Shift dynamics near T-point heteroclinic cycles

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Abstract

We consider nonwandering dynamics near heteroclinic cycles between two hyperbolic equilibria. The constituting heteroclinic connections are assumed to be such that one of them is transverse and isolated. In the literature such heteroclinic cycles have been associated with the termination of a branch of homoclinic solutions, and have been named *T-points* in this context. We study the nonwandering dynamics near T-point heteroclinic cycles and their unfoldings, both in the general setting (where a T-point heteroclinic cycle has codimension two) and in the reversible setting (where it has codimension one). We focus in particular on the existence of shift dynamics in the local nonwandering set.

Contents

1 Introduction

Homoclinic bifurcations lie at the heart of our understanding of complicated (chaotic) recurrent behaviour in dynamical systems. The history goes back to Poincaré, with major subsequent contributions by the schools of Andronov, Shilnikov, Smale and Palis. The successes of the latter schools has been founded on a combination of analytical and geometrical tools, typical for the field of dynamical systems. In this paper we follow an alternative approach for studying local dynamics near connecting orbits based on a functional analytic approach developed by the school of Hale. The latter approach, named *Lin's method* after a paper by Lin [19], employs a reduction on an appropriate Banach space of piecewise continuous functions approximating the initial heteroclinic cycle to yield bifurcation equations whose solutions represent orbits of the nonwandering set.

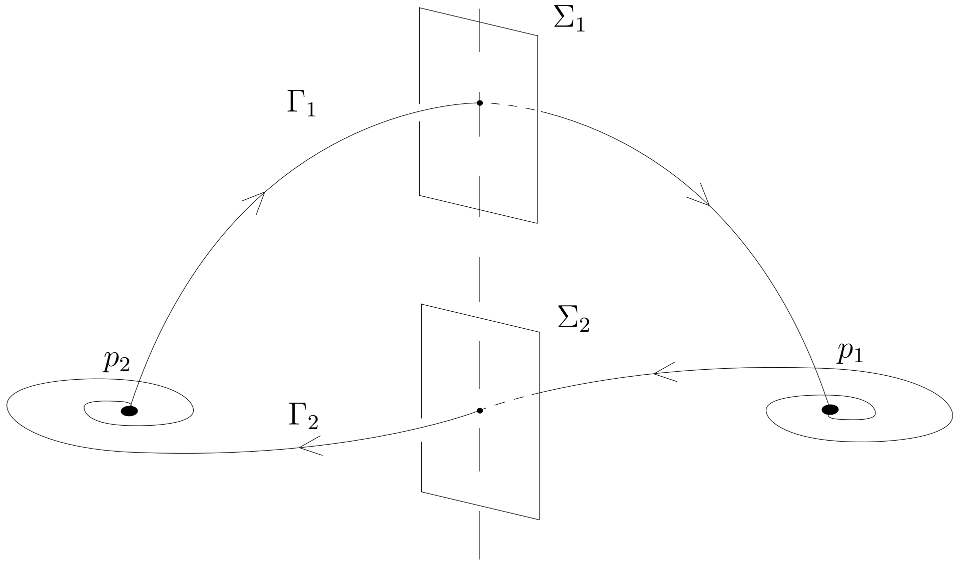

Figure 1: Sketch of an example of a T-point heteroclinic cycle in $\mathbb{R}^3$ between two saddle-foci, with robust heteroclinic orbit Γ_2 and non-robust heteroclinic orbit Γ_1.

We consider a parameter family of vector fields $f : \mathbb{R}^n \times \mathbb{R}^p \to \mathbb{R}^n$ $(n \geq 3)$, f smooth:

$$\dot{x} = f(x, \mu). \tag{1.1}$$

The aim of this paper is to study the nonwandering dynamics in the neighbourhood of a *T-point* [8]: a heteroclinic cycle between two hyperbolic equilibria of saddle type p_1 and p_2, where one of the connections is transverse and isolated, see Figure 1 for a sketch of a T-point heteroclinic cycle in $\mathbb{R}^3$. T-points have been found to appear in many applications of interest, ranging from the Lorentz [8] and Kuramoto-Sivashinsky [18] systems, to electronic oscillators [1, 2, 10, 11, 12], semiconductor lasers [9, 29], magnetoconvection [22] and travelling waves in reaction-diffusion dynamics [17, 26, 21, 13].

Without loss, we assume that the cycle exists at $\mu = 0$ and that the hyperbolic equilibria do not change position while μ is close to 0. Throughout this paper we denote the stable manifold of the equilibrium p_i at parameter(s) μ by $W^s(p_i,\mu)$. For brevity, we also denote $W^s(p_i,0)$ by $W^s(p_i)$. In the same manner we use $W^u(p_i,\mu)$ and $W^u(p_i)$ to denote the corresponding unstable manifolds.

Let $\Gamma_1 = \{q_1(t) \mid t \in \mathbb{R}\}$ denote the heteroclinic solution between p_2 and p_1, and $\Gamma_2 = \{q_2(t) \mid t \in \mathbb{R}\}$ the heteroclinic orbit from p_1 to p_2. The corresponding heteroclinic cycle is a T-point if Γ_2 is an isolated transversal intersection, ie

$$W^s(p_2) \pitchfork_{q_2(0)} W^u(p_1), \quad \text{and} \quad \dim T_{q_2(0)}W^s(p_2) + \dim T_{q_2(0)}W^u(p_1) = n+1, \tag{1.2}$$

where n is the dimension of the phase space. Further we assume that Γ_1 is nondegenerate, in the sense that

$$T_{q_1(0)}W^s(p_1) \cap T_{q_1(0)}W^u(p_2) = \operatorname{span}\{f(q_1(0),0)\}. \tag{1.3}$$

Note that due to this assumption the hyperbolic equilibria p_1 and p_2 have different saddle point indices.

The above conditions specify the simplest case of a heteroclinic cycle between two fixed points where one heteroclinic orbit is structurally stable and the other is not.

The nonrobust connection Γ_1 has codimension two in general and and codimension one if we assume the cycle is (setwise) invariant with respect to a time-reversal symmetry. Our main focus is with the general case and discuss the situation in the reversible case afterwards in Section 5.

We use Lin's method to analyse the nonwandering dynamics near a T-point heteroclinic cycle (and its unfolding). We recall the essentials of Lin's method in Section 2. The problem of finding particular orbits in the local nonwandering set reduces to that of solving a (infinite) set of bifurcation equations. We show how these bifurcation equations can be viewed in a geometric way as describing the set of intersections of logarithmic spirals and lines in the plane. These geometrical objects are determined by the principal eigenvalues of the linearisation at the equilibria.

We finally assume that the principal stable eigenvalue λ_2^s of $D_1f(p_2,0)$ and the principal unstable eigenvalue λ_1^u of $D_1f(p_1,0)$ are simple. We write

$$\lambda_1^u(\mu) = \rho_1(\mu) + i\phi_1(\mu), \quad \lambda_2^s(\mu) = -\rho_2(\mu) + i\phi_2(\mu),$$

where in the case that $\lambda_1^u(\mu)$ or $\lambda_2^s(\mu)$ are real, $\phi_1(\mu)$ or $\phi_2(\mu)$ are set equal to zero. We refer again to Figure 1 for a visualisation of the situation in $\mathbb{R}^3$ in case the addressed principal eigenvalues are complex. Therefore there are three cases to consider:

(CC) Both $\lambda_1^u(\mu)$ and $\lambda_2^s(\mu)$ are complex,

(RC) One of $\lambda_1^u(\mu)$, $\lambda_2^s(\mu)$ is real and the other is complex,

(RR) Both $\lambda_1^u(\mu)$ and $\lambda_2^s(\mu)$ are real.

It turns out that corresponding information about the leading eigenvalues λ_1^s and λ_2^u are not important for the purpose of this paper. Before we describe the nearby dynamics of the heteroclinic cycle in each case we define some terminology. For that we introduce hyperplanes Σ_j which are transversal to Γ_j at $q_j(0)$ (which may be located "in the middle" of Γ_j), $j = 1, 2$.

Definition 1.1. *Let $\mathcal{U}$ be a sufficiently small neighbourhood of the primary heteroclinic cycle. All orbits which we shall describe in the following are assumed to be subsets of $\mathcal{U}$.*

- *A periodic orbit is called k-periodic, and a homoclinic orbit (to p_1 or p_2) is called k-homoclinic, if it passes through Σ_1 and Σ_2 k times in each case.*
- *A heteroclinic orbit connecting p_i to p_j (in forward time) is called a (i,j)-heteroclinic orbit. A (i,j)-heteroclinic orbit that passes k times through Σ_j is called a k-(i,j)-heteroclinic orbit.*

Our main result concerning shift dynamics is the following.

Theorem 1.2. *Consider the system* (1.1) *under the assumptions as stated above. Assume either eigenvalue case (CC) or (RC), then the following is typically true.*

When $\mu = 0$, for each $N \geq 2$ there exists a set $\mathcal{S}_0^N \subset \Sigma_1$ which is invariant under the first-return-map $\Pi : \Sigma_1 \to \Sigma_1$ (defined by the flow), and $(\mathcal{S}_0^N, \Pi)$ is topologically conjugated to the full shift on N symbols.

Moreover, for fixed $N \geq 2$ there exists $\mu_N > 0$, such that if $|\mu| < \mu_N$ there is a set $\mathcal{S}_\mu^N$ such that $(\mathcal{S}_\mu^N, \Pi(\mu))$ is topologically conjugated to a full shift on N symbols, and $\mathcal{S}_\mu^N \to \mathcal{S}_0^N$ in the Hausdorff-metric as $\mu \to 0$.

The proof of this theorem is given in Section 4.1. In the course of this proof we introduce several assumptions that define an open and dense set of vector fields in $\mathcal{H}$ for which this theorem holds. Here $\mathcal{H}$ is the space of two parameter vector fields containing a heteroclinic cycle under the assumptions stated above for $f(\cdot, 0)$. This space is endowed with the C^1 topology. This set of conditions comprises a generic unfolding of the heteroclinic cycle, standard non-orbit flip and non-inclination flip assumptions, as well as a more technical assumption in case (CC), which guarantees the existence of infinitely many transversal intersections of the spirals displayed in Figure 3.

Here our strategy is to use a Homoclinic Liapunov-Schmidt reduction, better known in this context as Lin's method [19], to determine the existence of orbits close to a given heteroclinic or homoclinic cycle. Lin's method consists of the search for *piecewise continuous orbits* of (1.1), which we call *Lin orbits*, cf [14]. These orbits consist of pieces of actual solution curves which remain close to the original heteroclinic cycle, but are allowed to have discontinuities in the section Σ_1. This discontinuity, or *jump*, is restricted to lie in a certain direction, see Section 2 for more details. For a given parameter μ and sequence $\boldsymbol{\omega} = (\omega_{1,i}, \omega_{2,i})_{i\in\mathbb{Z}}$ of transition times (from one hyperplane to the next) there is a unique Lin orbit. We refer to Figure 4 for a visualisation. Let $\Xi_i(\boldsymbol{\omega}, \mu)$ be the ith jump of the Lin orbit corresponding to $(\boldsymbol{\omega}, \mu)$. Splitting $\Xi_i(\boldsymbol{\omega}, \mu)$ into the leading order term plus a remainder term, we find that the leading order term only depends on $\omega_{1,i+1}$, $\omega_{2,i}$ and μ. We find orbits staying all time close to the initial heteroclinic cycle by *closing the jumps*, or in other words by solving the bifurcation equation

$$(\Xi_i(\boldsymbol{\omega}, \mu))_{i\in\mathbb{Z}} = 0, \quad \Xi_i(\boldsymbol{\omega}, \mu) = L((\omega_{1,i+1}, \omega_{2,i}), \mu) + r_i(\boldsymbol{\omega}, \mu), \tag{1.4}$$

where $L((\omega_{1,i+1}, \omega_{2,i}), \mu)$ denotes leading order terms and $r_i(\boldsymbol{\omega}, \mu)$ denotes higher order terms. Under certain conditions, (H 2) and (H 3), the leading order terms of the bifurcation equations are determined by $\lambda_1^u(\mu)$ and $\lambda_2^s(\mu)$. The leading order terms for the cases (CC) and (RC) are as follows:

Case (CC) (Complex-Complex)

$$L((\omega_{1,i+1},\omega_{2,i}),\mu) = \begin{pmatrix} \mu_1 + e^{-2\rho_1\omega_{1,i+1}}c_{11}\sin(2\phi_1\omega_{1,i+1}+\varphi_{11}) - e^{-2\rho_2\omega_{2,i}}c_{21}\sin(2\phi_2\omega_{2,i}+\varphi_{21}) \\ \mu_2 + e^{-2\rho_1\omega_{1,i+1}}c_{12}\sin(2\phi_1\omega_{1,i+1}+\varphi_{12}) - e^{-2\rho_2\omega_{2,i}}c_{22}\sin(2\phi_2\omega_{2,i}+\varphi_{22}) \end{pmatrix} \tag{1.5}$$

Case (RC) (Real-Complex)

$$L((\omega_{1,i+1},\omega_{2,i}),\mu) = \begin{pmatrix} \mu_1 + e^{-2\rho_1\omega_{1,i+1}}c_{11} - e^{-2\rho_2\omega_{2,i}}c_{21}\sin(2\phi_2\omega_{2,i}+\varphi_{21}) \\ \mu_2 + e^{-2\rho_1\omega_{1,i+1}}c_{12} - e^{-2\rho_2\omega_{2,i}}c_{22}\sin(2\phi_2\omega_{2,i}+\varphi_{22}) \end{pmatrix} \tag{1.6}$$

Here $\mu = (\mu_1,\mu_2)$, $c_{kl} \neq 0$ and the c_{kl} and φ_{kl} are functions of of μ. It turns out that solutions are extractable from the leading order terms. For that we now observe how the problem of solving the bifurcation equation reduces to a geometric problem of studying intersections of logarithmic spirals and lines in the plane. For each i the leading order term L of Ξ_i has the same form

$$L((\omega_1,\omega_2),\mu) = \begin{pmatrix} \mu_1 + L_{11}(\omega_1,\mu) - L_{21}(\omega_2,\mu) \\ \mu_2 + L_{12}(\omega_1,\mu) - L_{22}(\omega_2,\mu) \end{pmatrix};$$

Now it is clear that when $\lambda_1^u(\mu)$ is complex, $\mu + (L_{11}(\omega_1,\mu), L_{12}(\omega_1,\mu))$ parametrises (for fixed μ and ω_1 as parameter) a logarithmic spiral in the plane. When $\lambda_1^u(\mu)$ is real, $\mu + (L_{11}(\omega_1,\mu), L_{12}(\omega_1,\mu))$ parametrises a line in the plane. A similar statement is true for $\lambda_2^s(\mu)$ and $(L_{21}(\omega_2,\mu), L_{22}(\omega_2,\mu))$. Therefore, for each i solutions of the truncated equation $L((\omega_1,\omega_2),\mu) = 0$ may be represented by intersections of these logarithmic spirals and lines, corresponding to the cases (CC) and (RC). See Figure 3 for the situation for $\mu = 0$. Finally sequences $\hat{\boldsymbol{\omega}} = (\hat{\omega}_{1,i},\hat{\omega}_{2,i})_{i\in\mathbb{Z}}$ related to transversal intersections of these curves serve as first estimates of solutions of the full bifurcation equation $(\Xi(\boldsymbol{\omega},\mu))_{i\in\mathbb{Z}} = 0$. Note that the sequences $\hat{\boldsymbol{\omega}}$ may be built up by only finitely many different $(\hat{\omega}_{1,i},\hat{\omega}_{2,i})$.

The above described shift dynamics can be interpreted in the context of transversal intersections of stable and unstable manifolds of 1-periodic orbits. We find a set $\mathcal{S}_\mu^N$ of 1-periodic orbits $\mathcal{O}_1,\ldots,\mathcal{O}_N$ with periods (close to) $\hat{\omega}_1,\ldots,\hat{\omega}_N$ (which are found from the intersections of lines and spirals, cf. Section 4). A sequence $\mathbf{s} := (s_i)_{i\in\mathbb{Z}}$, $s_i \in \{1,\ldots,N\}$ corresponds to an orbit which is closely following a sequence of orbits in $\{\mathcal{O}_1,\ldots,\mathcal{O}_N\}$ as prescribed by the sequence $\mathbf{s}$. In our analysis we take advantage of the fact that there are infinitely many transversal intersections, see Section 4.1.2. This guarantees that there are intersections related to arbitrarily large transition times.

For fixed N and μ there may be many choices for the set of periodic solutions $\mathcal{S}_\mu^N$. For $\mu = 0$ there exist countably many of them, corresponding to the infinitely many intersections of the spirals we referred to already. These intersections correspond to an infinite set of 1-periodic orbits $\mathcal{O}_i$, $i \in \mathbb{N}$, which potentially can serve as symbols. For $\mu = 0$ the tips of the spirals coincide – a situation we exploit to verify infinitely many intersections of them. For $\mu \neq 0$ the tips of the spirals are separated. Generically the tip of one spiral will not be located on the other spiral. Hence generically only for finitely many $N \in \mathbb{N}$ there exist corresponding sets $\mathcal{S}_\mu^N$, and for fixed N only finitely many different choices for these sets exist.

The verification of topological conjugacy is mainly based on the continuous dependence of Ξ (see (2.1)) on sequences $\boldsymbol{\omega}$ in spaces of sequences equipped with the product topology. Those ideas trace back to similar considerations in [15, 23].

The shift dynamics referred to in Theorem 1.2 does not concern k-homoclinic or k-heteroclinic orbits. Our second result, whose proof is discussed in Section 4.2. describes the existence of such connecting orbits.

Lemma 1.3. *Assume the eigenvalue case (CC) or (RC), and let $f(\cdot,\mu))$ be a two parameter family of vector fields for which Theorem 1.2 holds. Then we have the following:*

1. *For each $k \geq 2$, $k \in \mathbb{N}$ there is a countable set $\mathcal{M}_k$ of parameter values accumulating at $\mu = 0$, for which there exists a k-(2,1)-heteroclinic orbit.*
2. *At $\mu = 0$ there exist a countable infinity of k-(1,2)-heteroclinic orbits for each $k \geq 2$. Moreover, for fixed k, $q_1(0)$ is an accumulation point of the intersections of the k-(1,2)-heteroclinic orbits with Σ_1. Each such k-(1,2)-heteroclinic orbit can be continued for $\mu \neq 0$ sufficiently small, but for fixed $\mu \neq 0$ and fixed k there are only finitely many k-(1,2)-heteroclinic orbits.*
3. *For each $k \geq 2$, $k \in \mathbb{N}$, and each $i \in \{1, 2\}$, there is a curve $\mathcal{L}^{hom}_{k,i}$ in μ-space which is either a logarithmic spiral centred at $(0,0)$ or a line terminating at $(0,0)$, for which each point on $\mathcal{L}^{hom}_{k,i}$ there exists a k-homoclinic orbit to the fixed point p_i.*

The dynamics in case (RR) is simple in comparison to the previous two cases.

Theorem 1.4. *Consider the system* (1.1), *and assume the eigenvalue case (RR). Then typically we have the following. There exist open sets Q and $\tilde{Q}$ in μ-space, whose boundaries ∂Q and $\partial\tilde{Q}$ consists of two lines in each case that terminate at the origin, and where ∂Q and $\partial\tilde{Q}$ are tangent at the origin, for which the following is true:*

1. *There exists exactly one 1-periodic orbit at each point $\mu \in Q$, see Figure 2. The period of these periodic orbits tends to infinity as $\mu \to \partial Q$. There are no k-periodic orbits for k larger than one.*
2. *There are no k-(2,1)-heteroclinic orbits for each $k \geq 2$, $k \in \mathbb{N}$, for any parameter value μ.*
3. *At $\mu = 0$ there are no k-(1,2)-heteroclinic orbits for $k \geq 2$. But for each $\mu \in \tilde{Q}$ there exists a k-(1,2)-heteroclinic orbit for $k \geq 2$.*
4. *For each $\mu \in \partial Q$, there exists a 1-homoclinic orbit to one of the fixed points p_i. There are no k-homoclinic orbits for $k \geq 2$.*

The homoclinic orbits addressed in point 4 may naturally be thought of as limit points for the periodic orbits stated in point 1. The period of these periodic orbits tends to infinity as $\mu \to \partial Q$.

