

## Stability and robust stability of positive Volterra systems

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### SUMMARY

We study positive linear Volterra integro-differential systems with infinitely many delays. Positivity is characterized in terms of the system entries. A generalized version of the Perron-Frobenius Theorem is shown; this may be interesting in its own right but is exploited here for stability results: explicit spectral criteria for  $L^1$ -stability and exponential asymptotic stability. Also the concept of stability radii, determining the maximal robustness with respect to additive perturbations to  $L^1$ -stable system, is introduced and it is shown that the complex, real and positive stability radii coincide and can be computed by an explicit formula. Copyright © 2009 John Wiley & Sons, Ltd.

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### Nomenclature

$e_i$  :  $i^{\text{th}}$  unit vector in  $\mathbb{R}^n$ ,  $n$  clear from the context

$\mathbb{N}_0 := \mathbb{N} \cup \{0\}$

$\mathbb{R}_+^{\ell \times q} := \{M = (m_{ij}) \in \mathbb{R}^{\ell \times q} \mid m_{ij} \geq 0 \forall i, j\}$

$\mathbb{C}_\lambda := \{s \in \mathbb{C} \mid \Re s \geq \lambda\}$ ,  $\lambda \in \mathbb{R}$

$\mathcal{B}_r^n(x) := \{y \in \mathbb{R}^n \mid \|x - y\| < r\}$ , open ball of radius  $r > 0$  centered at  $x \in \mathbb{R}^n$

$\underline{m} := \{1, \dots, m\}$ ,  $m \in \mathbb{N}$

$M^{\mathbb{N}}$  := the set of all mappings from  $\mathbb{N}$  to a set  $M$

$\text{spec}(A) := \{\lambda \in \mathbb{C} \mid \det(\lambda I_n - A) = 0\}$ , the *spectrum* of  $A \in \mathbb{C}^{n \times n}$

$\mu(A) := \max \{\Re s \mid s \in \text{spec}(A)\}$ , the *spectral abscissa* of  $A \in \mathbb{C}^{n \times n}$

$A$  Metzler iff  $a_{ij} \geq 0$  for all  $i, j \in \{1, \dots, n\}$  with  $i \neq j$ ,  $A = (a_{ij}) \in \mathbb{R}^{n \times n}$

$|A| := (|a_{ij}|)$  for  $A = (a_{ij}) \in \mathbb{R}^{\ell \times q}$

$A \geq B := \Leftrightarrow a_{ij} \geq b_{ij}$  for all entries of  $A = (a_{ij})$ ,  $B = (b_{ij}) \in \mathbb{R}^{\ell \times q}$

$\|\cdot\| : \mathbb{C}^n \rightarrow \mathbb{R}_+$  monotone norm; monotone means:  $\forall x, y \in \mathbb{C}^n$  with  $|x| \leq |y| : \|x\| \leq \|y\|$

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$\|A\| := \max\{\|Ax\| \mid \|x\| = 1\}$ , an operator norm induced by monotone norms on  $\mathbb{C}^\ell$  and  $\mathbb{C}^q$  resp.,  $A \in \mathbb{C}^{\ell \times q}$

$C(J, \mathbb{R}^{\ell \times q})$  the vector space of continuous functions  $f : J \rightarrow \mathbb{R}^{\ell \times q}$ ,  $J \subset \mathbb{R}$  an interval, with norm  $\|f\|_\infty := \sup_{t \in J} \|f(t)\|$

$L^p(J, \mathbb{R}^{\ell \times q})$  the space of functions  $f : J \rightarrow \mathbb{R}^{\ell \times q}$ ,  $J \subset \mathbb{R}$  an interval, with  $\int_J \|f(t)\|^p dt < \infty$ ,  $p \in [1, \infty)$

$L^\infty(J, \mathbb{R}^{\ell \times q})$  space of measurable essentially bounded functions  $f : J \rightarrow \mathbb{R}^{\ell \times q}$ ,  $J \subset \mathbb{R}$  an interval, with norm  $\|f\|_\infty := \text{ess-sup}_{t \in J} \|f(t)\|$

$\phi \geq 0$  iff  $\phi : J \rightarrow \mathbb{R}^{\ell \times q}$  satisfies  $\phi(t) \in \mathbb{R}_+^{\ell \times q}$  for a.a.  $t \in J \subset \mathbb{R}$ , in this case  $\phi$  is called *non-negative*.

## 1. INTRODUCTION

We study positive linear Volterra integro-differential systems with infinitely many delays of the form

$$\dot{x}(t) = A_0 x(t) + \sum_{i \geq 1} A_i x(t - h_i) + \int_0^t B(t-s)x(s) ds, \quad \text{for a.a. } t \geq 0, \quad (1.1)$$

where  $((A_i)_{i \in \mathbb{N}_0}, (h_i)_{i \in \mathbb{N}_0}, B(\cdot))$  satisfy

$$(A1) \quad \forall i \in \mathbb{N}_0 : A_i \in \mathbb{R}^{n \times n} \text{ with } \sum_{i \geq 0} \|A_i\| < \infty$$

$$(A2) \quad 0 = h_0 < h_1 < h_2 < \dots < h_k < h_{k+1} < \dots$$

$$(A3) \quad B(\cdot) \in L^1(\mathbb{R}_+, \mathbb{R}^{n \times n})$$

and, for  $(\varphi, x^0) \in L^1((-\infty, 0); \mathbb{R}^n) \times \mathbb{R}^n$ , the solution of (1.1) may satisfy the initial data

$$(x|_{(-\infty, 0)}, x(0)) = (\varphi, x^0). \quad (1.2)$$

Roughly speaking, a system is called *positive* if, and only if, for any nonnegative initial condition, the corresponding solution of the system is also nonnegative. In particular, a dynamical system with state space  $\mathbb{R}^n$  is positive if, and only if, any trajectory of the system starting at an initial state in the positive orthant  $\mathbb{R}_+^n$  remains in  $\mathbb{R}_+^n$ . Positive dynamical systems play an important role in the modelling of dynamical phenomena whose variables are restricted to be nonnegative.

The mathematical theory of positive systems is based on the theory of nonnegative matrices founded by Perron and Frobenius, see for example [4], [10], [22]. Recently, problems of positive systems have attracted a lot of attention from researchers, see for example stability [25, 30], robustness [30, 31], Perron-Frobenius Theorem [28].

Positive systems have been studied in many applications, such as Economics and Population Dynamics [10, Sec. 13], [22, Sec. 6], Biology and Chemistry [3, 6], Biology and Physiology [11, 12], Nuclear Reactors [7, p. 298].

By setting  $B(\cdot) \equiv 0$ , (1.1) encompasses the subclass

$$\dot{x}(t) = A_0 x(t) + \sum_{i \geq 1} A_i x(t - h_i), \quad t \geq 0, \quad (1.3)$$

of linear differential systems with infinitely many delays. In particular, the subclass of linear time-delay differential systems with discrete delays (i.e.  $\exists m \in \mathbb{N} \forall i > m : A_i = 0$ ) is well understood and numerous results are available on positivity and stability [29], robust stability [33, 31, 26] and the Perron-Frobenius Theorem [27]. However, to the best of our knowledge, the subclass of positive systems with infinitely many delays (1.3) has not been studied in the literature.

By setting  $A_i = 0$  for all  $i \in \mathbb{N}$ , (1.1) encompasses the subclass

$$\dot{x}(t) = A_0 x(t) + \int_0^t B(t-s) x(s) ds, \quad a.a. t \geq 0, \quad (1.4)$$

of linear Volterra integro-differential system of convolution type. Also this subclass is well understood and numerous results on positivity, Perron-Frobenius theorem, stability and robust stability have been given recently, see [30].

The purpose of the present paper is to develop a complete theory of positive systems (1.1) which includes the definition of positivity and characterizations thereof, a Perron-Frobenius Theorem, explicit criteria for stability and robust stability. Several results are also new for the subclasses (1.3) and (1.4).

The paper is organized as follows. In Section 2 we collect some well known results on solution theory of (1.1). In Section 3 we characterize positivity in terms of the system data. A generalized version of the classical Perron-Frobenius Theorem is shown in Section 4. This result may be worth knowing in its own right as a result in Linear Algebra. However, we utilize it for proving stability results in the following sections. In Section 5 we investigate various stability concepts and give, beside other characterizations, explicit spectral criteria for  $L^1$ -stability and exponential asymptotic stability of positive linear Volterra integro-differential systems with delays (1.1). Finally, Section 6 is on robustness of  $L^1$ -stability of (1.1). For this we introduce the concept of complex, real, and positive stability radius, show that, for positive systems, all three are equal and present a simple formula to determine the stability radius.

## 2. SOLUTION THEORY

In this section we recall well known facts on the solution theory of equations of the form (1.1).

**Definition 2.1.** Let  $(\varphi, x^0) \in L^1((-\infty, 0); \mathbb{R}^n) \times \mathbb{R}^n$ . Then a function  $x : \mathbb{R} \rightarrow \mathbb{R}^n$  is said to be a solution of the initial value problem (1.1), (1.2) if, and only if,

- $x$  is locally absolutely continuous on  $[0, \infty)$ ,
- $x$  satisfies the initial condition (1.2) on  $(-\infty, 0]$ ,
- $x$  satisfies (1.1) for almost all  $t \in [0, \infty)$ .

This solution is denoted by  $x(\cdot; 0, \varphi, x^0)$ .

A fundamental solution for (1.1) is given as follows.

**Proposition 2.2.** [7, p. 301] *Consider, for  $((A_i)_{i \in \mathbb{N}_0}, (h_i)_{i \in \mathbb{N}_0}, B(\cdot))$  satisfying (A1)-(A3), the matrix initial value problem*

$$\left. \begin{aligned} \dot{X}(t) &= A_0 X(t) + \sum_{i \geq 1} A_i X(t - h_i) + \int_0^t B(t-s) X(s) ds, \quad a.a. t \geq 0 \\ X(t) &= 0 \quad \forall t < 0, \quad X(0) = I_n. \end{aligned} \right\} \quad (2.1)$$

*Then there exists a solution  $X(\cdot) : \mathbb{R} \rightarrow \mathbb{R}^{n \times n}$  of (2.1); this solution is unique and called fundamental solution.*

**Remark 2.3.** *In [20, p. 55] it is claimed that the fundamental solution  $X$  of (1.1) satisfies the semigroup property*

$$\forall \tau \geq 0 \quad \forall t \geq \tau : X(t) = X(t - \tau) X(\tau).$$

*Unfortunately, this is in general not true. A counterexample is*

$$\dot{x}(t) = -\frac{1}{2}x(t) + \frac{1}{4} \int_0^t e^{-(t-s)/2} x(s) ds, \quad t \geq 0.$$

*which satisfies (A1)-(A3) but the fundamental solution*

$$X(t) = \frac{e^{-t} + 1}{2}, \quad t \geq 0$$

*does not satisfy the semigroup property.*

The following proposition gives the Variation of Constants formula for (1.1).

**Proposition 2.4.** [7, p. 300]

*Consider  $((A_i)_{i \in \mathbb{N}_0}, (h_i)_{i \in \mathbb{N}_0}, B(\cdot))$  satisfying (A1)-(A3), and augment (1.1) by  $g \in L^1_{\text{loc}}(\mathbb{R}_+, \mathbb{R}^n)$  to a non-homogeneous system*

$$\dot{x}(t) = A_0 x(t) + \sum_{i \geq 1} A_i x(t - h_i) + \int_0^t B(t-s) x(s) ds + g(t), \quad a.a. t \geq 0. \quad (2.2)$$

*For any initial data  $(\varphi, x^0) \in L^1((-\infty, 0); \mathbb{R}^n) \times \mathbb{R}^n$ , there exists a solution  $x(\cdot; 0, \varphi, x^0, g) : \mathbb{R} \rightarrow \mathbb{R}^n$  of the initial value problem (2.2), (1.2); this solution is unique and, invoking the fundamental solution  $X$  of (1.1), satisfies, for all  $t \geq 0$ ,*

$$x(t; 0, \varphi, x^0, g) = X(t)x^0 + \sum_{i \geq 1} \int_{-h_i}^0 X(t - h_i - u) A_i \varphi(u) du + \int_0^t X(t-u) g(u) du. \quad (2.3)$$

In what follows, we need the following modification of the Variation of Constants formula for homogeneous systems (1.1) with shifted initial time.

**Remark 2.5.** *Let  $((A_i)_{i \in \mathbb{N}_0}, (h_i)_{i \in \mathbb{N}_0}, B(\cdot))$  satisfy (A1)-(A3), and  $(\sigma, \varphi, x^0) \in \mathbb{R}_+ \times L^1((-\infty, \sigma); \mathbb{R}^n) \times \mathbb{R}^n$ . Then the initial value problem*

$$\left. \begin{aligned} \dot{x}(t) &= A_0 x(t) + \sum_{i \geq 1} A_i x(t - h_i) + \int_0^t B(t-s) x(s) ds, \quad a.a. t \geq \sigma \\ (x|_{(-\infty, \sigma)}, x(\sigma)) &= (\varphi, x^0) \end{aligned} \right\} \quad (2.4)$$

has a unique solution  $x(\cdot; \sigma, \varphi, x^0) : \mathbb{R} \rightarrow \mathbb{R}^n$  and this solution satisfies, invoking the fundamental solution  $X$  of (1.1),

$$x(t; \sigma, \varphi, x^0) = X(t - \sigma)x^0 + \sum_{i \geq 1} \int_{-h_i}^0 X(t - \sigma - h_i - u) A_i \varphi(u) du + \int_0^{t-\sigma} X(t - \sigma - u) \int_0^\sigma B(u + \sigma - s) \varphi(s) ds du, \quad t \geq \sigma. \quad (2.5)$$

For notational convenience, some further notation is introduced.

