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ROBUSTNESS OF FUNNEL CONTROL IN THE GAP METRIC*

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Abstract. For m-input, m-output, finite-dimensional, linear systems satisfying the classical assumptions of adaptive control (i.e., (i) minimum phase, (ii) relative degree one, and (iii) positive high-frequency gain), the well-known funnel controller $k(t) = \frac{\varphi(t)}{1-\varphi(t)\|e(t)\|}$, $u(t) = -k(t)e(t)$ achieves output regulation in the following sense: all states of the closed-loop system are bounded, and, most importantly, transient behavior of the tracking error $e = y - y_{\text{ref}}$ is ensured such that the evolution of $e(t)$ remains in a performance funnel with prespecified boundary $1/\varphi(t)$, where y_{ref} denotes a reference signal bounded with an essentially bounded derivative. As opposed to classical adaptive high-gain output feedback, neither system identification nor the internal model is invoked and the gain $k(\cdot)$ is not monotone. Invoking the conceptual framework of the nonlinear gap metric, we show that the funnel controller is robust in the following sense: the funnel controller copes with bounded input and output disturbances, and, more importantly, it may even be applied to a system not satisfying any of the classical conditions (i)–(iii) as long as the initial conditions and the disturbances are "small" and the system is "close" (in terms of a "small" gap) to a system satisfying (i)–(iii).

Key words. funnel control, gap metric, robust stabilization, tracking, output feedback control

AMS subject classifications. 93D21, 93D15, 93B52, 34D23

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1. Introduction.

In the early 1980s, a novel feature in classical adaptive control was introduced: adaptive control without identifying the entries of the system being controlled. Pioneering contributions in this area include [1, 13, 14, 16, 19] (see also the survey [8] and the references therein). The classical assumptions on such a system class—rather than a single system—of linear m-input, m-output systems are (i) minimum phase, (ii) strict relative degree one, and (iii) positive definite high-frequency gain matrix. Then the simple output feedback $u(t) = -k(t)\, y(t)$ stabilizes each system belonging to the above class and $k(\cdot)$ adapted by $\dot{k}(t) = \|y(t)\|^2$ and variations thereof. Two major drawbacks of the latter strategy (and its variations) are, first, the gain $k(t)$ is, albeit bounded, monotonically increasing which might finally become too large whence amplifying measurement noise, and second, while asymptotic performance is guaranteed, transient behavior is not taken into account (apart from [15], where the issue of prescribed transient behavior is successfully addressed).

A fundamentally different approach, the so-called *funnel controller*, was introduced in [9] in the context of the following output regulation problem: this controller ensures prespecified transient behavior of the tracking error, has a nonmonotone gain, is simpler than the above adaptive controller (actually it is not adaptive in so far as the gain is not dynamically generated), and does not invoke any internal model. Funnel control has been applied to a large class of systems described by functional differential equations including nonlinear and/or infinite-dimensional systems and systems with higher relative degree [10], it has been successfully applied in experiments controlling

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the speed of electric devices [11] (see [8] for further applications and a survey), and recently it has been shown that funnel control copes with input constraints if a certain feasibility inequality holds [6].

The contribution of the present paper is to show that the funnel controller is *robust* in the sense that the control objectives (bounded signals and tracking within a prespecified performance funnel) are still met if the funnel controller is applied to any system "close" (in terms of the gap metric) to a system satisfying the classical assumptions (i)–(iii) and if initial conditions and disturbances are "sufficiently small." This will be achieved by exploiting the concept of (nonlinear) gap metric and graph topology from [5, 2]. We present an example which suggests that there is a tight trade-off between uncertainty and allowable initial condition and disturbances: initial conditions and disturbances might be "very small" in some cases. The results are analogous in structure to their precursors: robustness of the common adaptive controller [4] and of the λ-tracker [7]. However, some care must be exercised in finding the appropriate signal spaces, mainly in proving the existence of a gain function, and applying the known robustness results from [5, 2].

1.1. System class. We consider the class of linear n-dimensional, m-input, m-output systems ($n, m \in \mathbb{N}$ with $n \geq m$)

$$\begin{cases} \dot{x}(t) = A\,x(t) + B\,u_1(t), \qquad x(0) = x^0 \in \mathbb{R}^n, \\ y_1(t) = C\,x(t), \end{cases} \tag{1.1}$$

which satisfy the classical assumptions in high-gain adaptive control, that is, minimum phase with relative degree one and positive definite high-frequency gain matrix; i.e., they belong to

$$\widetilde{\mathcal{M}}_{n,m} := \left\{ \begin{array}{l} (A,B,C) \\ \in \mathbb{R}^{n\times n} \times \mathbb{R}^{n\times m} \times \mathbb{R}^{m\times n} \end{array} \,\middle|\, \begin{array}{l} CB + (CB)^\top > 0 \\ \forall\, s \in \overline{\mathbb{C}}_+ \,:\, \det \begin{bmatrix} sI_n - A & B \\ C & 0 \end{bmatrix} \neq 0 \end{array} \right\}.$$

The state space dimension $n \in \mathbb{N}$ need not be known—only the dimension $m \in \mathbb{N}$ of the input/output space. Most importantly, only structural assumptions are required, but the system entries may be completely unknown.

Note that for any $(A, B, C) \in \widetilde{\mathcal{M}}_{n,m}$ with $\det CB \neq 0$ we may choose $V \in \mathbb{R}^{n\times(n-m)}$ with $\operatorname{rk} V = n - m$ and $\operatorname{im} V = \ker C$; then $T := [B(CB)^{-1}, V]$ is invertible and

$$T^{-1}AT = \begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix}, \quad T^{-1}B = \begin{bmatrix} CB \\ 0 \end{bmatrix}, \quad CT = \begin{bmatrix} I_m & 0_{m\times(n-m)} \end{bmatrix}.$$

Moreover, if (A, B, C) is minimum-phase, then A_4 has spectrum in the open left half complex plane $\mathbb{C}_-$. Therefore, we replace $\widetilde{\mathcal{M}}_{n,m}$ by

$$\mathcal{M}_{n,m} := \left\{ \begin{array}{l} (A,B,C) \\ \in \mathbb{R}^{n\times n} \times \mathbb{R}^{n\times m} \times \mathbb{R}^{m\times n} \end{array} \,\middle|\, \begin{array}{l} A = \begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix},\ B = \begin{bmatrix} B_1 \\ 0 \end{bmatrix},\ C = \begin{bmatrix} I & 0 \end{bmatrix}, \\ B_1, A_1 \in \mathbb{R}^{m\times m},\ \operatorname{spec}(A_4) \subset \mathbb{C}_-, \\ B_1 + B_1^\top > 0 \end{array} \right\}$$

and restrict our attention to systems $(A, B, C) \in \mathcal{M}_{n,m}$ in Byrnes–Isidori normal form (see, for example, [12, Sec. 4]), i.e.,

$$\begin{cases} \dot{y}_1 = A_1 y_1 + A_2 z + CB\,u_1, \quad y_1(0) = y_1^0 \in \mathbb{R}^m, \\ \ \dot{z} = A_3 y_1 + A_4 z, \qquad\qquad\quad\ z(0) = z^0 \in \mathbb{R}^{n-m}. \end{cases} \tag{1.2}$$

We will study the initial value problem (1.1) or (1.2) as *plant* P mapping the interior input signal u_1 to the interior output signal y_1, in conjunction with the *controller* C (the funnel controller (1.4) in our setup), mapping the interior output signal y_2 to the interior input signal u_2, and in the presence of additive input/output disturbances u_0, y_0 so that

$$u_0 \;=\; u_1 + u_2, \qquad y_0 \;=\; y_1 + y_2, \tag{1.3}$$

as depicted in Figure 1.1.

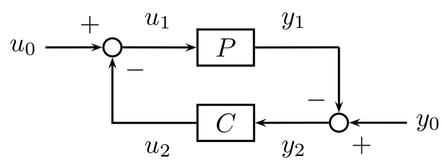


FIG. 1.1. *The closed-loop system* $[P, C]$.

1.2. Performance funnel and funnel control. The control objective, defined in the following subsection, will be captured in terms of the *performance funnel*

$$\mathcal{F}_\varphi := \{(t,e) \in \mathbb{R}_{\geq 0} \times \mathbb{R}^m \mid \varphi(t)\|e\| < 1\},$$

determined by $\varphi(\cdot)$ belonging to

$$\Phi := \left\{ \varphi\colon \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0} \;\middle|\; \begin{array}{l} \varphi \in W^{1,\infty}(\mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}),\ \varphi(0) = 0, \\ \forall\, t > 0 \ :\ \varphi(t) > 0,\ \liminf_{t\to\infty} \varphi(t) > 0, \\ \forall\, \varepsilon > 0 \ :\ \varphi|_{[\varepsilon,\infty)}(\cdot)^{-1} \text{ is globally Lipschitz continuous} \end{array} \right\}.$$

Note that the funnel boundary is given by $1/\varphi(t)$, $t > 0$; see Figure 1.2. The concept of the performance funnel was introduced in [9]. There, it is not assumed that $\varphi(\cdot)$ has the Lipschitz condition as given in Φ; we incorporate this mild assumption for technical reasons. The assumption $\varphi(0) = 0$ allows us to start with arbitrarily large initial conditions x_0 and output disturbances y_0. If for special applications the initial value and y_0 are known, then $\varphi(0) = 0$ may be relaxed by $\varphi(0)\|y_0(0) - Cx^0\| < 1$; see also the simulations in Example 4.6.

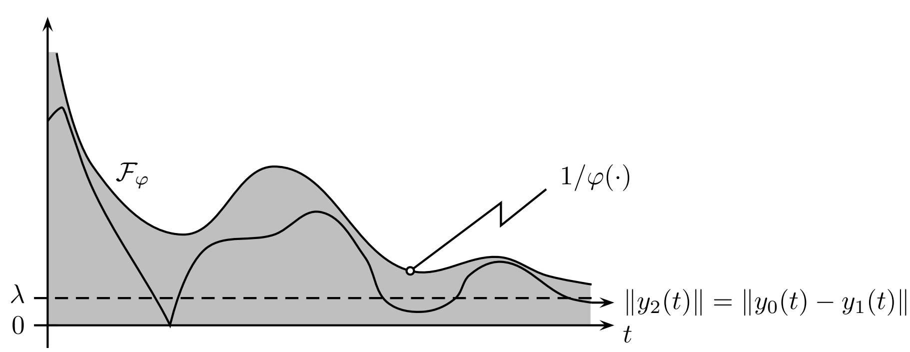


FIG. 1.2. *Funnel* $\mathcal{F}_\varphi$ *with* $\varphi \in \Phi$ *and* $\inf_{t>0} \varphi(t)^{-1} = \lambda$.

The funnel controller, for prespecified $\varphi(\cdot) \in \Phi$, is given by

$$u_2(t) = -k(t)y_2(t), \qquad k(t) = \frac{\varphi(t)}{1 - \varphi(t)\|y_2(t)\|} \tag{1.4}$$

and will be applied to (1.1) or (1.2). Note that the funnel controller (1.4) is actually not an adaptive controller in the sense that it is not dynamic. The gain $k(t)$ is the reciprocal of the distance between $y_2 = y_0 - y_1$ (i.e., the difference of a reference signal y_0 and the output of (1.1)) and the funnel boundary $1/\varphi(t)$; and, loosely speaking, if the error approaches the funnel boundary, then $k(t)$ becomes large, thereby exploiting the high-gain properties of the system and precluding boundary contact.

We will study properties of the closed-loop system generated by the application of the funnel controller (1.4) to systems (1.1) of class $\mathcal{M}_{n,m}$ or of class $\mathcal{P}_{n,m}$ (see below) in the presence of disturbances $(u_0, y_0) \in L^\infty(\mathbb{R}_{\geq 0} \to \mathbb{R}^m) \times W^{1,\infty}(\mathbb{R}_{\geq 0} \to \mathbb{R}^m)$ satisfying the interconnection equations (1.3). The closed-loop system (1.2), (1.4), (1.3) is depicted in Figure 1.3.