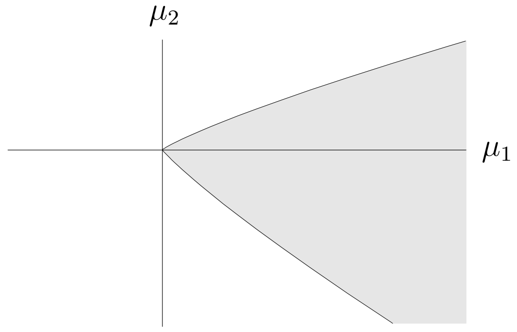

Figure 2: The open set Q with boundary ∂Q in (μ_1, μ_2)-space. The set $\tilde{Q}$ has approximately the same appearance.

The proof is given in Section 4.3. The assumptions used in the course of this proof again define an open and dense set of vector fields in $\mathcal{H}$, and consist of the same standard generic unfolding, non-orbit flip and non-inclination flip conditions used previously, as well as a more technical hypothesis that guarantees the linear independence of the tangents at the origin of the two lines displayed in Figure 3 at $\mu = 0$.

Note that in the case (RR) the leading order term L reads

$$L((\omega_{1,i+1}, \omega_{2,i}), \mu) = \begin{pmatrix} \mu_1 + e^{-2\rho_1\omega_{1,i+1}} c_{11} - e^{-2\rho_2\omega_{2,i}} c_{21} \\ \mu_2 + e^{-2\rho_1\omega_{1,i+1}} c_{12} - e^{-2\rho_2\omega_{2,i}} c_{22} \end{pmatrix}. \tag{1.7}$$

Therefore solutions of $L = 0$ correspond to the intersection of two lines. It is obvious that there is at most one such intersection and hence at most one solution of $L = 0$. From that we can conclude the statement of the theorem in a similar way as the one of Theorem 1.2.

We remark that our Hypothesis (H 1), see Section 3, which prescribes that the stable manifold of p_1 and the unstable manifold of p_2 split with positive speed, implies that in all cases there is just one 1-(2,1)-heteroclinic orbit, and this is the original one that exists at $\mu = 0$. In $\mathbb{R}^3$ it is clear that there can only be one k-(2,1)-heteroclinic orbit for each parameter value, because in this case the stable manifold of p_1 is one-dimensional.

We now outline the organisation of this paper. In Section 2 we introduce Lin's method in more detail.

In Section 3 we derive an expression for the i^{th} jump of a Lin orbit, depending on μ and the

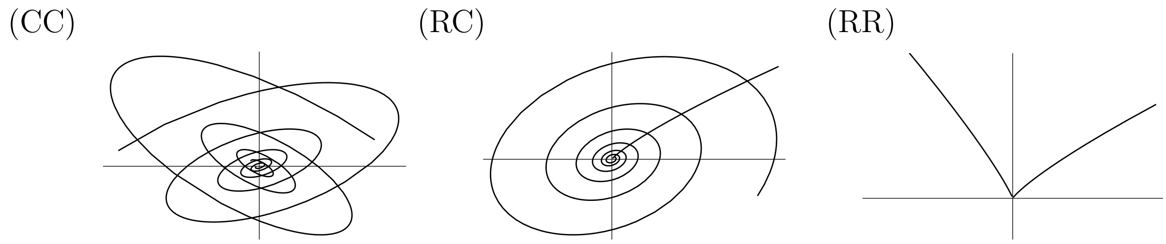

Figure 3: Graphical representations of the solutions of $L((\omega_1, \omega_2), 0) = 0$.

sequence of transition times $\boldsymbol{\omega}$. In particular we determine their leading order terms.

Theorem 1.2 is proven in Section 4.1. As can be seen in equation (1.4), the bifurcation equation consists of a leading order term and higher order remainder terms. The strategy for studying the bifurcation equation is to first study solutions to the equation truncated at leading order. We prove in Section 4 that nondegenerate solutions to this equation persist under addition of the higher order terms.

In Section 5 we discuss the time-reversible case, which has attracted interest since it arises in the context of the analysis of travelling waves in the Kuramoto-Sivashinsky PDE [18]. The main differences between the general case and reversible case are discussed, the bifurcation equations are modified, and the corresponding results stated and proved.

2 Lin's method

In this section we outline Lin's method, and explain the properties of Lin orbits corresponding to the system under consideration. Lin's method centres around the existence of 'piecewise continuous orbits' X of (1.1) which we call *Lin orbits*, after [14]. In the present context such orbits consist of pieces of actual orbits X_i, $X := (X_i)_{i\in\mathbb{Z}}$; the orbit X_i starts in Σ_1, follows Γ_1 until it reaches a neighbourhood of p_1 follows then Γ_2, meets Σ_2, stays further close to Γ_2 until it reaches a neighbourhood of p_2, follows then Γ_1 again, and terminates finally in Σ_1. Between two consecutive orbits X_{i-1} and X_i there may be a jump Ξ_i in a distinguished direction Z. We refer to Figure 4 for a visualisation. Note that in the present context all jumps in Σ_2 are equal to zero. This is due to the transversal intersection of the unstable manifold of p_1 and the stable manifold of p_2 in Γ_2.

Let $2\omega_{1,i}$ and $2\omega_{2,i}$ be (prescribed) transition times of X_i from Σ_1 to Σ_2 and Σ_2 to Σ_1, respectively. It can be proved that for each μ which is sufficiently close to 0, and each sequence $\boldsymbol{\omega} := ((\omega_{1,i}, \omega_{2,i}))_{i\in\mathbb{Z}}$, where $\omega_{j,i}$ are sufficiently large, there exists a unique Lin orbit $X(\boldsymbol{\omega}, \mu)$, see Theorem 2.2. By making the jumps Ξ_i to zero one finds real orbits staying for all time close to the heteroclinic cycle Γ. Therefore the bifurcation equation for orbits staying close to Γ reads

$$\boldsymbol{\Xi} := (\Xi_i(\boldsymbol{\omega}, \mu))_{i\in\mathbb{Z}} = 0. \tag{2.1}$$

To begin the actual analysis, fix an inner product $\langle\cdot,\cdot\rangle$. Let, with respect to $\langle\cdot,\cdot\rangle$,

$$Y_i := \{f(q_i(0), 0)\}^{\perp}, \quad i = 1, 2.$$

With that we construct the cross-sections Σ_1 and Σ_2 as follows

$$\Sigma_i := q_i(0) + Y_i, \quad i = 1, 2. \tag{2.2}$$

Also, consistent with the standard theory, we define a subspace $Z \subset Y_1$:

$$Z := (T_{q_1(0)}W^s(p_1) + T_{q_1(0)}W^u(p_2))^{\perp}.$$

By construction we have $\dim Z = 2$. Assigned to $q_1(0)$ and $q_2(0)$ we consider the following orthogonal direct sum decomposition of $\mathbb{R}^n$:

$$\begin{aligned} \mathbb{R}^n &= \operatorname{span}\{f(q_1(0), 0)\} \oplus W_1^+ \oplus W_1^- \oplus Z, \\ \mathbb{R}^n &= \operatorname{span}\{f(q_2(0), 0)\} \oplus W_2^+ \oplus W_2^-, \end{aligned} \tag{2.3}$$

where $W_i^+ = T_{q_i(0)}W^s(p_i) \cap Y_i$ and $W_i^- = T_{q_i(0)}W^u(p_j) \cap Y_i$, $i = 1, 2$, $j \neq i$.

A further assumption, that simplifies the analysis, is that the local stable/unstable manifolds of the fixed points p_1, p_2 are flat; that is:

$$W^s_{loc}(p_i, \mu) \subset T_{p_i}W^s(p_i), \quad W^u_{loc}(p_i, \mu) \subset T_{p_i}W^u(p_i). \tag{2.4}$$

We can bring the local stable/unstable manifolds into this form by means of local transformations based around each of the fixed points p_i.

2.1 Splitting of the stable and unstable Manifolds

The first step of Lin's method is to study the splitting of the stable and unstable manifolds in Σ_i with respect to the parameter μ. Because of (1.2) the situation in Σ_2 is clear. For each μ which is sufficiently close to 0 there is exactly one point $q_{2,\mu} \in \Sigma_2 \cap W^u(p_1, \mu) \cap W^s(p_2, \mu)$ such that the orbit $\Gamma_{2,\mu}$ through $q_{2,\mu}$ is a 1-(1,2)-heteroclinic orbit. The corresponding solution with $q_2(\mu)(0) = q_{2,\mu}$ we denote by $q_2(\mu)(\cdot)$; their restriction on $\mathbb{R}^\pm$ we denote by $q_2^\pm(\mu)(\cdot)$.

In $\mathbb{R}^3$ also the situation in Σ_1 is rather simple. In this case both the stable manifold of p_1 and the unstable manifold of p_2 are one-dimensional, and the intersection with the two-dimensional hyperplane Σ_1 consists of single points in each case. The heteroclinic connection Γ_1 generally splits up under perturbation. Let $q^+_{1,\mu}$ and $q^-_{1,\mu}$ be determined by the 'first hit' of the stable manifold of p_1 and the unstable manifold of p_2, respectively. Of course $q^+_{1,\mu} - q^-_{1,\mu} \in Z$, recall that $Z = Y_1$ in this case. So, in a trivial way, for each μ we find a unique pair of orbits in the stable manifold of p_1 and the unstable manifold of p_2, respectively, such that the difference of their first hits in Σ_1 is in Z.

The main goal of this section is to show that this property persists in higher dimensions. More precisely we prove the following lemma:

Lemma 2.1. *For each μ which is sufficiently close to 0 there is a unique pair $(q_1^+(\mu)(\cdot), (q_1^-(\mu)(\cdot))$ of solutions of (1.1) such that:*

(i) $q_1^+(\mu)(0) \in \Sigma_1 \cap W^s(p_1, \mu)$, $\quad q_1^-(\mu)(0) \in \Sigma_1 \cap W^u(p_2, \mu)$,

(ii) $|q_1^+(\mu)(t) - q_1(t)|$ *small* $\forall t \in \mathbb{R}^+$ *and* $|q_1^-(\mu)(t) - q_1(t)|$ *small* $\forall t \in \mathbb{R}^-$,

(iii) $q_1^+(\mu)(0) - q_1^-(\mu)(0) \in Z$.

Proof. To investigate the splitting of Γ_1 we set

$$q_1^\pm(t) = q_1(t) + v^\pm(t). \tag{2.5}$$

This gives the following equations for $v^\pm$

$$\dot{v}^\pm = A(t)v^\pm + g(t, v^\pm, \mu), \tag{2.6}$$

which we consider on $\mathbb{R}^+$ or $\mathbb{R}^-$ respectively. Here $A(t) = D_x f(q_1(t), 0)$, and

$$g(t, v, \mu) = f(q_1(t) + v, \mu) - f(q_1(t), 0) - A(t)v. \tag{2.7}$$

With the above setting, we are looking for bounded solutions $v^\pm(\cdot)$ of (2.6). Hence $v^\pm(\cdot) \in C_b(\mathbb{R}^\pm)$, the space of continuous on $\mathbb{R}^\pm$ bounded functions equipped with the norm $\|v^\pm\|_\infty := \sup_{t\in\mathbb{R}^\pm} \|v^\pm(t)\|$. If $\|v^\pm\|_\infty$ is close to zero, then by the theory of stable and unstable manifolds, $q^\pm(t) := q_1(t) + v^\pm(t)$ is in the desired stable/unstable manifold. Further, if $v^\pm(0) \in Y_1$ and $v^+(0) - v^-(0) \in Z$, then (i) - (iii) of the lemma holds true.

In what follows we show that (2.6) has unique solutions $v^\pm$ with the outlined boundary conditions. An important role is played by the properties of the linear nonautonomous equations

$$\dot{v} = A(t)v, \tag{2.8}$$

which we consider either on $\mathbb{R}^+$ or on $\mathbb{R}^-$. Denote by $\Phi(t, s)$ the transition matrix for (2.8). We have that $\lim_{t\to\infty} q_1(t) = p_1$, so then $\lim_{t\to\infty} A(t) = D_x f(p_1, 0)$. Similarly $\lim_{t\to-\infty} A(t) = D_x f(p_2, 0)$. Also $D_x f(p_1, \mu) = -R \circ D_x f(p_2, \mu) \circ R$.

By the theory of exponential dichotomies [5], due to the fact that p_1, p_2 are hyperbolic, equation (2.8) has an exponential dichotomy on both $\mathbb{R}^+$ and $\mathbb{R}^-$. For $t \in \mathbb{R}_0^\pm$ we define projections $P^\pm(t)$ and $Q^\pm(t)$ as follows

$$\operatorname{im} P^+(0) = T_{q_1(0)} W^s(p_1), \quad \ker P^+(0) = W_1^- \oplus Z,$$

and

$$\ker P^-(0) = T_{q_1(0)} W^u(p_2) \quad \operatorname{im} P^-(0) = W_1^+ \oplus Z,$$

and for $t, s \geq 0$ or $t, s \leq 0$ let

$$P^\pm(t)\Phi(t, s) = \Phi(t, s)P^\pm(s).$$

Finally we define $Q^+ := I - P^+$ and $Q^- := I - P^-$.

Using the properties of exponential dichotomy we get that for each function $g(\cdot)$ bounded on $\mathbb{R}^+$:

$$\int_0^t \Phi(t, s)P^+(s)g(s)ds - \int_t^\infty \Phi(t, s)Q^+(s)g(s)ds$$

is well defined on $\mathbb{R}^+$ and solves $\dot{v} = A(t)v + g(t)$. Therefore we see that the solutions $v^+(\cdot)$ of (2.6) that are bounded on $\mathbb{R}^+$ solve the following fixed point problem in $C_b^0(\mathbb{R}^+, \mathbb{R}^n)$, and vice versa:

$$v^+(t) = \Phi(t, 0)\nu + \int_0^t \Phi(t, s)P^+(s)g(s, v^+(s), \mu)ds - \int_t^\infty \Phi(t, s)Q^+(s)g(s, v^+(s), \mu)ds,$$

where $\nu = P^+(0)v^+(0)$. In a similar way it can be shown that the solutions $v^-(\cdot)$ of (2.6) that are bounded on $\mathbb{R}^-$ solve the following fixed point problem in $C_b^0(\mathbb{R}^-, \mathbb{R}^n)$, and vice versa:

$$v^-(t) = \Phi(t, 0)\eta + \int_{-\infty}^t \Phi(t, s)P^-(s)g(s, v^-(s), \mu)ds - \int_t^0 \Phi(t, s)Q^-(s)g(s, v^-(s), \mu)ds,$$

where $\eta = Q^-(0)v^-(0)$. Both of these fixed point equations can be solved for $v^+ = v^+(\nu, \mu)$ and $v^- = v^-(\eta, \mu)$, near $(v^+, \nu, \mu) = (0, 0, 0)$ and $(v^-, \eta, \mu) = (0, 0, 0)$ in each case. We find that $v^\pm(0)$ can be written in the form

$$v^+(\nu, \mu)(0) = \nu + w^-(\nu, \mu) + z^+(\nu, \mu), \quad v^-(\eta, \mu)(0) = \eta + w^+(\eta, \mu) + z^-(\eta, \mu),$$

where $w^\pm \in W_1^\pm$ and $z^\pm \in Z$, and moreover $w^\pm(0,0) = 0$, $D_1 w^\pm(0,0) \neq 0$. In view of (iii) of the lemma we consider

$$\nu = w^+(\eta, \mu), \quad \eta = w^-(\nu, \mu).$$

This system can, near $(\nu, \eta, \mu) = (0,0,0)$, be solved for $\nu = \nu(\mu)$, $\eta = \eta(\mu)$.

The functions $q_1^+(\mu)(\cdot) := q_1(\cdot) + v^+(\nu(\mu), \mu)(\cdot)$ and $q_1^-(\mu)(\cdot) := q_1(\cdot) + v^-(\eta(\mu), \mu)(\cdot)$ are the wanted solutions of (1.1). ∎

2.2 Construction of Lin orbits

The next step in the method is to search for orbits $X_{j,i}$, $j = 1,2$, $i \in \mathbb{Z}$, composing the Lin orbits $X = (X_i)_{i\in\mathbb{Z}}$ which we introduced in Section 1; more precisely $X_i = X_{1,i} \cup X_{2,i}$. Here $X_{j,i}$ is an orbit of the vector field starting in a point in Σ_j, staying close to Γ_j until it reaches a neighbourhood of p_j, and continuing close to Γ_{j+1} until it reaches Σ_{j+1}; throughout the term "$j+1$" is computed *modulo* 2. By $x_{j,i}(\cdot)$ we denote solutions of (1.1) corresponding to the orbits $X_{j,i}$ with $x_{j,i}(0) \in \Sigma_j$ and $x_{j,i}(2\omega_{j,i}) \in \Sigma_{j+1}$. Actually $x_{1,i}(\cdot)$ is composed of solutions $x_{1,i}^+(\cdot)$ and $x_{2,i}^-(\cdot)$ which are defined on $[0, \omega_{1,i}]$ and $[-\omega_{1,i}, 0]$, respectively. Similarly $x_{2,i}(\cdot)$ is composed of solutions $x_{2,i}^+(\cdot)$ and $x_{1,i}^-(\cdot)$ which are defined on $[0, \omega_{2,i}]$ and $[-\omega_{2,i}, 0]$, respectively. This demands coupling conditions

$$x_{j,i}^+(\omega_{j,i}) = x_{j+1,i}^-(-\omega_{j,i}),\ j = 1,2, \tag{2.9}$$

and the jump conditions

$$\Xi_i := x_{1,i+1}^+(0) - x_{1,i}^-(0) \in Z, \quad x_{2,i}^+(0) = x_{2,i}^-(0), \tag{2.10}$$

for $i \in \mathbb{Z}$ in each case. Figure 4 visualises this situation.

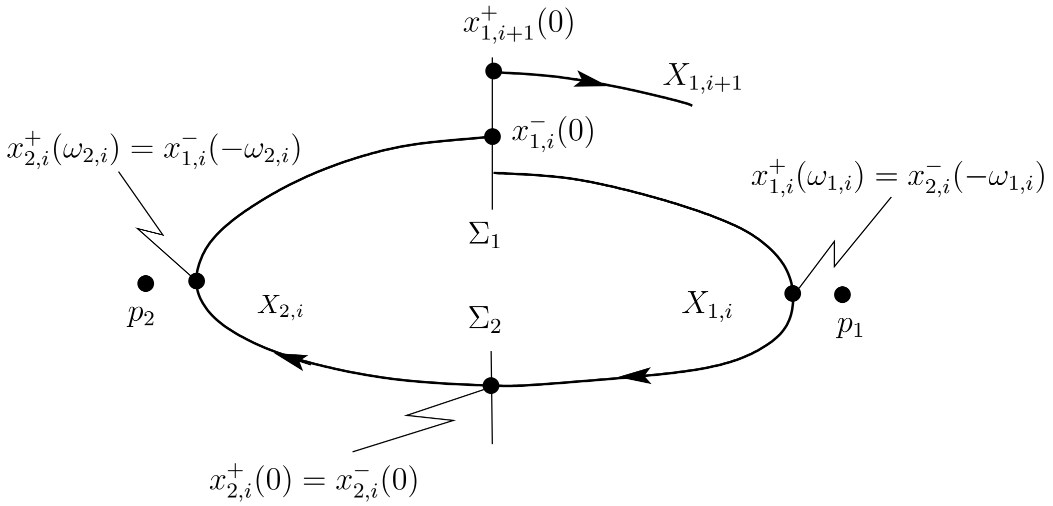

Figure 4: Ingredients of Lin orbits; $X_i = X_{1,i} \cup X_{2,i}$.

It is a specific feature of T-points that there is no jump in Σ_2, $x_{2,i}^+(0) - x_{2,i}^-(0) = 0$. However, large parts of the procedure run parallel to the exposition in [23], [15], [19] and [28]. In our presentation we confine to explain only those parts of Lin's method in more detail which differ from the standard scheme.