**Definition 2.6.**

(i) The Laplace transform of a function  $F : \mathbb{R}_+ \rightarrow \mathbb{R}^{\ell \times q}$  is given by

$$\hat{F} : \mathcal{S} \rightarrow \mathbb{C}^{\ell \times q}, \quad z \mapsto \hat{F}(z) := \int_0^\infty e^{-zt} F(t) dt,$$

on a set  $\mathcal{S} \subset \mathbb{C}$  where it exists, see e.g. [14, p. 742].

(ii) For  $((A_i)_{i \in \mathbb{N}_0}, (h_i)_{i \in \mathbb{N}_0}, B(\cdot))$  satisfying (A1)-(A3), the function

$$\mathcal{H} : \mathcal{S} \rightarrow \mathbb{C}^{\ell \times q}, \quad z \mapsto \mathcal{H}(z) := zI_n - A_0 - \sum_{i \geq 1} A_i e^{-h_i z} - \hat{B}(z),$$

defined on a set  $\mathcal{S} \subset \mathbb{C}$  where it exists, is called characteristic matrix of (1.1). In this case we also use

$$\mathcal{R}(z) := zI_n - A_0 - \mathcal{H}(z) = \sum_{i \geq 1} e^{-h_i z} A_i + \hat{B}(z).$$

**Remark 2.7.** Suppose  $((A_i)_{i \in \mathbb{N}_0}, (h_i)_{i \in \mathbb{N}_0}, B(\cdot))$  satisfy (A1)-(A3). Then an application of the Gronwall inequality to (2.3) for  $g \equiv 0$  yields for the fundamental solution  $X$  of (1.1)

$$\exists M > 0 \exists \lambda \in \mathbb{R} \forall t \geq 0 : \|X(t)\| \leq M e^{\lambda t}.$$

Thus applying Laplace transform to the first equation in (2.1) gives

$$\mathcal{H}(z) \hat{X}(z) = \left( zI_n - \sum_{i \geq 0} A_i e^{-zh_i} - \hat{B}(z) \right) \hat{X}(z) = X(0) = I_n \quad \forall z \in \mathring{\mathcal{C}}_\lambda.$$

### 3. POSITIVITY

Although recently problems of positive systems have attracted a lot of attention, see [11]-[12], [15], [16], [28]-[30] and references therein, the general class of Volterra-integro differential systems (1.1) has not been investigated. This will be done in the present section.

**Definition 3.1.** System (1.1) is said to be positive if, and only if, for every nonnegative initial data  $(\sigma, \varphi, x^0) \in \mathbb{R}_+ \times L^1((-\infty, \sigma); \mathbb{R}_+^n) \times \mathbb{R}_+^n$ , the solution of the initial value problem (2.4) is non-negative.

We are now in the position to state the main result of this section.

**Theorem 3.2.** *Let  $((A_i)_{i \in \mathbb{N}_0}, (h_i)_{i \in \mathbb{N}_0}, B(\cdot))$  satisfy (A1)-(A3). Then the system (1.1) is positive if, and only if,*

- (i)  $A_0$  is a Metzler matrix,
- (ii)  $A_i \geq 0$  for all  $i \in \mathbb{N}$ ,
- (iii)  $B(\cdot) \geq 0$ .

**Proposition 3.3.** *Suppose  $((A_i)_{i \in \mathbb{N}_0}, (h_i)_{i \in \mathbb{N}_0}, B(\cdot))$  satisfy (A1)-(A3) and (1.1) is positive. Then for any non-negative initial data  $(\varphi, x^0) \in L^1((-\infty, 0); \mathbb{R}_+^n) \times \mathbb{R}_+^n$  and any non-negative inhomogeneity  $g: \mathbb{R} \rightarrow \mathbb{R}_+^n$ , the solution of the initial value problem (2.2), (1.2) is non-negative:  $x(\cdot; 0, \varphi, x^0, g): \mathbb{R} \rightarrow \mathbb{R}_+^n$ .*

**Remark 3.4.** *It may be worth noting that positivity of (1.1) implies monotonicity in the sense that if*

$$(\varphi^k, x^k, g^k) \in L^1((-\infty, 0); \mathbb{R}_+^n) \times \mathbb{R}_+^n \times L^1(\mathbb{R}_+, \mathbb{R}^n), \quad k = 1, 2$$

satisfy

$$\varphi^1 \leq \varphi^2, \quad x^1 \leq x^2, \quad g^1 \leq g^2,$$

then

$$x(t; 0, \varphi^1, x^1, g^1) \leq x(t; 0, \varphi^2, x^2, g^2) \quad \forall t \geq 0.$$

This follows immediately from (2.3) and since  $X(t) \geq 0$  for all  $t \geq 0$  by Proposition 3.3.

In the remainder of this section we prove Theorem 3.2 and Proposition 3.3. For this, some technical lemmata are needed. Throughout, we assume that  $((A_i)_{i \in \mathbb{N}_0}, (h_i)_{i \in \mathbb{N}_0}, B(\cdot))$  satisfy (A1)-(A3).

**Lemma 3.5.** *Let  $\sigma \geq 0$ , and consider, for non-negative  $B(\cdot) \in L^1([0, \sigma], \mathbb{R}_+^{n \times n})$ ,  $g \in L^1([0, \sigma], \mathbb{R}_+^n)$ ,  $x^0 \in \mathbb{R}_+^n$ , and Metzler matrix  $A_0 \in \mathbb{R}^{n \times n}$  the initial value problem*

$$\left. \begin{aligned} \dot{x}(t) &= A_0 x(t) + \int_0^t B(t-s)x(s) ds + g(t), & \text{a.a. } t \in [0, \sigma], \\ x(0) &= x^0. \end{aligned} \right\} \quad (3.1)$$

Then its solution  $x(\cdot; 0, x^0, g)$  is non-negative on  $[0, \sigma]$ .

Proof:

Applying the Variation of Constants formula and writing

$$\begin{aligned} T &: C([0, \sigma], \mathbb{R}^n) \longrightarrow C([0, \sigma], \mathbb{R}^n) \\ \phi &\mapsto e^{A_0 \cdot} x^0 + \int_0^\cdot e^{A_0(\cdot-s)} g(s) ds + \int_0^\cdot e^{A_0(\cdot-s)} \left( \int_0^s B(s-\tau) \phi(\tau) d\tau \right) ds, \end{aligned}$$

the solution  $x$  of (3.1) satisfies

$$x(t) = (Tx)(t), \quad \forall t \in [0, \sigma].$$

For

$$M := \sup_{t \in [0, \sigma]} \|e^{At}\| \int_0^\sigma \|B(s)\| ds,$$

a simple induction argument shows that

$$\|T^k \hat{\phi}(t) - T^k \phi(t)\| \leq \frac{M^k t^k}{k!} \|\hat{\phi} - \phi\|_\infty \quad \forall t \in [0, \sigma] \quad \forall \phi, \hat{\phi} \in C([0, \sigma], \mathbb{R}^n) \quad \forall k \in \mathbb{N}.$$

This implies the existence of some  $k^* \in \mathbb{N}$  so that  $T^{k^*}$  is a contraction. By the contraction mapping principle, the sequence  $(T^{\ell k^*} \phi)_{\ell \in \mathbb{N}}$  converges in the space  $C([0, \sigma], \mathbb{R}^n)$ , for arbitrary  $\phi \in C([0, \sigma], \mathbb{R}^n)$  to the unique solution  $x(\cdot; 0, x^0, g)$  of  $x = Tx$ . Choose  $\phi \equiv x^0 \in C([0, \sigma], \mathbb{R}_+^n)$ . Since  $A_0$  is a Metzler matrix,  $\alpha I_n + A_0$  is nonnegative for some  $\alpha \geq 0$ . This implies that  $e^{\alpha t} e^{A_0 t} = e^{(\alpha I_n + A_0)t} \geq 0$  for all  $t \in [0, \sigma]$ . Hence,  $e^{A_0 t} \geq 0$  for all  $t \in [0, \sigma]$  and non-negativity of  $g$  and  $B$  yields

$$(T\phi)(t) \geq 0, \quad \forall t \in [0, \sigma].$$

Thus,  $T^{\ell k^*} \phi \geq 0$  for all  $\ell \in \mathbb{N}$  and we arrive at  $x(t) \geq 0$  for all  $t \in [0, \sigma]$ . This completes the proof.

**Lemma 3.6.** *If  $A_0 \in \mathbb{R}^{n \times n}$  is a Metzler matrix,  $A_i \in \mathbb{R}_+^{n \times n}$  for all  $i \in \mathbb{N}$  and  $B(\cdot) \geq 0$ , then the fundamental solution of (1.1) is non-negative:  $X(\cdot) \geq 0$ .*

Proof:

The initial value problem (2.1) may be written as

$$\left. \begin{aligned} \dot{X}(t) &= A_0 X(t) + \int_0^t B(t-s) X(s) ds + G(t), \quad a.a. \ t \geq 0 \\ (X|_{(-\infty, 0)}, X(0)) &= (0, I_n), \end{aligned} \right\} \quad (3.2)$$

where

$$G(t) = \sum_{i \geq 1} A_i X(t - h_i).$$

Since  $G(t) = 0$  for all  $t \in [0, h_1)$ , Lemma 3.5 gives  $X(t) \geq 0$  for all  $t \in [0, h_1)$ . Hence  $G(t) \geq 0$  for all  $t \in [0, 2h_1)$ , and a repeated application of Lemma 3.5 gives  $X(t) \geq 0$  for all  $t \in [0, 2h_1)$ . Proceeding in this way, we arrive at  $X(t) \geq 0$  for all  $t \in [0, k h_1)$ , for all  $k \in \mathbb{N}$ . This completes the proof.

Proof of Theorem 3.2:

“ $\Leftarrow$ ”: This direction follows immediately from Lemma 3.6 and (2.5).

“ $\Rightarrow$ ”: STEP 1: We show that  $A_0$  is a Metzler matrix.

By Remark 2.7 and Lemma 3.6 we have, for some  $\lambda \in \mathbb{R}$ ,

$$\left( sI_n - \sum_{i \geq 0} A_i e^{-sh_i} - \hat{B}(s) \right)^{-1} = \hat{X}(s) = \int_0^\infty e^{-st} X(t) dt \geq 0 \quad \forall s > \lambda$$

and, since  $B(\cdot) \in L^1(\mathbb{R}_+, \mathbb{R}^{n \times n})$  yields  $\lim_{s \rightarrow \infty} \hat{B}(s) = 0$ , there exists  $p > \lambda$  so that

$$\begin{aligned} \hat{X}(s) &= s^{-1} \left( I_n - s^{-1} \left( \sum_{i \geq 0} A_i e^{-sh_i} + \hat{B}(s) \right) \right)^{-1} \\ &= s^{-1} I_n + \sum_{k \geq 1} s^{-(k+1)} \left( \sum_{i \geq 0} A_i e^{-sh_i} + \hat{B}(s) \right)^k \quad \forall s > p, \end{aligned}$$

and thus

$$sI_n + \sum_{k \geq 1} s^{-(k-1)} \left( \sum_{i \geq 0} A_i e^{-sh_i} + \hat{B}(s) \right)^k \geq 0 \quad \forall s > p.$$

Since

$$\lim_{s \rightarrow \infty} \sum_{k \geq 1} s^{-(k-1)} \left( \sum_{i \geq 0} A_i e^{-sh_i} + \hat{B}(s) \right)^k = A_0,$$

it follows that, for all  $i, j \in \underline{n}$  with  $i \neq j$ ,

$$\lim_{s \rightarrow \infty} e_i^T \left[ sI_n + \sum_{k \geq 1} s^{-(k-1)} \left( \sum_{i \geq 0} A_i e^{-sh_i} + \hat{B}(s) \right)^k \right] e_j = e_i^T A_0 e_j \geq 0.$$

Thus,  $A_0$  is a Metzler matrix.

STEP 2: We show that  $A_\ell \geq 0$  for all  $\ell \in \mathbb{N}$ .