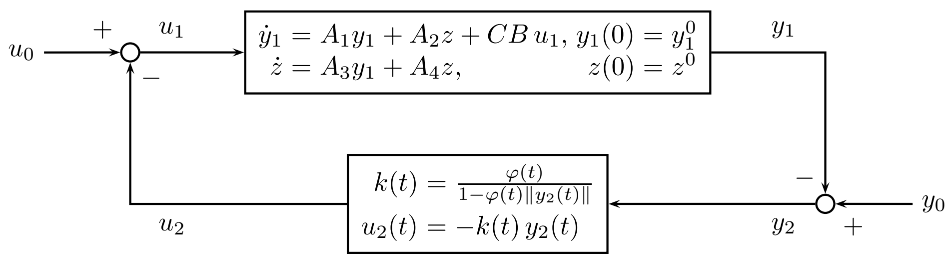


FIG. 1.3. *The "funnel controlled" closed-loop system.*

1.3. Control objectives. We are ready to formulate the control objectives. If the funnel controller (1.4), for prespecified $\varphi \in \Phi$ determining the funnel boundary, is applied to any system (1.1), belonging to the class $\mathcal{M}_{n,m}$, in the presence of disturbances $(u_0, y_0) \in L^\infty(\mathbb{R}_{\geq 0} \to \mathbb{R}^m) \times W^{1,\infty}(\mathbb{R}_{\geq 0} \to \mathbb{R}^m)$ satisfying the interconnection equations (1.3), then the closed-loop system (1.2), (1.4), (1.3), as depicted in Figure 1.3, is supposed to meet the following control objectives:
- all signals are bounded;
- the output error $y_2(t) = y_0(t) - y_1(t)$ of the output disturbance and the output of the linear system evolve in the funnel; in other words,

$$\forall\, t \geq 0 \;:\; (t, y_2(t)) \in \mathcal{F}_\varphi = \{(t,y) \in \mathbb{R}_{\geq 0} \times \mathbb{R}^m \mid \varphi(t)\|y\| < 1\}\,.$$

1.4. Main result: Robustness. The main result of the present paper is to show robustness of the funnel controller in the following sense: The control objectives should still be met if $(A,B,C) \in \mathcal{M}_{n,m}$ is replaced by some system $(\widetilde{A}, \widetilde{B}, \widetilde{C})$ belonging to the system class

$$\mathcal{P}_{q,m} := \left\{ (A,B,C) \in \mathbb{R}^{q\times q} \times \mathbb{R}^{q\times m} \times \mathbb{R}^{m\times q} \;\middle|\; \begin{array}{l} (A,B,C) \text{ is stabilizable} \\ \text{and detectable} \end{array} \right\} \supsetneq \mathcal{M}_{q,m},$$

where $q, m \in \mathbb{N}$ with $q \geq m$, and $(\widetilde{A}, \widetilde{B}, \widetilde{C})$ is close (in terms of the gap metric) to a system belonging to $\mathcal{M}_{n,m}$ and the initial conditions and the disturbances are "small."

For the purpose of illustration, we will further show that a minimal realization $(\widetilde{A}, \widetilde{b}, \widetilde{c})$ of the transfer function

$$s \mapsto \frac{N(M-s)}{(s-\alpha)(s+N)(s+M)}, \qquad \alpha, N, M > 0 \tag{1.5}$$

(which obviously does not satisfy any of the classical assumptions since it is not minimum-phase and has relative degree 2 and negative high-frequency gain), becomes arbitrarily close to a system belonging to $\mathcal{M}_{n,m}$ as N and M tend to infinity.

The paper is organized as follows. In section 2 we show that the funnel controller achieves all control objectives if applied to a linear system (1.1) belonging to class $\mathcal{M}_{n,m}$, in the presence of $L^\infty(\mathbb{R}_{\geq 0} \to \mathbb{R}^m) \times W^{1,\infty}(\mathbb{R}_{\geq 0} \to \mathbb{R}^m)$ input/output disturbances; see Figure 1.3. In section 3, we collect the basics of the framework of gap metric and graph topology from [5, 2, 4] necessary for our setup. Finally, section 4 contains the main result, i.e., robustness of funnel control.

Nomenclature.

$\mathbb{C}_+$, $\mathbb{C}_-$	=	$\{s \in \mathbb{C} \mid \operatorname{Re} s > 0\}$, $\{s \in \mathbb{C} \mid \operatorname{Re} s < 0\}$, respectively
$M > 0$		if and only if $x^\top M x > 0$ for all $x \in \mathbb{R}^n \setminus \{0\}$, where $M \in \mathbb{R}^{n \times n}$
$\|x\|$	=	$\sqrt{x^\top x}$, the Euclidean norm of $x \in \mathbb{R}^n$
$\|M\|$	=	$\max\{\|M x\| \mid x \in \mathbb{R}^m, \|x\| = 1\}$, induced matrix norm of $M \in \mathbb{R}^{n \times m}$
$\|v\|_{\mathcal{V}}$		the norm of $v \in \mathcal{V}$ for any normed vector space $\mathcal{V}$
$L^p(\mathbb{R}_{\geq 0} \to \mathbb{R}^\ell)$		the space of p-integrable functions $y\colon \mathbb{R}_{\geq 0} \to \mathbb{R}^\ell$, $1 \leq p < \infty$, with norm
$\|y\|_{L^p(\mathbb{R}_{\geq 0} \to \mathbb{R}^\ell)}$	=	$\left(\int_0^\infty \|y(t)\|^p \, \mathrm{d}t\right)^{1/p}$
$L^p_{\mathrm{loc}}(I \to \mathbb{R}^\ell)$		the space of locally p-integrable functions $y\colon I \to \mathbb{R}^\ell$, with $\int_K \|y(t)\|^p \, \mathrm{d}t < \infty$ for all compact $K \subset I$, where $1 \leq p < \infty$ and $I \subset \mathbb{R}_{\geq 0}$ is an interval
$L^\infty(\mathbb{R}_{\geq 0} \to \mathbb{R}^\ell)$		the space of essentially bounded functions $y\colon \mathbb{R}_{\geq 0} \to \mathbb{R}^\ell$ with norm
$\|y\|_{L^\infty(\mathbb{R}_{\geq 0} \to \mathbb{R}^\ell)}$	=	$\operatorname{ess\,sup}_{t \geq 0} \|y(t)\|$
$L^\infty_{\mathrm{loc}}(I \to \mathbb{R}^\ell)$		the space of locally bounded functions $y\colon I \to \mathbb{R}^\ell$, with $\operatorname{ess\,sup}_{t \in K} \|y(t)\| < \infty$ for all compact $K \subset I$, where $I \subset \mathbb{R}_{\geq 0}$ is an interval
$W^{1,\infty}(\mathbb{R}_{\geq 0} \to \mathbb{R}^\ell)$		the Sobolev space of absolutely continuous functions $y\colon \mathbb{R}_{\geq 0} \to \mathbb{R}^\ell$ with $y, \dot{y} \in L^\infty(\mathbb{R}_{\geq 0} \to \mathbb{R}^\ell)$ and norm
$\|y\|_{W^{1,\infty}(\mathbb{R}_{\geq 0} \to \mathbb{R}^\ell)}$	=	$\|y\|_{L^\infty(\mathbb{R}_{\geq 0} \to \mathbb{R}^\ell)} + \|\dot{y}\|_{L^\infty(\mathbb{R}_{\geq 0} \to \mathbb{R}^\ell)}$

2. Funnel control.

In this section we show that the funnel controller (1.4) applied to any linear system (A, B, C) of class $\mathcal{M}_{n,m}$ achieves, in the presence of input/output disturbances $(u_0, y_0) \in L^\infty(\mathbb{R}_{\geq 0} \to \mathbb{R}^m) \times W^{1,\infty}(\mathbb{R}_{\geq 0} \to \mathbb{R}^m)$, the following control objectives: y_2 is forced to evolve within a performance funnel $\mathcal{F}_\varphi$ for prespecified $\varphi \in \Phi$, and all signals and states of the closed-loop system (1.2), (1.3), (1.4), as depicted in Figure 1.3, remain essentially bounded. Moreover, it is shown that the derivatives of the output signals y_1, y_2 and the state $\binom{y_1}{\eta}$ are essentially bounded, too.

Write, for notational convenience,

$$\mathcal{D}_{n,m} := \mathcal{M}_{n,m} \times (\mathbb{R}^m \times \mathbb{R}^{n-m}) \times \Phi \times L^\infty(\mathbb{R}_{\geq 0} \to \mathbb{R}^m) \times W^{1,\infty}(\mathbb{R}_{\geq 0} \to \mathbb{R}^m),$$

$n, m \in \mathbb{N}$, $n \geq m$, the set of all tuples of systems, initial values y_1^0, η^0 of the linear system, functions φ describing the funnel $\mathcal{F}_\varphi$, and input/output disturbances (u_0, y_0).

Proposition 2.1. *Let $n, m \in \mathbb{N}$, $n \geq m$, and $\varphi \in \Phi$. Then there exists a continuous map $\nu\colon \mathcal{D}_{n,m} \to \mathbb{R}_{\geq 0}$ such that, for all $d = \left(\left[\begin{smallmatrix} A_1 & A_2 \\ A_3 & A_4 \end{smallmatrix}\right], B, C, (y_1^0, \eta^0), \varphi, u_0, y_0\right) \in \mathcal{D}_{n,m}$, the associated closed-loop initial value problem* (1.2), (1.3), (1.4) *satisfies*

$$\|(k, u_2, y_2, \eta)\|_{L^\infty(\mathbb{R}_{\geq 0} \to \mathbb{R}^{1+m}) \times W^{1,\infty}(\mathbb{R}_{\geq 0} \to \mathbb{R}^{m+n-m})} \leq \nu(d) \tag{2.1}$$

and

$$\forall\, t \geq 0 \;:\; (t, y_2(t)) \in \mathcal{F}_\varphi = \{(t, y) \in \mathbb{R}_{\geq 0} \times \mathbb{R}^m \mid \varphi(t)\|y\| < 1\}. \tag{2.2}$$

Note that Proposition 2.1 also yields that all control objectives are met if the funnel controller (1.4) is applied to $(A, B, C) \in \widetilde{\mathcal{M}}_{n,m}$. This was already shown for $u_0 = 0$ in [9]; the essential difference between the proof in [9] and our proof given here is that we prove the result by the construction of a continuous function ν so that (2.1) holds. The latter is crucial for the robustness analysis of funnel control in section 4. The proof of Proposition 2.1 uses ideas from [4] and [6].

Proof of Proposition 2.1. Let $d = \left(\left[\begin{smallmatrix} A_1 & A_2 \\ A_3 & A_4 \end{smallmatrix}\right], B, C, (y_1^0, \eta^0), \varphi, u_0, y_0\right) \in \mathcal{D}_{n,m}$. Then the closed-loop initial value problem (1.2), (1.3), (1.4) may be written as

$$\tfrac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} y_2 \\ \eta \end{pmatrix} = f(t, y_2, \eta), \qquad \begin{pmatrix} 0 \\ y_2(0) \\ \eta(0) \end{pmatrix} = \begin{pmatrix} 0 \\ y_0(0) - y_1^0 \\ \eta^0 \end{pmatrix} \in \mathcal{F}_\varphi \times \mathbb{R}^{n-m}, \tag{2.3}$$

where the right-hand side is given by

$$\begin{aligned} &f\colon \mathcal{F}_\varphi \times \mathbb{R}^{n-m} \to \mathbb{R}^n, \\ &\qquad (t, y_2, \eta) \mapsto \begin{pmatrix} A_1\, y_2 - A_2\, \eta - CB \frac{\varphi(t)}{1 - \varphi(t)\|y_2\|} y_2 + \dot{y}_0(t) - A_1\, y_0(t) - CB\, u_0(t) \\ -A_3\, y_2 + A_4\, \eta + A_3\, y_0(t) \end{pmatrix}. \end{aligned}$$

We proceed in several steps.

Step 1. We show that the initial value problem (2.3) has an absolutely continuous solution $(y_2, \eta)\colon [0, \omega) \to \mathbb{R}^m \times \mathbb{R}^{n-m}$ for maximal $\omega \in (0, \infty]$; this solution satisfies $(t, y_2(t), \eta(t)) \in \mathcal{F}_\varphi \times \mathbb{R}^{n-m}$ for all $t \in [0, \omega)$ and is unique, and the maximality of ω means that the solution is extended up to the boundary of $\mathcal{F}_\varphi \times \mathbb{R}^{n-m}$: the closure of $\operatorname{graph}((y_2, \eta)|_{[0,\omega)})$ is not a compact subset of $\mathcal{F}_\varphi \times \mathbb{R}^{n-m}$; i.e., for every compact $\mathcal{K} \subset \mathcal{F}_\varphi \times \mathbb{R}^{n-m}$ there exists $t \in [0, \omega)$ such that $(t, y_2(t), \eta(t)) \notin \mathcal{K}$.

Since $\varphi|_{[\varepsilon,\infty)}(\cdot)^{-1}$ is globally Lipschitz for every $\varepsilon > 0$ and $\varphi(0) = 0$, it follows that f is locally Lipschitz on the relatively open set $\mathcal{F}_\varphi \times \mathbb{R}^{n-m}$ in the sense that, for all $(\tau, \xi, \zeta) \in \mathcal{F}_\varphi \times \mathbb{R}^{n-m}$, there exists an open neighborhood O of (τ, ξ, ζ) and a constant $L > 0$ such that

$$\forall\, (t, y, \eta) \in O \;:\; \|f(t, y, \eta) - f(t, \xi, \zeta)\| \leq L(\|y - \xi\| + \|\eta - \zeta\|).$$

Now by the standard theory of ordinary differential equations (see, for example, [18, Th. III.11.III]), the initial value problem (2.3) has the desired properties.

Step 2. We collect some definition and technicalities.