To begin, we look for solutions of the form

$$x^{\pm}_{j,i}(t) = q^{\pm}_{j}(\mu)(t) + v^{\pm}_{j,i}(t). \tag{2.11}$$

Then

$$\dot v^{\pm}_{j,i} = A^{\pm}_{j}(t,\mu)v^{\pm}_{j,i} + g^{\pm}_{j}(t, v^{\pm}_{j,i}, \mu) \tag{2.12}$$

where $A^{\pm}_{j}(t,\mu) = D_x f(q^{\pm}_{j}(\mu)(t), \mu)$, and

$$g^{\pm}_{j}(t,v,\mu) = f(q^{\pm}_{j}(\mu)(t) + v, \mu) - f(q^{\pm}_{j}(\mu)(t), \mu) - A^{\pm}_{j}(t,\mu)v. \tag{2.13}$$

Let $\Phi^{\pm}_{j}(\mu,t,s)$ be the transition matrix for the equation

$$\dot v = A^{\pm}_{j}(t,\mu)v. \tag{2.14}$$

As before, these operators have an exponential dichotomy on $\mathbb{R}^+$ or $\mathbb{R}^-$, respectively, with corresponding projections $P^{+}_{j}(\mu,t)$, $Q^{+}_{j}(\mu,t) = I - P^{+}_{j}(\mu,t)$, and $Q^{-}_{j}(\mu,t)$, $P^{-}_{j}(\mu,t) = I - Q^{-}_{j}(\mu,t)$; therefore we have for $j = 1, 2$

$$\operatorname{im} P^{+}_{j}(\mu,0) = T_{q^{+}_{j}(\mu)(0)}W^s(p_j), \quad \operatorname{im} Q^{-}_{j}(\mu,0) = T_{q^{-}_{j}(\mu)(0)}W^u(p_{j+1}).$$

Moreover we commit to

$$\ker P^{+}_{1}(\mu,0) = W^{-}_{1} \oplus Z, \quad \ker P^{+}_{2}(\mu,0) = W^{-}_{2}$$

and

$$\ker Q^{-}_{1}(\mu,0) = W^{+}_{1} \oplus Z, \quad \ker Q^{-}_{2}(\mu,0) = W^{+}_{2}.$$

Our main existence result in this respect is the following:

Theorem 2.2. *Consider the system (1.1) in the present setting. Then there are constants c, Ω such that for each μ with $|\mu| < c$ and each $\boldsymbol{\omega}$ with $\omega_{j,i} > \Omega$, $j = 1, 2$, $i \in \mathbb{Z}$, there is a unique sequence of solutions $x^{\pm}_{j,i}(\boldsymbol{\omega},\mu)(\cdot)$, $j = 1, 2$, $i \in \mathbb{Z}$, of (1.1) satisfying the coupling condition (2.9) and the jump condition (2.10).*

In other words this theorem says that for each μ and for each sequence $\boldsymbol{\omega}$ there exists a unique Lin orbit $X(\boldsymbol{\omega},\mu)$.

Next we explain the steps leading to the proof of this theorem. We consider (instead of the original equation (1.1) with boundary conditions (2.9) and (2.10)) the differential equation (2.12) with corresponding boundary conditions (according to (2.11))

$$v^{+}_{j,i}(\omega_{j,i}) - v^{-}_{j+1,i}(-\omega_{j,i}) = q^{-}_{j+1}(-\omega_{j,i}) - q^{+}_{j}(\omega_{j,i}),\ j = 1, 2, \tag{2.15}$$

$$v^{\pm}_{j,i}(0) \in Y_j, \quad v^{+}_{1,i+1}(0) - v^{-}_{1,i}(0) \in Z, \quad v^{+}_{2,i}(0) = v^{-}_{2,i}(0), \tag{2.16}$$

for $i \in \mathbb{Z}$ in each case. We fix a sequence $\boldsymbol{\omega}$ with sufficiently large $\omega_{j,i}$. Assigned to this sequence we denote by $\mathcal{V}_{\boldsymbol{\omega}}$ the space of all sequences $\mathbf{v} := (v^{+}_{1,i}, v^{-}_{2,i}, v^{+}_{2,i}, v^{-}_{1,i})_{i\in\mathbb{Z}}$, where $v^{+}_{j,i} \in C([0,\omega_{j,i}], \mathbb{R}^n)$ and $v^{-}_{j,i} \in C([-\omega_{j+1,i}, 0], \mathbb{R}^n)$.

In a first step we consider a "linearised" equation

$$\dot{v}^{\pm}_{j,i} = A^{\pm}_{j}(t,\mu)v^{\pm}_{j,i} + h^{\pm}_{j,i}(t), \tag{2.17}$$

with continuous $h^{\pm}_{j,i}(\cdot)$, and with boundary conditions (2.16) and

$$Q^{+}_{j}(\mu,\omega_{j,i})v^{+}_{j,i}(\omega_{j,i}) = a^{+}_{j,i}, \quad P^{-}_{j}(\mu,-\omega_{j+1,i})v^{-}_{j,i}(-\omega_{j+1,i}) = a^{-}_{j+1,i}, \tag{2.18}$$

for any given $a^{+}_{j,i} \in \operatorname{Im} Q^{+}_{j}(\mu,\omega_{j,i})$, $a^{-}_{j+1,i} \in \operatorname{Im} P^{-}_{j}(\mu,-\omega_{j+1,i})$.

Lemma 2.3. *The boundary value problem ((2.17), (2.16), (2.18)) has a unique solution* $\mathbf{v}_{\boldsymbol{\omega}} \in \mathcal{V}_{\boldsymbol{\omega}}$.

Proof. Solutions of (2.17) can be written in the form

$$v^{\pm}_{j,i}(t) = \Phi^{\pm}_{j}(\mu,t,0)v^{\pm}_{j,i}(0) + \int_{0}^{t} \Phi^{\pm}_{j}(\mu,t,s)h^{\pm}_{j,i}(s)ds. \tag{2.19}$$

Working in the boundary condition (2.18) gives for $i \in \mathbb{Z}$

$$\begin{aligned} Q^{+}_{j}(\mu,0)v^{+}_{j,i}(0) &= \Phi^{+}_{j}(\mu,0,\omega_{j,i})a^{+}_{j,i} - \int_{0}^{\omega_{j,i}} \Phi^{+}_{j}(\mu,0,s)Q^{+}_{j}(\mu,s)h^{+}_{j,i}(s)ds, \\ P^{-}_{j}(\mu,0)v^{-}_{j,i}(0) &= \Phi^{-}_{j}(\mu,0,-\omega_{j+1,i})a^{-}_{j,i} + \int_{-\omega_{j+1,i}}^{0} \Phi^{-}_{j}(\mu,0,s)P^{-}_{j}(\mu,s)h^{-}_{j,i}(s)ds, \end{aligned} \tag{2.20}$$

Because of (2.16) we have

$$v^{+}_{1,i+1}(0) = w^{+}_{1,i} + w^{-}_{1,i} + z^{+}_{i},\ v^{-}_{1,i}(0) = w^{+}_{1,i} + w^{-}_{1,i} + z^{-}_{i} \quad \text{and} \quad v^{\pm}_{2,i}(0) = w^{+}_{2,i} + w^{-}_{2,i},$$

where $w^{\pm}_{j,i} \in W^{\pm}_{j}$ and $z^{\pm}_{i} \in Z$. For fixed j the left-hand sides of these equations are decoupled over i. Moreover, for each single $i \in \mathbb{Z}$ the left-hand side of (2.20) can be seen as a linear mapping

$$W^{+}_{1} \times W^{-}_{1} \times Z \times Z \to (W^{-}_{1} \oplus Z) \times (W^{+}_{1} \oplus Z) \quad \text{or} \quad W^{+}_{2} \times W^{-}_{2} \to W^{+}_{2} \times W^{-}_{2}$$

depending on $j = 1$ or $j = 2$, respectively. These mappings are invertible. So the equations under consideration can be solved for

$$(w^{\pm}_{1,i}, z^{\pm}_{i}) = (w^{\pm}_{1,i}, z^{\pm}_{i})(\mu, h^{+}_{1,i+1}, h^{-}_{2,i}, a^{+}_{1,i+1}, a^{-}_{2,i}), \quad w^{\pm}_{2,i} = w^{\pm}_{2,i}(\mu, h^{-}_{1,i}, h^{+}_{2,i}, a^{-}_{1,i}, a^{+}_{2,i}).$$

With (2.19) we get eventually the lemma. ∎

In the next step we "replace" the boundary conditions (2.18) by

$$v^{+}_{j,i}(\omega_{j,i}) - v^{-}_{j+1,i}(-\omega_{j,i}) = d_{j,i}, \quad d_{j,i} \in \mathbb{R}^{n}, \quad j = 1,2, \tag{2.21}$$

and consider the boundary value problem ((2.17), (2.16), (2.21)).

Lemma 2.4. *The boundary value problem ((2.17), (2.16), (2.21)) has a unique solution $\hat{\mathbf{v}}_{\boldsymbol{\omega}} \in \mathcal{V}_{\boldsymbol{\omega}}$.*

Proof. First we claim that for sufficiently large $\omega > 0$ and sufficiently small μ it holds that $\operatorname{im} Q_j^+(\mu, \omega) \oplus \operatorname{im} P_{j+1}^-(\mu, -\omega) = \mathbb{R}^n$, $j = 1, 2$. This is due to asymptotic behaviour (as $\omega \to \infty$) of the involved projections and due to the hyperbolicity of the equilibria p_1 and p_2; we refer to [28] for more details.

The rest of the proof runs along the lines of the corresponding assertions in [23] or [15]: For each given sequence $(d_{j,i})$ one proves the existence of sequences $(a_{j,i}^+)$ and $(a_{j,i}^-)$ such that the corresponding solutions of the boundary value problem ((2.17), (2.16), (2.18)) solve the boundary value problem ((2.17), (2.16), (2.21)). ∎

Altogether we find $\hat{\mathbf{v}}_{\boldsymbol{\omega}} = \hat{\mathbf{v}}_{\boldsymbol{\omega}}(\mu, \mathbf{h}, \mathbf{d})$, where $\mathbf{h} := (h_{1,i}^+, h_{2,i}^-, h_{2,i}^+, h_{1,i}^-)_{i\in\mathbb{Z}}$ and $\mathbf{d} := (d_{1,i}, d_{2,i})_{i\in\mathbb{Z}}$. It is worth to mention that actually each entry of $\hat{\mathbf{v}}_{\boldsymbol{\omega}}$ depends only on a finite part of the sequences $\mathbf{h}$ and $\mathbf{d}$.

A coupling of the gained solutions can be achieved by setting

$$d_{j,i} = d_{\boldsymbol{\omega}_{j,i}}(\mu) := q_{j+1}^-(\mu)(-\omega_{j,i}) - q_j^+(\mu)(\omega_{j,i}),\ j = 1, 2.$$

Now, the nonlinear boundary value problem ((2.12),(2.15),(2.16)) is equivalent to the following fixed point equation in $\mathcal{V}_{\boldsymbol{\omega}}$:

$$\mathbf{v} = \hat{\mathbf{v}}_{\boldsymbol{\omega}}(\mu, \mathcal{G}(\mathbf{v}, \mu), \mathbf{d}_{\boldsymbol{\omega}}(\mu)), \tag{2.22}$$

where

$$\begin{array}{rccl} \mathcal{G}: & \mathcal{V}_{\boldsymbol{\omega}} \times \mathbb{R} & \to & \mathcal{V}_{\boldsymbol{\omega}} \\ & (\mathbf{v}, \mu) & \mapsto & (h_{1,i}^+, h_{2,i}^-, h_{2,i}^+, h_{1,i}^-)_{i\in\mathbb{Z}}, \quad h_{j,i}^\pm(\cdot) := g_j^\pm(\cdot, v_{j,i}^\pm(\cdot), \mu). \end{array}$$

The following Lemma provides solutions to the fixed point problem (2.22).

Lemma 2.5. *For fixed $\boldsymbol{\omega}$, $\omega_{j,i}$ sufficiently large, and $|\mu|$ sufficiently small, the fixed point problem (2.22) has a unique solution $\bar{\mathbf{v}}_{\boldsymbol{\omega}}$ in a sufficiently small neighbourhood of $0 \in \mathcal{V}_{\boldsymbol{\omega}}$. Moreover, the mapping $\mu \mapsto \bar{\mathbf{v}}_{\boldsymbol{\omega}}(\mu)$ is smooth.*

Note that the necessary considerations have been done for fixed $\boldsymbol{\omega}$ in spaces $\mathcal{V}_{\boldsymbol{\omega}}$. Define

$$\bar{\mathbf{v}}(\boldsymbol{\omega}, \mu) = (\bar{v}_{1,i}^+, \bar{v}_{2,i}^-, \bar{v}_{2,i}^+, \bar{v}_{1,i}^-)_{i\in\mathbb{Z}} := \bar{\mathbf{v}}_{\boldsymbol{\omega}}(\mu), \quad \bar{v}_{j,i}^\pm = \bar{v}_{j,i}^\pm(\boldsymbol{\omega}, \mu). \tag{2.23}$$

Lemma 2.6. *The mappings $l^\infty_{\mathbb{R}^2} \times \mathbb{R} \to l^\infty_{\mathbb{R}^n}$, $(\boldsymbol{\omega}, \mu) \mapsto \bar{v}_{j,i}^\pm(\boldsymbol{\omega}, \mu)(0)$ are smooth.*

For both Lemma 2.5 and Lemma 2.6 the proof of the corresponding statements in [23] or [15] can easily be adapted for the present situation.

3 The Bifurcation Equations

In this section we derive the leading order terms for $\Xi_i(\boldsymbol{\omega},\mu)$ given in (1.5), (1.6) and (1.7).

For given $\boldsymbol{\omega}$ and μ we find a unique Lin orbit $X(\boldsymbol{\omega},\mu)$, see Theorem 2.2, and a corresponding sequence $(\Xi_i(\boldsymbol{\omega},\mu))_{i\in\mathbb{Z}}$ of jumps, see (2.10). The Lin orbit becomes a real orbit if all jumps are equal to zero; this leads to the set of bifurcation equations

$$\Xi_i(\boldsymbol{\omega},\mu)=0,\quad i\in\mathbb{Z}, \tag{3.1}$$

where $\boldsymbol{\Xi}:=(\Xi_i)_{i\in\mathbb{Z}}$ can be read as a mapping

$$\boldsymbol{\Xi}:(l^\infty\times l^\infty)\times\mathbb{R}^2\to l^\infty\times l^\infty, \tag{3.2}$$

bearing in mind that $\boldsymbol{\omega}=(\omega_{1,i},\omega_{2,i})_{i\in\mathbb{Z}}$ and $\dim Z=2$. As in [23] or [15] we find (as a consequence of Lemma 2.6)

Lemma 3.1. $\boldsymbol{\Xi}$ *depends smoothly on* $\boldsymbol{\omega}$ *and* μ. ■

By solving the bifurcation equations (3.1) we find all kinds of orbits staying close to the primary heteroclinic cycle Γ. For instance, a k-periodic orbit can be seen as a k-periodic Lin orbit where all jumps are zero. A Lin orbit is k-periodic if and only if $\boldsymbol{\omega}$ is k-periodic. So there arise only k different pairs $(\omega_{1,i},\omega_{2,i})$, $i\in\{1,\dots,k\}$. Therefore the bifurcation equation for detecting k-periodic orbits consists only of k equations

$$\Xi_i(\boldsymbol{\omega},\mu)=0,\quad i\in\{1,\dots,k\}, \tag{3.3}$$

and the corresponding $\boldsymbol{\Xi}$ can be read as a mapping $\mathbb{R}^{2k}\times\mathbb{R}^2\to\mathbb{R}^{2k}$.

Further it is worth to mention that our considerations remain true if some $\omega_{j,i}$ are formally put to infinity. If in addition the sequence $\boldsymbol{\omega}$ is periodic then this leads to the bifurcation equation for k-homoclinic orbits or heteroclinic cycles which are composed of k-heteroclinic connections. for instance a k-homoclinic orbit to p_1 corresponds to a k-periodic sequence $\boldsymbol{\omega}$ with $\omega_{1,1}=\infty$.

From (2.11) we see that the jumps Ξ_i have the form

$$\Xi_i(\boldsymbol{\omega},\mu):=\xi^\infty(\mu)+\xi_i(\boldsymbol{\omega},\mu), \tag{3.4}$$

where

$$\xi^\infty(\mu):=q_1^+(\mu)(0)-q_1^-(\mu)(0), \tag{3.5}$$

and

$$\xi_i(\boldsymbol{\omega},\mu)=\bar{v}_{1,i+1}^+(\boldsymbol{\omega},\mu)(0)-\bar{v}_{1,i}^-(\boldsymbol{\omega},\mu)(0). \tag{3.6}$$

First we consider $\xi^\infty(\mu)$. By introducing appropriate coordinates, $\xi^\infty(\cdot)$ can be seen as a mapping

$$\xi^\infty(\cdot):\mathbb{R}^2\to\mathbb{R}^2.$$

Of course $\xi^\infty(0)=0$, which represents that the unstable manifold of p_2 and the stable manifold of p_1 intersect (in the primary heteroclinic orbit Γ_1). The following hypothesis is a transversality condition which ensures that these manifolds split with positive speed by varying μ.

(H 1) $D\xi^\infty(0)$ is non-singular.

This justifies our setting $\mu \in \mathbb{R}^2$, $\mu := (\mu_1, \mu_2)$. Due to this hypothesis we may assume

$$\xi^\infty(\mu) = \mu. \tag{3.7}$$

We remark that hypothesis (H 1) is in fact not strictly necessary in order for our results to hold: we have made this hypothesis for the sake of convenience. Our results would still hold in the case that the manifolds $W^u(p_2)$ and $W^s(p_1)$ do not split with positive speed; in this case, only accumulation rates of sets with respect to the parameter μ are affected.

We would also like to mention that Hypothesis (H 1) can be made more explicit by using a representation of ξ^∞ by means of a Melnikov-type integral. For such a consideration we refer to Remark 5.11 below.

Next we turn to the representation of $\xi_i(\boldsymbol{\omega}, \mu)$:

$$\xi_i(\boldsymbol{\omega}, \mu) = \sum_{j=1}^{2} \langle \psi_j, \xi_i^\omega(\mu) \rangle \psi_j, \tag{3.8}$$

where $\{\psi_1, \psi_2\}$ is a orthonormal basis of Z. We can write

$$\xi_i(\boldsymbol{\omega}, \mu) = \sum_{j=1}^{2} \Big(\langle \psi_j, Q_1^+(\mu, 0)\bar{v}_{1,i+1}^+(\boldsymbol{\omega}, \mu)(0) \rangle - \langle \psi_j, P_1^-(\mu, 0)\bar{v}_{1,i}^-(\boldsymbol{\omega}, \mu)(0) \rangle \Big) \psi_j.$$

As in [23] or [15] we find that under certain conditions, see (H 2) and (H 3) below, the leading order term (as $\boldsymbol{\omega} \to \infty$)) for the expression at the right-hand side can be extracted from

$$\sum_{j=1}^{2} \Big(S_1(\omega_{1,i+1}, \psi_j, \mu) - S_2(\omega_{2,i}, \psi_j, \mu) \Big) \psi_j,$$

where

$$S_1(t, \psi_j, \mu) := \langle \Psi_1^+(\mu, t, 0) Q_1^{+*}(\mu, 0)\psi_j, \tilde{Q}_1(\mu, t) q_2^-(\mu)(-t) \rangle$$

and

$$S_2(t, \psi_j, \mu) := \langle \Psi_1^-(\mu, -t, 0) P_1^{-*}(\mu, 0)\psi_j, (id - \tilde{Q}_2(\mu, t)) q_2^+(\mu)(t) \rangle.$$

Here $\Psi_j^\pm(\mu, \cdot, \cdot)$ is the transition matrix of the adjoint of (2.14), and $\tilde{Q}_j(\mu, \omega)$ is the projection which projects on $\operatorname{im} Q_j^+(\mu, \omega)$ along $\operatorname{im} P_{j+1}^-(\mu, -\omega)$, see also in the proof of Lemma 2.4. By P^* we denote the adjoint of the projection P.