Let  $\ell \in \mathbb{N}$  be fixed and consider  $A_\ell := (c_{ij}) \in \mathbb{R}^{n \times n}$ . Fix  $i, j \in \underline{n}$ . Define an  $L^1$ -function

$$\phi: (-\infty, 0] \rightarrow \mathbb{R}_+^n, \quad t \mapsto \phi(t) := \begin{cases} 0, & t < -h_\ell \\ \left( \frac{t}{h_{\ell-1}-h_\ell} + \frac{h_{\ell-1}}{h_{\ell-1}-h_\ell} \right) e_j, & t \in [-h_\ell, -h_{\ell-1}] \\ 0, & t \in (-h_{\ell-1}, 0]. \end{cases}$$

Since (1.1) is positive, the initial value problem (1.1),  $(x|_{(-\infty, 0)}, x(0)) = (\phi, 0)$  has, by Proposition 2.4, a unique solution  $x(\cdot) = x(\cdot; 0, \phi, 0)$  with  $x(t) \geq 0$  for all  $t \geq 0$ . Note that for  $k \in \mathbb{N}$ ,  $x(\cdot) = (x_1(\cdot), \dots, x_n(\cdot))^T$  is locally absolutely continuous on  $(0, 1/k)$ ,  $x(\cdot) \geq 0$ ,  $x(0+) = 0$  and  $x(\cdot)$  satisfies (1.1) almost everywhere on  $(0, 1/k)$ . Thus, invoking the Newton-Leibniz's formula, we may choose  $t_k \in (0, 1/k)$  such that  $\dot{x}_i(t_k) \geq 0$  and  $x(\cdot)$  satisfies (1.1) at  $t_k$ . Since  $\lim_{k \rightarrow \infty} \dot{x}(t_k) = A_\ell \phi(-h_\ell) = A_\ell e_j \geq 0$ , we get, in particular,  $\lim_{k \rightarrow \infty} \dot{x}_i(t_k) = e_i^T A_\ell e_j = c_{ij} \geq 0$ . Since  $i, j \in \underline{n}$  are arbitrary, it follows that  $A_\ell \in \mathbb{R}_+^{n \times n}$ .

STEP 3: We show that

$$\forall i, j \in \underline{n} \quad \text{for a.a. } t \in \mathbb{R}_+ : e_i^T B(t) e_j \geq 0.$$

Fix  $i, j \in \underline{n}$ . Choose

$$\xi \in L^1((-\infty, h_1), \mathbb{R}_+) \quad \text{with} \quad \xi|_{(-\infty, 0]} = 0$$

and set

$$\varphi: (-\infty, h_1) \rightarrow \mathbb{R}_+^n, \quad t \mapsto \xi(t) e_j.$$

By positivity, the solution  $x(\cdot; h_1, \varphi, 0)$  of the initial value problem

$$\left. \begin{aligned} \dot{x}(t) &= A_0 x(t) + \sum_{i=1}^{\infty} A_i x(t - h_i) + \int_0^t B(t-s) x(s) ds, \quad \text{a.a. } t \geq h_1, \\ (x|_{(-\infty, h_1)}, x(h_1)) &= (\varphi, 0) \end{aligned} \right\} \quad (3.3)$$

satisfies

$$x(t) := x(t; h_1, \varphi, 0) \geq 0 \quad \forall t \geq h_1.$$

Thus, invoking the Newton-Leibniz's formula, for every  $k \in \mathbb{N}$  there exists  $t_k \in (h_1, h_1 + 1/k)$  such that  $e_i^T \dot{x}(t_k) \geq 0$  and the differential equation in (3.3) is satisfied at  $t = t_k$ . This implies that

$$0 \leq \lim_{k \rightarrow \infty} e_i^T \dot{x}(t_k) = \lim_{k \rightarrow \infty} \int_0^{t_k} e_i^T B(t_k - s) x(s) ds = \int_0^{h_1} e_i^T B(h_1 - s) e_j \xi(s) ds. \quad (3.4)$$

Assume on the contrary that

$$\exists \mathcal{N} \subset [0, h_1] \text{ with } \text{mess}(\mathcal{N}) > 0 \quad \forall t \in \mathcal{N} : e_i^T B(t) e_j < 0.$$

We may specify  $\xi$  to satisfy  $\xi|_{[0, h_1]} = \chi_{\mathcal{N}}$ , where  $\chi_{\mathcal{N}}$  denotes the indicator function of  $\mathcal{N}$ . Then,

$$\int_{\mathcal{N}} e_i^T B(s) e_j \, ds = \int_0^{h_1} e_i^T B(h_\ell - s) e_j \xi(s) \, ds \leq 0. \tag{3.5}$$

It follows from (3.4) and (3.5) that

$$\int_{\mathcal{N}} -e_i^T B(s) e_j \, ds = 0.$$

However, since  $\text{mess}(\mathcal{N}) > 0$ , this contradicts  $-e_i^T B(t) e_j > 0 \ t \in \mathcal{N}$ . Hence,  $B(t) \geq 0$  for a.a.  $t \in [0, h_1]$ .

By a similar argument, we can show that  $B(t) \geq 0$  for a.a.  $t \in [h_1, 2h_1]$ . Proceeding in this way, we get  $B(t) \geq 0$  for a.a.  $t \in [kh_1, (k + 1)h_1]$  and for arbitrary  $k \in \mathbb{N}$ . This completes the proof of the theorem.  $\square$

**Proof of Proposition 3.3:**

The proof of Proposition 3.3 is an immediate consequence of Lemma 3.6 combined with (2.3) and Theorem 3.2.  $\square$

#### 4. PERRON-FROBENIUS THEOREM

It is well-known that Perron-Frobenius type theorems are principal tools for analyzing stability and robust stability of positive systems. There are many extensions of the classical Perron-Frobenius theorem, see e.g. [1, 18, 28, 27, 30] and references therein.

In this section, we present a Perron-Frobenius Theorem for positive systems (1.1). This may also be interesting in its own right as a result in Linear Algebra. However, we will apply the Perron-Frobenius Theorem to prove stability and robustness results in Sections 5 and 6. Note that the assumption (A1)-(A3) are relaxed in the present section.

**Theorem 4.1.** *If  $((A_i)_{i \in \mathbb{N}_0}, (h_i)_{i \in \mathbb{N}_0}, B(\cdot))$  satisfy*

$$(\tilde{A}1) \quad \forall i \in \mathbb{N} : A_i \in \mathbb{R}_+^{n \times n}, \text{ and } A_0 \in \mathbb{R}^{n \times n} \text{ is a Metzler matrix}$$

$$(\tilde{A}2) \quad \forall i \in \mathbb{N} : h_i \geq 0$$

$$(\tilde{A}3) \quad B(\cdot) : \mathbb{R}_+ \rightarrow \mathbb{R}^{n \times n} \text{ is Lebesgue measurable and, for a.a. } t \in \mathbb{R}_+, B(t) \in \mathbb{R}_+^{n \times n} \text{ then,}$$

$$(\tilde{A}4) \quad \beta := \inf \left\{ \gamma \in \mathbb{R} \mid \sum_{i \geq 0} e^{-h_i \gamma} \|A_i\| + \int_0^\infty e^{-\gamma t} \|B(t)\| \, dt < \infty \right\} < \infty,$$

using the notation introduced in Definition 2.6,

$$\mu [A_0, (A_i)_{i \in \mathbb{N}}, B(\cdot)] := \sup \left\{ \Re z \mid \begin{array}{l} z \in \mathbb{C} \text{ with } \det \mathcal{H}(z) = 0 \text{ and} \\ \sum_{i \geq 1} e^{-h_i \Re z} \|A_i\| + \int_0^\infty e^{-t \Re z} \|B(t)\| \, dt < \infty \end{array} \right\} < \infty.$$

Moreover, if  $-\infty < \mu_0 := \mu[A_0, (A_i)_{i \in \mathbb{N}}, B(\cdot)]$ , we have, for  $\beta < \alpha < \infty$  that

- (i)  $\exists x \in \mathbb{R}_+^n \setminus \{0\} : \left( A_0 + \sum_{i \geq 1} e^{-h_i \mu_0} A_i + \int_0^\infty e^{-\mu_0 t} B(t) dt \right) x = \mu_0 x$ ,
- (ii)  $\alpha \leq \mu_0 \iff \left[ \exists x \in \mathbb{R}_+^n \setminus \{0\} : \left( A_0 + \sum_{i \geq 1} e^{-h_i \alpha} A_i + \int_0^\infty e^{-\alpha t} B(t) dt \right) x \geq \alpha x \right]$ ,
- (iii)  $\alpha > \mu_0 \iff \mathcal{H}(\alpha)^{-1} \geq 0$ .

Theorem 4.1 is a generalization of the Perron-Frobenius Theorem for positive

- linear time delay differential systems, proved in [27];
- linear Volterra integro-differential system of convolution type (1.4), proved in [30];
- linear systems  $\dot{x}(t) = A_0 x(t)$ , proved in [15].

The latter case is quoted next because it will be used in several proofs.

**Proposition 4.2.** *For a Metzler matrix  $A_0 \in \mathbb{R}^{n \times n}$ , and  $A_i = 0$  for all  $i \in \mathbb{N}$  and  $B \equiv 0$  in (1.1), the spectral abscissa (see Nomenclature) satisfies*

- (i)  $\mu(A_0) = \mu[A_0, 0, 0]$
- (ii)  $\exists x \in \mathbb{R}_+^n \setminus \{0\} : A_0 x = \mu(A_0) x$
- (iii)  $\alpha \leq \mu(A_0) \iff [\exists x \in \mathbb{R}_+^n \setminus \{0\} : A_0 x \geq \alpha x]$
- (iv)  $\alpha > \mu(A_0) \iff (\alpha I_n - A_0)^{-1} \geq 0$
- (v)  $\forall P \in \mathbb{C}^{n \times n} \forall Q \in \mathbb{R}_+^{n \times n} : [P \leq Q \implies \mu(A_0 + P) \leq \mu(A_0 + Q)]$ .

In the remainder of this section we prove Theorem 4.1. First, a technical lemma is proved.

**Lemma 4.3.** *Suppose  $((A_i)_{i \in \mathbb{N}_0}, (h_i)_{i \in \mathbb{N}_0}, B(\cdot))$  satisfy  $(\tilde{A}1)$ – $(\tilde{A}3)$  and  $\mathcal{R}$  (see Definition 2.6) is defined, for some  $\lambda \in \mathbb{R}$ , on  $\mathbb{C}_\lambda$ . Then*

- (i)  $\forall s > \lambda : \mathcal{R}(s) \geq 0$ ,
- (ii)  $\forall z \in \mathbb{C}_\lambda : |\mathcal{R}(z)| \leq \mathcal{R}(\Re z)$ ,
- (iii)  $\forall z \in \mathbb{C}_\lambda : \mu(A_0 + \mathcal{R}(z)) \leq \mu(A_0 + \mathcal{R}(\lambda))$ .

*Proof:*

The claims follow immediately from the assumptions and Proposition 4.2(v).

*Proof of Theorem 4.1:*

STEP 1: We show  $\mu_0 < \infty$ .

Seeking a contradiction, suppose that  $\mu_0 = \infty$ . Then

$$\forall k \in \mathbb{N} \exists z_k \in \mathbb{C}_k : \det \mathcal{H}(z_k) = 0, \sum_{i \geq 1} e^{-h_i z_k} \|A_i\| + \int_0^\infty e^{-t z_k} \|B(t)\| dt < \infty$$

and thus, invoking Lemma 4.3,

$$\forall k \in \mathbb{N} : k \leq \Re z_k \leq \mu(A_0 + \mathcal{R}(z_k)) \leq \mu(A_0 + \mathcal{R}(\Re z_k)) \leq \mu(A_0 + \mathcal{R}(k)),$$

which gives, by  $k \rightarrow \infty$ ,

$$\infty \leq \lim_{k \rightarrow \infty} \mu(A_0 + \mathcal{R}(k)) = \mu(A_0) < \infty,$$

which is a contradiction.

STEP 2: We show that

$$\mu_0 \leq \mu(A_0 + \mathcal{R}(\mu_0)). \quad (4.1)$$

Since  $\mu_0 > -\infty$ , there exists  $z \in \mathbb{C}$  :

$$\det \mathcal{H}(z) = 0, \quad \sum_{i \geq 1} e^{-h_i \Re z} \|A_i\| + \int_0^\infty e^{-t \Re z} \|B(t)\| dt < \infty.$$

Then

$$\beta \leq \Re z \leq \mu_0.$$

If  $\Re z = \mu_0$ , then Lemma 4.3(iii) yields (4.1).

If  $\Re z < \mu_0$ , then by definition of  $\mu_0$ ,

$$\exists (z_k)_{k \in \mathbb{N}} : \beta < \Re z_k < \mu_0, \quad \det \mathcal{H}(z_k) = 0, \quad \lim_{k \rightarrow \infty} \Re z_k = \mu_0$$

and again by Lemma 4.3(iii) we arrive at

$$\mu_0 = \lim_{k \rightarrow \infty} \Re z_k \leq \mu(A_0 + \mathcal{R}(\mu_0)).$$

This proves (4.1).

STEP 3: Set

$$J := \begin{cases} [\beta, \infty), & \text{if } \sum_{i \geq 0} e^{-h_i \beta} \|A_i\| + \int_0^\infty e^{-\beta t} \|B(t)\| dt < \infty \\ (\beta, \infty), & \text{otherwise.} \end{cases}$$

We show that the continuous function

$$f : J \rightarrow \mathbb{R}, \quad \theta \mapsto \theta - \mu(A_0 + \mathcal{R}(\theta))$$

satisfies  $f(\mu_0) = 0$ .