By Step 1 and the properties of φ it follows that

$$\text{(2.4)}\quad \exists\,\delta=\delta(d)>0\ \forall\,t\in[0,\delta]\ :\ \|y_2(t)\|\le\|y_2(0)\|+1\ \wedge\ 1-\varphi(t)\|y_2(t)\|\ge\max\{1/2,\varphi(t)\}.$$

Let $L_\delta>0$ denote a global Lipschitz constant of $\varphi|_{[\delta,\infty)}(\cdot)^{-1}$ (which exists by definition of Φ), and set $\lambda:=\inf\{\varphi(t)^{-1}\,|\,t>0\}$. Note that $(t,y_2(t))\in\mathcal{F}_\varphi$ for all $t\in[0,\omega)$ yields

$$\text{(2.5)}\qquad \forall\,t\in[0,\omega)\ :\ \|y_2(t)\|\le\max\left\{\|\varphi|_{[\delta,\infty)}(\cdot)^{-1}\|_{L^\infty},\|y_0(0)-y_1^0\|+1\right\}.$$

By the minimum phase property of (1.2), i.e., $\operatorname{spec}A_4\subset\mathbb{C}_-$,

$$\text{(2.6)}\qquad \exists\,\alpha,\beta>0\ \forall\,t\ge0\ :\ \|e^{A_4}t\|\le\beta e^{-\alpha t}.$$

In view of the positive definiteness of CB, let $\gamma_{CB}>0$ denote the smallest singular value of $CB+(CB)^\top$, and thus

$$\forall\,v\in\mathbb{R}^m\setminus\{0\}\ :\ \langle v,CBv\rangle\ge\gamma_{CB}\|v\|^2.$$

Step 3. We show that

$$\text{(2.7)}\qquad \forall\,t\in[\delta,\omega)\ :\ \varphi(t)^{-1}-\|y_2(t)\|\ge\varepsilon,$$

where $\delta>0$ is defined by (2.4) and, for $\gamma_{CB},\lambda,L_\delta,\alpha,$ and β defined in Step 2,

$$\text{(2.8)}\quad \varepsilon:=\min\left\{\frac{1}{2},\frac{\lambda}{2},\frac{\gamma_{CB}\,\lambda}{2},\left[L_\delta+\left(\|A_1\|+\|A_2\|\,\|A_3\|\frac{\beta}{\alpha}\right)\cdot\left(\|y_0\|_{L^\infty}+\|\varphi|_{[\delta,\infty)}(\cdot)^{-1}\|_{L^\infty}\right)+\|A_2\|\beta\|\eta^0\|+\|\dot y_0\|_{L^\infty}+\|CB\|\,\|u_0\|_{L^\infty}\right]^{-1}\right\}.$$

Seeking a contradiction, suppose that

$$\exists\,t_1\in[\delta,\omega)\ :\ \varphi(t_1)^{-1}-\|y_2(t_1)\|<\varepsilon.$$

Since $t\mapsto\varphi(t)\|y_2(t)\|$ is continuous on $[0,\omega)$ and in view of (2.4), it follows that

$$\exists\,t_0\ge\delta\ :\ t_0=\max\left\{t\in[\delta,t_1)\,\middle|\,\varphi(t)^{-1}-\|y_2(t)\|=\varepsilon\right\}.$$

Thus, by the definition of Φ,

$$\forall\,t\in[t_0,t_1]\ :\ \varphi(t)^{-1}-\|y_2(t)\|\le\varepsilon\ \wedge\ \|y_2(t)\|\ge\varphi(t)^{-1}-\varepsilon\ge\lambda-\lambda/2,$$

and hence

$$\text{(2.9)}\qquad \forall\,t\in[t_0,t_1]\ :\ \frac{\|y_2(t)\|}{\varphi(t)^{-1}-\|y_2(t)\|}\ge\frac{\lambda}{2\,\varepsilon}.$$

By variation of constants, the second line of the differential equation (2.3) yields

$$\forall\, t \geq 0 \;:\; \eta(t) = e^{A_4 t}\eta^0 + \int_0^t e^{A_4(t-s)} A_3 \left(y_0(s) - y_2(s)\right) \mathrm{d}s; \tag{2.10}$$

thus the first line of the differential equation (2.3) writes, for almost all $t \geq 0$,

$$\begin{aligned}\dot y_2(t) = -A_1(y_0(t) - y_2(t)) + A_2 \int_0^t e^{A_4(t-s)} A_3 \left(y_0(s) - y_2(s)\right) \mathrm{d}s \\ - A_2 e^{A_4 t}\eta^0 + \dot y_0(t) - CBu_0(t) + CB\frac{-\varphi(t)}{1-\varphi(t)\|y_2(t)\|} y_2(t).\end{aligned}$$

Hence, by (2.5), (2.6), (2.9), and (2.8), we conclude that, for almost all $t \in [t_0, t_1]$,

$$\begin{aligned}&\langle y_2(t), \dot y_2(t)\rangle \\ &\leq \|y_2(t)\| \Bigg[\left(\|A_1\| + \|A_2\|\,\|A_3\|\frac{\beta}{\alpha}\right)\left[\|y_0\|_{L^\infty} + \left\|\varphi|_{[\delta,\infty)}(\cdot)^{-1}\right\|_{L^\infty}\right] \\ &\quad + \|A_2\|\beta\|\eta^0\| + \|\dot y_0\|_{L^\infty} + \|CB\|\,\|u_0\|_{L^\infty}\Bigg] - \frac{\varphi(t)\,\langle y_2(t), CBy_2(t)\rangle}{1-\varphi(t)\|y_2(t)\|} \\ &\leq \|y_2(t)\| \Bigg[\left(\|A_1\| + \|A_2\|\,\|A_3\|\frac{\beta}{\alpha}\right)\left[\|y_0\|_{L^\infty} + \left\|\varphi|_{[\delta,\infty)}(\cdot)^{-1}\right\|_{L^\infty}\right] \\ &\quad + \|A_2\|\beta\|\eta^0\| + \|\dot y_0\|_{L^\infty} + \|CB\|\,\|u_0\|_{L^\infty}\Bigg] - \frac{\varphi(t)\gamma_{CB}\|y_2(t)\|}{\varphi(t)\left(\varphi(t)^{-1} - \|y_2(t)\|\right)}\|y_2(t)\| \\ &\leq \|y_2(t)\| \Bigg[\left(\|A_1\| + \|A_2\|\,\|A_3\|\frac{\beta}{\alpha}\right)\left[\|y_0\|_{L^\infty} + \left\|\varphi|_{[\delta,\infty)}(\cdot)^{-1}\right\|_{L^\infty}\right] \\ &\quad + \|A_2\|\beta\|\eta^0\| + \|\dot y_0\|_{L^\infty} + \|CB\|\,\|u_0\|_{L^\infty}\Bigg] - \frac{\gamma_{CB}\lambda}{2\varepsilon}\|y_2(t)\| \\ &\leq -L_\delta\|y_2(t)\|.\end{aligned}$$

Thus

$$\begin{aligned}\|y_2(t_1)\| - \|y_2(t_0)\| &= \int_{t_0}^{t_1} \frac{\langle y_2(\tau), \dot y_2(\tau)\rangle}{\|y_2(\tau)\|}\,\mathrm{d}\tau \\ &\leq -L_\delta(t_1 - t_0) \leq -|\varphi(t_1)^{-1} - \varphi(t_0)^{-1}| \leq \varphi(t_1)^{-1} - \varphi(t_0)^{-1},\end{aligned}$$

whence the contradiction $\varepsilon = \varphi(t_0)^{-1} - \|y_2(t_0)\| \leq \varphi(t_1)^{-1} - \|y_2(t_1)\| < \varepsilon$. This proves (2.7).

Step 4. We show that $\omega = \infty$.

Let $\sigma := \min\left\{1, \inf_{t\in[\delta,\omega)}\varphi(t)\right\} > 0$. By (2.7) it follows, for $\varepsilon > 0$ as defined in (2.8), that

$$\forall\, t \in [\delta, \omega) \;:\; 1 - \varphi(t)\|y_2(t)\| \geq \varepsilon\varphi(t) \geq \varepsilon\sigma,$$

and so, in view of (2.4),

$$\forall\, t \in [0, \omega) \;:\; 1 - \varphi(t)\|y_2(t)\| \geq \varepsilon\sigma.$$

Seeking a contradiction, suppose that $\omega < \infty$. By (2.5) and (2.10) it follows that $\eta \in L^\infty([0,\omega) \to \mathbb{R}^{n-m})$ with $\|\eta|_{[0,\omega)}\|_{L^\infty} \leq c$ for some $c > 0$. Then

$$\mathcal{K} := \left\{(t, y, z) \in \mathcal{F}_\varphi \times \mathbb{R}^{n-m} \,\middle|\, t \in [0, \omega],\ 1 - \varphi(t)\|y\| \geq \varepsilon\sigma,\ \|z\| \leq c\right\}$$

is a compact subset of $\mathcal{F}_\varphi \times \mathbb{R}^{n-m}$ with $(t, y_2(t), \eta(t)) \in \mathcal{K}$ for all $t \in [0, \omega)$, which contradicts the fact that the closure of graph$((y_2, \eta)|_{[0,\omega)})$ is not a compact set; see Step 1. Therefore, $\omega = \infty$.

Step 5. We show (2.1).

Step 4 yields $\omega = \infty$. Then Step 3 and (2.4) guarantee that $(t, y_2(t)) \in \mathcal{F}_\varphi$ for all $t \geq 0$. Moreover, for some $\delta > 0$ as in (2.4), $\|y_2(t)\| \leq \varphi(t)^{-1} - \varepsilon$ for all $t \geq \delta$, and, in view of (2.4), we have $\|y_2(t)\| \leq \|y_2(0)\| + 1 \leq \|y_0(0)\| + \|y_1^0\| + 1$ for all $t \in [0, \delta]$. Thus $y_2 \in L^\infty(\mathbb{R}_{\geq 0} \to \mathbb{R}^m)$ is uniformly bounded in terms of $d = \left(\left[\begin{smallmatrix} A_1 & A_2 \\ A_3 & A_4 \end{smallmatrix}\right], B, C, (y_1^0, \eta^0), \varphi, u_0, y_0\right)$. Moreover, (2.7) and (2.4) yield

$$\forall\, t \geq 0 \ : \ 1 - \varphi(t)\|y_2(t)\| \geq \varepsilon\varphi(t),$$

and so

$$\forall\, t \geq 0 \ : \ k(t) = \frac{\varphi(t)}{1 - \varphi(t)\|y_2(t)\|} \leq \varepsilon^{-1},$$

which gives $k \in L^\infty(R_{\geq 0} \to \mathbb{R})$ and, in view of (2.4), $\|k\|_{L^\infty} \leq \frac{1}{\varepsilon}$; thus k is uniformly bounded in terms of d. Hence, $u_2 = -k\, y_2 \in L^\infty(\mathbb{R}_{\geq 0} \to \mathbb{R}^m)$ is also uniformly bounded in terms of d. By (2.10) we have, for all $t \geq 0$,

$$\begin{aligned}
\|\eta(t)\| &= \left\| e^{A_4 t}\eta^0 + \int_0^t e^{A_4(t-s)} A_3 \left(y_0(s) - y_2(s)\right) \mathrm{d}s \right\| \\
&\leq \beta e^{-\alpha t}\|\eta^0\| + \int_0^t \beta\|A_3\| e^{-\alpha(t-s)} \left(\|y_0\|_{L^\infty} - \|y_2\|_{L^\infty}\right) \mathrm{d}s \\
&\leq \beta\|\eta^0\| e^{-\alpha t} + \frac{\beta}{\alpha}\|A_3\| \left(\|y_0\|_{L^\infty} - \|y_2\|_{L^\infty}\right) \left(1 + e^{-\alpha t}\right);
\end{aligned}$$

hence $\eta \in L^\infty\left(\mathbb{R}_{\geq 0} \to \mathbb{R}^{n-m}\right)$, and, moreover, η is uniformly bounded in terms of the system matrices and the L^∞-norms of y_0 and y_2, which yields that η is uniformly bounded in terms of $d = \left(\left[\begin{smallmatrix} A_1 & A_2 \\ A_3 & A_4 \end{smallmatrix}\right], B, C, (y_1^0, \eta^0), \varphi, u_0, y_0\right)$.

Finally, in view of (2.3), it follows that the derivatives of y_2 and η are also uniformly bounded in terms of d, which yields that $(y_2, \eta) \in W^{1,\infty}\left(\mathbb{R}_{\geq 0} \to \mathbb{R}^m \times \mathbb{R}^{n-m}\right)$. Moreover, this proves the existence of a continuous function $\nu\colon \mathcal{D}_{n,m} \to \mathbb{R}_{\geq 0}$ such that (2.1) holds true.

Step 6. Finally, we show (2.2).

By Step 5 we have $k \in L^\infty(\mathbb{R}_{\geq 0} \to \mathbb{R})$. Thus, and since y_2 is continuous, it follows that, for all $t \geq 0$, $1 - \varphi(t)\|y_2(t)\| > 0$, which shows (2.2) and completes the proof. ◻

3. The concept of the gap metric.

The material in this section is based on [5, Sec. II], [4, Sec. 2], [2, Sec. 2], and mainly [3, Sec. 2]. Definitions for extended and ambient spaces, well posedness, and the nonlinear gap can be found in [3, Sec. 2]; however, gain-functions and gain-function stability, which are required for the robust stability results in section 4, are not defined in [3]. A section about the basic concept of the gap metric needed in the setup of robustness is in [7]; however, the latter contains a technical flaw: extended and ambient spaces are defined there as in [4, Sec. 2] and [2, Sec. 2] and are not applicable to function spaces of continuous functions. Therefore, in the following we correct this flaw when defining extended and ambient spaces and well posedness more carefully. The results in [7] hold true if this minor correction is applied; only the proof of [7, Prop. 4.4] is affected: one has to apply [17, Th. 6.5.3 and Th. 6.5.4], which are revisions of [2, Th. 5.2 and Th. 5.3]; see also subsection 4.3 for more details.