We consider $S_1(t, \psi_1, \mu)$ somewhat closer. The following estimates can be found in [24, 25] or [15], respectively. We first make the following hypotheses about our system:

(H 2) $\Gamma_i \not\subset W^{ss}(p_i), \quad \Gamma_i \not\subset W^{uu}(p_{i+1}), \quad i = 1, 2.$

Here $W^{ss}(p)$ and $W^{uu}(p)$ denotes the strong stable manifold and the strong unstable manifold, respectively, of the equilibrium p. This is a standard non-orbit flip condition. Further we assume a slight modification of the non-inclination flip condition for Γ_1, which says that the stable manifold of p_1 and the local centre-unstable manifold of p_2, $W^{cu}_{loc}(p_2)$ (this is an invariant manifold whose tangent space at p_2 comprises the unstable and weakest stable directions), have a non-degenerate intersection (and *vice versa* for $W^{cs}_{loc}(p_1)$ and $W^u(p_2)$). This modification is required due to the different fixed point indices.

(H 3) $$W^{cu}_{loc}(p_2) \cap_{q_1(-t)} W^s(p_1) = \mathrm{span}\{f(q_1(-t), 0)\},$$
$$W^{cs}_{loc}(p_1) \cap_{q_1(t)} W^u(p_2) = \mathrm{span}\{f(q_1(t), 0)\}, \qquad t \gg 0.$$

As a consequence of Hypothesis (H 2) $q_2^-(\mu)(t)$ behaves asymptotically, as $t \to -\infty$, like $e^{D_1 f(p_1,\mu)t}\eta^u$, where $\eta^u = \eta^u(\mu)$ is in the generalised (real) eigenspace X^u of the principal unstable eigenvalue $\lambda_1^u(\mu)$ of $D_1 f(p_1, \mu)$. The same holds true for the behaviour of $\tilde{Q}_1(\mu, t)q_2^-(\mu)(-t)$. Here our analysis is dependent on the nature of the eigenvalues $\lambda_1^u(\mu)$ and $\lambda_2^s(\mu)$. In the case where $\lambda_1^u(\mu)$ is complex, this generalised eigenspace has dimension two, otherwise it has dimension one. Similarly, due to Hypothesis (H 3), we may deduce that when $\lambda_1^u(\mu)$ is real, $\Psi_1^+(\mu, t, 0)Q_1^{+*}(\mu, 0)\psi_1$ and $\Psi_1^+(\mu, t, 0)Q_1^{+*}(\mu, 0)\psi_2$ do not both approach p_1 in the strong stable subspaces of the adjoint of equation (2.14) as $t \to \infty$. When $\lambda_1^u(\mu)$ is complex, it is always the case that both terms do not approach p_1 in the strong stable subspace. The corresponding statement also holds for $\Psi_1^-(\mu, -t, 0)P_1^{-*}(\mu, 0)\psi_j$, $j = 1, 2$ and $\lambda_2^s(\mu)$. Then we can assume without loss of generality, for instance, that $\Psi_1^+(\mu, t, 0)Q_1^{+*}(\mu, 0)\psi_j$ behaves asymptotically as $t \to \infty$, like $e^{-(D_1 f(p_1,\mu))^* t}\eta^+$, where $\eta^+ = \eta^+(\psi_1, \mu)$ belongs to the generalised (real) eigenspace $(X^u)^\perp$ of the principal stable eigenvalue(s) $-\lambda_1^u(\mu)$, $(-\overline{\lambda}_1^u(\mu))$, of $-(D_1 f(p_1, \mu))^*$. So, the leading term $L_{11}(t, \mu)$ of $S_1(t, \psi_1, \mu)$ reads

$$L_{11}(t, \mu) = \langle e^{-D_1 f(p_1,\mu)t}\eta^u,\, e^{-(D_1 f(p_1,\mu))^* t}\eta^+\rangle = \langle e^{-D_1 f(p_1,\mu)2t}\eta^u,\, \eta^+\rangle.$$

We introduce an appropriate basis in $\mathbb{R}^n$ whose elements lie in X^u and $(X^u)^\perp$, respectively. We consider first the case where $\lambda_1^u(\mu) = \rho_1(\mu) + i\phi_1(\mu)$ is complex. Then η^u has a coordinate representation $(\eta_1^u, \eta_2^u, 0, \ldots, 0)$ where $(\eta_1^u, \eta_2^u) \neq (0, 0)$, and $e^{-D_1 f(p_1,\mu)2t}\eta^u$ acts like

$$e^{-2\rho_1(\mu)t}\begin{pmatrix} \begin{pmatrix} \cos(2\phi_1(\mu)t) & \sin(2\phi_1(\mu)t) \\ -\sin(2\phi_1(\mu)t) & \cos(2\phi_1(\mu)t) \end{pmatrix}\begin{pmatrix} \eta_1^u \\ \eta_2^u \end{pmatrix} \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

Let $(\eta_1^+, \ldots, \eta_n^+)$ be the coordinate representation of η^+ with respect to the chosen basis. Since $\eta^+ \in X^\perp$, where X denotes the generalised eigenspace of the other (the strong unstable and the stable) eigenvalues of $D_1 f(p_1, \mu)$, we have $(\eta_1^+, \eta_2^+) \neq (0, 0)$. This gives finally that $L_{11}(t, \mu)$ takes the form

$$L_{11}(t, \mu) = e^{-2\rho_1(\mu)t}\Big((\eta_2^u\eta_1^+ - \eta_1^u\eta_2^+)\sin(2\phi_1(\mu)t) + (\eta_1^u\eta_1^+ + \eta_2^u\eta_2^+)\cos(2\phi_1(\mu)t)\Big).$$

Since both (η_1^u, η_2^u) and (η_1^+, η_2^+) are different from $(0, 0)$ it holds that

$$(\eta_1, \eta_2) := (\eta_2^u\eta_1^+ - \eta_1^u\eta_2^+,\, \eta_1^u\eta_1^+ + \eta_2^u\eta_2^+) \neq (0, 0). \tag{3.9}$$

So there is an angle $\varphi_{11} = \varphi_{11}(\psi_1, \mu)$ such that

$$\sin\varphi_{11} = \eta_1 \left(\eta_1^2 + \eta_2^2\right)^{-1/2}, \quad \cos\varphi_{11} = \eta_2 \left(\eta_1^2 + \eta_2^2\right)^{-1/2}. \tag{3.10}$$

Hence in the case where $\lambda_1^u(\mu)$ is complex, $L_{11}(t,\mu)$ reads

$$L_{11}(t,\mu) = e^{-2\rho_1(\mu)t} c_{11}(\mu) \sin(2\phi_1(\mu)t + \varphi_{11}), \tag{3.11}$$

where $c_{11}(\mu) = (\eta_1^2 + \eta_2^2)^{1/2}$. By construction $c_{11}(\cdot)$ is smooth and in this case $c_{11}(0) \neq 0$.

In the case where $\lambda_1^u(\mu) = \rho_1(\mu)$ is real, we have that η^u has a coordinate representation $(\eta_1^u, 0, 0, \ldots, 0)$ where $\eta_1^u \neq 0$, and $e^{-D_1 f(p_1,\mu)2t}\eta^u$ acts like

$$e^{-2\rho_1(\mu)t} \begin{pmatrix} \eta_1^u \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

Similarly, if we let $(\eta_1^+, \ldots, \eta_n^+)$ be the coordinate representation of η^+ with respect to the chosen basis, we have $\eta_1^+ \neq 0$. This gives that in the case where $\lambda_1^u(\mu)$ is real, $L_{11}(t,\mu)$ takes the form

$$L_{11}(t,\mu) = e^{-2\rho_1(\mu)t} c_{11}(\mu),$$

where we have written $c_{11}(\mu) = \eta_1^u \eta_1^+$ for consistency.

In the same way we find constants $c_{ij}(\mu)$ and φ_{ij} such that the leading term $L_{ij}(t,\mu)$ of $S_i(t,\psi_j,\mu)$ reads

$$L_{ij}(t,\mu) = e^{-2\rho_i(\mu)t} c_{ij}(\mu) \sin(2\phi_i(\mu)t + \varphi_{ij}), \tag{3.12}$$

in the case where λ_1^u $(i=1)$, or λ_2^s $(i=2)$ is complex, where $c_{ij}(\mu) \neq 0$ and $\varphi_{i1} - \varphi_{i2} \neq 0 \pmod{\pi}$. The latter inequality is proved in Lemma 4.3. Also

$$L_{ij}(t,\mu) = e^{-2\rho_i(\mu)t} c_{ij}(\mu), \tag{3.13}$$

in the case where λ_1^u $(i=1)$, or λ_2^s $(i=2)$ is real, where $(c_{i1}(\mu), c_{i2}(\mu)) \neq (0,0)$.

With that we obtain the following representation of $\xi_i(\boldsymbol{\omega},\mu)$:

Lemma 3.2. *For the system under consideration the jump $\xi_i(\boldsymbol{\omega},\mu)$ can be written in the form*

$$\begin{aligned} \xi_i(\boldsymbol{\omega},\mu) = \;& \psi_1 \left(L_{11}(\omega_{1,i+1},\mu) - L_{21}(\omega_{2,i},\mu) + r_{1,i}(\boldsymbol{\omega},\mu)\right) \\ & + \psi_2 \left(L_{12}(\omega_{1,i+1},\mu) - L_{22}(\omega_{2,i},\mu) + r_{2,i}(\boldsymbol{\omega},\mu)\right), \end{aligned}$$

where for the quantities $r_{j,i}(\boldsymbol{\omega},\mu)$ it holds that

$$r_{j,i}(\boldsymbol{\omega},\mu) = o(e^{-2\rho_1(\mu)\omega_{1,i+1}}) + o(e^{-2\rho_2(\mu)\omega_{2,i}}).$$

The functions $c_{ij}(\cdot)$ which arise in the definition of L_{ij} are smooth, and $(c_{i1}(\mu), c_{i2}(\mu)) \neq (0,0)$, $i = 1,2$ in the case where λ_1^u $(i=1)$, or λ_2^s $(i=2)$ is real, and $c_{ij}(\mu) \neq 0$, $i,j = 1,2$ in the case where λ_1^u $(i=1)$, or λ_2^s $(i=2)$ is complex. ■

All together, we have derived the leading order terms to $\Xi_i(\boldsymbol{\omega},\mu) = \xi^\infty(\mu) + \xi_i(\boldsymbol{\omega},\mu)$ as in equations (1.5), (1.6) and (1.7). We note that in cases (CC) and (RR), further generic assumptions are necessary to guarantee the existence of nondegenerate solutions to these equations, see sections 4.1.1 and 4.3.

For the derivatives of the jumps, see also Lemma 3.1, there hold similar estimates:

Lemma 3.3. *The derivatives of the jumps ξ_i have the following form*

$$\begin{aligned} D_j\xi_i(\boldsymbol{\omega},\mu) = \; & \psi_1\Big(D_j\left(L_{11}(\omega_{1,i+1},\mu) - L_{21}(\omega_{2,i},\mu)\right) + \tilde{r}_{1,i}(\boldsymbol{\omega},\mu)\Big) \\ & + \psi_2\Big(D_j\left(L_{12}(\omega_{1,i+1},\mu) - L_{22}(\omega_{2,i},\mu)\right) + \tilde{r}_{2,i}(\boldsymbol{\omega},\mu)\Big), \end{aligned}$$

where for the quantities $\tilde{r}_{j,i}(\boldsymbol{\omega},\mu)$ it holds that

$$\tilde{r}_{j,i}(\boldsymbol{\omega},\mu) = o(e^{-2\rho_1(\mu)\omega_{1,i+1}}) + o(e^{-2\rho_2(\mu)\omega_{2,i}}).$$

$\blacksquare$

For the proof we refer again to the corresponding statements in [23] or [15].

4 Proof of Theorems

The bifurcation equation for orbits staying for all time close to the primary cycle reads:

$$\Xi_i(\boldsymbol{\omega},\mu) = \begin{pmatrix} \mu_1 + L_{11}(\omega_{1,i+1},\mu) - L_{21}(\omega_{2,i},\mu) + r_{1,i}(\boldsymbol{\omega},\mu) \\ \mu_2 + L_{12}(\omega_{1,i+1},\mu) - L_{22}(\omega_{2,i},\mu) + r_{2,i}(\boldsymbol{\omega},\mu) \end{pmatrix} = 0, \quad i \in \mathbb{Z}. \tag{4.1}$$

The strategy for proving the main theorems is to first study the solutions of the truncated bifurcation equation

$$L((\omega_{1,i+1},\omega_{2,i}),\mu) := \begin{pmatrix} \mu_1 + L_{11}(\omega_{1,i+1},\mu) - L_{21}(\omega_{2,i},\mu) \\ \mu_2 + L_{12}(\omega_{1,i+1},\mu) - L_{22}(\omega_{2,i},\mu) \end{pmatrix} = 0. \tag{4.2}$$

We denote $\mathbf{L}(\boldsymbol{\omega},\mu) := (L((\omega_{1,i+1},\omega_{2,i}),\mu))_{i\in\mathbb{Z}}$. Then $\mathbf{L} : (l^\infty \times l^\infty) \times \mathbb{R} \to (l^\infty \times l^\infty)$. Note that $(L((\omega_{1,i+1},\omega_{2,i}),\mu))_{i\in\mathbb{Z}}$ decouples with respect to $\omega_{1,i+1}$, $\omega_{2,i}$ over i.

The proof makes use of geometrical properties of the truncated bifurcation equations. We note that for $t \in \mathbb{R}^+$

$$\mathfrak{S}_1(t,\mu) := \begin{pmatrix} \mu_1 + L_{11}(t,\mu) \\ \mu_2 + L_{12}(t,\mu) \end{pmatrix} \quad \text{and} \quad \mathfrak{S}_2(t,\mu) := \begin{pmatrix} L_{21}(t,\mu) \\ L_{22}(t,\mu) \end{pmatrix} \tag{4.3}$$

define geometrical objects in $\mathbb{R}^2$. In the case of $\lambda_1^u(\mu)$ (resp. $\lambda_2^s(\mu)$) being a complex eigenvalue, $\mathfrak{S}_1(t,\mu)$ (resp. $\mathfrak{S}_2(t,\mu)$) defines a smooth spiral with the tip at $(\mu_1,\mu_2)^T$ (resp. $(0,0)^T$). Note that $\mathfrak{S}_i(\cdot,\mu)$, are indeed spirals since $\varphi_{i1} - \varphi_{i2} \neq 0(\mod \pi)$. A proof of the latter fact is presented in Lemma 4.3. Similarly, in the case of real eigenvalues, $\mathfrak{S}_i(\cdot,\mu)$ define lines in the plane that terminate at $(\mu_1, \mu_2)^T$, respectively $(0,0)^T$. These lines are smooth, and their derivative is bounded close to their endpoint; see Figure 3.

We study intersections of the $\mathfrak{S}_i(\cdot, \mu)$, which correspond to solutions of the truncated bifurcation equation. Such solutions that we find, may be nondegenerate (transversal) with respect to $(\omega_{1,i+1}, \omega_{2,i})$. The following Lemma proves that such nondegenerate solutions of $\mathbf{L}(\boldsymbol{\omega}, \mu) = 0$, are indeed approximations of solutions to the full bifurcation equation $\Xi_i = 0$, $i \in \mathbb{Z}$. They depend on the solution set having the property that each $\omega_{j,i}$ is sufficiently large, and μ sufficiently small.

Lemma 4.1. *Suppose the equation $L((\omega_{1,i+1}, \omega_{2,i}), 0) = 0$ has a set of nondegenerate solutions $\mathcal{S}$; nondegenerate in the sense that $D_1 L((\hat{\omega}_{1,i+1}, \hat{\omega}_{2,i}), 0)$ is invertible at points $((\hat{\omega}_{1,i+1}, \hat{\omega}_{2,i}), 0) \in \mathcal{S}$. Let $((\hat{\omega}_{1,i+1}, \hat{\omega}_{2,i}), 0)_{i \in \mathbb{Z}}$ be a solution to the truncated equations $\mathbf{L}(\boldsymbol{\omega}, 0) = 0$, where each $((\hat{\omega}_{1,i+1}, \hat{\omega}_{2,i}), 0) \in \mathcal{S}$ and there is a constant d such that $|\hat{\omega}_{j,i} - \hat{\omega}_{k,l}| < d$ for all $j, k \in \{1, 2\}$, $i, l \in \mathbb{Z}$.*

There are constants Ω, $\delta > 0$ depending on d, and $c(\hat{\boldsymbol{\omega}}, \mu) > 0$ such that for each $\hat{\boldsymbol{\omega}}$ with $\hat{\omega}_{j,i} > \Omega$, $j = 1, 2$, $i \in \mathbb{Z}$ and $|\mu| < \delta$, the full bifurcation equation $\Xi_i = 0$, $i \in \mathbb{Z}$ has a unique solution $(\boldsymbol{\omega}_{\hat{\boldsymbol{\omega}}}(\mu), \mu)$ with $||\hat{\boldsymbol{\omega}} - \boldsymbol{\omega}_{\hat{\boldsymbol{\omega}}}(\mu)|| < c(\hat{\boldsymbol{\omega}}, \mu)$ and $c(\hat{\boldsymbol{\omega}}, \mu) \to 0$ as $\hat{\omega}_{j,i} \to \infty$, $j = 1, 2$, $i \in \mathbb{Z}$ and $\mu \to 0$.

Sketch of proof. The equation $\Xi(\boldsymbol{\omega}, \mu) = 0$ is equivalent to the fixed point equation

$$\boldsymbol{\omega} = \boldsymbol{\omega} - \Big[D_1 \mathbf{L}(\hat{\boldsymbol{\omega}}, 0)\Big]^{-1} \Xi(\boldsymbol{\omega}, \mu) =: \boldsymbol{\mathcal{A}}(\boldsymbol{\omega}, \mu),$$

We would like to show that $\boldsymbol{\mathcal{A}}(\cdot, \mu)$ is, for μ close to 0, a contractive mapping of some closed neighbourhood of $\hat{\boldsymbol{\omega}}$ into itself. We control the higher order terms in the full bifurcation equation by requiring that the $\hat{\omega}_{j,i}$ are all sufficiently large. For that we consider

$$D_1 \boldsymbol{\mathcal{A}}(\boldsymbol{\omega}, \mu) = \Big[D_1 \mathbf{L}(\hat{\boldsymbol{\omega}}, 0)\Big]^{-1} \Big(D_1 \mathbf{L}(\hat{\boldsymbol{\omega}}, 0) - D_1 \Xi(\boldsymbol{\omega}, \mu)\Big).$$

The main observation in this respect is that the norm of $D_1 \mathbf{L}(\hat{\boldsymbol{\omega}}, 0)$ is uniformly bounded for all $\hat{\boldsymbol{\omega}}$ sufficiently large (but bounded from above) and that the norm of $D_1 \mathbf{L}(\hat{\boldsymbol{\omega}}, 0) - D_1 \Xi(\boldsymbol{\omega}, \mu)$ can be made arbitrarily small only by choosing the $\hat{\omega}_{j,i}$ large enough and $\|\hat{\boldsymbol{\omega}} - \boldsymbol{\omega}\|$ sufficiently small. We also refer to [20] where similar considerations (verification of shift dynamics in the Shilnikov scenario by using Lin's method) have been worked out in more detail. ∎

A more detailed proof for a similar statement can be found in [16].

Remark 4.2. *For $\mu \neq 0$ there are only finitely many different $(\hat{\omega}_1, \hat{\omega}_2)$ which solve the truncated equation $L((\hat{\omega}_1, \hat{\omega}_2), \mu) = 0$. Therefore, for those μ there is an upper bound for the number of symbols with which the sequence $\hat{\boldsymbol{\omega}}$ and hence, due to Lemma 3.2, $\boldsymbol{\omega}_{\hat{\boldsymbol{\omega}}}(\mu)$ can be constructed.*

4.1 Proof of Theorem 1.2

In order to prove Theorem 1.2 we show that the truncated form of the bifurcation equation, $L((\omega_1, \omega_2), 0) = 0$, has an infinite set of nondegenerate (transversal) solutions $(\hat{\omega}_1, \hat{\omega}_2)_n$, $(n \in \mathbb{Z})$ such that $(\hat{\omega}_1, \hat{\omega}_2)_n \to (\infty, \infty)$ as $n \to \infty$.