By (4.1),  $f(\mu_0) \leq 0$ . Seeking a contradiction, suppose that  $f(\mu_0) < 0$ . Then we may choose

$$\theta_0 > \mu_0 : f(\theta_0) = 0. \quad (4.2)$$

Hence  $\theta_0 = \mu(A_0 + \mathcal{R}(\theta_0))$ . Since  $A_0 + \mathcal{R}(\theta_0)$  is a Metzler matrix, we may apply Proposition 4.2(i) to conclude that  $\det \mathcal{H}(\theta_0) = 0$   $\theta_0 > \mu_0$ , which contradicts the definition of  $\mu_0$ .

STEP 4: We are now ready to show (i)-(iii).

By Step 3, we have  $\mu_0 = \mu(A_0 + \mathcal{R}(\mu_0))$ , and an application of Proposition 4.2(i) yields (i).

By Proposition 4.2(v), we conclude, for  $\beta < \theta_1 \leq \theta_2$ ,  $\mu(A_0 + \mathcal{R}(\theta_2)) \leq \mu(A_0 + \mathcal{R}(\theta_1))$ , and so  $\theta \mapsto f(\theta)$  as in Step 3 is increasing. Since  $f(\mu_0) = 0$ , (ii) and (iii) follow from Proposition 4.2(iii) and (iv).  $\square$

## 5. STABILITY

In this section we investigate various stability concepts which are standard for linear integro-differential systems, see e.g. [5, p. 37], [21], [24]. We characterize them and specialize them also to positive systems.

**Definition 5.1.** A system (1.1) (more precisely, its zero solution) satisfying (A1)-(A3) is said to be

*stable* : $\iff \forall \varepsilon > 0 \forall \sigma \geq 0 \exists \delta > 0 \forall (\varphi, x^0) \in L^1((-\infty, \sigma); \mathbb{R}^n) \times \mathbb{R}^n$  with  
 $\|\varphi\|_{L^1} + \|x^0\| < \delta \forall t \geq \sigma : \|x(t; \sigma, \varphi, x^0)\| < \varepsilon$

*uniformly stable* : $\iff$  stable and  $\delta > 0$  can be chosen independently of  $\sigma$

*asymptotically stable* : $\iff$  stable and  $\forall (\sigma, \varphi, x^0) \in \mathbb{R}_+ \times L^1((-\infty, \sigma); \mathbb{R}^n) \times \mathbb{R}^n$  :  
 $\lim_{t \rightarrow \infty} x(t; \sigma, \varphi, x^0) = 0$

*uniformly asymptotically stable* : $\iff$  uniformly stable and  $\exists \delta > 0 \forall \varepsilon > 0$   
 $\exists T(\varepsilon) > 0 \forall (\sigma, \varphi, x^0) \in \mathbb{R}_+ \times L^1((-\infty, \sigma); \mathbb{R}^n) \times \mathbb{R}^n$  with  
 $\|\varphi\|_{L^1} + \|x^0\| < \delta \forall t \geq T(\varepsilon) + \sigma : \|x(t; \sigma, \varphi, x^0)\| < \varepsilon$

*exponentially asymptotically stable* : $\iff \exists M, \lambda > 0 \forall (\sigma, \varphi, x^0) \in \mathbb{R}_+ \times L^1((-\infty, \sigma); \mathbb{R}^n) \times \mathbb{R}^n$   
 $\forall t \geq \sigma : \|x(t; \sigma, \varphi, x^0)\| \leq M e^{-\lambda(t-\sigma)} [\|\varphi\|_{L^1} + \|x^0\|]$

*$L^p$ -stable,  $p \in [1, \infty]$*  : $\iff X \in L^p(\mathbb{R}_+; \mathbb{R}^{n \times n})$  where  $X$  denotes the fundamental solution of (1.1).

The following proposition shows how these stability concepts are related and how they can be characterized in term of the fundamental solution  $X(\cdot)$  and characteristic matrix  $\mathcal{H}(\cdot)$ , see Definition 2.6.

**Proposition 5.2.** Consider a system (1.1) satisfying (A1)-(A3). Then the statements

- (i) (1.1) is asymptotically stable
- (ii) (1.1) is  $L^1$ -stable
- (iii) (1.1) is  $L^p$ -stable for all  $p \in [1, \infty]$
- (iv)  $\forall z \in \mathbb{C}_0 : \det \mathcal{H}(z) \neq 0$ , where  $\mathcal{H}$  is the characteristic matrix of (1.1)
- (v) (1.1) is uniformly asymptotically stable
- (vi)  $\exists M, \lambda > 0 \forall t \geq 0 : \|X(t)\| \leq M e^{-\lambda t}$ , where  $X$  is the fundamental solution of (1.1),
- (vii) (1.1) is exponentially asymptotically stable

are related as follows:

$$(v) \iff (vii) \implies (vi) \implies (iv) \iff (iii) \iff (ii) \implies (i).$$

Proof: By definition, it is easy to see that (v)  $\iff$  (vii)  $\implies$  (vi) and (vi)  $\implies$  (iii). The proof of implications (iv)  $\iff$  (iii)  $\iff$  (ii) can be found in [7, p. 303]. It remains to show that (ii)  $\implies$  (i). Since the convolution of an  $L^1$ -function with an  $L^p$ -function belongs to  $L^p$  (see e.g. [19, p. 172]), it follows from (1.1) that  $\dot{X} \in L^1(\mathbb{R}_+; \mathbb{R}_+^{n \times n})$ . Therefore,  $X, \dot{X} \in L^1(\mathbb{R}_+; \mathbb{R}_+^{n \times n})$  and [17, Lem. 2.1.7] yields  $\lim_{t \rightarrow \infty} X(t) = 0$ . Finally, an application of the Variation of Constants formula (2.5) shows (i).

**Remark 5.3.** In particular, for linear Volterra integro-differential systems (1.4) without delay and  $B(\cdot) \in L^1(\mathbb{R}_+, \mathbb{R}^{n \times n})$ , [23] have shown

$$(iv) \iff (v).$$

For (1.1), by a standard argument, we can show that (iv) implies that (1.1) is uniform stable and asymptotically stable. Conversely, (v) implies that  $\det \mathcal{H}(z) \neq 0$  for  $z \in \mathring{\mathbb{C}}_0$  and in

particular (v) implies (iv) provided  $\sup_{i \in \mathbb{N}} h_i < \infty$ . However, in general, it is an open problem whether the implications

$$(iv) \implies (v) \quad \text{or} \quad (v) \implies (iv)$$

hold true.

Finally, [24] showed that even for (1.4),

$$(iv) \not\Rightarrow (vi) \quad \text{and} \quad (v) \not\Rightarrow (vi).$$

These implications are claimed in [20, Th. 2.2.2]; the proof in [20] however is based on the semigroup property of the fundamental solution; in Remark 2.3 we showed that this property does not hold.

Next we present a sufficient condition under which most of statements in Proposition 5.2 are equivalent.

**Theorem 5.4.** *Consider a system (1.1).*

(i) *If  $((A_i)_{i \in \mathbb{N}_0}, (h_i)_{i \in \mathbb{N}_0}, B(\cdot))$  satisfy (A2) and*

$$\exists \alpha > 0 : \sum_{i \geq 1} \|A_i\| e^{\alpha h_i} + \int_0^\infty e^{\alpha t} \|B(t)\| dt < \infty, \quad (5.1)$$

*then the statements (ii), (iii), (iv), (vi) and (vii) in Proposition 5.2 are equivalent.*

(ii) *If  $((A_i)_{i \in \mathbb{N}_0}, (h_i)_{i \in \mathbb{N}_0}, B(\cdot))$  satisfy (A1)-(A3),  $A_i \in \mathbb{R}_+^{n \times n}$  for all  $i \in \mathbb{N}$ ,  $B(\cdot) \geq 0$  and (vi) in Proposition 5.2 holds, then (5.1) is valid.*

Theorem 5.4 extends [24, Th. 1, Th. 2] and [2, Th. 3.1, Th. 3.2], where linear integro-differential systems without delay (1.4) are considered, to the linear Volterra integro-differential systems with delays (1.1). The proof of Theorem 5.4(i) is different from [24], the proof of Theorem 5.4(ii) is based on ideas of the proof of [2, Th. 3.2].

Proof of Theorem 5.4:

(i)

In view of Proposition 5.2, it suffices to show “(iv)  $\implies$  (vii)”. Let  $X(\cdot)$  and  $\mathcal{H}(\cdot)$  be the fundamental solution and the characteristic matrix of (1.1), resp.

We show

$$\exists K, \varepsilon > 0 \forall t \geq 0 : \|X(t)\| \leq K e^{-\varepsilon t}. \quad (5.2)$$

Note that, by (5.1),

$$\begin{aligned} \det \mathcal{H}(z) = 0 \quad \text{for some } z \in \mathbb{C}_{-\alpha} \\ \implies |z| \leq T_0 := \|A_0\| + \sum_{i \geq 1} e^{\alpha h_i} \|A_i\| + \int_0^\infty e^{\alpha t} \|B(t)\| dt \end{aligned}$$

and hence

$$\forall z \in \mathbb{C} \text{ with } -\alpha \leq \Re z \leq 0 \text{ and } |\Im z| \geq T_0 + 1 : \det \mathcal{H}(z) \neq 0.$$

Since  $\det \mathcal{H}(\cdot)$  is analytic on  $\mathring{\mathbb{C}}_{-\alpha}$ , it has at most a finite number of zeros in

$$\mathcal{D} := \{z \in \mathbb{C} \mid -\alpha/2 \leq \Re z \leq 0, \quad |\Im z| \leq T_0 + 1\},$$

and thus  $\det \mathcal{H}(z) \neq 0$  for all  $z \in \mathbb{C}_0$  yields

$$c_0 := \sup \{\Re z \mid z \in \mathbb{C}, \det \mathcal{H}(z) = 0\} < 0.$$

Choose  $\varepsilon \in (0, \min\{-c_0, \alpha\})$ . Then, it is easy to check that

$$Y(\cdot) = e^{\varepsilon \cdot} X(\cdot), \quad \mathcal{H}_\varepsilon(\cdot) = \mathcal{H}(\cdot - \varepsilon)$$

are, respectively, the fundamental solution and the characteristic matrix of

$$\begin{aligned} \dot{y}(t) = (A_0 + \varepsilon I_n) y(t) + \sum_{i \geq 1} e^{\varepsilon h_i} A_i y(t - h_i) \\ + \int_0^t e^{\varepsilon(t-s)} B(t-s) y(s) ds, \quad \text{for a.a. } t \geq 0. \end{aligned} \quad (5.3)$$

Since  $\det \mathcal{H}(z) \neq 0$  for all  $z \in \mathbb{C}_{-\varepsilon}$ , it follows that  $\det \mathcal{H}_\varepsilon(z) \neq 0$  for all  $z \in \mathbb{C}_0$ . Applying Proposition 5.2 to (5.3), we may conclude that  $Y(\cdot) = e^{\varepsilon \cdot} X(\cdot) \in L^\infty(\mathbb{R}_+; \mathbb{R}^{n \times n})$ . This gives (5.2).

We show that

$$\begin{aligned} \exists K_1 > 0, \forall (\sigma, \varphi) \in \mathbb{R}_+ \times L^1((-\infty, \sigma); \mathbb{R}^n) \forall t \geq \sigma : \\ \left\| \sum_{i \geq 1} \int_{-h_i}^0 X(t - \sigma - h_i - u) A_i \varphi(u) du \right. \\ \left. + \int_0^{t-\sigma} X(t - \sigma - u) \int_0^\sigma B(u + \sigma - s) \varphi(s) ds du \right\| \\ \leq K_1 e^{-\varepsilon(t-\sigma)} \|\varphi\|_{L^1}. \end{aligned} \quad (5.4)$$

Then exponential asymptotic stability of (1.1) follows from (5.2) and the Variation of Constants formula (2.5).

By (5.2), we have, for all  $t \geq \sigma \geq 0$ ,

$$\begin{aligned} \left\| \sum_{i \geq 1} \int_{-h_i}^0 X(t - \sigma - h_i - u) A_i \varphi(u) du \right\| \\ \leq \sum_{i \geq 1} \int_{-h_i}^0 K e^{-\varepsilon(t-\sigma-h_i-u)} \|A_i\| \|\varphi(u)\| du \\ \leq \left( K \sum_{i \geq 1} e^{\varepsilon h_i} \|A_i\| \right) e^{-\varepsilon(t-\sigma)} \|\varphi\|_{L^1} \end{aligned}$$

and

$$\begin{aligned} \left\| \int_0^{t-\sigma} X(t - \sigma - u) \int_0^\sigma B(u + \sigma - s) \varphi(s) ds du \right\| \\ \leq \int_0^{t-\sigma} K e^{-\varepsilon(t-\sigma-u)} \int_0^\sigma \|B(u + \sigma - s)\| \|\varphi(s)\| ds du \\ \leq K e^{-\varepsilon(t-\sigma)} \int_0^\sigma \|\varphi(s)\| \int_0^{t-\sigma} e^{\varepsilon u} \|B(u + \sigma - s)\| du ds \\ \leq K e^{-\varepsilon(t-\sigma)} \int_0^\sigma \|\varphi(s)\| \int_{\sigma-s}^{t-s} e^{\varepsilon(s-\sigma)} e^{\varepsilon u} \|B(u)\| du ds \\ \leq K \left( \int_0^\infty e^{\varepsilon u} \|B(u)\| du \right) e^{-\varepsilon(t-\sigma)} \|\varphi\|_{L^1}. \end{aligned}$$

Now combining the above two chains of inequalities gives (5.4). This completes the proof of Assertion (i).