3.1. Generalized signal spaces. Let $\mathcal{X}$ be a nonempty set. For $0 < \omega \le \infty$, let $\mathcal{S}_\omega$ denote the set of all locally integrable maps in $\mathrm{map}([0,\omega) \to \mathcal{X})$. For ease of notation define $\mathcal{S} := \mathcal{S}_\infty$. For $0 < \tau < \omega \le \infty$, define the *truncation operator* T_τ and the *restriction* of maps as follows:

$$T_\tau : \mathcal{S}_\omega \to \mathcal{S}, \qquad v \mapsto T_\tau v := \left(t \mapsto \begin{cases} v(t), & t \in [0,\tau), \\ 0, & t \in [\tau,\infty) \end{cases} \right),$$
$$(\cdot)\big|_{[0,\tau)} : \mathcal{S}_\omega \to \mathcal{S}_\tau, \quad v \mapsto v\big|_{[0,\tau)} := (t \mapsto v(t), \quad t \in [0,\tau)).$$

Consider next a space $\mathcal{V} \subset \mathcal{S}$ of maps defined on $[0,\infty)$ with norm $\|\cdot\|_{\mathcal{V}} : \mathcal{V} \to \mathbb{R}_{\ge 0}$. Note that $T_\tau v$ may not belong to $\mathcal{V}$, for example, if $\mathcal{V}$ contains continuous functions. Therefore, we introduce the norm $\|\cdot\|_{\mathcal{V}|_{[0,\tau)}} : \{v|_{[0,\tau)} \,|\, v \in \mathcal{V}\} \to \mathbb{R}_{\ge 0}$, where $\|v|_{[0,\tau)}\|_{\mathcal{V}|_{[0,\tau)}}$ denotes the norm on the restriction $[0,\tau) \subset \mathbb{R}_{\ge 0}$, and we write, for ease of notation, $\|T_\tau v\|_{\mathcal{V}} = \|v|_{[0,\tau)}\|_{\mathcal{V}|_{[0,\tau)}}$ for $v \in \mathcal{V}$.

We associate with $\mathcal{V}$ spaces as follows:

$$\begin{aligned}
\mathcal{V}[0,\tau) &= \left\{ v \in \mathcal{S}_\tau \,\middle|\, \exists\, w \in \mathcal{V} \text{ with } \|T_\tau w\|_{\mathcal{V}} < \infty \ : \ v = w\big|_{[0,\tau)} \right\} \text{ for } \tau > 0\,; \\
\mathcal{V}_e &= \left\{ v \in \mathcal{S} \,\middle|\, \forall\, \tau > 0 \ : \ v\big|_{[0,\tau)} \in \mathcal{V}[0,\tau) \right\}, \ \textit{the extended space}\,; \\
\mathcal{V}_\omega &= \left\{ v \in \mathcal{S}_\omega \,\middle|\, \forall\, \tau \in (0,\omega) \ : \ v\big|_{[0,\tau)} \in \mathcal{V}[0,\tau) \right\} \text{ for } 0 < \omega \le \infty\,; \\
\mathcal{V}_a &= \textstyle\bigcup_{\omega \in (0,\infty]} \mathcal{V}_\omega, \ \textit{the ambient space.}
\end{aligned}$$

If $v, w \in \mathcal{V}_a$ with $v|_I = w|_I$ on $I = \mathrm{dom}(v) \cap \mathrm{dom}(w)$, then write $v = w$. For $(u,y) \in \mathcal{V}_a \times \mathcal{V}_a$, the domains of u and y may be different; adopt the convention

$$\mathrm{dom}(u,y) \ := \ \mathrm{dom}(u) \cap \mathrm{dom}(y).$$

The set $\mathcal{V} \subset \mathcal{S}$ is said to be a *signal space* if and only if it is (a) a normed vector space and (b) $\sup_{\tau \ge 0} \|T_\tau v\|_{\mathcal{V}} < \infty$ implies $v \in \mathcal{V}$.

For the purpose of illustration, consider $\mathcal{V} = L^\infty(\mathbb{R}_{\ge 0} \to \mathbb{R}^m)$, which obviously satisfies the aforementioned assumptions (a) and (b): $L^\infty(\mathbb{R}_{\ge 0} \to \mathbb{R}^m)$ is a normed space, and, if $\sup_{\tau \ge 0} \|T_\tau v\|_{L^\infty} < \infty$, then $v \in L^\infty(\mathbb{R}_{\ge 0} \to \mathbb{R}^m)$. Note that this also holds for the Sobolev space $W^{1,\infty}(\mathbb{R}_{\ge 0} \to \mathbb{R}^m)$. For $\mathcal{V} = L^\infty(\mathbb{R}_{\ge 0} \to \mathbb{R}^m)$ it follows that $\mathcal{V}_e = L^\infty_{\mathrm{loc}}(\mathbb{R}_{\ge 0} \to \mathbb{R}^m)$, $\mathcal{V}_\omega = L^\infty_{\mathrm{loc}}([0,\omega) \to \mathbb{R}^m)$ for $\omega \in (0,\infty]$, and $\mathcal{V}_a = \cup_{0<\omega\le\infty} L^\infty_{\mathrm{loc}}([0,\omega) \to \mathbb{R}^m)$. It is important to note that $\mathcal{V}_\omega \supsetneq L^\infty([0,\omega) \to \mathbb{R}^m)$.

For a normed signal space $\mathcal{U}$ and the Euclidean space $\mathbb{R}^l$, $l \in \mathbb{N}$, subsets of $\mathcal{V} = \mathbb{R}^l \times \mathcal{U}$ will also be considered, which, on identifying each $\theta \in \mathbb{R}^l$ with the constant signal $t \mapsto \theta$, can be thought of as a normed signal space with norm given by $\|(\theta,x)\|_{\mathcal{V}} = \sqrt{|\theta|^2 + \|x\|^2_{\mathcal{U}}}$.

3.2. Well posedness. A mapping $Q : \mathcal{U}_a \to \mathcal{Y}_a$ is said to be *causal* if and only if

$$\forall\, x, y \in \mathcal{U}_a \ \forall\, \tau \in \mathrm{dom}(x,y) \cap \mathrm{dom}(Qx,Qy) \ : \quad \left[\ x|_{[0,\tau)} = y|_{[0,\tau)} \implies (Qx)|_{[0,\tau)} = (Qy)|_{[0,\tau)}\ \right].$$

Consider $P\colon \mathcal{U}_a \to \mathcal{Y}_a$, $u_1 \mapsto y_1$, and $C\colon \mathcal{Y}_a \to \mathcal{U}_a$, $y_2 \mapsto u_2$ being causal mappings representing the plant and the controller, respectively, and satisfying the closed-loop equations

$$[P, C] \ : \quad y_1 = Pu_1, \quad u_2 = Cy_2, \quad u_0 = u_1 + u_2, \quad y_0 = y_1 + y_2, \tag{3.1}$$

corresponding to the closed-loop system shown in Figure 1.1.

For $w_0 = (u_0, y_0) \in \mathcal{W} := \mathcal{U} \times \mathcal{Y}$, a pair $(w_1, w_2) = ((u_1, y_1), (u_2, y_2)) \in \mathcal{W}_a \times \mathcal{W}_a$, $\mathcal{W}_a := \mathcal{U}_a \times \mathcal{Y}_a$, is a *solution* if and only if (3.1) holds on $\operatorname{dom}(w_1, w_2)$. The (possibly empty) set of solutions is denoted by

$$\mathcal{X}_{w_0} := \{(w_1, w_2) \in \mathcal{W}_a \times \mathcal{W}_a \mid (w_1, w_2) \text{ solves } (3.1)\}.$$

The closed-loop system $[P, C]$, given by (3.1), is said to have

- the *existence property* if and only if $\mathcal{X}_{w_0} \neq \emptyset$;
- the *uniqueness property* if and only if

$$\forall\, w_0 \in \mathcal{W} \ : \ \Big[\ (\hat{w}_1, \hat{w}_2), (\tilde{w}_1, \tilde{w}_2) \in \mathcal{X}_{w_0} \Longrightarrow (\hat{w}_1, \hat{w}_2) = (\tilde{w}_1, \tilde{w}_2) \quad \text{on} \quad \operatorname{dom}(\hat{w}_1, \hat{w}_2) \cap \operatorname{dom}(\tilde{w}_1, \tilde{w}_2) \ \Big].$$

Assume that $[P, C]$ has the existence and uniqueness properties. For each $w_0 \in \mathcal{W}$, define $\omega_{w_0} \in (0, \infty]$ by the property

$$[0, \omega_{w_0}) := \bigcup_{(\hat{w}_1, \hat{w}_2) \in \mathcal{X}_{w_0}} \operatorname{dom}(\hat{w}_1, \hat{w}_2),$$

and define $(w_1, w_2) \in \mathcal{W}_a \times \mathcal{W}_a$, with $\operatorname{dom}(w_1, w_2) = [0, \omega_{w_0})$, by the property $(w_1, w_2)|_{[0,t)} \in \mathcal{X}_{w_0}$ for all $t \in [0, \omega_{w_0})$. This construction induces the closed-loop operator

$$H_{P,C}\colon \mathcal{W} \to \mathcal{W}_a \times \mathcal{W}_a, \ w_0 \mapsto (w_1, w_2).$$

The closed-loop system $[P, C]$, given by (3.1), is said to be

- *locally well posed* if and only if it has the existence and uniqueness properties and the operator $H_{P,C}\colon \mathcal{W} \to \mathcal{W}_a \times \mathcal{W}_a$, $w_0 \mapsto (w_1, w_2)$, is causal;
- *globally well posed* if and only if it is locally well posed and $H_{P,C}(\mathcal{W}) \subset \mathcal{W}_e \times \mathcal{W}_e$;
- $\mathcal{W}$*-stable* if and only if it is locally well posed and $H_{P,C}(\mathcal{W}) \subset \mathcal{W} \times \mathcal{W}$;
- *regularly well posed* if and only if it is locally well posed and

$$\forall\, w_0 \in \mathcal{W} \ : \ \Big[\ \omega_{w_0} < \infty \ \Rightarrow \ \big\|(H_{P,C}w_0)|_{[0,\tau)}\big\|_{\mathcal{W}_\tau \times \mathcal{W}_\tau} \to \infty \text{ as } \tau \to \omega_{w_0} \ \Big]. \tag{3.2}$$

If $[P, C]$ is globally well posed, then for each $w_0 \in \mathcal{W}$ the solution $H_{P,C}(w_0)$ exists on the half line $\mathbb{R}_{\geq 0}$. Regular well posedness means that if the closed-loop system has a finite escape time $\omega_{w_0} > 0$ for some disturbance $w_0 \in \mathcal{W}$, then at least one of the components u_1, u_2 or y_1, y_2 is not a restriction to $[0, \omega_{w_0})$ of a function in $\mathcal{U}$ or $\mathcal{Y}$, respectively. If $[P, C]$ is regularly well posed and satisfies

$$\forall\, w_0 \in \mathcal{W} \ : \ \Big[\ \omega_{w_0} < \infty \ \Rightarrow \ H_{P,C}(w_0)\big|_{[0,\omega_{w_0})} \in \mathcal{W}[0, \omega_{w_0}) \times \mathcal{W}[0, \omega_{w_0}) \Big],$$

there does not exist a solution of $[P, C]$ with a finite escape time, and therefore $[P, C]$ is globally well posed. However, global well posedness does not guarantee that each solution belongs to $\mathcal{W} \times \mathcal{W}$; the latter is ensured by $\mathcal{W}$-stability of $[P, C]$. Note also that neither regular nor global well posedness implies the other.

3.3. Graphs, the nonlinear gap metric, and gain-function stability. To measure the distance between two plants P and P_1, it is necessary to find sets associated with the plant operators within some space where one may define a map which identifies the gap. These sets are the *graphs* of the operators: for the plant operator $P\colon \mathcal{U}_a \to \mathcal{Y}_a$ and the controller operator $C\colon \mathcal{Y}_a \to \mathcal{U}_a$ define the *graph* $\mathcal{G}_P$ of the plant and the *graph* $\mathcal{G}_C$ of the controller, respectively, as follows:

$$\mathcal{G}_P := \left\{ \begin{pmatrix} u \\ Pu \end{pmatrix} \,\middle|\, u \in \mathcal{U},\, Pu \in \mathcal{Y} \right\} \subset \mathcal{W}, \quad \mathcal{G}_C := \left\{ \begin{pmatrix} Cy \\ y \end{pmatrix} \,\middle|\, Cy \in \mathcal{U},\, y \in \mathcal{Y} \right\} \subset \mathcal{W}.$$

Note that $\mathcal{G}_P$ and $\mathcal{G}_C$ are, strictly speaking, not subsets of $\mathcal{W}$; however, abusing the notation, one may identify $\mathcal{G}_P \ni \left(\begin{smallmatrix} u \\ Pu \end{smallmatrix}\right) = (u, Pu) \in \mathcal{W}$ and $\mathcal{G}_C \ni \left(\begin{smallmatrix} Cy \\ y \end{smallmatrix}\right) = (Cy, y) \in \mathcal{W}$.

The essence of section 4 is the study of robust stability of funnel control in a specific control context. Robust stability is the property that the stability properties of a globally well posed closed-loop system $[P, C]$ persist under "sufficiently small" perturbations of the plant. In other words, robust stability is the property that $[P_1, C]$ inherits the stability properties of $[P, C]$, when the plant P is replaced by any plant P_1 sufficiently "close" to P. In the present context, plants P and P_1 are deemed to be close if and only if their respective graphs are *close* in the gap sense of [5]. The nonlinear gap is defined as follows.