In Case (RC) (real-complex case, see equation (1.6)), we observe that $\mathfrak{S}_1(\cdot,0)$, $\mathfrak{S}_2(\cdot,0)$ (see equation (4.3)) represent a logarithmic spiral and a line, with tips both centred at the origin. Since both objects are smooth, it is clear that there exists an infinite set of transversal intersections which accumulate to the origin, and therefore have times $(\hat{\omega}_1,\hat{\omega}_2)_n$ that tend to infinity as $n\to\infty$.

The following section deals with the less trivial case of two complex eigenvalues, see equation (1.5). In this case $\mathfrak{S}_1(\cdot,0)$, $\mathfrak{S}_2(\cdot,0)$ represent two logarithmic spirals centred at the origin.

4.1.1 Case (CC): Analysis of spirals

First we make clear that the curves $\mathfrak{S}_i(\cdot,\mu)$ which we defined in (4.3) are indeed spirals. For that, as already mentioned, we only have to prove

Lemma 4.3. $\varphi_{i1}-\varphi_{i2}\neq 0(\mod \pi)$*, $i=1,2$.*

Proof. We show that $\sin(\varphi_{i1}-\varphi_{i2})\neq 0$. We illustrate this in the case $i=1$. The quantity φ_{11} is defined by (3.10). Similarly φ_{12} is associated to a pair $(\hat{\eta}_1,\hat{\eta}_2)$, which again, in accordance with (3.9) is defined by

$$(\hat{\eta}_1,\hat{\eta}_2):=(\eta_2^u\hat{\eta}_1^+-\eta_1^u\hat{\eta}_2^+,\ \eta_1^u\hat{\eta}_1^++\eta_2^u\hat{\eta}_2^+)\neq(0,0). \tag{4.4}$$

Note that the η_i^u appearing in (3.9) and (4.4) are the same in both equations. Exploiting $\sin(\varphi_{11}-\varphi_{12})=\sin\varphi_{11}\cos\varphi_{12}-\sin\varphi_{12}\cos\varphi_{11}$ we find that $\sin(\varphi_{11}-\varphi_{12})=0$ if and only if (η_1,η_2) and $(\hat{\eta}_1,\hat{\eta}_2)$ are linearly dependent; and because of (3.9) and (4.4) this is the case if and only if (η_1^+,η_2^+) and $(\hat{\eta}_1^+,\hat{\eta}_2^+)$ are linearly dependent. But these vectors are linearly independent because ψ_1 and ψ_2 are linearly independent. ∎

We continue with analysing the solutions of equations (4.2). We note again that these equations are equal to the real bifurcation equations (4.1) modulo the rest terms. For ease of notation in this section, we drop the μ-dependence of variables inside (4.2), and rename $\omega_{1,i+1}$ and $\omega_{2,i}$ as ω_1, ω_2 respectively. Then the equation reads

$$\begin{pmatrix} \mu_1+e^{-2\rho_1\omega_1}c_{11}\sin(2\phi_1\omega_1+\varphi_{11})-e^{-2\rho_2\omega_2}c_{21}\sin(2\phi_2\omega_2+\varphi_{21}) \\ \mu_2+e^{-2\rho_1\omega_1}c_{12}\sin(2\phi_1\omega_1+\varphi_{12})-e^{-2\rho_2\omega_2}c_{22}\sin(2\phi_2\omega_2+\varphi_{22}) \end{pmatrix}=0 \tag{4.5}$$

The following Lemma defines a renormalisation of equations (4.5).

Lemma 4.4. *There exists an invertible change of coordinates such that equations* (4.5) *take the form*

$$\begin{pmatrix} \mu_1+e^{-\rho_1\omega_1}\sin\omega_1 & -e^{-\rho_2\omega_2}c_{21}\sin\omega_2 \\ \mu_2+e^{-\rho_1\omega_1}\sin(\omega_1+\varphi_{12}) & -e^{-\rho_2\omega_2}c_{22}\sin(\omega_2+\varphi_{22}) \end{pmatrix}=0 \tag{4.6}$$

Proof. We define the variables

$$\begin{aligned}
\hat{\omega}_1 &= 2\phi_1\omega_1 + \varphi_{11}, & \hat{c}_{11} &= e^{\rho_1\varphi_{11}/\phi_1}c_{11}, & \hat{c}_{21} &= e^{\rho_2\varphi_{21}/\phi_2}c_{21},\\
\hat{\omega}_2 &= 2\phi_2\omega_2 + \varphi_{21}, & \hat{c}_{12} &= e^{\rho_1\varphi_{11}/\phi_1}c_{12}, & \hat{c}_{22} &= e^{\rho_2\varphi_{21}/\phi_2}c_{22},\\
\hat{\varphi}_{12} &= \varphi_{12} - \varphi_{11}, & \hat{\varphi}_{22} &= \varphi_{22} - \varphi_{21}, & \hat{\rho}_1 &= \rho_1/\phi_1,\\
\hat{\rho}_2 &= \rho_2/\phi_2,
\end{aligned}$$

under which (4.5) becomes

$$\begin{pmatrix} \mu_1 + e^{-\hat{\rho}_1\hat{\omega}_1}\hat{c}_{11}\sin\hat{\omega}_1 & -e^{-\hat{\rho}_2\hat{\omega}_2}\hat{c}_{21}\sin\hat{\omega}_2 \\ \mu_2 + e^{-\hat{\rho}_1\hat{\omega}_1}\hat{c}_{12}\sin(\hat{\omega}_1+\hat{\varphi}_{12}) & -e^{-\hat{\rho}_2\hat{\omega}_2}\hat{c}_{22}\sin(\hat{\omega}_2+\hat{\varphi}_{22}) \end{pmatrix} = 0$$

Now divide the first equation by $\hat{c}_{11}$ and the second equation by $\hat{c}_{12}$. Defining $\bar{c}_{21} = \hat{c}_{21}/\hat{c}_{11}$, $\bar{c}_{22} = \hat{c}_{22}/\hat{c}_{12}$, $\hat{\mu}_1 = \mu_1/\hat{c}_{11}$, $\hat{\mu}_2 = \mu_2/\hat{c}_{12}$ and dropping hats/bars provides equation (4.6).

Note that the new variables in equation (4.6) have the properties $\varphi_{12}, \varphi_{22} \neq 0 \,(\mathrm{mod}\ \pi)$ and $c_{21}, c_{22} \neq 0$. ∎

Given the above change of variables, we define the spirals $\mathfrak{S}_1(t,\mu)$, $\mathfrak{S}_2(t,\mu)$ in the plane as before:

$$\mathfrak{S}_1(t,\mu) := \begin{pmatrix} \mu_1 + e^{-\rho_1 t}\sin t \\ \mu_2 + e^{-\rho_1 t}\sin(t+\varphi_{12}) \end{pmatrix}, \quad \mathfrak{S}_2(t,\mu) := \begin{pmatrix} e^{-\rho_2 t}c_{21}\sin t \\ e^{-\rho_2 t}c_{22}\sin(t+\varphi_{22}) \end{pmatrix}.$$

We first consider solutions to the equations (4.6) where $\mu_1 = \mu_2 = 0$. We now prove the following Lemma, which proves the existence of infinitely many transversal intersections of $\mathfrak{S}_1(\omega_1, 0)$ and $\mathfrak{S}_2(\omega_2, 0)$ in the case where the ratio ρ_1/ρ_2 is irrational.

Lemma 4.5. *Assume that $\mu_1 = \mu_2 = 0$ and the ratio ρ_1/ρ_2 is irrational. Then equation (4.6) has infinitely many solutions for ω_1, ω_2 arbitrarily large. Moreover, at each of these solutions, the Jacobian of $(\omega_1,\omega_2) \mapsto \mathfrak{S}_1(\omega_1,0) - \mathfrak{S}_2(\omega_2,0)$ is nonsingular.*

Equivalently, given that the ratio ρ_1/ρ_2 is irrational, there are infinitely many transversal intersections of $\mathfrak{S}_1(\omega_1,0)$ and $\mathfrak{S}_2(\omega_2,0)$ for arbitrarily large ω_1, ω_2.

Proof. Central to the proof is the fact that numbers of the form $k_1\rho_1 - k_2\rho_2$ (where $k_1, k_2 \in \mathbb{N}$) are dense in the real line. This may be easily seen from the fact that numbers of the form

$$k_1\frac{\rho_1}{\rho_2} \pmod 1, \qquad k_1 \in \mathbb{N}$$

are dense in the unit interval (since ρ_1/ρ_2 is irrational). Therefore numbers of the form $k_1\frac{\rho_1}{\rho_2} - k_2$ (where besides k_1 also $k_2 \in \mathbb{N}$) are dense in the real line, and hence so are numbers of the form $k_1\rho_1 - k_2\rho_2$. We may make $k_1\rho_1 - k_2\rho_2$ arbitrarily close to any real number, by taking appropriate k_1 and k_2 (which, if necessary, have to be chosen sufficiently large).

Now consider the spirals $\mathfrak{S}_1(\omega_1, 0)$ and $\mathfrak{S}_2(\omega_2, 0)$. These spirals are centred on the origin in the plane. They also have the property that they are invariant under transformations $\omega \mapsto \omega - 2n_j\pi$

($n_j \in \mathbb{N}$). Such a transformation has the effect of enlarging the spiral $\mathfrak{S}_i(\omega, 0)$ by a factor of $e^{2\rho_i n_j \pi}$, which recovers the original spiral. Then it is clear that a straight line through the origin in the plane intersects the spiral $\mathfrak{S}_i(\omega, 0)$ in an infinite set of points $\mathfrak{S}_i(\omega_i^* + 2n_j\pi, 0)$ for some ω_i^*. Moreover, the tangents at $\mathfrak{S}_i(\cdot, 0)$ at the points $\omega_i^* + 2n_j\pi$, $j \in \mathbb{N}$, are all parallel.

Now we would like to chose one such straight line such that the tangent directions to the spirals at $\mathfrak{S}_1(\omega_1^*, 0)$ and $\mathfrak{S}_2(\omega_2^*, 0)$ are different. Such a straight line exists because the exponential rates ρ_1 and ρ_2 are different. However note that $\mathfrak{S}_1(\omega_1^*, 0)$ and $\mathfrak{S}_2(\omega_2^*, 0)$ are not necessarily intersection points of the two spirals, see Figure 5.

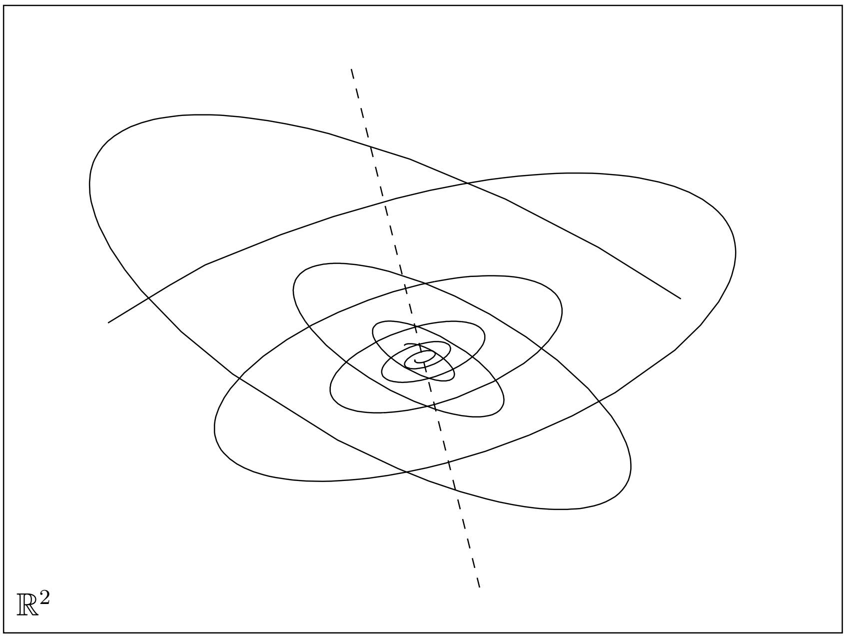

Figure 5: The spirals $\mathfrak{S}_1(\omega_1, 0)$ and $\mathfrak{S}_2(\omega_2, 0)$ together with a straight line through the origin that intersects both spirals at points that have different tangent direction.

Now let the constant M_0 be defined by

$$\mathfrak{S}_1(\omega_1^*, 0) = M_0 \mathfrak{S}_2(\omega_2^*, 0).$$

Then (ω_1^*, ω_2^*) is a point of transversal intersection of the two spirals $\mathfrak{S}_1(\omega_1^*, 0)$ and $M_0\mathfrak{S}_2(\omega_2^*, 0)$. Equivalently, $((\omega_1^*, \omega_2^*), M_0)$ is a solution of the equation

$$\mathfrak{S}((\omega_1, \omega_2), M) := (\mathfrak{S}_1(\omega_1, 0) - M\mathfrak{S}_2(\omega_2, 0) = 0 \tag{4.7}$$

and that the Jacobian $D_1\mathfrak{S}((\omega_1^*, \omega_2^*), M_0)$ is nonsingular. Then by the Implicit Function Theorem, we may find functions $\omega_1^*(M)$, $\omega_2^*(M)$ for M close to M_0, such that $\omega_1^*(M_0) = \omega_1^*$, $\omega_2^*(M_0) = \omega_2^*$, and

$$\mathfrak{S}((\omega_1^*(M), \omega_2^*(M)), M) \equiv 0.$$

Now we can make $M := e^{(k_1\rho_1 - k_2\rho_2)2\pi}$ arbitrarily close to M_0 by choosing appropriate k_1, k_2 arbitrarily large. Hence we find for $\omega_i = \omega_i^*(M)$

$$\begin{pmatrix} e^{-2k_1\rho_1\pi} e^{-\rho_1\omega_1} \sin\omega_1 & -e^{-2k_2\rho_2\pi} e^{-\rho_2\omega_2} c_{21} \sin\omega_2 \\ e^{-2k_1\rho_1\pi} e^{-\rho_1\omega_1} \sin(\omega_1 + \varphi_{12}) & -e^{-2k_2\rho_2\pi} e^{-\rho_2\omega_2} c_{22} \sin(\omega_2 + \varphi_{22}) \end{pmatrix} = 0. \tag{4.8}$$

The 2π-periodicity of $\sin(\cdot)$ gives

$$\mathfrak{S}_1(\omega_1^*(M) + 2k_1\pi, 0) = \mathfrak{S}_2(\omega_2^*(M) + 2k_2\pi, 0).$$

Now, we may choose sequences $(k_i(n))_{n\in\mathbb{N}}$ tending to infinity as n tends to infinity, such that the sequence (M_n), where $M_n := e^{(k_1(n)\rho_1 - k_2(n)\rho_2)2\pi}$ tends to M_0 (as $k_i \to \infty$). Therefore we find an infinite number of solutions to equation (4.8), and hence an infinite number of transversal intersections of the spirals $\mathfrak{S}_1(\omega_1, 0)$ and $\mathfrak{S}_2(\omega_2, 0)$. This proves the Lemma. ∎

It is clear that under perturbation of μ, (4.6) retains only finitely many transversal intersections, but this number of transversal intersections may be arbitrarily large by taking μ sufficiently close to zero.

The following Lemma states that in the case where ρ_1/ρ_2 is rational, the spirals $\mathfrak{S}_1(\omega_1, 0)$ and $\mathfrak{S}_2(\omega_2, 0)$ are together self-similar.

Lemma 4.6. *Assume that that $\mu_1 = \mu_2 = 0$ and ρ_1/ρ_2 is rational, $\rho_1/\rho_2 = m/n$. Then equation (4.6) is periodic in (ω_1, ω_2).*

Proof. Under the transformations

$$\begin{aligned} \omega_1 &\to \omega_1 + 2n\pi \\ \omega_2 &\to \omega_2 + 2m\pi \end{aligned}$$

equation (4.6) becomes

$$\begin{pmatrix} e^{-2\rho_1 n\pi} e^{-\rho_1\omega_1} \sin\omega_1 & +e^{-2\rho_2 m\pi} e^{-\rho_2\omega_2} c_{21} \sin\omega_2 \\ e^{-2\rho_1 n\pi} e^{-\rho_1\omega_1} \sin(\omega_1 + \varphi_{12}) & +e^{-2\rho_2 m\pi} e^{-\rho_2\omega_2} c_{22} \sin(\omega_2 + \varphi_{22}) \end{pmatrix} = 0$$

Multiplying through both equations by $e^{2\rho_1 n\pi}$ and using $\rho_1 n - \rho_2 m = 0$ recovers the original equations (4.6). ∎

Lemma 4.7. *Let $\mathcal{H}$ be defined as in the introduction. Assume that $\mu_1 = \mu_2 = 0$. There exists an open and dense set $\mathcal{D} \subset \mathcal{H}$, such that for each $f(\cdot, 0) \in \mathcal{D}$, equation (4.5) has an infinite number of non-degenerate solutions, or equivalently the spirals $\mathfrak{S}_1(\omega_1, 0)$ and $\mathfrak{S}_2(\omega_2, 0)$ have an infinite number of transversal intersections.*

Proof. First we note that since $\mathcal{H}$ is endowed with the C^1 topology, the constants in equation (4.5) varies continuously in this topology, meaning that they depend continuously on the vector field. Therefore Lemma 4.5 implies that there is a dense set $\tilde{\mathcal{D}} \subset \mathcal{H}$ such that equation (4.5) has an infinite number of transversal intersections.

We now consider a vector field f in $\tilde{\mathcal{D}}$, and take a small neighbourhood $B(f; \epsilon)$ around it. It is clear that if ϵ is sufficiently small, the constants in (4.5) undergoes an arbitrarily small perturbation, and at least one transversal intersection of $\mathfrak{S}_1(\omega_1, 0)$ and $\mathfrak{S}_2(\omega_2, 0)$ persists. By Lemma 4.5, for every point in $B(f; \epsilon)$ where ρ_1/ρ_2 is irrational, there exist infinitely many transversal intersections. Also, since there is as least one transversal intersection at every point in $B(f;\epsilon)$, we may use Lemma 4.6 to show that where ρ_1/ρ_2 is rational, there must also be infinitely many transversal intersections.

This provides an open and dense set in $\mathcal{H}$ where equation (4.5) has infinitely many non-degenerate solutions, and completes the Lemma. ∎

Remark 4.8. *For μ different from zero each $f(\cdot,\mu) \in \mathcal{D}$ equation* (4.5) *has only a finite number of non-degenerate solutions, or equivalently the spirals $\mathfrak{S}_1(\omega_1,\mu)$ and $\mathfrak{S}_2(\omega_2,\mu)$ have only a finite number of transversal intersections. See also Remark 4.2.*

4.1.2 Symbolic dynamics

We have proved the existence of an infinite set of nondegenerate (transversal) solutions $(\hat{\omega}_1,\hat{\omega}_2)_n$, $(n \in \mathbb{Z})$ for $L((\omega_1,\omega_2),0) = 0$ such that $(\hat{\omega}_1,\hat{\omega}_2)_n \to (\infty,\infty)$ as $n \to \infty$ for the Cases (RC) and (CC) (in an open and dense set $\mathcal{D} \subset \mathcal{H}$). We now let

$$\mathbf{\Omega}_{n_0} := \{(\hat{\omega}_1,\hat{\omega}_2)_n,\ n \geq n_0\},$$

for a sufficiently large n_0, such that $(\hat{\omega}_1,\hat{\omega}_2)_n$ are large enough to invoke Lemma 4.1. We may take any finite collection of these solutions, say $\mathbf{\Omega}_{n_0,N} \subset \mathbf{\Omega}_{n_0}$, which comprises N elements, $|\mathbf{\Omega}_{n_0,N}| = N$, for some $N \in \mathbb{N}$. We may then define the following sets of sequences $\boldsymbol{\omega}$:

$$\mathbf{\Omega}^{\mathbb{Z}}_{n_0,N} := \{\boldsymbol{\omega} : (\omega_{1,i+1},\omega_{2,i}) \in \mathbf{\Omega}_{n_0,N}\}.$$

Then for any sequence $\hat{\boldsymbol{\omega}} \in \mathbf{\Omega}^{\mathbb{Z}}_{n_0,N}$ we may use Lemma 4.1 to solve $\Xi(\boldsymbol{\omega},\mu) = 0$ near $(\hat{\boldsymbol{\omega}},0)$ to find $\boldsymbol{\omega}_{\hat{\boldsymbol{\omega}}}(\mu)$. Note that moreover $\operatorname{dom}\boldsymbol{\omega}_{\hat{\boldsymbol{\omega}}}(\cdot)$ does not depend on $\hat{\boldsymbol{\omega}} \in \mathbf{\Omega}^{\mathbb{Z}}_{n_0,N}$.