(ii)

Assume that  $((A_i)_{i \in \mathbb{N}_0}, (h_i)_{i \in \mathbb{N}_0}, B(\cdot))$  satisfy (A1)-(A3),  $A_i \in \mathbb{R}_+^{n \times n}$  for all  $i \in \mathbb{N}$ ,  $B(\cdot) \geq 0$  and

$$\exists M, \lambda > 0 \forall t \geq 0 : \|X(t)\| \leq M e^{-\lambda t}. \quad (5.5)$$

Choose  $\alpha \in (0, \lambda)$ . Then, (5.5) implies that  $\hat{X}(\cdot)$  is analytic on  $\mathbb{C}_{-\alpha}$ . Clearly,  $\mathcal{H}(z)\hat{X}(z) = I_n$ ,  $z \in \mathbb{C}_0$ . Thus,  $\det \hat{X}(0) \neq 0$ . Since the function  $z \mapsto \det \hat{X}(z)$  is continuous at  $z = 0$ , there exists  $\alpha_0 \in (0, \alpha)$  such that  $\det \hat{X}(z) \neq 0$  for all  $z \in \mathcal{B}_{\alpha_0}(0)$ . Thus  $\hat{X}(\cdot)^{-1}$  exists on  $\mathcal{B}_{\alpha_0}(0)$ . Since the entries of  $\hat{X}(\cdot)$  are analytic on  $\mathcal{B}_{\alpha_0}(0)$ , so must be the entries of  $\hat{X}(\cdot)^{-1}$ . Therefore,

$$\mathcal{R}: \mathcal{B}_{\alpha_0}(0) \rightarrow \mathbb{C}^{n \times n}, \quad z \mapsto \mathcal{R}(z) := zI_n - A_0 - \hat{X}(z)^{-1}$$

is analytic on  $\mathcal{B}_{\alpha_0}(0)$ . Note that  $\mathcal{R}(z) = \sum_{i \geq 1} e^{-zh_i} A_i + \hat{B}(z)$ ,  $z \in \mathbb{C}_0$ . Since  $A_1, A_3$  hold, by standard properties of the Laplace transform and of sequences of analytic functions [34, p. 230], we have

$\forall m \in \mathbb{N} \forall s \in \mathcal{B}_{\alpha_0}(0) \cap \dot{\mathbb{C}}_0 :$

$$\mathcal{R}^{(m)}(s) = (-1)^m \left( \int_0^\infty t^m e^{-st} B(t) dt + \sum_{i \geq 1} h_i^m e^{-sh_i} A_i \right). \quad (5.6)$$

We may consider in the following, without restriction of generality, the norm

$$\|U\| := \sum_{i,j=1}^n |u_{ij}| \quad \text{for } U := (u_{ij}) \in \mathbb{C}^{n \times n}.$$

STEP 1: We show, by induction, that

$$\forall m \in \mathbb{N} : (t \mapsto t^m B(t)) \in L^1(\mathbb{R}_+, \mathbb{R}^{n \times n}) \quad \text{and} \quad \sum_{i \geq 1} h_i^m \|A_i\| < \infty. \quad (5.7)$$

Set

$$M := \|\mathcal{R}'(0)\|.$$

$m = 1$ : Seeking a contradiction, suppose that at least one of the following holds

$$\exists T > 1 : \int_0^T \|B(t)\|(t-1) dt > M; \quad \exists N_1 \in \mathbb{N} : \sum_{i=1}^{N_1} h_i \|A_i\| > M + \sum_{i \geq 1} \|A_i\|. \quad (5.8)$$

Choose  $\delta_0 > 0$  sufficiently small such that

$$\forall h \in (0, \delta_0) \forall t \in [0, T] : \frac{1 - e^{-ht}}{h} \geq t - 1; \quad \forall i \in \underline{N}_1 : \frac{1 - e^{-hh_i}}{h} \geq h_i - 1. \quad (5.9)$$

Invoking the properties  $A_i = (A(i)_{p,q}) \in \mathbb{R}_+^{n \times n}$  for all  $i \in \mathbb{N}$  and  $B(\cdot) \geq 0$  yields, for  $h > 0$  sufficiently small,

$$\begin{aligned} \left\| \frac{\mathcal{R}(h) - \mathcal{R}(0)}{h} \right\| &= \left\| \frac{\hat{B}(h) - \hat{B}(0)}{h} + \sum_{i \geq 1} \frac{e^{-hh_i} - 1}{h} A_i \right\| \\ &= \sum_{p,q=1}^n \left| \int_0^\infty B_{pq}(t) \frac{e^{-ht} - 1}{h} dt + \sum_{i \geq 1} \frac{e^{-hh_i} - 1}{h} (A(i)_{p,q}) \right| \\ &= \sum_{p,q=1}^n \int_0^\infty B_{pq}(t) \frac{1 - e^{-ht}}{h} dt + \sum_{i \geq 1} \frac{1 - e^{-hh_i}}{h} (A(i)_{p,q}). \end{aligned} \quad (5.10)$$

If the first inequality in (5.8) is valid, then, by invoking the first inequality in (5.9), (5.10) and continuity of the norm, we arrive at the contradiction

$$M = \|\mathcal{R}'(0)\| = \lim_{h \rightarrow 0^+} \left\| \frac{\mathcal{R}(h) - \mathcal{R}(0)}{h} \right\| \geq \int_0^T \|B(t)\|(t-1) dt > M.$$

If the second inequality in (5.8) is valid, then, by invoking the second inequality in (5.9) and (5.10), we arrive at the contradiction

$$\begin{aligned} M &\geq \lim_{h \rightarrow 0^+} \sum_{p,q=1}^n \sum_{i \geq 1} \frac{1 - e^{-hh_i}}{h} (A(i)_{p,q}) \geq \lim_{h \rightarrow 0^+} \sum_{p,q=1}^n \sum_{i=1}^{N_1} \frac{1 - e^{-hh_i}}{h} (A(i)_{p,q}) \\ &\geq \sum_{i=1}^{N_1} (h_i - 1) \|A_i\| > M. \end{aligned}$$

Therefore, (5.7) holds for  $m = 1$ .

If (5.7) holds for  $m$ , then it can be shown analogously as in the previous paragraph for  $m = 1$  that (5.7) holds for  $m + 1$  by replacing  $B(t)$ ,  $A_i$ ,  $\mathcal{R}(\cdot)$  by  $t^m B(t)$ ,  $h_i^m A_i$ ,  $\mathcal{R}^{(m)}(\cdot)$ , resp. This proves Step 1.

STEP 2: We show (5.1).

By (5.6) and (5.7), we have for any  $m \in \mathbb{N}$

$$\mathcal{R}^{(m)}(0) = \lim_{s \rightarrow 0^+} \mathcal{R}^{(m)}(s) = (-1)^m \left( \int_0^\infty t^m B(t) dt + \sum_{i \geq 1} h_i^m A_i \right).$$

Since  $\mathcal{R}(\cdot)$  is analytic on  $\mathcal{B}_{\alpha_0}(0)$ , the Maclaurin's series

$$\sum_{k=0}^{\infty} \frac{\mathcal{R}^{(k)}(0)}{k!} s^k,$$

is, for some  $\alpha_1 > 0$ , absolutely convergent in  $\overline{\mathcal{B}_{\alpha_1}(0)}$ . Therefore,

$$\sum_{k=0}^{\infty} \frac{\alpha_1^k}{k!} \left( \int_0^\infty t^k B_{pq}(t) dt + \sum_{i \geq 1} h_i^k (A(i)_{p,q}) \right) = \sum_{k=0}^{\infty} \frac{|\mathcal{R}_{pq}^{(k)}(0)|}{k!} \alpha_1^k < \infty,$$

and so, in view of nonnegativity of  $B(\cdot)$ ,  $A_i$  for all  $i \in \mathbb{N}$ , and Fatou's Lemma,

$$\begin{aligned} & \int_0^\infty e^{\alpha_1 t} \|B(t)\| dt + \sum_{i \geq 1} e^{\alpha_1 h_i} \|A_i\| \\ &= \int_0^\infty e^{\alpha_1 t} \sum_{p,q=1}^n B_{pq}(t) dt + \sum_{i \geq 1} e^{\alpha_1 h_i} \sum_{p,q=1}^n (A(i)_{p,q}) \\ &= \sum_{p,q=1}^n \left( \int_0^\infty e^{\alpha_1 t} B_{pq}(t) dt + \sum_{i \geq 1} e^{\alpha_1 h_i} (A(i)_{p,q}) \right) \\ &= \sum_{p,q=1}^n \left( \int_0^\infty \sum_{k=0}^\infty \frac{(\alpha_1 t)^k}{k!} B_{pq}(t) dt + \sum_{i \geq 1} \sum_{k=0}^\infty \frac{(\alpha_1 h_i)^k}{k!} (A(i)_{p,q}) \right) \\ &= \sum_{p,q=1}^n \left( \sum_{k=0}^\infty \frac{\alpha_1^k}{k!} \left( \int_0^\infty t^k B_{pq}(t) dt + \sum_{i \geq 1} h_i^k (A(i)_{p,q}) \right) \right) \\ &< \infty. \end{aligned}$$

This completes the proof. □.

The following is immediate from Theorem 5.4 and Proposition 5.2.

**Corollary 5.5.** *If  $((A_i)_{i \in \mathbb{N}_0}, (h_i)_{i \in \mathbb{N}_0}, B(\cdot))$  satisfy (A1)-(A3),  $A_i \in \mathbb{R}_+^{n \times n}$  for all  $i \in \mathbb{N}$ ,  $B(\cdot) \geq 0$  and (iv) in Proposition 5.2 holds, then the following statements are equivalent:*

- (i) (1.1) is exponentially asymptotically stable,
- (ii)  $\exists M, \lambda > 0 \forall t \geq 0 : \|X(t)\| \leq M e^{-\lambda t}$ , where  $X$  is the fundamental solution of (1.1),
- (iii) (5.1) holds.

We now deal with stability of positive systems. Exploiting positivity, we get explicit criteria for  $L^p$ -stability and exponential stability of positive system (1.1), invoking the spectral abscissa.

**Theorem 5.6.** *For a positive system (1.1) satisfying (A1)-(A3) we have:*

- (i) (1.1) is  $L^p$ -stable for all  $p \in [1, \infty]$  if, and only if,

$$\mu \left( A_0 + \sum_{i \geq 1} A_i + \int_0^\infty B(t) dt \right) < 0.$$

- (ii) (1.1) is exponentially asymptotically stable if, and only if, (5.1) holds for some  $\alpha_0 > 0$  and

$$\mu \left( A_0 + \sum_{i \geq 1} e^{\alpha_0 h_i} A_i + \int_0^\infty e^{\alpha_0 t} B(t) dt \right) < -\alpha_0.$$

Proof:

- (i)

In view of the equivalences “(ii)  $\iff$  (iii)  $\iff$  (iv)” in Proposition 5.2, and using the notation introduced in Definition 2.6, it suffices to show that

$$[ \forall z \in \mathbb{C}_0 : \det(zI_n - A_0 - \mathcal{R}(z)) \neq 0 ] \iff \mu(A_0 + \mathcal{R}(0)) < 0.$$

“ $\Leftarrow$ ”: Suppose

$$\exists z \in \mathbb{C}_0 : \det(zI_n - A_0 - \mathcal{R}(z)) = 0.$$

By Lemma 4.3,

$$0 \leq \Re z \leq \mu(A_0 + \mathcal{R}(z)) \leq \mu(A_0 + \mathcal{R}(0)).$$

This proves the claim.

“ $\Rightarrow$ ”: Suppose

$$\mu(A_0 + \mathcal{R}(0)) \geq 0.$$

Then the continuous function

$$f : [0, \infty) \rightarrow \mathbb{R}, \quad \theta \mapsto \theta - \mu(A_0 + \mathcal{R}(\theta))$$

satisfies  $f(\mu_0) \leq 0$  and  $\lim_{\theta \rightarrow \infty} f(\theta) = \infty$  and so we may choose  $\theta \geq 0$  such that  $f(\theta) = 0$ . The latter is equivalent to  $\theta = \mu(A_0 + \mathcal{R}(\theta))$  and, by Proposition 4.2(i), to  $\det(\theta I_n - A_0 - \mathcal{R}(\theta)) = 0$ . This is a contradiction and completes the proof of Assertion (i).

(ii)

“ $\Leftarrow$ ”: This follows directly from (i) and Theorem 5.4(i).