Let, for signal spaces $\mathcal{U}$ and $\mathcal{Y}$,

$$\Gamma(\mathcal{U}, \mathcal{Y}) := \left\{ P : \mathcal{U}_a \to \mathcal{Y}_a \,\middle|\, P \text{ is causal} \right\},$$

and, for $P_1, P_2 \in \Gamma$, define the (possibly empty) set

$$\mathcal{O}_{P_1,P_2} := \left\{ \Phi :\, \mathcal{G}_{P_1} \to \mathcal{G}_{P_2} \,\middle|\, \Phi \text{ is causal and surjective, and } \Phi(0) = 0 \right\}.$$

The *directed nonlinear gap* is given by

$$\begin{aligned} &\vec{\delta}\colon \Gamma(\mathcal{U}, \mathcal{Y}) \times \Gamma(\mathcal{U}, \mathcal{Y}) \to [0, \infty], \\ &\quad (P_1, P_2) \mapsto \inf_{\Phi \in \mathcal{O}_{P_1,P_2}} \sup_{x \in \mathcal{G}_{P_1} \setminus \{0\},\, \tau > 0} \left(\frac{\| T_\tau (\Phi - I)|_{\mathcal{G}_{P_1}}(x) \|_{\mathcal{U}\times\mathcal{Y}}}{\| T_\tau x \|_{\mathcal{U}\times\mathcal{Y}}} \right), \end{aligned}$$

with the convention that $\vec{\delta}(P_1, P_2) := \infty$ if $\mathcal{O}_{P_1,P_2} = \emptyset$, and the *nonlinear gap* δ is

$$\delta\colon \Gamma(\mathcal{U}, \mathcal{Y}) \times \Gamma(\mathcal{U}, \mathcal{Y}) \to [0, \infty], (P_1, P_2) \mapsto \max\{\vec{\delta}(P_1, P_2), \vec{\delta}(P_2, P_1)\}.$$

The following definition of gain-function stability goes back to [5]: A causal operator $F\colon \mathcal{X} \to \mathcal{V}_a$, where $\mathcal{X}, \mathcal{V}$ are subsets of normed signal spaces, is said to be *gain-function stable* if and only if $F(\mathcal{X}) \subset \mathcal{V}$ and the following nonlinear so-called *gain-function* is well defined:

$$\begin{aligned} &g[F] : (r_0, \infty) \to \mathbb{R}_{\geq 0}, \\ &\quad r \mapsto g[F](r) = \sup \left\{ \| T_\tau F x \|_{\mathcal{V}} \,\middle|\, x \in \mathcal{X},\ \| T_\tau x \|_{\mathcal{X}} \in (r_0, r],\ \tau > 0 \right\}, \end{aligned}$$

where $r_0 := \inf_{x \in \mathcal{X}} \|x\|_{\mathcal{X}} < \infty$.

A closed-loop system $[P, C]$ is said to be *gain-function stable* if and only if it is globally well posed and $H_{P,C}\colon \mathcal{W} \to \mathcal{W}_e \times \mathcal{W}_e$ is gain-function stable.

Observe that $\|T_\tau F x\|_{\mathcal{V}} \leq g[F](\|T_\tau x\|_{\mathcal{X}})$ and note the following facts:

(i) global well posedness of $[P, C]$ implies that $\operatorname{im} H_{P,C} \subset \mathcal{W}_e \times \mathcal{W}_e$;
(ii) gain-function stability of $[P, C]$ implies $\mathcal{W}$-stability of $[P, C]$;
(iii) if $[P, C]$ is $\mathcal{W}$-stable, then $H_{P,C} : \mathcal{W} \to \mathcal{G}_P \times \mathcal{G}_C$ is a bijective operator with inverse $H_{P,C}^{-1} : (w_1, w_2) \mapsto w_1 + w_2$.

To see (iii), note that $H_{P,C}(\mathcal{W}) \subset \mathcal{W} \times \mathcal{W}$ implies that $H_{P,C}(\mathcal{W}) \subset \mathcal{G}_P \times \mathcal{G}_C$, and since, for any $w_1 \in \mathcal{G}_P \subset \mathcal{W}$, $w_2 \in \mathcal{G}_C \subset \mathcal{W}$, one has $w_1 + w_2 \in \mathcal{W}$, it follows that $H_{P,C}(\mathcal{W}) \supset \mathcal{G}_P \times \mathcal{G}_C$. Therefore, think of a gain-function stable $H_{P,C}$ as a surjective operator $H_{P,C} \colon \mathcal{W} \to \mathcal{G}_P \times \mathcal{G}_C$. The inverse of $H_{P,C} \colon \mathcal{W} \to \mathcal{G}_P \times \mathcal{G}_C$ is obviously $H_{P,C}^{-1} : (w_1, w_2) \mapsto w_1 + w_2$.

Finally, we associate with the closed-loop system $[P, C]$ given by (3.1) the following two parallel projection operators:

$$\Pi_{P//C} \colon \mathcal{W} \to \mathcal{W}_a,\ w_0 \mapsto w_1, \qquad \text{and} \qquad \Pi_{C//P} \colon \mathcal{W} \to \mathcal{W}_a,\ w_0 \mapsto w_2.$$

Clearly, $H_{P,C} = \left(\Pi_{P//C}, \Pi_{C//P}\right)$ and $\Pi_{P//C} + \Pi_{C//P} = I$. Therefore, gain-function stability of one of the operators $\Pi_{P//C}$ and $\Pi_{C//P}$ implies the gain-function stability of the other, and so gain-function stability of either operator implies gain-function stability of the closed-loop system $[P, C]$.

We close this section with an example. Define for $\alpha > 0$ and $x^0 \in \mathbb{R}$ the plant operator

$$\begin{aligned} P_{\alpha;x^0} \colon L_e^\infty(\mathbb{R}_{\geq 0} \to \mathbb{R}) &\to W_e^{1,\infty}(\mathbb{R}_{\geq 0} \to \mathbb{R}), \\ u_1 &\mapsto y_1 = x,\ \dot{x} = \alpha x + u_1,\ x(0) = x^0. \end{aligned} \tag{3.3}$$

Note that the transfer function (1.5) has a minimal realization $(\widetilde{A}, \widetilde{b}, \widetilde{c})$ of form (1.1), where

$$\widetilde{A} := \begin{bmatrix} 0 & 1 & 0 \\ \alpha N + 2M(\alpha - M - N), & \alpha - 2M - N, & 2M(NM + M^2 - \alpha M - \alpha N) \\ -1 & 0 & -M \end{bmatrix}, \quad \widetilde{b} := \begin{bmatrix} 0 \\ -N \\ 0 \end{bmatrix}, \quad \widetilde{c} := [1, 0, 0], \tag{3.4}$$

and $N, M > 0$. This defines, for $\widetilde{x}^0 \in \mathbb{R}^3$, the plant operator

$$\begin{aligned} P_{N,M,\alpha;\widetilde{x}^0} \colon L_e^\infty(\mathbb{R}_{\geq 0} \to \mathbb{R}) &\to W_e^{1,\infty}(\mathbb{R}_{\geq 0} \to \mathbb{R}), \\ u_1 &\mapsto y_1 = \widetilde{c}\,x,\ \dot{x} = \widetilde{A}\,x + \widetilde{b}\,u_1,\ x(0) = \widetilde{x}^0. \end{aligned} \tag{3.5}$$

In [7, Ex. 3.5] it is shown that, for $x^0 = 0$, $\widetilde{x}^0 = 0$, $\alpha > 0$, and sufficiently large $M > 0$ and $N = 2M$, $P_{\alpha;0}$ is close to $P_{N,M,\alpha;0}$ in the sense that

$$\limsup_{M \to \infty} \vec{\delta}(P_{\alpha;0}, P_{2M,M,\alpha;0}) = 0. \tag{3.6}$$

4. Robustness of the funnel controller.

4.1. Well posedness of the nominal closed-loop system. For $n, m \in \mathbb{N}$ with $n \geq m$, consider $\mathcal{P}_{n,m}$ as a subspace of the Euclidean space $\mathbb{R}^{n^2+2mn}$ by identifying a plant $\theta = (A, B, C)$ with a vector θ consisting of the elements of the plant matrices, ordered lexicographically. With normed signal spaces $\mathcal{U}$ and $\mathcal{Y}$ and $(\theta, x^0) \in \mathcal{P}_{n,m} \times \mathbb{R}^n$, where $x^0 \in \mathbb{R}^n$ is the initial value of a linear system (1.1), we associate the causal plant operator

$$P(\theta, x^0) \colon \mathcal{U}_a \to \mathcal{Y}_a, \quad u_1 \mapsto P(\theta, x^0)(u_1) := y_1, \tag{4.1}$$

where, for $u_1 \in \mathcal{U}_a$ with $\mathrm{dom}(u_1) = [0,\omega)$, we have $y_1 = cx$, x being the unique solution of (1.1) on $[0,\omega)$. Note that P is a map from $\bigcup_{n\geq m}(\mathcal{P}_{n,m}\times\mathbb{R}^n)$ to the space of maps $\mathcal{U}_a \to \mathcal{Y}_a$. Consider, for $\varphi\in\Phi$, the control strategy (1.4) and associate the causal control operator, parameterized by φ, i.e.,

$$C(\varphi)\colon \mathcal{Y}_a \to \mathcal{U}_a, \qquad y_2 \mapsto C(\varphi)(y_2) := u_2. \tag{4.2}$$

Note that C is a map from the set of inverse funnel boundary functions Φ to the space of causal maps $\mathcal{Y}_a \to \mathcal{U}_a$.

In this subsection we show that, for $\mathcal{U} = L^\infty(\mathbb{R}_{\geq 0}\to\mathbb{R}^m)$ and $\mathcal{Y} = W^{1,\infty}(\mathbb{R}_{\geq 0}\to\mathbb{R}^m)$, the closed-loop system $[P(\theta,x^0), C(\varphi)]$ of any plant of the form (1.1) (with associated operator $P(\theta,x^0)$) and controller (1.4) (with associated operator $C(\varphi)$), where $(\theta,x^0)\in\mathcal{P}_{n,m}\times\mathbb{R}^n$ and $\varphi\in\Phi$, is regularly well posed. Furthermore, we show that, for $\theta\in\mathcal{M}_{n,m}$, the closed-loop system $[P(\theta,x^0),C(\varphi)]$ is globally well posed and $(\mathcal{U}\times\mathcal{Y})$-stable.

Proposition 4.1. *Let $n,m\in\mathbb{N}$ with $n\geq m$, $\varphi\in\Phi$, $(\theta,x^0)\in\mathcal{M}_{n,m}\times\mathbb{R}^n$, and $(u_0,y_0)\in L^\infty(\mathbb{R}_{\geq 0}\to\mathbb{R}^m)\times W^{1,\infty}(\mathbb{R}_{\geq 0}\to\mathbb{R}^m)$. Then, for plant operator $P(\theta,x^0)$ and funnel control operator $C(\varphi)$, given by* (4.1) *and* (4.2)*, respectively, the closed-loop initial value problem $[P(\theta,x^0),C(\varphi)]$, given by* (1.2), (1.3), (1.4)*, is globally well posed and, moreover, $[P(\theta,x^0),C(\varphi)]$ is $\big(L^\infty(\mathbb{R}_{\geq 0}\to\mathbb{R}^m)\times W^{1,\infty}(\mathbb{R}_{\geq 0}\to\mathbb{R}^m)\big)$-stable.*

Proof. The proposition is a direct consequence of Proposition 2.1. ∎

In the following subsection we show that an application of the funnel controller to any stabilizable and detectable linear system (A,B,C) yields a closed-loop system which is regularly well posed. This is required for the robustness analysis in subsection 4.3, namely, the application of [17, Th. 6.5.3 and Th. 6.5.4].