By $x_{\hat{\boldsymbol{\omega}}}(\mu)(\cdot)$ we denote the corresponding solution of (1.1), see also (2.11) and (2.23),

$$x_{\hat{\boldsymbol{\omega}}}(\mu)(0) = q_1^+(\mu)(0) + v_{1,1}^+(\boldsymbol{\omega}_{\hat{\boldsymbol{\omega}}}(\mu),\mu)(0),$$

and define

$$\mathcal{S}^N_\mu := \{x_{\hat{\boldsymbol{\omega}}}(\mu)(0) : \hat{\boldsymbol{\omega}} \in \mathbf{\Omega}^{\mathbb{Z}}_{n_0,N}\}.$$

We introduce a shift operator on $\mathbf{\Omega}^{\mathbb{Z}}_{n_0,N}$:

$$\zeta : \mathbf{\Omega}^{\mathbb{Z}}_{n_0,N} \to \mathbf{\Omega}^{\mathbb{Z}}_{n_0,N}, \quad \boldsymbol{\omega} \mapsto \boldsymbol{\tau}, \quad (\tau_{1,i},\tau_{2,i}) := (\omega_{1,i+2},\omega_{2,i+1}).$$

Now the system $(\mathbf{\Omega}^{\mathbb{Z}}_{n_0,N},\zeta)$ is conjugated to the full shift on N symbols. All that remains in order to prove Theorem 1.2 is to show that this relation is in fact a topological conjugacy.

Lemma 4.9. *The system $(\mathbf{\Omega}^{\mathbb{Z}}_{n_0,N},\zeta)$ is topologically conjugated to the full shift on N symbols, where $\mathbf{\Omega}^{\mathbb{Z}}_{n_0,N}$ is equipped with the product topology coming from $\mathbf{\Omega}^{\mathbb{Z}}_{n_0,N} \cong \{(\omega_1^k,\omega_2^k),\ k = 1,\dots,N\}^{\mathbb{Z}}$.*

Proof. There is a canonical one-to-one mapping

$$h_\mu : \mathbf{\Omega}_{n_0,N}^{\mathbb{Z}} \to \mathcal{S}_\mu^N, \quad \hat{\boldsymbol{\omega}} \mapsto x_{\hat{\boldsymbol{\omega}}}(\mu)(0).$$

By Π_μ we denote the first-return map on Σ_1; $x \in \operatorname{dom} \Pi_\mu$, if there is a $t^* > 0$ such that $\phi_\mu^{t^*}(x) \in \Sigma_1$, ($\{\phi_\mu^{t^*}(\cdot)\}$ flow of (1.1)), and $\phi_\mu^t(x) \notin \Sigma_1$, $\phi_\mu^t(x) \in \mathcal{U}$, for all $t \in (0, t^*)$, see Definition 1.1 for the meaning of $\mathcal{U}$. We claim that $\mathcal{S}_\mu^N \subset \operatorname{dom} \Pi_\mu$, that $\mathcal{S}_\mu^N$ is Π_μ invariant, and moreover

$$\Pi_\mu \circ h_\mu = h_\mu \circ \zeta. \tag{4.9}$$

The latter equation can be concluded from the uniqueness part of Theorem 2.2 and the construction of $\boldsymbol{\omega}_{\hat{\boldsymbol{\omega}}}$ as follows.We have

$$\Pi_\mu x_{\hat{\boldsymbol{\omega}}}(\mu)(0) = x_{\hat{\boldsymbol{\omega}}}(\mu)(2(\omega_{1,1}(\mu) + \omega_{2,1}(\mu))) = q_1^+(0) + \bar{v}_{1,2}^+(\boldsymbol{\omega}(\mu), \mu)(0),$$

and hence

$$\Pi_\mu x_{\hat{\boldsymbol{\omega}}}(\mu)(0) = x_{\zeta\hat{\boldsymbol{\omega}}}(\mu)(0).$$

This is just another representation of (4.9).

Equation (4.9) means that $(\mathcal{S}_\mu^N, \Pi_\mu)$ is conjugated to $(\mathbf{\Omega}_{n_0,N}^{\mathbb{Z}}, \zeta)$. So, in order to prove topological conjugacy, as claimed in the theorem, it remains to prove that h_μ is a homeomorphism. For that purpose we consider h_μ as a composition of mappings

$$\mathfrak{H} : \mathbf{\Omega}_{n_0,N}^{\mathbb{Z}} \to \mathfrak{O} := \{\boldsymbol{\omega}_{\hat{\boldsymbol{\omega}}}(\mu) : \hat{\boldsymbol{\omega}} \in \mathbf{\Omega}_{n_0,N}^{\mathbb{Z}}\}, \quad \hat{\boldsymbol{\omega}} \mapsto \boldsymbol{\omega}_{\hat{\boldsymbol{\omega}}}(\mu),$$

and

$$\mathfrak{h} : \mathfrak{O} \to \mathcal{S}_\mu^N, \quad \boldsymbol{\omega} \mapsto q_1^+(\mu)(0) + v_{1,1}^+(\boldsymbol{\omega}_{\hat{\boldsymbol{\omega}}}(\mu), \mu)(0).$$

First we show that $\mathfrak{H}$ is a homeomorphism, where both $\mathbf{\Omega}_{n_0,N}^{\mathbb{Z}}$ and $\mathfrak{O}$ are considered to be equipped with the product topology. We start with showing that $\mathfrak{O}$ is compact. Let ρ be small enough that for any two (different) elements of $\mathbf{\Omega}_{n_0,N}$ the closed balls with radius ρ centred at these elements do not intersect:

$$B[(\omega_1^i, \omega_2^i), \rho] \cap B[(\omega_1^j, \omega_2^j), \rho] = \emptyset, \quad (\omega_1^i, \omega_2^i), (\omega_1^j, \omega_2^j) \in \mathbf{\Omega}_{n_0,N}, \quad i \neq j.$$

By construction $\mathfrak{O} \subset \left(\cup_{i=1}^N B[(\omega_1^i, \omega_2^i), \rho]\right)^{\mathbb{Z}}$. Since $\cup_{i=1}^N B[(\omega_1^i, \omega_2^i), \rho]$ is compact, also the set of sequences $\left(\cup_{i=1}^N B[(\omega_1^i, \omega_2^i), \rho]\right)^{\mathbb{Z}}$ is compact by the Tychonoff theorem, see [7].

So, in order to verify the compactness of $\mathfrak{O}$ it remains to show that $\mathfrak{O}$ is closed. The set $\mathfrak{O}$ however is the set of zeros of the mapping $\Xi(\cdot, \mu) : \left(\cup_{i=1}^N B[(\omega_1^i, \omega_2^i), \rho]\right)^{\mathbb{Z}} \to l_{\mathbb{R}^2}^\infty$. By [19, Lemma 3.4] this mapping is continuous, where the spaces are equipped with product topology. Altogether this proves that $\mathfrak{O}$ is compact.

Consider now the one-to-one map

$$\mathfrak{H}^{-1} : \mathfrak{O} \to \mathbf{\Omega}_{n_0,N}^{\mathbb{Z}}.$$

Obviously $\mathfrak{H}^{-1}$ is continuous. Due to the compactness of $\mathfrak{O}$ also $\mathfrak{H}$ is continuous – hence $\mathfrak{H}$ is a homeomorphism.

Next we consider $\mathfrak{h}$. Again by invoking [19, Lemma 3.4] or its proof respectively, we find that $\mathfrak{h}$ is continuous. (Again both $\mathfrak{O}$ and $\mathbf{\Omega}^{\mathbb{Z}}_{n_0,N}$ should be equipped with the product topology.) Once more the compactness of $\mathfrak{O}$ gives that $\mathfrak{h}$ is a homeomorphism.

All in all $h_\mu = \mathfrak{h} \circ \mathfrak{H}$ is a homeomorphism and therefore $(\mathcal{S}^N_\mu, \Pi_\mu)$ is topologically conjugated to the full shift on N symbols. ∎

To conclude we remark that the sets $\mathcal{S}^N_\mu$ have the properties which we claimed in theorem. Further, for fixed $\mu \neq 0$ the spirals have only finitely many intersections. So there is an N_μ such that for $N > N_\mu$ there are neither N-periodic orbits nor shift dynamics in N symbols. This completes the proof of Theorem 1.2.

4.2 Proof of Lemma 1.3

4.2.1 k-$(2,1)$ heteroclinic orbits

A k-(2,1) heteroclinic connection may be considered as part of a heteroclinic cycle, together with a 1-(1,2) heteroclinic orbit. In terms of the $\boldsymbol{\omega}$-sequence, this corresponds to a k-periodic sequence with $\omega_{1,1} = \omega_{2,1} = \infty$. Then the bifurcation equation for these orbits is as follows, see also Section 3, equation (3.3).

$$
\begin{aligned}
\Xi_1(\boldsymbol{\omega},\mu) &= \begin{pmatrix} \mu_1 + L_{11}(\omega_{1,2},\mu) + r_{1,1}(\boldsymbol{\omega},\mu) \\ \mu_2 + L_{12}(\omega_{1,2},\mu) + r_{2,1}(\boldsymbol{\omega},\mu) \end{pmatrix} = 0 \\
\Xi_i(\boldsymbol{\omega},\mu) &= \begin{pmatrix} \mu_1 + L_{11}(\omega_{1,i+1},\mu) - L_{21}(\omega_{2,i},\mu) + r_{1,i}(\boldsymbol{\omega},\mu) \\ \mu_2 + L_{12}(\omega_{1,i+1},\mu) - L_{22}(\omega_{2,i},\mu) + r_{2,i}(\boldsymbol{\omega},\mu) \end{pmatrix} = 0 \\
& \qquad i = 2,\dots,k-1 \\
\Xi_k(\boldsymbol{\omega},\mu) &= \begin{pmatrix} \mu_1 - L_{21}(\omega_{2,k},\mu) + r_{1,k}(\boldsymbol{\omega},\mu) \\ \mu_2 - L_{22}(\omega_{2,k},\mu) + r_{2,k}(\boldsymbol{\omega},\mu) \end{pmatrix} = 0.
\end{aligned} \tag{4.10}
$$

We begin by considering the truncated form of equations $\Xi_1(\boldsymbol{\omega},\mu) = 0$ and $\Xi_k(\boldsymbol{\omega},\mu) = 0$:

$$
\begin{aligned}
L_1(\omega_{1,2},\mu) &= \begin{pmatrix} \mu_1 + L_{11}(\omega_{1,2},\mu) \\ \mu_2 + L_{12}(\omega_{1,2},\mu) \end{pmatrix} = 0 \\
L_k(\omega_{2,k},\mu) &= \begin{pmatrix} \mu_1 - L_{21}(\omega_{2,k},\mu) \\ \mu_2 - L_{22}(\omega_{2,k},\mu) \end{pmatrix} = 0.
\end{aligned} \tag{4.11}
$$

Solutions of the subsystem (4.11) are related to intersections of spirals which we discussed in section 4.1 as follows: we solve $L_1(\omega_{1,2},\mu) = 0$ for $\mu = \tilde{L}_1(\omega_{1,2})$ (by applying for sufficiently large $\omega_{1,2}$ the Banach fixed point theorem on $(\mu_1,\mu_2) = (L_{11},L_{12})(\omega_{1,2},\mu)$). In the same way $L_k = 0$ can be solved for $\mu = \tilde{L}_k(\omega_{2,k})$.

Both $\tilde{L}_1(\omega_{1,2})$ and $\tilde{L}_k(\omega_{2,k})$ are either a logarithmic spiral centred at (0,0) or a half-line terminating at (0,0), depending on the eigenvalues cases (CC) or (RC).

In the same way that we previously studied intersections of $\mathfrak{S}_1(\cdot, 0)$ and $\mathfrak{S}_2(\cdot, 0)$ (see section 4.1) we find a countable set of transversal intersections of $\tilde{L}_1(\omega_{1,2})$ and $\tilde{L}_k(\omega_{2,k})$ related to parameter values $(\hat{\omega}_{1,2}, \hat{\omega}_{2,k})_m$, $m \in \mathbb{Z}$. Then $(\hat{\omega}_{1,2}, \hat{\omega}_{2,k}, \hat{\mu})_m$, with $\hat{\mu}_m = \tilde{L}_1(\hat{\omega}_{1,2_m}) = \tilde{L}_k(\hat{\omega}_{2,k_m})$ are solutions of (4.11). Note that $(\hat{\omega}_{1,2}, \hat{\omega}_{2,k}, \hat{\mu})_m \to (\infty, \infty, 0)$ as $m \to \infty$.

Next we consider for sufficiently large m

$$L((\omega_{1,i+1}, \omega_{2,i}), \hat{\mu}_m) = 0, \quad i = 2, \ldots, k-1, \tag{4.12}$$

see (4.2) for the definition of L. As a consequence of Remark 4.8 we find solutions $(\hat{\omega}_{1,i+1}, \hat{\omega}_{2,i})_m$, $i = 2, \ldots, k-1$ of (4.12) such that $D_1 L((\hat{\omega}_{1,i+1}, \hat{\omega}_{2,i})_m, \hat{\mu}_m)$ is nonsingular.

Altogether we find for the truncated form

$$\mathbf{L}(\boldsymbol{\omega}, \mu) = (L_1(\omega_{1,2}, \mu), L(\omega_{1,3}, \omega_{2,2}, \mu), \ldots, L(\omega_{1,k}, \omega_{2,k-1}, \mu), L_k(\omega_{2,k}, \mu)),$$

of (4.10) that $\mathbf{L}(\hat{\boldsymbol{\omega}}_m, \hat{\mu}_m) = 0$ and that $D\mathbf{L}(\hat{\boldsymbol{\omega}}_m, \hat{\mu}_m)$ is invertible.

A similar proof to that of Lemma 4.1 may be used to show that we can solve the full equation (4.10) near $(\hat{\boldsymbol{\omega}}_m, \hat{\mu}_m)$ for $(\boldsymbol{\omega}_m, \mu_m)$. Note that $\mu_m \to 0$ as $m \to \infty$. As we have obtained a countable set of solutions with $\hat{\mu}$ accumulating to zero, this proves the existence of the parameter set $\mathcal{M}_k$ as stated in Lemma 1.3.

4.2.2 k-(1,2) heteroclinic orbits

We now search for k-(1,2) heteroclinic orbits, for $k \geq 2$. For that we consider the concatenation of a k-(1,2) heteroclinic orbit with Γ_1, which corresponds to a k-periodic Lin orbit with a k-periodic sequence $\boldsymbol{\omega}$ with $\omega_{1,1} = \omega_{2,k} = \infty$. Since we are only interested in solving for k-(1,2) heteroclinic orbits, we need not solve the equation $\Xi_k(\boldsymbol{\omega}, \mu) = \xi^\infty(\mu) = 0$, which is only related to Γ_1.

The bifurcation equations to be solved are then as follows:

$$\Xi_i(\boldsymbol{\omega}, \mu) = \begin{pmatrix} \mu_1 + L_{11}(\omega_{1,i+1}, \mu) - L_{21}(\omega_{2,i}, \mu) + r_{1,i}(\boldsymbol{\omega}, \mu) \\ \mu_2 + L_{12}(\omega_{1,i+1}, \mu) - L_{22}(\omega_{2,i}, \mu) + r_{2,i}(\boldsymbol{\omega}, \mu) \end{pmatrix} = 0 \qquad i = 1, \ldots, k-1. \tag{4.13}$$

Similar to the previous section, we begin by considering the truncated equations

$$L((\omega_{1,i+1}, \omega_{2,i}), 0) = 0, \qquad i = 1, \ldots, k-1.$$

As in section 4.1.2, the sequence $\hat{\boldsymbol{\omega}}$ solves these equations, with $(\hat{\omega}_{1,i+1}, \hat{\omega}_{2,i}) \in \Omega_{n_0}$. We may now use Lemma 4.1 to solve (4.13) to find $\boldsymbol{\omega} = \boldsymbol{\omega}_{\hat{\boldsymbol{\omega}}}(\mu)$ for $(\boldsymbol{\omega}, \mu)$ near $(\hat{\boldsymbol{\omega}}, 0)$.

Since there are countably many different $\boldsymbol{\omega}$ at which we can solve $\mathbf{L}(\boldsymbol{\omega}, 0) = 0$, this proves that at $\mu = 0$, there is a countable infinity of k-(1,2) heteroclinic orbits for each $k \geq 2$. The set of intersections of these orbits with Σ_1 accumulate to $q_1(0)$.

With an argument similar to that given in Remark 4.2 we find that for $\mu \neq 0$ the equations (4.13) have only finitely many solutions, and hence there are only finitely many k-(1,2) heteroclinic orbits.

4.2.3 k-homoclinic orbits

Here we consider homoclinic orbits to p_1, since the proof is similar for homoclinic orbits to p_2. In this case, such a homoclinic orbit corresponds to a k-periodic $\boldsymbol{\omega}$ sequence with $\omega_{1,1} = \infty$. The bifurcation equations then read as follows:

$$
\begin{aligned}
\Xi_i(\boldsymbol{\omega}, \mu) &= \begin{pmatrix} \mu_1 + L_{11}(\omega_{1,i+1}, \mu) - L_{21}(\omega_{2,i}, \mu) + r_{1,i}(\boldsymbol{\omega}, \mu) \\ \mu_2 + L_{12}(\omega_{1,i+1}, \mu) - L_{22}(\omega_{2,i}, \mu) + r_{2,i}(\boldsymbol{\omega}, \mu) \end{pmatrix} = 0 \\
& \qquad i = 1, \ldots, k-1 \\
\Xi_k(\boldsymbol{\omega}, \mu) &= \begin{pmatrix} \mu_1 - L_{21}(\omega_{2,k}, \mu) + r_{1,k}(\boldsymbol{\omega}, \mu) \\ \mu_2 - L_{22}(\omega_{2,k}, \mu) + r_{2,k}(\boldsymbol{\omega}, \mu) \end{pmatrix} = 0.
\end{aligned}
\tag{4.14}
$$

These equations may be solved in a similar way to those for k-(2,1) heteroclinic orbits, see section 4.2.1. The difference is that in solving the truncated form of the equation $\Xi_k(\boldsymbol{\omega}, \mu) = 0$, we obtain a one parameter family of (μ_1, μ_2)-values parameterised by $\omega_{2,k}$. If $\lambda_2^s(\mu)$ is complex, this is a logarithmic spiral centred on (0,0) in the (μ_1, μ_2)-plane; otherwise it is a line terminating at the origin in the (μ_1, μ_2)-plane. The solutions of (4.14) can also be parameterised by $\omega_{2,k}$: $(\omega_{1,2}, \ldots, \omega_{1,k}, \omega_{2,1}, \ldots, \omega_{2,k-1}, \mu)(\omega_{2,k})$. The curve $\mathcal{L}_k^{hom}$ as stated in Lemma 1.3 is given by $\mu = \mu(\omega_{2,k})$.