“ $\Rightarrow$ ”: Since (1.1) is positive, we have  $A_i \in \mathbb{R}_+^{n \times n}$  for all  $i \in \mathbb{N}$  and  $B(\cdot) \geq 0$ . Moreover, (vi) of Proposition 5.2 holds because (1.1) is exponentially stable. Now (5.1) follows from Theorem 5.4(ii). Since (1.1) is exponentially asymptotically stable, (i) gives

$$\mu\left(A_0 + \sum_{i \geq 1} A_i + \int_0^\infty B(t) dt\right) < 0.$$

Consider the continuous function

$$g : [0, \alpha] \rightarrow \mathbb{R}, \quad s \mapsto s + \mu\left(A_0 + \sum_{i \geq 1} e^{h_i s} A_i + \int_0^\infty e^{st} B(t) dt\right).$$

Since  $g(0) < 0$ , it follows that  $g(\alpha_0) < 0$  for some  $\alpha_0 \in [0, \alpha]$ . Thus,

$$\mu\left(A_0 + \sum_{i \geq 1} e^{\alpha_0 h_i} A_i + \int_0^\infty e^{\alpha_0 t} B(t) dt\right) < -\alpha_0$$

and

$$\sum_{i \geq 1} \|A_i\| e^{\alpha_0 h_i} + \int_0^\infty e^{\alpha_0 t} \|B(t)\| dt < \infty.$$

This completes the proof.

The following is immediate from Theorem 5.4 and Theorem 5.6.

**Corollary 5.7.** *For a positive system (1.1) satisfying (A2) and (5.1), the following statements are equivalent:*

- (i) (1.1) is  $L^1$ -stable,
- (ii) (1.1) is  $L^p$ -stable for all  $p \in [1, \infty]$ ,
- (iii)  $\exists M, \lambda > 0 \forall t \geq 0 : \|X(t)\| \leq M e^{-\lambda t}$ , where  $X$  is the fundamental solution of (1.1),
- (iv) (1.1) is exponentially asymptotically stable,
- (v)  $\mu\left(A_0 + \sum_{i \geq 1} A_i + \int_0^\infty B(t) dt\right) < 0$ ,
- (vi)  $\exists \alpha_0 > 0 : \mu\left(A_0 + \sum_{i \geq 1} e^{\alpha_0 h_i} A_i + \int_0^\infty e^{\alpha_0 t} B(t) dt\right) < -\alpha_0$ .

In the following example we show that even for positive equations, exponential asymptotic stability and  $L^p$ -stability (for all  $p \geq 1$ ) of (1.1) do not coincide.

**Example 5.8.** *Consider a linear Volterra integro-differential equation with delay given by*

$$\dot{x}(t) = -2x(t) + \sum_{i=1}^{\infty} a_i x(t - h_i) + \int_0^t b(t - \tau)x(\tau) d\tau, \quad t \geq 0, \quad (5.11)$$

for parameters  $h_i \geq 0$  and  $a_i \in \mathbb{R}$ ,  $i \in \mathbb{N}$  and  $b(\cdot) : \mathbb{R}_+ \rightarrow \mathbb{R}$  as specified below:

(a) For

$$a_i := 1/2^{i+1}, \quad i \in \mathbb{N} \quad \text{and} \quad b(t) := \frac{1}{(t+1)^2}, \quad t \geq 0,$$

we have  $-2 + \sum_{i=1}^{\infty} a_i + \int_0^{\infty} b(s) ds = -1/2 < 0$ . By Theorem 5.6(i), the system (5.11) is  $L^p$ -stable for all  $p \geq 1$ . However, since

$$\int_0^{\infty} \frac{e^{\alpha t}}{(t+1)^2} dt = \infty,$$

for any  $\alpha > 0$ , (5.1) does not hold. Thus, (5.11) is not exponentially asymptotically stable, by Theorem 5.6 (ii).

(b) For  $m \in \mathbb{N}$  and

$$a_i := 1/2^{i+1} \quad i = 1, 2, \dots, m, \quad a_i = 0 \quad i > m \quad \text{and} \quad b(t) := e^{-t} \quad t \geq 0,$$

both conditions of Theorem 5.6(ii) hold. Therefore, (5.11) is exponentially asymptotically stable.

**Remark 5.9.** Consider a linear functional differential equation of the form

$$\dot{x}(t) = \int_{-h}^0 d\eta(\theta)x(t+\theta), \quad t \geq 0, \quad (5.12)$$

where  $\eta : [-h, 0] \rightarrow \mathbb{R}^{n \times n}$  is of bounded variation. It is well-known (see e.g. [8]) that (5.12) is exponentially asymptotically stable if, and only if,

$$\det \left( sI_n - \int_{-h}^0 e^{st} d\eta(t) \right) \neq 0 \quad \forall s \in \mathbb{C}_0.$$

Example 5.8 shows that even for positive equations, exponential asymptotic stability and  $L^p$ -stability (for all  $p \geq 1$ ) of (1.1) do not coincide. Taking into account Proposition 5.2 ((iii)  $\Leftrightarrow$  (iv)), this also means that the condition

$$\det \left( sI_n - A_0 - \sum_{i \geq 1} e^{-h_i s} - \int_0^{\infty} e^{-st} B(t) dt \right) \neq 0 \quad \forall s \in \mathbb{C}_0$$

does not ensure that (1.1) is exponentially asymptotically stable. This is an essential difference between the exponential asymptotic stability of (1.1) and that of linear functional differential equations (5.12).

## 6. ROBUSTNESS OF STABILITY, STABILITY RADII

In 1986 Hinrichsen and Pritchard introduced the concept of structured stability radius [13]: Consider an asymptotically stable linear differential system  $\dot{x}(t) = Ax(t)$  and determine the maximal  $r > 0$  for which all systems of the form

$$\dot{x}(t) = (A + D\Delta E)x(t),$$

are asymptotically stable as long as  $\|\Delta\| < r$ . Here,  $\Delta$  is unknown disturbance matrix, and  $D$  and  $E$  are given matrices defining the structure of perturbations. This so called stability radius  $r$  was characterized and a formula had been given. Beside many other generalizations (see [14, Sect. 5.3] and the references therein), the concept of stability radius has been analyzed for positive systems  $\dot{x}(t) = Ax(t)$  in [15]. Recently, the main results of [15] have been extended to various classes of positive systems such as positive linear time-delay differential systems [33], [26], positive linear discrete time-delay systems [16, 31] and positive linear functional differential systems [32].

### 6.1. Structured perturbations

In this sub-section we study robustness of positive,  $L^1$ -stable systems (1.1) satisfying (A1)-(A3). Assume that the system (1.1) is subjected to additive perturbations of the form

$$\begin{aligned} \dot{x}(t) = (A_0 + D_0\Delta_0E)x(t) + \sum_{i \geq 1} (A_i + D_i\Delta_iE)x(t - h_i) \\ + \int_0^t (B(s) + D\delta(s)E)x(t - s) ds, \end{aligned} \quad (6.1)$$

where the matrices  $(D_i)_{i \in \mathbb{N}_0}$ ,  $D$ ,  $E$  specify the *structure* of the perturbation and belong to the class

$$\mathcal{S}_{\mathbb{K}} := \left\{ ((D_i)_{i \in \mathbb{N}_0}, D, E) \in (\mathbb{K}^{n \times \ell_i})^{\mathbb{N}_0} \times \mathbb{K}^{n \times \ell} \times \mathbb{K}^{q \times n} \mid \sup_{i \in \mathbb{N}_0} \|D_i\| < \infty \right\}$$

and the *perturbation class* is

$$\mathcal{P}_{\mathbb{K}} := \left\{ (\Delta, \delta) = ((\Delta_i)_{i \in \mathbb{N}_0}, \delta) \in (\mathbb{K}^{\ell_i \times q})^{\mathbb{N}_0} \times L^1(\mathbb{R}_+, \mathbb{K}^{\ell \times q}) \mid \|(\Delta, \delta)\| < \infty \right\}$$

endowed with the norm

$$\|(\Delta, \delta)\| := \sum_{i=0}^{\infty} \|\Delta_i\| + \int_0^{\infty} \|\delta(t)\| dt, \quad \mathbb{K} = \mathbb{R}, \mathbb{C}, \mathbb{R}_+, \text{ resp.}$$

The aim is to determine the maximal  $r > 0$  such that for any  $(\Delta, \delta) \in \mathcal{P}_{\mathbb{K}}$  the perturbed system (6.1) remains  $L^1$ -stable whenever  $\|(\Delta, \delta)\| < r$ . More precisely, we study the following *complex, real, and positive stability radius*

$$r_{\mathbb{K}} := \inf \left\{ \|(\Delta, \delta)\| \mid (\Delta, \delta) \in \mathcal{P}_{\mathbb{K}}, \text{ (6.1) is not } L^1\text{-stable} \right\}, \quad \mathbb{K} = \mathbb{R}, \mathbb{C}, \mathbb{R}_+, \text{ resp.}$$

While it is obvious that

$$0 < r_{\mathbb{C}} \leq r_{\mathbb{R}} \leq r_{\mathbb{R}_+} \leq \infty, \quad (6.2)$$

we will show equality of all three stability radii and present a formula.

**Theorem 6.1.** *Suppose  $((A_i)_{i \in \mathbb{N}_0}, (h_i)_{i \in \mathbb{N}_0}, B(\cdot))$  satisfy (A1)-(A3) and the system (1.1) is positive and  $L^1$ -stable. Then for any perturbation structure  $((D_i)_{i \in \mathbb{N}_0}, D, E) \in \mathcal{S}_{\mathbb{R}_+}$ , the stability radii satisfy, for characteristic matrix  $\mathcal{H}(\cdot)$  as defined in Definition 2.6,*

$$r_{\mathbb{C}} = r_{\mathbb{R}} = r_{\mathbb{R}_+} = \frac{1}{\max\{\sup_{i \in \mathbb{N}_0} \|E\mathcal{H}(0)^{-1}D_i\|, \|E\mathcal{H}(0)^{-1}D\|\}}. \quad (6.3)$$

We illustrate Theorem 6.1 by a simple example.

**Example 6.2.** Consider a linear Volterra integro-differential system with delays given by

$$\dot{x}(t) = A_0x(t) + \sum_{i \geq 1} A_i x(t - h_i) + \int_0^t B(t - \tau)x(\tau) d\tau, \quad t \geq 0, \quad (6.4)$$

where

$$A_0 = \begin{pmatrix} -3 & 1/2 \\ 0 & -3 \end{pmatrix}, \quad A_i = \begin{pmatrix} 0 & 1/2^{i+1} \\ 0 & 0 \end{pmatrix}, \quad B(t) = \begin{pmatrix} e^{-t} & 0 \\ (t+1)^{-2} & e^{-t} \end{pmatrix}, \quad i \in \mathbb{N}, \quad t \geq 0.$$

By Theorem 3.2, (6.4) is a positive system and, invoking Theorem 5.6, it is easy to see that (6.4) is  $L^1$ -stable. Consider the perturbed system

$$\dot{x}(t) = A_{0\Delta}x(t) + \sum_{i \geq 1} A_{i\Delta}x(t - h_i) + \int_0^t B_\delta(\tau)x(t - \tau) d\tau, \quad t \geq 0 \quad (6.5)$$

where

$$A_{0\Delta} = \begin{pmatrix} -3 + a_1 & 2^{-1} + a_2 \\ 0 & -3 \end{pmatrix}, \quad A_{i\Delta} = \begin{pmatrix} a_1 2^{-i} & 2^{-(i+1)} + a_2 2^{-i} \\ 0 & 0 \end{pmatrix}, \\ B_\delta(t) = \begin{pmatrix} e^{-t} & 0 \\ (t+1)^{-2} + \delta_1(t) & e^{-t} + \delta_2(t) \end{pmatrix}, \quad i \in \mathbb{N}, \quad t \geq 0.$$

and  $a_1, a_2 \in \mathbb{R}$ ,  $\delta_1, \delta_2 \in L^1(\mathbb{R}_+, \mathbb{R})$  are unknown parameters. Setting

$$D_i = (1, 0)^T, \quad \Delta_0 = (a_1, a_2), \quad \Delta_i = (a_1 2^{-i}, a_2 2^{-i}), \quad i \in \mathbb{N}, \\ D = (0, 1)^T, \quad \delta = (\delta_1, \delta_2), \quad E = I_2,$$

and

$$A_{i\Delta} = A_i + D_i \Delta_i E, \quad i \in \mathbb{N}_0, \quad \text{and} \quad B_\delta(\cdot) = B(\cdot) + D\delta(\cdot)E,$$

we may recast (6.5) as a perturbed system of the form (6.1).

Theorem 6.1 yields that the stability radius of (6.4) is equal to  $3\sqrt{5}/5$ . Therefore, perturbed systems (6.5) remain  $L^1$ -stable if

$$\sum_{i \geq 0} \|\Delta_i\| + \int_0^\infty \|\delta(t)\| dt = 2\sqrt{a_1^2 + a_2^2} + \int_0^\infty \sqrt{\delta_1(t)^2 + \delta_2(t)^2} dt < \frac{3\sqrt{5}}{5}.$$

In the remainder of this sub-section we prove Theorem 6.1 and some technical lemmata.