4.2. Well posedness of the general closed-loop system. For $(A,B,C)\in\mathcal{P}_{n,m}$, $x^0\in\mathbb{R}^n$, and $\varphi\in\Phi$, the closed-loop initial value problem (1.1), (1.3), (1.4) may be written as

$$\begin{cases} \dot{x}(t) = Ax(t) + B[u_0(t) - u_2(t)], \qquad x(0) = x^0 \in\mathbb{R}^n,\\ k(t) = \dfrac{\varphi(t)}{1-\varphi(t)\|y_2(t)\|},\\ y_2(t) = y_0(t) - Cx(t),\\ u_2(t) = -k(t)y_2(t). \end{cases} \tag{4.3}$$

Proposition 4.2. *Let $n\in\mathbb{N}$ with $n\geq m$, $\varphi\in\Phi$, $(\theta,x^0)\in\mathcal{P}_{n,m}\times\mathbb{R}^n$, and $(u_0,y_0)\in L^\infty(\mathbb{R}_{\geq 0}\to\mathbb{R}^m)\times W^{1,\infty}(\mathbb{R}_{\geq 0}\to\mathbb{R}^m)$. Then, for plant operator $P(\theta,x^0)$ and funnel control operator $C(\varphi)$, given by* (4.1) *and* (4.2)*, respectively, the closed-loop initial value problem $[P(\theta,x^0),C(\varphi)]$, given by* (4.3)*, has the following properties:*

(i) *there exists a unique solution $x\colon[0,\omega)\to\mathbb{R}^n$ for some $\omega\in(0,\infty]$, and the solution is maximal in the sense that for every compact $\mathcal{K}\subset\mathbb{R}_{\geq 0}\times\mathbb{R}^n$ there exists $t\in[0,\omega)$ such that $(t,x(t))\notin\mathcal{K}$;*

(ii) *if $(u_2,y_2)\in L^\infty([0,\omega)\to\mathbb{R}^m)\times W^{1,\infty}([0,\omega)\to\mathbb{R}^m)$, then $\omega=\infty$, $k\in L^\infty(\mathbb{R}_{\geq 0}\to\mathbb{R})$, and y_2 is uniformly bounded away from the funnel boundary $1/\varphi(\cdot)$;*

(iii) *$[P(\theta,x^0),C(\varphi)]$ is regularly well posed.*

Proof. Set, for $\varphi\in\Phi$ and $y_0\in W^{1,\infty}(\mathbb{R}_{\geq 0}\to\mathbb{R}^m)$,

$$\mathcal{H}_{\varphi,y_0} := \left\{(t,x)\in\mathbb{R}_{\geq 0}\times\mathbb{R}^n \,\middle|\, \varphi(t)\|y_0(t) - C\,x\| < 1\right\}.$$

(i) The initial value problem (4.3) may be written as

$$\dot{x} = g(t,x), \qquad x(0) = x^0, \quad (0, y_0(0) - C\,x_0) \in \mathcal{H}_{\varphi,y_0}, \tag{4.4}$$

where

$$g\colon \mathcal{H}_{\varphi,y_0} \to \mathbb{R}^n, \quad (t,x) \mapsto Ax + Bu_0(t) + \frac{\varphi(t)}{1-\varphi(t)\|y_0(t) - C\,x\|} B(y_0(t) - C\,x)$$

satisfies, in view of $\varphi|_{[\varepsilon,\infty)}(\cdot)^{-1}$ being globally Lipschitz for every $\varepsilon > 0$ and $\varphi(0) = 0$ (see the definition of Φ in section 1), a local Lipschitz condition on the relatively open set $\mathcal{H}_{\varphi,y_0}$ in the sense that, for all $(\tau,\xi) \in \mathcal{H}_{\varphi,y_0}$, there exists an open neighborhood O of (τ,ξ) and a constant $L > 0$ such that

$$\forall\,(t,x) \in O \;:\; \|g(t,x) - g(t,\xi)\| \le L\|x - \xi\|.$$

Therefore, standard theory of ordinary differential equations (see, for example, [18, Th. III.11.III]) yields that (4.4), and therefore (4.3), has an absolutely continuous solution $x\colon [0,\omega) \to \mathbb{R}^n$ for some $\omega \in (0,\infty]$, which satisfies $(t,x(t)) \in \mathcal{H}_{\varphi,y_0}$. Moreover, the solution is unique and the solution can be extended up to the boundary of $\mathcal{H}_{\varphi,y_0}$. In other words, for every compact $\mathcal{K} \subset \mathcal{H}_{\varphi,y_0}$, there exists $t \in [0,\omega)$ such that $(t,x(t)) \notin \mathcal{K}$, as required.

(ii) Suppose $(u_2,y_2) \in L^\infty([0,\omega) \to \mathbb{R}^m) \times W^{1,\infty}([0,\omega) \to \mathbb{R}^m)$ and, for contradiction, $\omega < \infty$. By boundedness of φ (see the definition of Φ), it follows that there exists $\lambda > 0$ such that $\varphi(t) \le 1/\lambda$ for all $t \in [0,\omega)$. Thus

$$\forall\,t \in [0,\omega) \;:\; 1 - \varphi(t)\|y_2(t)\| \le \frac{1}{2} \;\;\Rightarrow\;\; \frac{1}{2} \le \varphi(t)\|y_2(t)\| \le \frac{\|y_2(t)\|}{\lambda} \;\;\Rightarrow\;\; \|y_2(t)\| \ge \frac{\lambda}{2},$$

which yields, in view of $y_2 \in L^\infty([0,\omega) \to \mathbb{R}^m)$ and $\frac{-\varphi}{1-\varphi\|y_2\|} y_2 = u_2 \in L^\infty([0,\omega) \to \mathbb{R})$, that

$$\forall\,t \in [0,\omega) \;:\; 1 - \varphi(t)\|y_2(t)\| \le \frac{1}{2}$$
$$\Rightarrow\;\; \|u_2\|_{L^\infty} \ge \frac{\varphi(t)\,\|y_2(t)\|}{1-\varphi(t)\|y_2(t)\|} \ge \frac{\lambda\varphi(t)}{2(1-\varphi(t)\|y_2(t)\|)};$$

thus $\frac{\varphi}{1-\varphi\|y_2\|}$ is bounded on $\{t \in [0,\omega) \,|\, 1 - \varphi(t)\|y_2(t)\| \le 1/2\}$. Moreover, for all $t \in [0,\omega)$ such that $1 - \varphi(t)\|y_2(t)\| > 1/2$, $\frac{\varphi(t)}{1-\varphi(t)\|y_2(t)\|} \le 2/\lambda$. Thus $k = \frac{\varphi}{1-\varphi\|y_2\|} \in L^\infty([0,\omega) \to \mathbb{R})$. Hence, by continuity of the solution

$$\exists\,\varepsilon > 0 \;\forall\,t \in [0,\omega) \;:\; 1 - \varphi(t)\|y_2(t)\| \ge \varepsilon. \tag{4.5}$$

Then, variation of constants applied to (4.3) yields the existence of constants $c_0 = c_0(B,\lambda,\varepsilon)$, $c_1 = c_1(A) > 0$ such that

$$\forall\; t \in [0,\omega) \;:\; \|x(t)\| \le c_0 \left(e^{c_1\omega} + \int_0^\omega e^{c_1(\omega-s)} \left(\|u_0(s)\| + \|y_2(s)\|\right) \mathrm{d}s \right). \tag{4.6}$$

Since $y_2 \in L^\infty([0,\omega) \to \mathbb{R}^m)$ and $u_0 \in L^\infty(\mathbb{R}_{\ge 0} \to \mathbb{R}^m)$, it follows from the convolution in (4.6) that the right-hand side of (4.6) is bounded by $c_3 = c_0\big(e^{c_1\omega} + (e^{c_1\omega}+1)(\|u_0\|_{L^\infty([0,\omega)\to\mathbb{R}^m)} + \|y_2\|_{L^\infty([0,\omega)\to\mathbb{R}^m)})/c_1\big) > 0$ on $[0,\omega)$, which gives that

$$\mathcal{K} := \{(t,x) \in \mathcal{H}_{\varphi,y_0} \,|\, t \in [0,\omega],\; \|x\| \le c_3\}$$

is a compact subset of $\mathcal{H}_{\varphi,y_0}$ with $(t,x(t)) \in \mathcal{K}$ for all $t \in [0,\omega)$, which contradicts the fact that the closure of $\operatorname{graph}(x|_{[0,\omega)})$ is not a compact set; see (i). Therefore, $\omega = \infty$, and in view of (4.5) we have k bounded and y_2 uniformly bounded away from the funnel boundary $\varphi(\cdot)^{-1}$.

(iii) By (i), the closed-loop initial value problem $[P(\theta,x^0),C(\varphi)]$ is locally well posed. To prove that $[P(\theta,x^0),C(\varphi)]$ is regularly well posed, it suffices to show that (3.2) holds. For arbitrary $w_0 = (u_0,y_0) \in \mathcal{W}$ consider $(w_1,w_2) = H_{P(\theta,x^0),C(\varphi)}(w_0)$, where $\operatorname{dom}(w_1,w_2) = [0,\omega)$ is maximal. Suppose, contrary to the right-hand side of (3.2), $\|(w_1,w_2)|_{[0,\omega)}\|_{\mathcal{W}_\omega\times\mathcal{W}_\omega} < \infty$. Then $(u_2,y_2) \in L^\infty([0,\omega) \to \mathbb{R}^m)\times W^{1,\infty}([0,\omega) \to \mathbb{R}^m)$, which, in view of (ii), yields $\omega = \infty$, i.e., the contrary of the left-hand side of (3.2). Hence the closed-loop system is regularly well posed and the proof is complete. □

4.3. Robustness of funnel control. In Proposition 4.1 we have established that, for $(\theta,x^0,\varphi) \in \mathcal{M}_{n,m}\times\mathbb{R}^n\times\Phi$ and $n,m \in \mathbb{N}$ with $n \geq m$, $(u_0,y_0) \in L^\infty(\mathbb{R}_{\geq 0} \to \mathbb{R}^m)\times W^{1,\infty}(\mathbb{R}_{\geq 0} \to \mathbb{R}^m)$, the closed-loop system $[P(\theta,x^0),C(\varphi)]$ is globally well posed and has certain stability properties.

The purpose of this subsection is to determine conditions under which these properties are maintained when the plant $P(\theta,x^0)$ is perturbed to a plant $P(\widetilde{\theta},\widetilde{x}^0)$, where $(\widetilde{\theta},\widetilde{x}^0) \in \mathcal{P}_{q,m}\times\mathbb{R}^q$ for some $q \in \mathbb{N}$, $q \geq m$, in particular when $\widetilde{\theta} \notin \mathcal{M}_{q,m}$. Proposition 4.2 shows that the closed-loop system $[P(\widetilde{\theta},\widetilde{x}^0),C(\varphi)]$ is regularly well posed. This provides the basis for our main result: Theorem 4.5 shows that stability properties of the funnel controller persist if (a) the plants $P(\widetilde{\theta},0)$ and $P(\theta,0)$ are sufficiently close (in the gap sense) and (b) the initial data $\widetilde{x}^0$ and disturbance $w_0 = (u_0,y_0)$ are sufficiently small.

To establish gap margin results, we will need to construct the augmented plant and controller operators as in [7] and [4]. Note that $0 \notin \mathcal{M}_{n,m}$. Define $\widetilde{\mathcal{U}} := \mathbb{R}^{n^2+2mn}\times\mathcal{U} = \mathbb{R}^{n^2+2mn}\times L^\infty(\mathbb{R}_{\geq 0} \to \mathbb{R}^m)$, and let $\widetilde{\mathcal{W}} := \widetilde{\mathcal{U}}\times\mathcal{Y} = \widetilde{\mathcal{U}}\times W^{1,\infty}(\mathbb{R}_{\geq 0} \to \mathbb{R}^m)$, which can be considered as signal spaces by identifying $\theta \in \mathbb{R}^{n^2+2mn}$ with the constant function $t \mapsto \theta$ and endowing $\widetilde{\mathcal{U}}$ with the norm $\|(\theta,u)\|_{\widetilde{\mathcal{U}}} := \sqrt{\|\theta\|^2 + \|u\|^2_{L^\infty(\mathbb{R}_{\geq 0}\to\mathbb{R}^m)}}$. For given $P(\theta,0)$ as in (4.1), we define the (augmented) plant operator as

$$(4.7)\quad \widetilde{P}\colon \widetilde{\mathcal{U}}_a \to W_a^{1,\infty}(\mathbb{R}_{\geq 0} \to \mathbb{R}^m), \quad (\theta,u_1) = \widetilde{u}_1 \mapsto y_1 = \widetilde{P}(\widetilde{u}_1) := P(\theta,0)(u_1).$$

Fix $\varphi \in \Phi$ and define, for $C(\varphi)$ as in (4.2), the (augmented) controller operator as

$$(4.8)\quad \widetilde{C}\colon W_a^{1,\infty}(\mathbb{R}_{\geq 0} \to \mathbb{R}^m) \to \widetilde{\mathcal{U}}_a, \quad y_2 \mapsto \widetilde{u}_2 = \widetilde{C}(y_2) := \big(0, C(\varphi)(y_2)\big) = (0,u_2)\,.$$

For each nonempty $\Omega \subset \mathcal{M}_{n,m}$, define

$$(4.9)\quad \mathcal{W}^\Omega := (\Omega\times L^\infty(\mathbb{R}_{\geq 0} \to \mathbb{R}^m))\times W^{1,\infty}(\mathbb{R}_{\geq 0} \to \mathbb{R}^m) \ \text{ and } \ H^\Omega_{\widetilde{P},\widetilde{C}} := H_{\widetilde{P},\widetilde{C}}|_{\mathcal{W}^\Omega}.$$

It follows from Proposition 4.1 that $H^\Omega_{\widetilde{P},\widetilde{C}}\colon \mathcal{W}^\Omega \to \widetilde{\mathcal{W}}\times\widetilde{\mathcal{W}}$ is a causal operator for any $\Omega \subset \mathcal{M}_{n,m}$. In Proposition 4.3 we show gain-function stability of $H^\Omega_{\widetilde{P},\widetilde{C}}$. This is a supposition of Theorem 5.2 in [2], the latter being used to show Proposition 4.4 and thus the main result, Theorem 4.5.

PROPOSITION 4.3. *Let $n,m \in \mathbb{N}$ with $n \geq m$, $\varphi \in \Phi$, and assume $\Omega \subset \mathcal{M}_{n,m}$ is closed. Then, for the closed-loop system $[\widetilde{P},\widetilde{C}]$ given by* (3.1), (4.7), *and* (4.8), *the operator $H^\Omega_{\widetilde{P},\widetilde{C}}$ given by* (4.9) *is gain-function stable.*

The proof for Proposition 4.3 is equivalent to the proof of [7, Prop. 4.3], when applying Proposition 2.1 instead of [7, Prop. 2.1], and is therefore omitted.