This completes the proof of Lemma 1.3. ∎

4.3 Proof of Theorem 1.4

We assume the eigenvalue Case (RR), where there are two real eigenvalues. We have that $L_{ij}(t, \mu) = e^{-2\rho_i(\mu)t} c_{ij}(\mu)$. Therefore the solutions of the truncated equation $L((\omega_1, \omega_2), \mu) = 0$ corresponds to intersections of the two lines $\mathcal{L}_1(\mu) = (\mu_1, \mu_2) + s_1(c_{11}(\mu), c_{12}(\mu))$ and $\mathcal{L}_2(\mu) = s_2(c_{21}(\mu), c_{22}(\mu))$, where $s_i > 0$ in each case, see Figure 3. In this case we make the following hypothesis:

(H 4) $\quad W^{cu}(p_2) \pitchfork_{q_1(0)} W^{cs}(p_1)$

It may be shown (see below) that this is a necessary and sufficient condition to guarantee that the vectors $(c_{11}(\mu), c_{12}(\mu))$ and $(c_{21}(\mu), c_{22}(\mu))$ are linearly independent. Now it is clear that there exists at most one solution $(\hat{\omega}_1(\mu), \hat{\omega}_2(\mu))$ to the equation $L((\omega_1, \omega_2), \mu) = 0$. Moreover, the set Q_L of parameter values for which $L((\omega_1, \omega_2), \mu) = 0$ has a solution is open, since by (H 4) each solution of $L((\omega_1, \omega_2), \mu) = 0$ corresponds to a transversal intersection of $\mathcal{L}_1(\mu)$ and $\mathcal{L}_2(\mu)$. The boundary ∂Q_L is the set of parameter values where the 'endpoint' of one of the lines lies on the other line, see also Figure 2.

We note that (H 4) in fact implies the earlier hypothesis (H 3). Now we outline the argument that the vectors $(c_{11}(\mu), c_{12}(\mu))$ and $(c_{21}(\mu), c_{22}(\mu))$ are linearly independent. Assuming (H 4), it may be shown that $\psi_1 = (T_{q_1(0)} W^{cu}(p_2) + T_{q_1(0)} W^s(p_1))^\perp$ and $\psi_2 = (T_{q_1(0)} W^{cs}(p_1) + T_{q_1(0)} W^u(p_2))^\perp$ form a basis of Z such that the corresponding vectors $(c_{11}(\mu), c_{12}(\mu))$ and

$(c_{21}(\mu), c_{22}(\mu))$ are linearly independent. Also any (nonsingular) linear transformation of this basis corresponds to a linear transformation of the vectors $\{(c_{i1}(\mu), c_{i2}(\mu))\}$, $i = 1, 2$. Similarly we show that where (H 4) does not hold, the vectors $\{(c_{i1}(\mu), c_{i2}(\mu))\}$, $i = 1, 2$ must be linearly dependent.

4.3.1 Periodic Orbits

The bifurcation equations for k-periodic orbits are as follows:

$$\Xi_i(\boldsymbol{\omega}, \mu) = \begin{pmatrix} \mu_1 + L_{11}(\omega_{1,i+1}, \mu) - L_{21}(\omega_{2,i}, \mu) + r_{1,i}(\boldsymbol{\omega}, \mu) \\ \mu_2 + L_{12}(\omega_{1,i+1}, \mu) - L_{22}(\omega_{2,i}, \mu) + r_{2,i}(\boldsymbol{\omega}, \mu) \end{pmatrix} = 0 \qquad i = 1, \dots, k. \tag{4.15}$$

We find easily that for $\mu \in Q_L$ the only solutions to the truncated equations $\mathbf{L}(\boldsymbol{\omega}, \mu) = 0$ are $(\omega_{1,i+1}(\mu), \omega_{2,i}(\mu)) = (\hat{\omega}_1(\mu), \hat{\omega}_2(\mu))$, $i = 1, \dots, k$.

We may then use Lemma 4.1 to solve the full bifurcation equations $\Xi_i(\boldsymbol{\omega}, \mu) = 0$ to find $\boldsymbol{\omega}_{\hat{\boldsymbol{\omega}}}(\mu)$ for $\mu \in Q$, where ∂Q and ∂Q_L are tangent at the origin. Due to the uniqueness statement of Lemma 4.1 we find $\boldsymbol{\omega}_{\hat{\boldsymbol{\omega}}}(\mu)$ with $(\omega_{1,i+1}(\mu), \omega_{2,i}(\mu)) =: (\omega_1(\mu), \omega_2(\mu))$, $i = 1, \dots, k$. Therefore, due to Theorem 2.2, the corresponding orbits are 1-periodic orbits for each $\mu \in Q$, and there are no k-periodic orbits for $k \geq 2$.

The period of these orbits is $2(\omega_1(\mu) + \omega_2(\mu))$. If $\mu \to \partial Q$ then $\hat{\omega}_1(\mu)$ or $\hat{\omega}_2(\mu)$ tends to infinity and hence $\omega_1(\mu)$ or $\omega_2(\mu)$ tends to infinity.

4.3.2 k-(2,1) heteroclinic orbits

The bifurcation equation for k-(2,1) heteroclinic orbits is given by equation (4.10). We will show that the subsystem $\Xi_1 = 0$, $\Xi_k = 0$ has no solution: assume that there is a solution $\boldsymbol{\omega}$, μ of these equations. Then this is also a solution of $\Xi_1 - \Xi_k = 0$, that means

$$\begin{pmatrix} L_{11} \\ L_{12} \end{pmatrix} (\omega_{1,2}, \mu) + \begin{pmatrix} L_{21} \\ L_{22} \end{pmatrix} (\omega_{2,k}, \mu) + \begin{pmatrix} r_{1,1} \\ r_{2,1} \end{pmatrix} (\boldsymbol{\omega}, \mu) - \begin{pmatrix} r_{1,k} \\ r_{2,k} \end{pmatrix} (\boldsymbol{\omega}, \mu) = 0. \tag{4.16}$$

Because of the linear independence of $(c_{11}(\mu), c_{12}(\mu))$ and $(c_{21}(\mu), c_{22}(\mu))$ the truncated form of (4.16) is different from zero. And therefore, due to the estimates given in Lemma 3.2 this equation has no solutions for sufficiently large $\omega_{1,2}$, $\omega_{2,k}$. Hence there are no k-(2,1) heteroclinic orbits for any μ.

4.3.3 k-(1,2) heteroclinic orbits

Recall from section 4.2.2 that the bifurcation equation for k-(1,2) heteroclinic orbits is given by equation (4.13).

For $\mu = 0$ the argument used in in Section 4.3.2 can be applied to each single equation $\Xi_i = 0$ to show that there are no solutions.

For $\mu \in \tilde{Q}$ the equations $\Xi_i(\boldsymbol{\omega}, \mu) = 0$, $i = 1, \ldots, k-1$ may be solved in the same way as in section 4.3.1 to give a unique k-(1,2) heteroclinic orbit for each $\mu \in \tilde{Q}$, $k \geq 2$.

4.3.4 Homoclinic Orbits

The bifurcation equation for homoclinic orbits to the fixed point p_1 is given by equation (4.14). Here, the truncated form of $\Xi_k(\boldsymbol{\omega}, \mu) = 0$ may be solved to give $\hat{\omega}_{2,k}(\mu)$, for some $\mu \in \partial Q_L$. For $k = 1$, we have $\Xi_i(\boldsymbol{\omega}, \mu) = \Xi_k(\boldsymbol{\omega}, \mu)$ for all i, and similar to section 4.2.3, we solve the full bifurcation equation to find a unique 1-homoclinic solution to p_1. Those homoclinic solutions correspond to parameter values μ which are located on ∂Q. Similarly we may find a unique 1-homoclinic solution to p_2.

This completes the proof of Theorem 1.4. ∎

5 Reversible systems

In this section we consider the system (1.1) in $\mathbb{R}^{2n+1}$. We suppose that the family (1.1) is reversible with respect to a linear involution R, thus

$$R\, f(x, \mu) = -f(Rx, \mu),$$

and we assume that the fixed point space of R is n-dimensional, dim Fix $R = n$. Further the fixed points p_1 and p_2 are non-symmetric but lie in the same group orbit, that is $R(p_1) = p_2$. As well we assume that the heteroclinic cycle Γ is symmetric, that means

$$R\Gamma = \Gamma.$$

Altogether an orbit $\mathcal{O}$ of a reversible vector field is called *symmetric* if $R(\mathcal{O}) = \mathcal{O}$. The symmetry of Γ implies that both Γ_1 and Γ_2 are symmetric heteroclinic orbits. Besides these conditions due to the symmetry we want, on the whole, to make the same assumptions as in the general situation. However, the reversibility which we imposed on the system enforces several particular properties and statements.

First of all we want to mention that such a symmetric heteroclinic cycle is a codimension-one configuration; so in the present case it is sufficient to consider a one-parameter family of vector fields, that means $\mu \in \mathbb{R}$ and $f : \mathbb{R}^{2n+1} \times \mathbb{R} \to \mathbb{R}^{2n+1}$.

The involution R maps the stable manifold of p_i onto the unstable manifold of Rp_i. We assume that the fixed point p_1 has an n-dimensional stable manifold, and an (n+1)-dimensional unstable (focus) manifold, and then due to the symmetry *vice versa* for p_2. This makes it possible to adopt the transversality assumption (1.2); then, by our assumptions, the second transversality assumption (1.3) is automatically fulfilled.

Due to the reversibility, the following relation for the principal stable and unstable eigenvalues holds true

$$-\lambda_2^s = \lambda_1^u =: \lambda^u \qquad -\lambda_2^u = \lambda_1^s =: \lambda^s.$$

We then only have two eigenvalue cases to consider: the complex-complex case (CC), and the real-real case (RR). So $-\lambda^u$ (also $-\overline{\lambda^u}$ in case (CC)), and $-\lambda^s$ are the principal stable and unstable eigenvalues respectively of $D_1 f(p_2, 0)$. In particular for the real and imaginary parts of the leading complex eigenvalues in case (CC), that means

$$\rho_1 = \rho_2 =: \rho, \quad \phi_1 = \phi_2 =: \phi.$$

We adopt the Hypotheses (H 2) and (H 3) as they are. We only want to remark that in the present case there is some redundancy in their formulation – the second assumption follows in each case from the first one. We also keep Hypothesis (H 1). But in the case under consideration it turns out that $\xi^\infty(\cdot)$ can be seen as a mapping $\mathbb{R} \to \mathbb{R}$, see Lemma 5.4.

In the discussion of general systems transversal intersections of certain spirals play a key role. In order to ensure transversal intersections of the corresponding spirals in case (CC) of the reversible context we assume:

(H 5) $\quad \frac{\phi(0)}{\rho(0)} \neq \tan(\varphi_2 - \varphi_1)$

The main results regarding reversible systems are summarised in the following Theorem, which generalises results of [18] to higher dimensions.

Theorem 5.1. *Consider a reversible system as described above. Assume the eigenvalue case (CC). Then*

1. *At $\mu = 0$, there exists for each $N \geq 2$ a set $\mathcal{S}_0^N \subset \Sigma_1$ which is invariant under the first-return-map Π (defined by the flow), and $(\mathcal{S}_0^N, \Pi)$ is topologically conjugated to the full shift on N symbols. Each set $\mathcal{S}_0^N$ is setwise symmetric, that is $R(\mathcal{S}_0^N) = \mathcal{S}_0^N$. The collection of all these sets exists only for $\mu = 0$, although each set is individually structurally stable meaning that for μ close to 0 there is a set $\mathcal{S}_\mu^N$ which converges to $\mathcal{S}_0^N$ in the Hausdorff-metric as $\mu \to 0$, and $(\mathcal{S}_\mu^N, \Pi(\mu))$ is topologically conjugated to a full shift on N symbols.*

2. *At $\mu = 0$, there is a countably infinite set of symmetric 1-periodic orbits, with difference in period asymptotically tending to $\pi/2\phi(\mu)$. For $\mu \neq 0$ there are finitely many symmetric 1-periodic orbits. With the addition of the parameter to the phase space these periodic orbits form a 1 parameter family parameterised by period, see Figure 6. The periodic orbits converge to the heteroclinic cycle as the period tends to infinity.*

3. *For each $k \geq 2$, $k \in \mathbb{N}$ there is a countable set $\mathcal{M}_k$ of parameter values, for which there exists a symmetric k-(2,1)-heteroclinic orbit. In particular there are sequences $(\mu_n^+(k))_{n\in\mathbb{N}}, (\mu_n^-(k))_{n\in\mathbb{N}} \subset \mathcal{M}_k$ with $\mu_n^+(k) \searrow 0$, $\mu_n^-(k) \nearrow 0$ as $n \to \infty$. There are no non-symmetric k-(2,1)-heteroclinic orbits.*

4. *At $\mu = 0$ there exist countably many k-(1,2)-heteroclinic orbits for each $k \geq 2$. Moreover, for fixed k, $q_1(0)$ is an accumulation point of the intersections of the k-(1,2)-heteroclinic orbits with Σ_1. Each such k-(1,2)-heteroclinic orbit can be continued for $\mu \neq 0$. But for fixed $\mu \neq 0$ and fixed k there are only finitely many k-(1,2)-heteroclinic orbits.*

5. *For each $k \geq 2$, $k \in \mathbb{N}$ there is a countable set $\mathcal{M}_k^{hom}$ of parameter values, for which there exists a k-homoclinic orbit. In particular there are sequences $(\mu_n^+(k))_{n\in\mathbb{N}}, (\mu_n^-(k))_{n\in\mathbb{N}} \subset \mathcal{M}_k^{hom}$ with $\mu_n^+(k) \searrow 0$, $\mu_n^-(k) \nearrow 0$ as $n \to \infty$.*

This theorem is proved in section 4.3. We also state the following theorem for the simpler case (RR) of real eigenvalues. We omit the proof as it is similar to the proof of Theorem 1.4.

Theorem 5.2. *Consider a reversible system as described above. Assume the eigenvalue case (RR). Then we have the following:*

1. *There exists an open set Q in parameter space, which is defined by either $\mu > 0$ or $\mu < 0$ (depending on the signs of the constants $c_{11}(\mu), c_{12}(\mu)$, see (5.2)), for which there exists a 1-periodic orbit at each point $\mu \in Q$. The period of these periodic orbits tend to infinity as $\mu \to 0$. There are no k-periodic orbits for $k \geq 2$.*
2. *There are no k-(2,1)-heteroclinic orbits for each $k \geq 2$, $k \in \mathbb{N}$, for any parameter value μ.*
3. *At $\mu = 0$ there are no k-(1,2)-heteroclinic orbits for $k \geq 2$. But for each $\mu \in Q$ there exists a k-(1,2)-heteroclinic orbit for $k \geq 2$.*
4. *There are no homoclinic orbits for all μ.*

Hereafter we assume the complex-complex eigenvalue case (CC). Before starting our analysis we want to remark on the discovered dynamics. The symmetry property of the sets $\mathcal{S}_0^N$ makes sense if we introduce cross-sections Σ_j of Γ_j, $j = 1, 2$, containing $\operatorname{Fix} R$. The existence of shift dynamics can be explained as in the general case. Here we have the additional feature that the corresponding 1-periodic orbits are symmetric. However, the sets $\mathcal{S}_0^N$ contain also nonsymmetric orbits.

Figure 6 depicts the solutions of the bifurcation equation for 1-periodic orbits, see Section 5.3.2. The oscillatory behaviour exhibited in Figure 6 is characteristic of the Shilnikov homoclinic

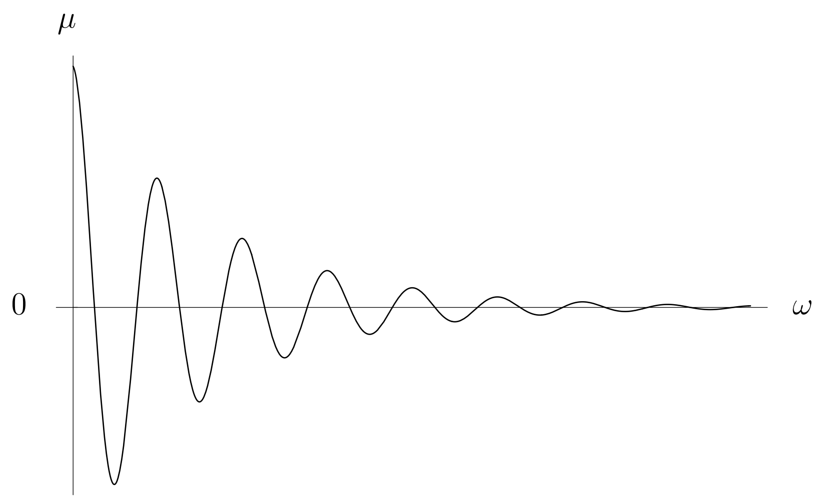

Figure 6: Bifurcation diagram for 1-periodic orbits. Here 2ω is the period of the corresponding 1-periodic orbits.

orbit to a saddle-focus under certain eigenvalue conditions (the non-real eigenvalues are closer to the imaginary axis than the real one), where there exists chaotic dynamics. This represents a marked difference in our analysis: we do not require any eigenvalue condition to have this oscillatory behaviour, which is due to the fact that only the complex eigenvalues of p_1 and p_2 appear in the bifurcation equations.

Note that the cylinder orbit theorem that states that symmetric periodic orbits generically come in 1-parameter families, see Devaney [6] or Vanderbauwhede [27], does not work in the present context – $\dim \mathrm{Fix}\, R \neq \dim \mathrm{Fix}\,(-R)$. So there may exist isolated symmetric periodic orbits as stated in the theorem.

We also note that the bifurcation equation for 1-periodic orbits may be solved to find non-symmetric 1-periodic orbits. Those do not exist at $\mu = 0$. They are associated to the families of homoclinic orbits to each fixed point, see point 5 of Theorem 5.1 and Section 5.3.5. Of course the homoclinic orbits are nonsymmetric – the R-image of a homoclinic orbit to p_i is a homoclinic orbit to Rp_i.

We remark that our Hypothesis (H 1), see Section 3, which prescribes that the stable manifold of p_1 and the unstable manifold of p_2 split with positive speed, implies that there is just one 1-(2,1)-heteroclinic orbit, and this is the original one that exists at $\mu = 0$. In $\mathbb{R}^3$, $n = 1$, the statement on k-(2,1)-heteroclinic orbits is partly obvious. It is clear that there can only be at most one k-(2,1)-heteroclinic orbit for each parameter value, because in this case the stable manifold of p_1 is one-dimensional. Consequently such a connection must be symmetric, because the R-image is also a k-(2,1)-heteroclinic orbit.

5.1 Special features of Lin's method for reversible systems

First we want to remark that there are R-invariant C^∞ bump function transformations that make $p_1(\mu), p_2(\mu)$ constant, and that there are R-invariant C^∞ bump function transformations, with disjoint supports, based around each of the fixed points p_i, that bring the local stable/unstable manifolds into the form (2.4).

In order to establish the direct sum decomposition (2.3) we use here an R-invariant inner product $\langle \cdot, \cdot \rangle$. Such an inner product exists because $\{R, id\}$ forms a finite group. Further fix the heteroclinic solutions so that $q_1(0), q_2(0) \in \mathrm{Fix}\, R$. The reversibility of the vector field f implies that $f(x) \in \mathrm{Fix}\,(-R)$ for $x \in \mathrm{Fix}\, R$. Therefore Y_i and hence also Σ_i, $i = 1, 2$, are R-invariant, see also (2.2). Moreover, Y_i, $i = 1, 2$, contain $\mathrm{Fix}\, R$ and an n-dimensional subspace of $\mathrm{Fix}\,(-R)$.

Lemma 5.3. *The space Z has a direct sum decomposition into one-dimensional subspaces of* $\mathrm{Fix}\, R$ *and* $\mathrm{Fix}\,(-R)$*.*

Proof. The vector field direction $f(q_1(0), 0)$ belongs to $\mathrm{Fix}\,(-R)$. The spaces W_1^+ and W_1^- are R-images of each other. So $W_1^+ \oplus W_1^-$ has a direct sum decomposition into $(n-1)$-dimensional subspaces of $\mathrm{Fix}\, R$ and $\mathrm{Fix}\,(-R)$, mind $\dim W_1^+ = \dim W_1^- = n-1$. Since Z is R-invariant it has a direct sum decomposition into subspaces of $\mathrm{Fix}\, R$ and $\mathrm{Fix}\,(-R)$. Now, counting dimensions gives the lemma. ∎

Next we turn to the splitting of the stable and unstable manifolds which we considered in Section 2.1 for the general case. In addition to the discussion in Section 2.1, we observe that also $Rq_{2,\mu} \in \Sigma_2 \cap W^u(p_1,\mu) \cap W^s(p_2,\mu)$. So, due to uniqueness we have $Rq_{2,\mu} = q_{2,\mu}$, hence $q_{2,\mu} \in \operatorname{Fix} R$; the 1-(1,2)-heteroclinic orbit through $q_{2,\mu}$ is symmetric.