**Lemma 6.3.** Suppose the system (1.1) satisfies (A1)-(A3) and is positive and  $L^1$ -stable. Then

- (i)  $\forall z \in \mathbb{C}_0$  :  $|\mathcal{H}(z)^{-1}| \leq \mathcal{H}(0)^{-1}$ ,
- (ii)  $\forall U \in \mathbb{R}_+^{n \times \ell} \forall V \in \mathbb{R}_+^{q \times n}$  :  $\max_{z \in \mathbb{C}_0} \|V\mathcal{H}(z)^{-1}U\| = \|V\mathcal{H}(0)^{-1}U\|$ .

Proof:

(i) By Theorem 5.6(i), we have

$$\mu \left( A_0 + \sum_{i \geq 1} A_i + \int_0^\infty B(t) dt \right) = \mu(A_0 + \mathcal{R}(0)) < 0$$

and so, invoking Lemma 4.3(iii),

$$\mu(A_0 + \mathcal{R}(z)) \leq \mu(A_0 + \mathcal{R}(0)) < 0, \quad \forall z \in \mathbb{C}_0.$$

Since  $A_0$  is a Metzler matrix, we may choose  $\alpha_0 > 0$  such that  $(A_0 + \alpha_0 I_n) \geq 0$ , and, invoking Lemma 4.3, it follows that

$$\begin{aligned} e^{\alpha_0 \theta} |e^{\theta(A_0 + \mathcal{R}(z))}| &= |e^{\alpha_0 \theta} e^{\theta(A_0 + \mathcal{R}(z))}| = |e^{\alpha_0 \theta I_n} e^{\theta(A_0 + \mathcal{R}(z))}| \\ &= |e^{\theta((A_0 + \alpha_0 I_n) + \mathcal{R}(z))}| \leq e^{\theta((\alpha_0 I_n + A_0) + \mathcal{R}(0))} = e^{\alpha_0 \theta} e^{\theta(A_0 + \mathcal{R}(0))}, \quad \forall \theta \geq 0, \end{aligned}$$

whence

$$|e^{\theta(A_0 + \mathcal{R}(z))}| \leq e^{\theta(A_0 + \mathcal{R}(0))} \quad \forall \theta \geq 0 \quad \forall z \in \mathbb{C}_0. \quad (6.6)$$

By the formula [9, p. 57]

$$\mathcal{H}(z)^{-1} = (zI_n - (A_0 + \mathcal{R}(z)))^{-1} = \int_0^\infty e^{-z\theta} e^{\theta(A_0 + \mathcal{R}(z))} d\theta, \quad \forall z \in \mathbb{C}_0, \quad (6.7)$$

(6.6) implies that

$$|\mathcal{H}(z)^{-1}| = \left| \int_0^\infty e^{-z\theta} e^{\theta(A_0 + \mathcal{R}(z))} d\theta \right| \stackrel{(6.6)}{\leq} \int_0^\infty e^{\theta(A_0 + \mathcal{R}(0))} d\theta \stackrel{(6.7)}{=} \mathcal{H}(0)^{-1}, \quad \forall z \in \mathbb{C}_0$$

and therefore Assertion (i) is proved.

(ii) For nonnegative  $U$  and  $V$  it follows that

$$|V\mathcal{H}(z)^{-1}U| \leq V\mathcal{H}(0)^{-1}U \quad \forall z \in \mathbb{C}_0.$$

By the monotonicity property of the vector norm and the definition of the induced matrix norm, we conclude

$$\|V\mathcal{H}(z)^{-1}U\| \leq \|V\mathcal{H}(0)^{-1}U\| \quad \forall z \in \mathbb{C}_0.$$

This shows Assertion (ii) and completes the proof.

**Lemma 6.4.** *Suppose the system (1.1) satisfies (A1)-(A3) and is positive and  $L^1$ -stable. Let  $((D_i)_{i \in \mathbb{N}_0}, D, E) \in \mathcal{S}_{\mathbb{R}_+}$  and suppose that, for  $\mathcal{H}(\cdot)$  as defined in Definition 2.6,*

$$\max \left\{ \sup_{i \in \mathbb{N}_0} \|E\mathcal{H}(0)^{-1}D_i\|, \|E\mathcal{H}(0)^{-1}D\| \right\} \neq 0.$$

*Then, for every  $\varepsilon > 0$ , there exists a nonnegative perturbation  $(\Delta, \delta) \in \mathcal{P}_{\mathbb{R}_+}$  such that the perturbed system (6.1) is not  $L^1$ -stable and*

$$\|(\Delta, \delta)\| = \frac{1}{\max \left\{ \sup_{i \in \mathbb{N}_0} \|E\mathcal{H}(0)^{-1}D_i\|, \|E\mathcal{H}(0)^{-1}D\| \right\}} + \varepsilon. \quad (6.8)$$

*Proof:*

Since (1.1) is positive and  $L^1$ -stable, by Lemma 6.3 (i),  $\mathcal{H}(0)^{-1} \geq 0$ , and therefore, in view of  $((D_i)_{i \in \mathbb{N}_0}, D, E) \in \mathcal{S}_{\mathbb{R}_+}$ , we conclude

$$E\mathcal{H}(0)^{-1}D_i \in \mathbb{R}_+^{q \times \ell_i}, \quad \forall i \in \mathbb{N}_0; \quad E\mathcal{H}(0)^{-1}D \in \mathbb{R}_+^{q \times \ell}.$$

We now consider two cases.

Case I:  $\sup_{i \in \mathbb{N}_0} \|E\mathcal{H}(0)^{-1}D_i\| > \|E\mathcal{H}(0)^{-1}D\|$ .

Let  $\varepsilon > 0$ . Since  $\sup_{i \in \mathbb{N}_0} \|E\mathcal{H}(0)^{-1}D_i\|$  is finite, there exists  $k \in \mathbb{N}_0$  such that

$$\frac{1}{\|E\mathcal{H}(0)^{-1}D_k\|} < \frac{1}{\sup_{i \in \mathbb{N}_0} \|E\mathcal{H}(0)^{-1}D_i\|} + \varepsilon.$$

Choose

$$u \in \mathbb{R}_+^{\ell_k} \quad \text{with} \quad \|u\| = 1 \quad \text{and} \quad \|E\mathcal{H}(0)^{-1}D_k\| = \|E\mathcal{H}(0)^{-1}D_k u\|$$

Since  $E\mathcal{H}(0)^{-1}D_k u \geq 0$ , there exists, by the Hahn-Banach theorem for positive linear functionals [35, p. 249], a positive linear functional  $y^* \in (\mathbb{C}^q)^*$  of dual norm  $\|y^*\| = 1$  such that

$$y^* E\mathcal{H}(0)^{-1}D_k u = \|E\mathcal{H}(0)^{-1}D_k u\|.$$

For

$$\Delta_k := \|E\mathcal{H}(0)^{-1}D_k\|^{-1} u y^* \in \mathbb{R}_+^{\ell_k \times q} \quad \text{and} \quad x_0 := \mathcal{H}(0)^{-1}D_k u,$$

we have

$$\|\Delta_k\| = \|E\mathcal{H}(0)^{-1}D_k\|^{-1} \quad \text{and} \quad \Delta_k E x_0 = u_0.$$

Therefore,

$$\left( A_0 + \sum_{i \geq 1} A_i + D_k \Delta_k E + \int_0^\infty B(t) dt \right) x_0 = 0, \quad x_0 = \mathcal{H}(0)^{-1}D_k \Delta_k E x_0 \neq 0.$$

Defining

$$(\Delta, \delta) := ((\Delta_i)_{i \in \mathbb{N}_0}, 0), \quad \text{where} \quad \Delta_i = \begin{cases} \Delta_k, & i = k, \\ 0, & i \neq k \end{cases}$$

yields  $(\Delta, \delta) \in \mathcal{P}_{\mathbb{R}_+}$  and

$$\|(\Delta, \delta)\| = \|\Delta_k\| = \frac{1}{\|E\mathcal{H}(0)^{-1}D_k\|} < \frac{1}{\sup_{i \in \mathbb{N}_0} \|E\mathcal{H}(0)^{-1}D_i\|} + \varepsilon,$$

whence, by Proposition 5.2, the perturbed system (6.1) is not  $L^1$ -stable.

Case II:  $\max\{\sup_{i \in \mathbb{N}_0} \|E\mathcal{H}(0)^{-1}D_i\|, \|E\mathcal{H}(0)^{-1}D\|\} \leq \|E\mathcal{H}(0)^{-1}D\|$ .

By a similar argument as in Case I, there exists a nonnegative matrix  $\Delta_D \in \mathbb{R}_+^{\ell \times q}$  such that

$$\|\Delta_D\| = \frac{1}{\|E\mathcal{H}(0)^{-1}D\|}$$

and

$$\left( A_0 + \sum_{i \geq 1} A_i + D \Delta_D E + \int_0^\infty B(t) dt \right) x = 0, \quad \text{for some} \quad x \in \mathbb{R}^n \setminus \{0\}.$$

For

$$\delta_D(\cdot) := (t \rightarrow e^{-t} \Delta_D) \in L^1(\mathbb{R}_+, \mathbb{R}_+^{\ell \times q})$$

we have

$$\left( A_0 + \sum_{i \geq 1} A_i + \int_0^\infty (B(t) + D \delta_D(t) E) dt \right) x = 0,$$

and

$$(\Delta, \delta) := ((0)_{i \in \mathbb{N}_0}, \delta_D) \in \mathcal{P}_{\mathbb{R}_+}$$

satisfies (6.8). Finally, Proposition 5.2 says that the perturbed system (6.1) is not  $L^1$ -stable. This completes the proof.

We are finally in a position to prove the main theorem of this sub-section.

Proof of Theorem 6.1:

Assume that  $r_{\mathbb{C}} < \infty$ . Let  $(\Delta, \delta) \in \mathcal{P}_{\mathbb{C}}$  be a destabilizing complex disturbance. By Proposition 5.2, there exist  $s \in \mathbb{C}_0$  and  $x \in \mathbb{C}^n \setminus \{0\}$  such that

$$\left( (A_0 + D_0 \Delta_0 E) + \sum_{i \geq 1} e^{-sh_i} (A_i + D_i \Delta_i E) + \int_0^\infty e^{-st} (B(t) + D\delta(t)E) dt \right) x = sx.$$

Since (1.1) is  $L^1$ -stable, it follows that

$$\mathcal{H}(s)^{-1} \left( D_0 \Delta_0 E + \sum_{i \geq 1} e^{-sh_i} D_i \Delta_i E + D \int_0^\infty e^{-st} \delta(t) dt \right) x = x,$$

and

$$E\mathcal{H}(s)^{-1} \left( D_0 \Delta_0 + \sum_{i \geq 1} e^{-sh_i} D_i \Delta_i + D \int_0^\infty e^{-st} \delta(t) dt \right) Ex = Ex \neq 0.$$

Taking norms, we derive

$$\left( \sum_{i \geq 0} \|E\mathcal{H}(s)^{-1} D_i\| \|\Delta_i\| + \|E\mathcal{H}(s)^{-1} D\| \int_0^\infty \|\delta(t)\| dt \right) \|Ex\| \geq \|Ex\|$$

and by Lemma 6.3 this implies that

$$\left( \sum_{i \geq 0} \|E\mathcal{H}(0)^{-1} D_i\| \|\Delta_i\| + \|E\mathcal{H}(0)^{-1} D\| \int_0^\infty \|\delta(t)\| dt \right) \|Ex\| \geq \|Ex\|.$$

Hence,

$$\max\{\sup_{i \in \mathbb{N}_0} \|E\mathcal{H}(0)^{-1} D_i\|, \|E\mathcal{H}(0)^{-1} D\|\} \left( \sum_{i \geq 0} \|\Delta_i\| + \int_0^\infty \|\delta(t)\| dt \right) \geq 1.$$

We thus obtain,  $\max\{\sup_{i \in \mathbb{N}_0} \|E\mathcal{H}(0)^{-1} D_i\|, \|E\mathcal{H}(0)^{-1} D\|\} > 0$  and

$$\|(\Delta, \delta)\| \geq \frac{1}{\max\{\sup_{i \in \mathbb{N}_0} \|E\mathcal{H}(0)^{-1} D_i\|, \|E\mathcal{H}(0)^{-1} D\|\}}.$$

Since this inequality holds true for any destabilizing complex perturbation, we conclude that

$$r_{\mathbb{C}} \geq \frac{1}{\max\{\sup_{i \in \mathbb{N}_0} \|E\mathcal{H}(0)^{-1} D_i\|, \|E\mathcal{H}(0)^{-1} D\|\}}.$$

Taking (6.2), (6.3) into account, it remains to show that

$$r_{\mathbb{R}_+} \leq \frac{1}{\max\{\sup_{i \in \mathbb{N}_0} \|E\mathcal{H}(0)^{-1} D_i\|, \|E\mathcal{H}(0)^{-1} D\|\}}. \quad (6.9)$$

Since  $\max\{\sup_{i \in \mathbb{N}_0} \|E\mathcal{H}(0)^{-1} D_i\|, \|E\mathcal{H}(0)^{-1} D\|\} > 0$ , it follows from Lemma 6.4 that

$$r_{\mathbb{R}_+} \leq \frac{1}{\max\{\sup_{i \in \mathbb{N}_0} \|E\mathcal{H}(0)^{-1} D_i\|, \|E\mathcal{H}(0)^{-1} D\|\}} + \varepsilon \quad \forall \varepsilon > 0.$$

Hence, (6.9) holds.

