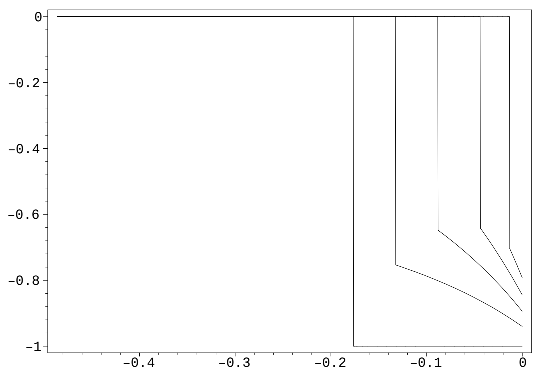
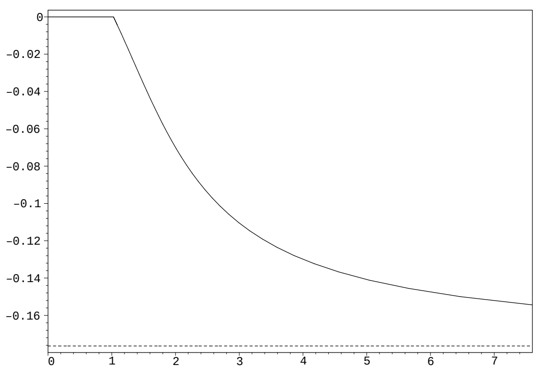


Figure 16: Inflated segment under pressure $p = \infty$, 4.18, 2.36, 1.61, 1.09, 0; (longitudinal cut). 1-point touch at $p_0 = 1.03$.



(a) $z(s,p)/p$ vs. s for $p=\infty$, 4.18, 2.36, 1.61, 1.20 .



(b) Junction point t_1 vs. p.

Figure 17: a,b

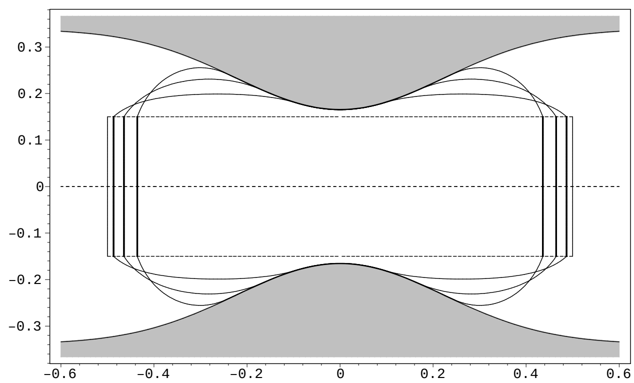
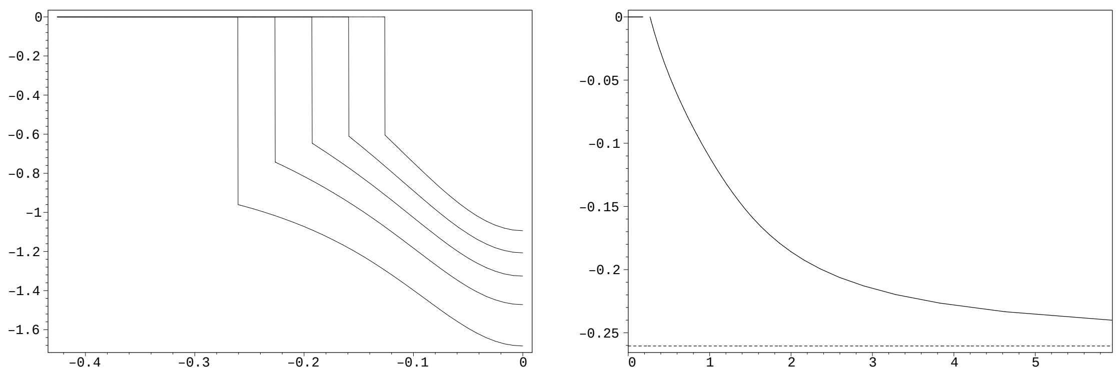


Figure 18: Inflated segment under pressure $p=\infty$, 1.86, .90; (longitudinal cut). 1-point touch at $p_0 = .18$.

Conclusion

The object of the investigations presented above is a compliant inflatable mechanical device '*segment*' (part of a worm or of a system in medical endoscopy) placed in a surrounding rigid tube. The primarily mathematical treatment based on the Principle of Minimal Potential Energy aims at a principal understanding of the quasistatic inflation process of the



(a) $z(s,p)/p$ vs. s for $p = \infty$, 3.83, 2.16, 1.53, 1.14 .

(b) Junction point t_1 vs. p.

Figure 19: a,b

segment when contacting the tube. The investigations go further than comparable ones found in literature. Not only does the theory avoid any presuppositions about the shape of the deformed segment but starts from clear assumptions concerning the rheology of the segment's material and, using adapted units of measurement for all physical quantities, it applies to any segment of fixed slenderness but of arbitrary absolute size, thickness and modulus of elasticity. The developed analytical machinery enables one to compute both the shape of the inflated segment and the force the segment exerts upon the tube under contact.

What has been done here with rigid tubes is to be extended to compliant tubes in next future.

There is a variety of *open problems* to be tackled as soon as possible.

(1) The evaluation of present theoretical results needs an improved numerical treatment aiming at an increased robustness (regarding profile and parameters like q) and application of direct methods.

(2) The theory remains incomplete without a stringent solution of the disjunction problem. In this context the state-control function Φ and the multiplier ρ, which are both intimately connected with the constraint force z, apparently deserve more careful inspection.

(3) Medical applications may need knowledge about very slender segments. Then a non-Hooke hyperelasticity must be envisaged and the effects of the nonlinearity of $\psi(\cdot)$ demand special observation.

Anyway, the rheology of the membrane should be extended by omitting the postulated meridional inextensibility and allowing for a general (or isotropic) Mooney-Rivlin constitution.

(4) With regard to producing segments the membrane stresses σ_1 and σ_2 should be determined (via n^{11} and n^{22}). Then, corresponding to the strength of the membrane material, an upper bound for practically feasible pressures could be found.

(5) Regarding the tube, wavy profiles (relaxing Assumption (16)), non-symmetric constric-

tions and symmetric constrictions combined with eccentrically placed segments ($\xi \neq 0$) should be investigated, both theoretically and by means of direct numerical methods.
(6) In the latter case ($\xi \neq 0$) one expects a non-zero *total longitudinal force* of contact. If this verifies then a link to dynamics shows up: propulsion of a worm with consecutively inflating/deflating segments within a multiply constricted tube.

Amazingly, the latter fact would, according to (38) and (39) again appear as a connecting link to the interpretation of the multiplier ρ that is crucial in a theory with state constraint.

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