The following proposition establishes $\big(L^\infty(\mathbb{R}_{\geq 0} \to \mathbb{R}^m) \times W^{1,\infty}(\mathbb{R}_{\geq 0} \to \mathbb{R}^m)\big)$-stability of the closed-loop system $[P(\widetilde{\theta}, \widetilde{x}^0), C(\varphi)]$ for a system $\widetilde{\theta}$ belonging to the system class $\mathcal{P}_{q,m}$ if, for a system θ belonging to $\mathcal{M}_{n,m}$, the gap between $P(\widetilde{\theta}, 0)$ and $P(\theta, 0)$, the initial value $\widetilde{x}^0 \in \mathbb{R}^q$, and the input/output disturbances $w_0 = (u_0, y_0)$ are sufficiently small. The proof uses the robustness results [17, Th. 6.5.3 and Th. 6.5.4].

PROPOSITION 4.4. *Let $n, q, m \in \mathbb{N}$ with $n, q \geq m$, $\mathcal{U} = L^\infty(\mathbb{R}_{\geq 0} \to \mathbb{R}^m)$, $\mathcal{Y} = W^{1,\infty}(\mathbb{R}_{\geq 0} \to \mathbb{R}^m)$, $\mathcal{W} = \mathcal{U} \times \mathcal{Y}$, and $\theta \in \mathcal{M}_{n,m}$. For $(\widetilde{\theta}, \widetilde{x}^0, \varphi) \in \mathcal{P}_{q,m} \times \mathbb{R}^q \times \Phi$, consider $P(\widetilde{\theta}, \widetilde{x}^0)\colon \mathcal{U}_a \to \mathcal{Y}_a$, and $C(\varphi)\colon \mathcal{Y}_a \to \mathcal{U}_a$ defined by* (4.1) *and* (4.2)*, respectively. Then there exist a continuous function $\eta\colon (0, \infty) \to (0, \infty)$ and a function $\psi\colon \mathcal{P}_{q,m} \to (0, \infty)$ such that the following holds:*

$$\forall\ \big(\widetilde{\theta}, \widetilde{x}^0, w_0, r\big) \in \mathcal{P}_{q,m} \times \mathbb{R}^q \times \mathcal{W} \times (0, \infty)\ : \quad \left.\begin{array}{l} \psi(\widetilde{\theta})|\widetilde{x}^0| + \|w_0\|_{\mathcal{W}} \leq r, \\ \vec{\delta}\big(P(\theta, 0), P(\widetilde{\theta}, 0)\big) \leq \eta(r) \end{array}\right\} \implies H_{P(\widetilde{\theta}, \widetilde{x}^0), C(\varphi)}(w_0) \in \mathcal{W} \times \mathcal{W}.$$

The proof of Proposition 4.4 is equivalent to the proof of [7, Prop. 4.4] if the gain-function stability result Proposition 4.3 for funnel control is applied instead of the corresponding result [7, Prop. 4.3]. Moreover, one has to choose signal spaces as in section 2, namely, $\mathcal{U} = L^\infty(\mathbb{R}_{\geq 0} \to \mathbb{R}^m)$ and $\mathcal{Y} = W^{1,\infty}(\mathbb{R}_{\geq 0} \to \mathbb{R}^m)$, instead of $\mathcal{U} = \mathcal{Y} = W^{1,\infty}(\mathbb{R}_{\geq 0} \to \mathbb{R}^m)$, and apply [17, Th. 6.5.3 and Th. 6.5.4] instead of [2, Th. 5.2 and Th. 5.3].

Finally, we are in the position to state and prove the main result of the present paper. Loosely speaking, we show that funnel control achieves the control objectives if applied to a system $\big(\widetilde{A}, \widetilde{B}, \widetilde{C}\big) \in \mathcal{P}_{q,m}$ as long as this system is sufficiently close—in the terms of the gap metric—to a system $(A, B, C) \in \widetilde{\mathcal{M}}_{n,m}$ and the initial value $\widetilde{x}^0 \in \mathbb{R}^q$ for $\big(\widetilde{A}, \widetilde{B}, \widetilde{C}\big)$ and the input/output disturbances (u_0, y_0) are sufficiently small. As a consequence $\big(\widetilde{A}, \widetilde{B}, \widetilde{C}\big) \in \mathcal{P}_{q,m}$ may not even satisfy any of the classical assumptions: minimum phase, relative degree one, and positive high-frequency gain.

THEOREM 4.5. *Let $n, q, m \in \mathbb{N}$ with $n, q \geq m$, $\mathcal{U} = L^\infty(\mathbb{R}_{\geq 0} \to \mathbb{R}^m)$, $\mathcal{Y} = W^{1,\infty}(\mathbb{R}_{\geq 0} \to \mathbb{R}^m)$, $\mathcal{W} = \mathcal{U} \times \mathcal{Y}$, $\varphi \in \Phi$, and $\theta \in \mathcal{M}_{n,m}$. For $(\widetilde{\theta}, \widetilde{x}^0) \in \mathcal{P}_{q,m} \times \mathbb{R}^q$ consider the associated operators $P(\widetilde{\theta}, \widetilde{x}^0)\colon \mathcal{U}_a \to \mathcal{Y}_a$ and $C(\varphi)\colon \mathcal{Y}_a \to \mathcal{U}_a$ defined by* (4.1) *and* (4.2)*, respectively, and the closed-loop initial value problem* (1.1)*,* (1.3)*,* (1.4)*. Then there exist a continuous function $\eta\colon (0, \infty) \to (0, \infty)$ and a function $\psi\colon \mathcal{P}_{q,m} \to (0, \infty)$ such that the following holds:*

$$\forall\ \big(\widetilde{\theta}, \widetilde{x}^0, w_0, r\big) \in \mathcal{P}_{q,m} \times \mathbb{R}^q \times \mathcal{W} \times (0, \infty)\ : \quad \left.\begin{array}{l} \psi(\widetilde{\theta})\|\widetilde{x}^0\| + \|w_0\|_{\mathcal{W}} \leq r, \\ \vec{\delta}\big(P(\theta, 0), P(\widetilde{\theta}, 0)\big) \leq \eta(r) \end{array}\right\} \implies \left\{\begin{array}{l} \forall\, t \geq 0\ :\ (t, y_2(t)) \in \mathcal{F}_\varphi, \\ k \in L^\infty(\mathbb{R}_{\geq 0} \to \mathbb{R}), \\ x \in W^{1,\infty}(\mathbb{R}_{\geq 0} \to \mathbb{R}^q), \end{array}\right. \tag{4.10}$$

where (x, k) and y_2 satisfy (4.3).

Proof. Step 1: We show

$$\big((u_1, y_1), (u_2, y_2)\big) = H_{P(\widetilde{\theta}, \widetilde{x}^0), C(\varphi)}(w_0) \in \mathcal{W} \times \mathcal{W}. \tag{4.11}$$

Choose functions $\eta\colon (0,\infty) \to (0,\infty)$ and $\psi\colon \mathcal{P}_{q,m} \to (0,\infty)$ from Proposition 4.4. Let

$$\big(\widetilde{\theta}, \widetilde{x}^0, w_0, r\big) \in \mathcal{P}_{q,m} \times \mathbb{R}^q \times \mathcal{W} \times (0,\infty) \ : \quad \psi(\widetilde{\theta})|\widetilde{x}^0| + \|w_0\|_{\mathcal{W}} \le r \ \wedge \ \vec{\delta}\big(P(\theta,0), P(\widetilde{\theta},0)\big) \le \ \eta(r).$$

Then Proposition 4.4 gives (4.11).

Step 2: By Proposition 4.2 it follows that (4.3) has a unique solution $x\colon [0,\omega) \to \mathbb{R}^q$ on a maximal interval of existence $[0,\omega)$ for some $\omega \in (0,\infty]$. Proposition 4.2(iii) yields $\omega = \infty$ and $k = \frac{\varphi}{1-\varphi\|y_2\|} \in L^\infty(\mathbb{R}_{\ge 0} \to \mathbb{R})$, the second assertion of (4.10).

Step 3: By Step 2 we have $k \in L^\infty(\mathbb{R}_{\ge 0} \to \mathbb{R})$, which, in view of continuity of $1-\varphi\|y_2\|$ on $(0,\infty)$, yields $1-\varphi(t)\|y_2(t)\| \ge \|\varphi\|_{L^\infty}\|k\|_{L^\infty}^{-1} > 0$. Thus, for all $t \ge 0$, $\varphi(t)\|y_2(t)\| < 1$, which yields the first assertion of (4.10).

Step 4: It remains to show that $x \in W^{1,\infty}(\mathbb{R}_{\ge 0} \to \mathbb{R}^q)$.

Let $\big(\widetilde{A}, \widetilde{B}, \widetilde{C}\big) \in \mathcal{P}_{q,m}$ associated with (1.1). Detectability of $\big(\widetilde{A}, \widetilde{B}, \widetilde{C}\big)$ yields the existence of a matrix $F \in \mathbb{R}^{q\times m}$ such that $\operatorname{spec}(\widetilde{A} + F\widetilde{C}) \subset \mathbb{C}_-$. Setting $g := -\big[F + k\,\widetilde{B}\big]\,(y_0 - y_2) + \widetilde{B}\,u_0 + \widetilde{B}\,ky_0$ gives

$$\dot{x} = \big[\widetilde{A} - k\,\widetilde{B}\widetilde{C}\big]x + \widetilde{B}\,u_0 + \widetilde{B}\,ky_0 = \big[\widetilde{A} + F\widetilde{C}\big]x + g. \tag{4.12}$$

By Proposition 4.4 and Step 3 we have $y_2 \in W^{1,\infty}(\mathbb{R}_{\ge 0} \to \mathbb{R}^m)$ and $k \in L^\infty(\mathbb{R}_{\ge 0} \to \mathbb{R})$, and since $w_0 = (u_0, y_0) \in L^\infty(\mathbb{R}_{\ge 0} \to \mathbb{R}^m) \times W^{1,\infty}(\mathbb{R}_{\ge 0} \to \mathbb{R}^m)$, it follows that $g \in L^\infty(\mathbb{R}_{\ge 0} \to \mathbb{R}^q)$. Hence, by (4.12) and variation of constants we obtain $x \in L^\infty(\mathbb{R}_{\ge 0} \to \mathbb{R}^q)$. The first equation in (4.3) then gives $\dot{x} \in L^\infty(\mathbb{R}_{\ge 0} \to \mathbb{R}^q)$, which shows the third assertion in (4.10), and the proof is complete. ∎

Example 4.6. We revisit, for $\alpha, N, M > 0$, the plant operators $P_{\alpha;x^0}$ and $P_{N,M,\alpha;\widetilde{x}^0}$ defined by (3.3) and (3.5), respectively. These plants will be studied in conjunction with the control operator $C(\varphi)$ defined by (4.2).

In passing, note that $P_{\alpha;x^0}$ has transfer functions $s \mapsto \frac{1}{s-\alpha}$; the plant $P_{N,M,\alpha;\widetilde{x}^0}$ has transfer function $s \mapsto N(M-s)/[(s-\alpha)(s+N)(s+M)]$; and, for $(\widetilde{A}, \widetilde{b}, \widetilde{c})$ as in (3.4), a minimal realization in normal form is

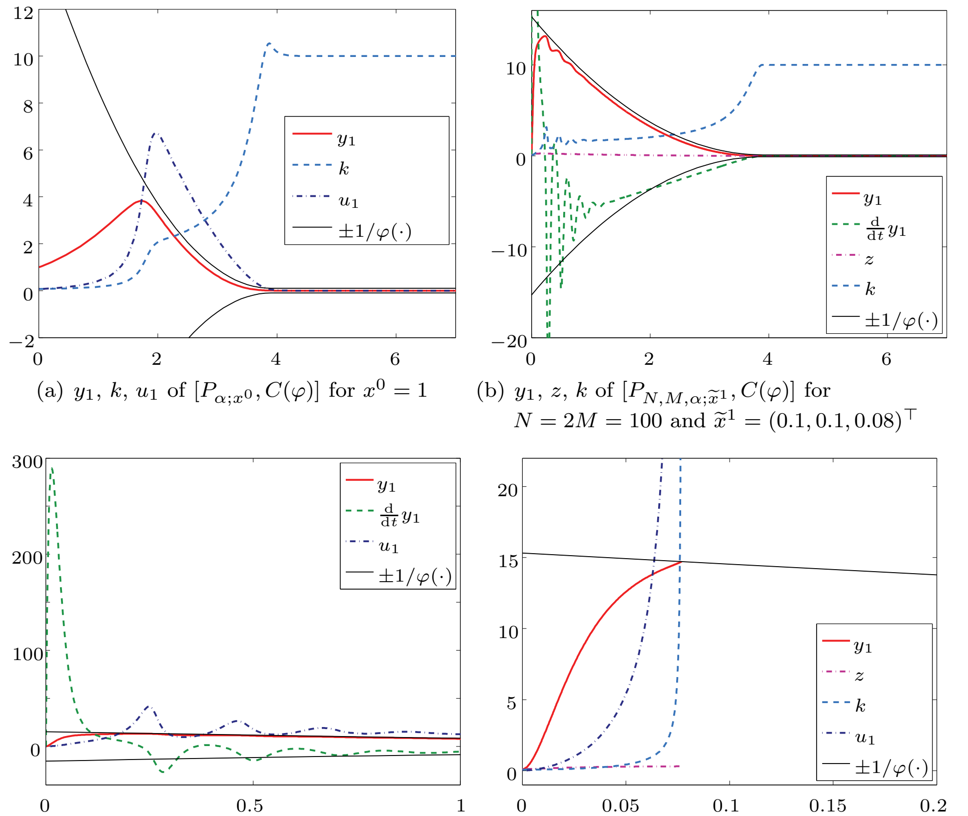
$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix}\xi_1\\ \xi_2\\ z\end{pmatrix} = \begin{bmatrix} 0 & 1 & 0\\ \alpha N + 2M(\alpha - M - N), & \alpha - M - N, & 2M(NM + M^2 - \alpha M - \alpha N)\\ -1 & 0 & -M\end{bmatrix}\begin{pmatrix}\xi_1\\ \xi_2\\ z\end{pmatrix} + \begin{pmatrix}0\\ -N\\ 0\end{pmatrix} u_1, \qquad y_1 = \xi_1. \tag{4.13}$$

We have already shown in (3.6) that for zero initial conditions the gap between the system $\big(\widetilde{A}, \widetilde{b}, \widetilde{c}\big) \in \mathcal{P}_{3,1} \setminus \mathcal{M}_{3,1}$ and $(\alpha, 1, 1) \in \mathcal{M}_{1,1}$ tends to zero as $N = 2M$ and M tend to infinity.