Finally, the above arguments also show that

$$r_{\mathbb{C}} = \infty \iff \max\{\sup_{i \in \mathbb{N}_0} \|E\mathcal{H}(0)^{-1} D_i\|, \|E\mathcal{H}(0)^{-1} D\|\} = 0.$$

This shows (6.3) and completes the proof of the theorem.  $\square$

### 6.2. Affine perturbations

In this sub-section, we study again positive  $L^1$ - stable systems (1.1) satisfying (A1)-(A3). In contrast to Sub-section 6.1, the system (1.1) is subjected to affine perturbations of the form

$$\begin{aligned} \dot{x}(t) = & \left( A_0 + \sum_{j=1}^N \alpha_{0j} A_{0j} \right) x(t) + \sum_{i \geq 1} \left( A_i + \sum_{j=1}^N \alpha_{ij} A_{ij} \right) x(t - h_i) \\ & + \int_0^t \left( B(s) + \sum_{j=1}^N \beta_j B_j(s) \right) x(t - s) ds, \end{aligned} \quad (6.10)$$

where the sequence of matrices  $(A_{ij})_{(i,j) \in \mathbb{N}_0 \times \underline{N}}$ ,  $(B_j(\cdot))_{j \in \underline{N}}$  specify the *structure* of the perturbation and belong to the class

$$\begin{aligned} \mathcal{S}_{\mathbb{K}}^a := & \left\{ ((A_{ij})_{(i,j) \in \mathbb{N}_0 \times \underline{N}}, (B_j(\cdot))_{j \in \underline{N}}) \in (\mathbb{K}^{n \times n})^{\mathbb{N}_0 \times \underline{N}} \times L^1(\mathbb{R}_+, \mathbb{K}^{n \times n})^N \right. \\ & \left. \left| \sum_{(i,j) \in \mathbb{N}_0 \times \underline{N}} \|A_{ij}\| < \infty \right. \right\} \end{aligned}$$

and the *perturbation class* is

$$\mathcal{P}_{\mathbb{K}}^a := \left\{ (\alpha, \beta) = (\alpha_{(i,j) \in \mathbb{N}_0 \times \underline{N}}, \beta_{j \in \underline{N}}) \in \mathbb{K}^{\mathbb{N}_0 \times \underline{N}} \times \mathbb{K}^{\underline{N}} \mid \|(\alpha, \beta)\| < \infty \right\}$$

endowed with the norm

$$\|(\alpha, \beta)\| := \max \left\{ \sup_{(i,j) \in \mathbb{N}_0 \times \underline{N}} |\alpha_{ij}|, \max_{j \in \underline{N}} |\beta_j| \right\} \quad \text{for } \mathbb{K} = \mathbb{R}, \mathbb{C}, \mathbb{R}_+, \text{ resp.}$$

Analogously to Sub-section 6.1, we study the *complex, real, and positive stability radius*

$$r_{\mathbb{K}}^a := \inf \left\{ \|(\alpha, \beta)\| \mid (\alpha, \beta) \in \mathcal{P}_{\mathbb{K}}^a, \text{ (6.10) is not } L^1\text{-stable} \right\}, \quad \mathbb{K} = \mathbb{R}, \mathbb{C}, \mathbb{R}_+, \text{ resp.}$$

While it is again obvious that

$$0 < r_{\mathbb{C}}^a \leq r_{\mathbb{R}}^a \leq r_{\mathbb{R}_+}^a \leq \infty,$$

we will show equality of all three stability radii and present a formula.

**Theorem 6.5.** *Suppose the system (1.1) satisfies (A1)-(A3) and is positive and  $L^1$ - stable. Then for any perturbation structure  $((A_{ij})_{(i,j) \in \mathbb{N}_0 \times \underline{N}}, (B_j(\cdot))_{j \in \underline{N}}) \in \mathcal{S}_{\mathbb{R}_+}^a$ , the stability radii satisfy, using  $\mathcal{H}(\cdot)$  as defined in Definition 2.6,*

$$r_{\mathbb{C}}^a = r_{\mathbb{R}}^a = r_{\mathbb{R}_+}^a = \frac{1}{\mu \left( \mathcal{H}(0)^{-1} \left( \sum_{(i,j) \in \mathbb{N}_0 \times \underline{N}} A_{ij} + \int_0^\infty \sum_{j=1}^N B_j(t) dt \right) \right)}. \quad (6.11)$$

*Proof:*

We first prove that

$$r_{\mathbb{R}_+}^a = \frac{1}{\mu(F)}, \quad \text{where } F := \mathcal{H}(0)^{-1} \left( \sum_{(i,j) \in \mathbb{N}_0 \times \underline{N}} A_{ij} + \sum_{j=1}^N \int_0^\infty B_j(t) dt \right).$$

Let  $(\alpha, \beta) = (\alpha_{(i,j) \in \mathbb{N}_0 \times \underline{N}}, \beta_{j \in \underline{N}}) \in \mathcal{P}_{\mathbb{R}_+}^a$  be a destabilizing perturbation so that (6.10) is not  $L^1$ -stable. By Theorem 5.2, there exist  $s \in \mathbb{C}_0$  and  $x \in \mathbb{C}^n \setminus \{0\}$  such that

$$\begin{aligned} & \left( A_0 + \sum_{j=1}^N \alpha_{0j} A_{0j} + \sum_{i \geq 1} e^{-sh_i} \left( A_i + \sum_{j=1}^N \alpha_{ij} A_{ij} \right) \right. \\ & \quad \left. + \int_0^\infty e^{-st} \left( B(t) + \sum_{j=1}^N \beta_j B_j(t) \right) dt \right) x = sx. \end{aligned}$$

Since (1.1) is  $L^1$ -stable, it follows that

$$\mathcal{H}(s)^{-1} \left( \sum_{i \geq 1} e^{-sh_i} \sum_{j=1}^N \alpha_{ij} A_{ij} + \sum_{j=1}^N \beta_j \int_0^\infty e^{-st} B_j(t) dt \right) x = x.$$

and by Lemma 6.3 (i),

$$\begin{aligned} |x| &= \left| \mathcal{H}(s)^{-1} \left( \sum_{i \geq 1} e^{-sh_i} \sum_{j=1}^N \alpha_{ij} A_{ij} + \sum_{j=1}^N \beta_j \int_0^\infty e^{-st} B_j(t) dt \right) x \right| \\ &\leq \mathcal{H}(0)^{-1} \left| \left( \sum_{i \geq 1} e^{-sh_i} \sum_{j=1}^N \alpha_{ij} A_{ij} + \sum_{j=1}^N \beta_j \int_0^\infty e^{-st} B_j(t) dt \right) x \right| \\ &\leq \mathcal{H}(0)^{-1} \left( \sum_{(i,j) \in \mathbb{N}_0 \times \underline{N}} \alpha_{ij} A_{ij} + \sum_{j=1}^N \beta_j \int_0^\infty B_j(t) dt \right) |x| \\ &\leq \|(\alpha, \beta)\| F |x|. \end{aligned}$$

Since  $F \geq 0$ , it follows from Proposition 4.2(iii) that  $\mu(F) \geq \|(\alpha, \beta)\|^{-1} > 0$ . Since this holds for arbitrary destabilizing nonnegative perturbation  $(\alpha, \beta)$ , we conclude that  $r_{\mathbb{R}_+}^a \geq \frac{1}{\mu(F)}$ .

Next we prove  $r_{\mathbb{R}_+}^a \leq \frac{1}{\mu(F)}$ . By Proposition 4.2(ii), there exists  $y \in \mathbb{R}_+^n \setminus \{0\}$  such that  $Fy = \mu(F)y$  and therefore

$$\begin{aligned} & \left( \left( A_0 + \sum_{j=1}^N \frac{1}{\mu(F)} A_{0j} \right) + \sum_{i \geq 1} \left( A_i + \sum_{j=1}^N \frac{1}{\mu(F)} A_{ij} \right) \right. \\ & \quad \left. + \int_0^\infty \left( B(t) + \sum_{j=1}^N \frac{1}{\mu(F)} B_j(t) \right) dt \right) y = 0. \end{aligned}$$

This means that the nonnegative perturbation

$$(\alpha, \beta) := (\alpha_{(i,j) \in \mathbb{N}_0 \times \underline{N}}, \beta_{j \in \underline{N}}) \in \mathcal{P}_{\mathbb{R}_+}^a$$

with

$$\alpha_{(i,j)} = \beta_j := 1/\mu(F), \quad (i, j) \in \mathbb{N}_0 \times \underline{N}, \quad j \in \underline{N},$$

is destabilizing. By definition of  $r_{\mathbb{R}_+}^a$  we get  $r_{\mathbb{R}_+}^a \leq \frac{1}{\mu(F)}$ . This proves the claim.

Finally, we are now ready to show that  $r_{\mathbb{C}}^a = r_{\mathbb{R}}^a = r_{\mathbb{R}_+}^a$ . Suppose  $(\alpha, \beta) = (\alpha_{(i,j) \in \mathbb{N}_0 \times \underline{N}}, \beta_{j \in \underline{N}}) \in \mathcal{P}_{\mathbb{C}}^a$  is a complex destabilizing perturbation so that (6.10) is not  $L^1$ -stable. By a similar argument as the above, we get

$$|x_0| \leq \mathcal{H}(0)^{-1} \left( \sum_{(i,j) \in \mathbb{N}_0 \times \underline{N}} |\alpha_{ij}| A_{ij} + \sum_{j=1}^N |\beta_j| \int_0^\infty B_j(t) dt \right) |x_0| \quad (6.12)$$

for some  $x_0 \in \mathbb{C}^n, x_0 \neq 0$ . Then, by Proposition 4.2(iii),

$$\mu \left( \mathcal{H}(0)^{-1} \left( \sum_{(i,j) \in \mathbb{N}_0 \times \underline{N}} |\alpha_{ij}| A_{ij} + \sum_{j=1}^N |\beta_j| \int_0^\infty B_j(t) dt \right) \right) \geq 1.$$

Since

$$C := \mathcal{H}(0)^{-1} \left( \sum_{(i,j) \in \mathbb{N}_0 \times \underline{N}} |\alpha_{ij}| A_{ij} + \sum_{j=1}^N |\beta_j| \int_0^\infty B_j(t) dt \right) \geq 0,$$

Proposition 4.2(ii) yields that  $Cx_1 = \mu(C)x_1$ , for some  $x_1 \in \mathbb{R}_+^n \setminus \{0\}$ . This gives

$$\begin{aligned} \left( \left( A_0 + \sum_{j=1}^N \frac{|\alpha_{0j}|}{\mu(C)} A_{0j} \right) + \sum_{i \geq 1} \left( A_i + \sum_{j=1}^N \frac{|\alpha_{ij}|}{\mu(C)} A_{ij} \right) \right. \\ \left. + \int_0^\infty \left( B(t) + \sum_{j=1}^N \frac{|\beta_j|}{\mu(C)} B_j(t) \right) dt \right) x_1 = 0, \end{aligned}$$

which means that

$$(|\alpha|, |\beta|) := \left( \left( \frac{|\alpha_{ij}|}{\mu(C)} \right)_{(i,j) \in \mathbb{N}_0 \times \underline{N}}, \left( \frac{|\beta_j|}{\mu(C)} \right)_{j \in \underline{N}} \right)$$

is a nonnegative destabilizing perturbation. Hence, it follows from the definition of  $r_{\mathbb{R}_+}^a$  that

$$\max \left( \sup_{(i,j) \in \mathbb{N}_0 \times \underline{N}} \left( \frac{|\alpha_{ij}|}{\mu(C)} \right), \max_{j \in \underline{N}} \left( \frac{|\beta_j|}{\mu(C)} \right) \right) \geq r_{\mathbb{R}_+}^a,$$

or

$$\max \left( \sup_{(i,j) \in \mathbb{N}_0 \times \underline{N}} |\alpha_{ij}|, \max_{j \in \underline{N}} |\beta_j| \right) \geq \mu(C) r_{\mathbb{R}_+}^a \geq r_{\mathbb{R}_+}^a,$$

which implies that  $r_{\mathbb{C}}^a \geq r_{\mathbb{R}_+}^a$ . Combined with the inequalities  $r_{\mathbb{C}}^a \leq r_{\mathbb{R}}^a \leq r_{\mathbb{R}_+}^a$ , this implies that  $r_{\mathbb{C}}^a = r_{\mathbb{R}}^a = r_{\mathbb{R}_+}^a$ . In addition, from the above arguments, we observe that  $r_{\mathbb{C}}^a = r_{\mathbb{R}}^a = r_{\mathbb{R}_+}^a = \infty$  if, and only if,  $\mu(F) = 0$ . This completes the proof.

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