Now, in view Theorem 4.5, there exist a continuous function $\eta\colon (0,\infty) \to (0,\infty)$ and a function $\psi\colon \mathcal{P}_{3,1} \to (0,\infty)$ such that, for all $(\widetilde{x}^0, w_0, r) \in \mathbb{R}^3 \times \mathcal{W} \times (0,\infty)$, we have

$$\left.\begin{array}{l} \psi\big((\widetilde{A}, \widetilde{b}, \widetilde{c})\big)\|\widetilde{x}^0\| + \|w_0\|_{\mathcal{W}} \le r,\\ \vec{\delta}\Big(P\big((\alpha,1,1),0\big), P\big((\widetilde{A}, \widetilde{b}, \widetilde{c}),0\big)\Big) \le \eta(r) \end{array}\right\} \Rightarrow \left\{\begin{array}{l} \forall\, t \ge 0 \ : \ (t, y_0(t) - y_1(t)) \in \mathcal{F}_\varphi,\\ k \in L^\infty(\mathbb{R}_{\ge 0} \to \mathbb{R}),\\ x \in W^{1,\infty}(\mathbb{R}_{\ge 0} \to \mathbb{R}^3), \end{array}\right.$$

where $\mathcal{W} = L^\infty(\mathbb{R}_{\ge 0} \to \mathbb{R}) \times W^{1,\infty}(\mathbb{R}_{\ge 0} \to \mathbb{R})$ and we use the notation from Theorem 4.5, namely, $P\big((\alpha,1,1),0\big) := P_{\alpha;0}$ and $P\big((\widetilde{A}, \widetilde{b}, \widetilde{c}),0\big) := P_{N,M,\alpha;0}$.



(a) y_1, k, u_1 of $[P_{\alpha;x^0}, C(\varphi)]$ for $x^0 = 1$

(b) y_1, z, k of $[P_{N,M,\alpha;\widetilde{x}^1}, C(\varphi)]$ for $N = 2M = 100$ and $\widetilde{x}^1 = (0.1, 0.1, 0.08)^\top$

(c) $\dot{y}_1$, u_1 of $[P_{N,M,\alpha;\widetilde{x}^1}, C(\varphi)]$ for $N = 2M = 100$ and $\widetilde{x}^1 = (0.1, 0.1, 0.08)^\top$

(d) y_1, z, k, and u_1 of $[P_{N,M,\alpha;\widetilde{x}^2}, C(\varphi)]$ for $N = 2M = 100$ and $\widetilde{x}^2 = (0.1, 0.1, 0.1)^\top$

FIG. 4.1. *Funnel control simulations: Nominal system $P_{\alpha;x^0}$ and general system $P_{N,M,\alpha;\widetilde{x}^i}$, $i = 1, 2$, with $N = 2M = 100$.*

Note that Theorem 4.5 shows only existence of the functions ψ and η; it is not straightforward to find these functions.

We are now in a position to discuss simulations for various values of $N, M > 0$ and $\widetilde{x}^i$; all simulations are performed by MATLAB for

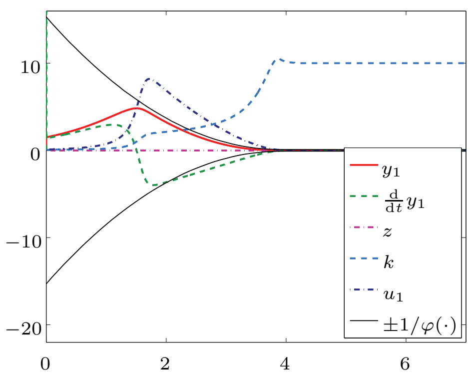
$$\alpha = 1, \quad u_0 = y_0 \equiv 0, \quad \lambda = 0.1$$

and funnel boundary

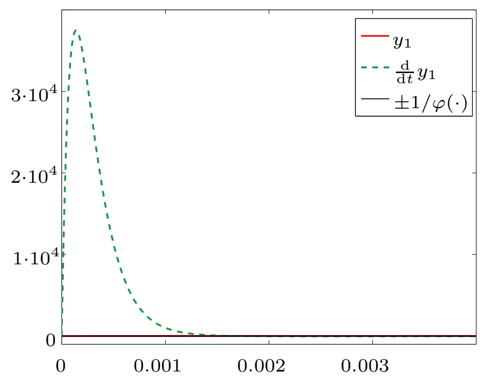
$$\varphi(\cdot)^{-1} \colon \mathbb{R}_{\geq 0} \to \mathbb{R}_{>0}, \quad t \mapsto \begin{cases} 15.31 - 7.8\,t + t^2 & \text{if } t \in [0, 3.9) \\ 0.1 & \text{if } t \geq 3.9. \end{cases}$$

The variables y_1, k, and u_1 of the nominal closed-loop system (3.3), (1.4), (1.3), i.e., the funnel controller $C(\varphi)$ applied to the linear system $P_{\alpha;x^0}$, are depicted in Figure 4.1(a) for initial value $x^0 = 1$.

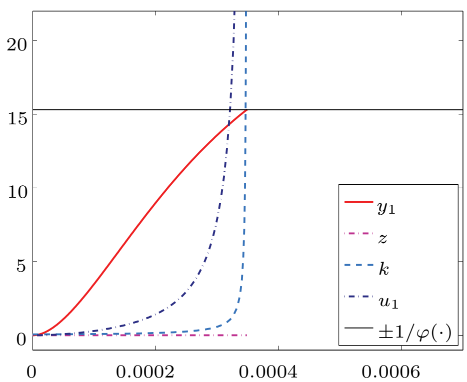
Consider next the closed-loop system (4.13), (1.4), (1.3) for $N = 2M = 100$. In [7, Sec. 3.5, p. 2737], it is shown that then $\vec{\delta}(P_\alpha, P_{N,M,\alpha}) \leq 8/51$. In Figures 4.1(b) and 4.1(c), we depict the simulations for initial value $\widetilde{x}^1 = (0.1, 0.1, 0.08)^\top$,



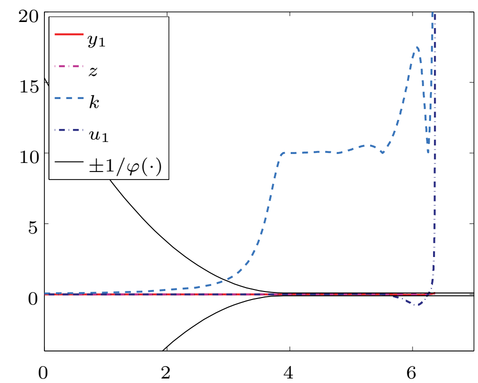
(a) y_1, z, k, and u_1 of $[P_{N,M,\alpha;\widetilde{x}^3}, C(\varphi)]$ for $N = 2M = 10{,}000$ and $\widetilde{x}^3 = (0.001, 0.001, 0.001)^\top$



(b) $\dot{y}_1$ of $[P_{N,M,\alpha;\widetilde{x}^3}, C(\varphi)]$ for $N = 2M = 10{,}000$ and $\widetilde{x}^3 = (0.001, 0.001, 0.001)^\top$



(c) y_1, z, k, and u_1 of $[P_{N,M,\alpha;\widetilde{x}^4}, C(\varphi)]$ for $N = 2M = 10{,}000$ and $\widetilde{x}^4 = (0.001, 0.001, 0.0015)^\top$



(d) y_1, z, k, and u_1 of $[P_{N,M,\alpha;\widetilde{x}^5}, C(\varphi)]$ for $N = 2M = 10$ and $\widetilde{x}^5 = (10^{-6}, 10^{-6}, 10^{-6})^\top$

FIG. 4.2. *Funnel control simulations: General system* $P_{N,M,\alpha;\widetilde{x}^i}$, $i = 3, 4$, *with* $N = 2M = 10{,}000$ *and* $P_{N,M,\alpha;\widetilde{x}^5}$ *with* $N = 2M = 10$.

which is sufficiently small to guarantee funnel control: all components of the solution $(\xi(\cdot)^\top, z(\cdot)) = (y_1(\cdot), \dot{y}_1(\cdot), z(\cdot))$ and $k(\cdot)$ and $u_1(\cdot)$ are bounded. However, a slight increase of the third component of the initial value to $\widetilde{x}^2 = (0.1, 0.1, 0.1)^\top$ leads to a finite escape time: the output y_1 tends to the funnel boundary in finite time $t_1 > 0$, and therefore $u_1(t)$ tends to infinity as $t \to t_1$; see Figure 4.1(d).

Consider the closed-loop system (4.13), (1.4), (1.3) for $N = 2M = 10{,}000$. Then the gap $\vec{\delta}(P_{\alpha;0}, P_{N,M,\alpha;0}) \leq 1/625$ is very small and $r > 0$ might be large such that the second inequality of the left-hand side of (4.10) holds. However, the system has very unstable zero dynamics; this indicates that $\psi(\widetilde{A}, \widetilde{b}, \widetilde{c})$ might be very large. Therefore, the initial value must be very small so that the first inequality of the left-hand side of (4.10) holds. Since ψ maps any system $(\widetilde{A}, \widetilde{b}, \widetilde{c})$ into $(0, \infty)$, then in view of (4.10) and given that the second inequality holds for r and $(\widetilde{A}, \widetilde{b}, \widetilde{c})$, it is always possible to choose a sufficiently small initial value not equal to zero such that the first inequality holds.

Figures 4.2(a) and 4.2(b) show that funnel control is achieved in case of the initial value $\widetilde{x}^3 = (0.001, 0.001, 0.001)^\top$ (note that $\dot{y}_1$ is very large), whereas funnel control is not achieved in case of the initial value $\widetilde{x}^4 = (0.001, 0.001, 0.0015)^\top$; see Figure 4.2(c).

If $N = 2M = 10$, then the gap between $P_{\alpha;0}$ and $P_{N,M,\alpha;0}$ is rather big. Figure 4.2(d) indicates that in this case the output tends to the funnel boundary for any initial value not equal to zero; the initial value for the simulation is $\widetilde{x}^5 = (10^{-6}, 10^{-6}, 10^{-6})^\top$. We conjecture that in this case the second inequality in (4.10) is not satisfied for small $N, M > 0$ since the gap is too large, whence the funnel controller will not achieve stabilization for any initial value not equal to zero.

Figure 4.2 shows a shortcoming of the main result: Theorem 4.5 ensures, as for λ-tracking [7, Th. 4.5], only existence of functions ψ and η in (4.10). For a given system $\widetilde{\theta}$ it may be hard to calculate the value $\psi(\widetilde{\theta})$. It could also be possible that the functions ψ and η counteract in some way, as seen, for example, for "huge" $N, M > 0$.

However, the simulations show that funnel control may be applied to system (3.5) despite the fact that it has unstable zero dynamics, relative degree two, and negative high-frequency gain. The only restrictions are that the zero is "far" in the right half complex plane, the initial condition $\widetilde{x}^0$ is "small," and the $L^\infty \times W^{1,\infty}$ input/output disturbances u_0 and y_0 are "small."

5. Conclusions. We have shown robustness of the funnel controller (1.4) for a class of linear systems which are close in the gap metric to minimum phase systems with (strict) relative degree one; moreover, funnel control copes with certain bounded input/output disturbances. The only shortcoming of the present approach is that the main result shows only existence of continuous functions ψ and η in (4.10). For a given system $\widetilde{\theta}$ it may be hard to calculate the value $\psi(\widetilde{\theta})$. It could also be possible that these functions counteract in some way. For example, given small $r > 0$ and $\widetilde{\theta} \in \mathcal{P}_{q,m}$ such that $\vec{\delta}\big(P(\theta,0), P(\widetilde{\theta},0)\big) \le \eta(r)$, it could be possible that $\psi(\widetilde{\theta})$ is very large, which requires then a very small initial value $\widetilde{x}_0 \in \mathbb{R}^q$ so that the left-hand side of (4.10) holds. However, in view of (4.10), given that the second inequality holds for r and $\widetilde{\theta}$, it is always possible to choose a sufficiently small initial value.

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