Further Lemma 2.1 remains true in the present case, with the following "reversible supplementation".

Lemma 5.4. *Due to the reversibility, it holds that*

$$q_1^+(\mu)(0) - q_1^-(\mu)(0) \in \operatorname{Fix}(-R).$$

Proof. Also the pair $(Rq_1^-(\mu)(\cdot), Rq_1^+(\mu)(\cdot))$ fulfils (i)-(iii) of Lemma 2.1. Therefore the uniqueness part of Lemma 2.1 provides $Rq_1^-(\mu)(0) = q_1^+(\mu)(0)$. ∎

In order to detect symmetric (periodic) orbits we use symmetric Lin orbits to construct the bifurcation equations. We define equality and periodicity of Lin orbits.

Definition 5.5. *1. Two Lin orbits $X(\boldsymbol{\omega}) = (X_i)_{i\in\mathbb{Z}}$ and $\hat{X}(\hat{\boldsymbol{\omega}}) = (\hat{X}_i)_{i\in\mathbb{Z}}$ are equal, $X(\boldsymbol{\omega}) = \hat{X}(\hat{\boldsymbol{\omega}})$, if there is an $i_0 \in \mathbb{Z}$ such that for all $i \in \mathbb{Z}$ it holds $X_i = \hat{X}_{i+i_0}$.*
2. A Lin orbit $X(\boldsymbol{\omega}) = (X_i)_{i\in\mathbb{Z}}$ is k-periodic, $k \in \mathbb{N}$, if for all $i \in \mathbb{Z}$ it holds $X_i = X_{i+k}$.

If $X(\boldsymbol{\omega}) = \hat{X}(\hat{\boldsymbol{\omega}})$ then

(i) $\bigcup_{i\in\mathbb{Z}} X_i = \bigcup_{i\in\mathbb{Z}} \hat{X}_i$;
(ii) for all $i \in \mathbb{Z}$ and $j \in \{1,2\}$ it holds $\omega_{j,i} = \hat{\omega}_{j,i+i_0}$; this condition is also sufficient for the equality of X and $\hat{X}$;
(iii) for all $i \in \mathbb{Z}$ it holds $\Xi_i = \hat{\Xi}_{i+i_0}$.

If $X(\boldsymbol{\omega})$ is an k-periodic Lin orbit then for all $i \in \mathbb{Z}$ we have $\Xi_i = \Xi_{i+k}$.

Now fix μ. Let $X := X(\boldsymbol{\omega},\mu) = (X_i(\boldsymbol{\omega},\mu))_{i\in\mathbb{Z}}$ be a Lin orbit associated to a given sequence $\boldsymbol{\omega} = (\omega_{1,i},\omega_{2,i})_{i\in\mathbb{Z}}$.

Lemma 5.6. *The R-image RX of X is a Lin orbit $\hat{X} := \hat{X}(\hat{\boldsymbol{\omega}},\mu) = (\hat{X}_i(\hat{\boldsymbol{\omega}},\mu))_{i\in\mathbb{Z}}$ associated to the sequence $\hat{\boldsymbol{\omega}} = (\hat{\omega}_{1,i},\hat{\omega}_{2,i})_{i\in\mathbb{Z}}$, with $\hat{\omega}_{j,i} = \omega_{j+1,-i}$.*

Proof. Let $x_{j,i}^{\pm}(\cdot)$ be the solutions of (1.1) which we introduced at the beginning of Section 2.2. With that we define solutions $\hat{x}_{j,i}^{\pm}(\cdot)$ of (1.1) by

$$\hat{x}_{j,i}^{+}(t) := Rx_{j,-i}^{-}(-t), \quad \hat{x}_{j,i}^{-}(t) := Rx_{j,-i}^{+}(-t).$$

By construction $\hat{x}_{1,i}^{+}$, $\hat{x}_{2,i}^{+}$, $\hat{x}_{1,i}^{-}$ and $\hat{x}_{2,i}^{-}$ are defined on $[0,\omega_{2,-i}]$, $[0,\omega_{1,-i}]$, $[-\omega_{1,-i},0]$ and $[-\omega_{2,-i},0]$, respectively. With

$$\hat{\omega}_{j,i} := \omega_{j+1,-i}$$

the functions $x_{j,i}^{\pm}(\cdot)$ satisfy the coupling conditions (2.9) and also the jump conditions (2.10). This shows that $\hat{X} = (\hat{X}_i)$ with $\hat{X}_i = RX_{-i}$ is indeed a Lin orbit associated to $\hat{\boldsymbol{\omega}}$; by construction $RX = \hat{X}$. ∎

Corollary 5.7. *Let X be a Lin orbit and let $(\Xi_i)_{i\in\mathbb{Z}}$ be the corresponding sequence of jumps (see (2.10)). For the sequence $(\hat{\Xi}_i)_{i\in\mathbb{Z}}$ associated to RX it holds that $R\Xi_i = -\hat{\Xi}_i$.* ■

Definition 5.8. *A Lin orbit $X(\boldsymbol{\omega})$ is symmetric if $X = RX$.*

Lemma 5.9. *A Lin orbit $X(\boldsymbol{\omega})$ is symmetric if and only if there is $i_0 \in \mathbb{Z}$ such that for all $i \in \mathbb{Z}$ and $j \in \{1,2\}$ it holds that $\omega_{j+1,-i} = \omega_{j,i-i_0}$.* ■

From Corollary 5.7 and Lemma 5.9 we get

Corollary 5.10. *Let $X(\boldsymbol{\omega})$ be a symmetric Lin orbit, and let $i_0 \in \mathbb{Z}$ be such $\omega_{j+1,-i} = \omega_{j,i-i_0}$, for all $i \in \mathbb{Z}$, $j \in \{1,2\}$. Then $R\,\Xi_i = -\Xi_{-i-(i_0+1)}$.* ■

5.2 The bifurcation equation

Lemma 5.4 says that $\xi^\infty(\mu) \in Z \cap \mathrm{Fix}\,(-R)$, and according to Lemma 5.3 this space is one-dimensional. So, by introducing appropriate coordinates, $\xi^\infty(\cdot)$ can be seen as a mapping

$$\xi^\infty(\cdot) : \mathbb{R} \to \mathbb{R}.$$

Hypothesis (H 1) says that the Jacobian $D\xi^\infty(0)$ is non-singular. Because the image of ξ^∞ is one-dimensional, a one-dimensional preimage space of ξ^∞ suffices to guarantee the non-singularity of $D\xi^\infty(0)$. This justifies our setting $\mu \in \mathbb{R}$.

Remark 5.11. Hypothesis (H 1) can be made more explicit by using a representation of ξ^∞ by means of a Melnikov-type integral.

Let ψ_1, ψ_2 be an orthonormal basis for Z, such that $Z \cap \mathrm{Fix}\ R = \mathrm{span}\{\psi_1\}$ and $Z \cap \mathrm{Fix}\ (-R) = \mathrm{span}\{\psi_2\}$. Set

$$\psi_j(t) := \Psi(t,0)\psi_j \qquad \forall t \in \mathbb{R},\ j = 1,2$$

where $\Psi(\cdot,\cdot)$ is the transition matrix of the adjoint (with respect to $\langle\cdot,\cdot\rangle$) of (2.8).

Representations (3.5) of ξ^∞ and (2.5) of $q_1^\pm$ give

$$\xi^\infty(\mu) = v^+(\mu)(0) - v^-(\mu)(0).$$

On the other hand we have, recall that $\xi^\infty \in \mathrm{Fix}\,(-R)$,

$$\xi^\infty(\mu) = \langle \psi_2, \xi^\infty(\mu)\rangle \psi_2.$$

Combining these two representations of ξ^∞, using further representations of $v^+(\mu)(0)$ or $v^-(\mu)(0)$ and exploiting the fact that that the involved direct sum decompositions are orthogonal with respect to the scalar product $\langle\cdot,\cdot\rangle$ we obtain

$$\xi^\infty(\mu) = -\psi_2 \int_{-\infty}^{\infty} \langle \psi_2(s), g_1(s, v(\mu)(s), \mu)\rangle ds,$$

where $v(\mu)(t) := v^-(\mu)(t)$, $t \le 0$, and $v(\mu)(t) := v^+(\mu)(t)$, $t > 0$. From this we find, see also (2.7),

$$D\xi^\infty(0) = -\psi_2 \int_{-\infty}^{\infty} \langle \psi_2(s), D_2 f(q_1(s), 0)\rangle ds.$$

□

Throughout let $\{\psi_1, \psi_2\}$ be the basis for Z which has been introduced in Remark 5.11.

Lemma 5.12. $S_1(t, \psi_1, \mu) = S_2(t, \psi_1, \mu), \quad S_1(t, \psi_2, \mu) = -S_2(t, \psi_2, \mu).$

Proof. The scalar product $\langle \cdot, \cdot \rangle$ is R-invariant. Therefore

$$S_1(t, \psi_j, \mu) := \langle R\, \Psi_1^+(\mu, t, 0) Q_1^{+*}(\mu, 0)\psi_j, R\, \tilde{Q}_1(\mu, t) q_2^-(\mu)(-t)\rangle.$$

Now the lemma follows immediately from the following symmetry properties:

(i) $R\, \Psi_1^+(\mu, t, 0) = \Psi_1^-(\mu, -t, 0)\, R$;
(ii) $R\, Q_1^{+*}(\mu, 0) = P_1^{-*}(\mu, 0)\, R$;
(iii) $R\psi_1 = \psi_1, \quad R\psi_2 = -\psi_2$;
(iv) $R\, \tilde{Q}_1(\mu, t) = (id - \tilde{Q}_{2,}(\mu, t))\, R$;
(v) $R\, q_2^-(\mu)(-t) = q_2^+(\mu)(t), t \in \mathbb{R}^+$.

Item (i) follows from the reversibility of (1.1) and $Rq_1^+(\mu)(0) = q_1^-(\mu)(0)$, see Lemma 2.1 and its proof.

The relation $Rq_1^+(\mu)(0) = q_1^-(\mu)(0)$ also implies $T_{q_1^+(\mu)(0)} W^s(p_1) = T_{q_1^-(\mu)(0)} W^u(p_2)$. This again implies

$$RQ_1^+(\mu, 0) = P_1^-(\mu, 0)R. \tag{5.1}$$

Since R is self-adjoint with respect to the R-invariant scalar product $\langle \cdot, \cdot \rangle$, so that $R = R^*$, item (ii) holds true.

The symmetry properties of ψ_1 and ψ_2 which are stated in (iii) follow immediately from their definition.

From (5.1) we can conclude $RQ_1^+(\mu, \omega) = P_1^-(\mu, -\omega)R$. In the same way we find $RQ_2^+(\mu, \omega) = P_2^-(\mu, -\omega)R$. For the latter equality we used that $q_2(\mu)(0) \in \operatorname{Fix} R$, see at the beginning of Section 2.1. So $R\left(\operatorname{im} Q_j^+(\omega)\right) = \operatorname{im} P_{j+1}^-(-\omega)$, $j = 1, 2$. This provides item (iv).

Finally item (v) is a simple consequence of $q_2(\mu)(0) \in \operatorname{Fix} R$. ∎

As a consequence of the previous lemma we get

$$\varphi_{11} = \varphi_{21} =: \varphi_1, \quad \varphi_{12} = \varphi_{22} =: \varphi_2,$$

From Lemma 5.12 we see that the leading terms of S_2 read

$$\begin{aligned} L_{21}(t, \mu) &= e^{-2\rho(\mu)t} c_{11}(\mu) \sin(2\phi(\mu)t + \varphi_1) = L_{11}(t, \mu), \\ L_{22}(t, \mu) &= -e^{-2\rho(\mu)t} c_{12}(\mu) \sin(2\phi(\mu)t + \varphi_2) = -L_{12}(t, \mu). \end{aligned}$$

Altogether our considerations show that the bifurcation equation for orbits staying for all time close to the primary cycle reads:

$$\Xi_i(\boldsymbol{\omega}, \mu) = \begin{pmatrix} L_{11}(\omega_{1,i+1}, \mu) - L_{11}(\omega_{2,i}, \mu) + r_{1,i}(\boldsymbol{\omega}, \mu) \\ \mu + L_{12}(\omega_{1,i+1}, \mu) + L_{12}(\omega_{2,i}, \mu) + r_{2,i}(\boldsymbol{\omega}, \mu) \end{pmatrix} = 0, \quad i \in \mathbb{Z}. \tag{5.2}$$

5.3 Proof of Theorem 5.1

In order to prove Theorem 5.1 we proceed in principle as in Section 4.1. But we have to take the particular features of reversible systems into account.

Our discussion in Section 4.1 was mainly based on the dicussion of the intersections of certain spirals. In particular we used that the spirals under consideration have infinitely many transversal intersections if the ratio of the real parts of the leading complex eigenvalues of the fixed points p_1 and p_2 is irrational, see Lemma 4.5. Here, due to the reversibility, this ratio is one. But in the current case we are able to state the necessary transversal intersections of the corresponding spirals explicitly. The remainder indeed runs parallel to our considerations in Section 4.1 – we have only to take care about additional symmetry statements.

As in Section 4.1 we introduce the truncated bifurcation equation:

$$L((\omega_{1,i+1}, \omega_{2,i}), \mu) := \begin{pmatrix} L_{11}(\omega_{1,i+1}, \mu) - L_{11}(\omega_{2,i}, \mu) \\ \mu + L_{12}(\omega_{1,i+1}, \mu) + L_{12}(\omega_{2,i}, \mu) \end{pmatrix} = 0. \tag{5.3}$$

Again we denote $\mathbf{L}(\boldsymbol{\omega}, \mu) := (L((\omega_{1,i+1}, \omega_{2,i}), \mu))_{i\in\mathbb{Z}}$. Then $\mathbf{L} : (l^\infty \times l^\infty) \times \mathbb{R} \to (l^\infty \times l^\infty)$.

We note that Lemma 4.1 remains valid in the symmetric case, and so our strategy for solving the bifurcation equations remains the same as in Section 4.

5.3.1 Symbolic Dynamics

As in Section 4.1.2 we first consider $\mathbf{L}(\boldsymbol{\omega}, 0) = 0$. Let

$$\Omega_{n_0} := \{(\omega, \omega) : \omega = (n\pi - \varphi_2)/2\phi(0),\, n \geq n_0\}. \tag{5.4}$$

Again $\Omega_{n_0,N}$ is a subset of Ω_{n_0} with N elements and

$$\boldsymbol{\Omega}^{\mathbb{Z}}_{n_0,N} := \{\boldsymbol{\omega} : (\omega_{1,i+1}, \omega_{2,i}) \in \Omega_{n_0,N}\}.$$

The set Ω_{n_0} corresponds to transversal intersections of the spirals $(L_{11}(\omega_{1,i+1}, 0), L_{12}(\omega_{1,i+1}, 0))$ and $(-L_{11}(\omega_{2,i}, 0), L_{12}(\omega_{2,i}, 0))$. Indeed, for $\omega_{1,i+1} = \omega_{2,i}$ we have $L_{11}(\omega_{1,i+1}, 0) - L_{11}(\omega_{2,i}, 0) = 0$, and it holds that $L_{12}((n\pi - \varphi_2)/2\phi(0), 0) = 0$. Hypothesis (H 5) implies that $\rho(0)\sin(n\pi - \varphi_2 + \varphi_1) + \phi(0)\cos(n\pi - \varphi_2 + \varphi_1) \neq 0$, and this ensures that $D_1\mathbf{L}(\hat{\boldsymbol{\omega}}, 0)$ is invertible, see also Lemma 3.3. So we can proceed as in Section 4.1.2. This includes the construction of the sets $\mathcal{S}^N_\mu$ and the first return map Π_μ as well as the topological conjugacy of $(\mathcal{S}^N_\mu, \Pi_\mu)$ to the full shift on N symbols.

It remains to show that $\mathcal{S}^N_\mu$ is setwise symmetric, that is $R\,\mathcal{S}^N_\mu = \mathcal{S}^N_\mu$. It can easily be checked that

$$\mathcal{R} : \boldsymbol{\Omega}^{\mathbb{Z}}_{n_0,N} \to \boldsymbol{\Omega}^{\mathbb{Z}}_{n_0,N}, \quad \boldsymbol{\omega} \mapsto \boldsymbol{\tau}, \quad \tau_{j,i} = \omega_{j+1,-i}$$

is an involution on $\boldsymbol{\Omega}^{\mathbb{Z}}_{n_0,N}$, see also Lemma 5.6. Again from the uniqueness part of Theorem 2.2 and from the construction of $\boldsymbol{\omega}_{\hat{\boldsymbol{\omega}}}(\mu)$ it follows that

$$R\,x_{\hat{\boldsymbol{\omega}}}(\mu)(0) = x_{\mathcal{R}\hat{\boldsymbol{\omega}}}(\mu)(-2(\omega_{1,0} + \omega_{2,0})) = x_{\zeta^{-1}(\mathcal{R}\hat{\boldsymbol{\omega}})}(\mu)(0).$$

This eventually proves the first statement of the theorem.

We would like to remark that the addressed spirals (which can be regarded as spirals in Z) are, for $\mu = 0$, R-images of each other, and the above introduced set Ω_{n_0} corresponds to intersections of these spirals in $\operatorname{Fix} R$; the first components (corresponding to $\psi_1 \in \operatorname{Fix} R$) coincide, $L_{11}(\omega_{1,i+1}, 0) = L_{11}(\omega_{2,i}, 0)$, and the second one vanishes in each case, $L_{12}((n\pi - \varphi_2)/2\phi(0), 0) = 0$.

5.3.2 1-periodic orbits

Let $X(\boldsymbol{\omega})$ be a symmetric 1-periodic Lin orbit. In accordance with our results concerning symmetric Lin orbits we find

Corollary 5.13. *1. A 1-periodic Lin orbit $X(\boldsymbol{\omega})$ is symmetric if and only if for all $i \in \mathbb{Z}$ it holds that $\omega_{1,i} = \omega_{2,i} := \omega$.*

2. For the jumps of a symmetric 1-periodic Lin orbit it holds that $\Xi_i \equiv \Xi$, $i \in \mathbb{Z}$, and moreover $\Xi \in \operatorname{Fix}(-R)$. ■

Then the bifurcation equation for symmetric 1-periodic orbits reduces down to a 1-dimensional equation:

$$\mu + 2c_{12}(\mu)e^{-2\rho(\mu)\omega}\sin(2\phi(\mu)\omega + \varphi^2) + o(e^{-2\rho(\mu)\omega}) = 0. \tag{5.5}$$

Equation (5.5) has at $\mu = 0$ a countably infinite set of solutions. Therefore we have at $\mu = 0$ a countably infinite set of symmetric 1-periodic orbits, with difference in period asymptotically tending to $\pi/2\phi(\mu)$. Using the Implicit Function Theorem each of these solutions can be continued for $\mu \neq 0$. However, for fixed $\mu \neq 0$ there are only finitely many solutions of (5.5) and hence only finitely many 1-periodic orbits. With the addition of the parameter to the phase space these periodic orbits form a 1 parameter family parameterised by period, see Figure 6.

We note that one can also find non-symmetric 1-periodic orbits. These are associated to sequences $\boldsymbol{\omega}$ with $\omega_{1,i} \equiv \omega_1$, $\omega_{2,i} \equiv \omega_2$, $i \in \mathbb{Z}$, and $\omega_1 \neq \omega_2$. The corresponding bifurcation equation is two-dimensional (the $\operatorname{Fix} R$-component does not vanish as in the above situation). Such 1-periodic orbits are associated to the families of homoclinic orbits to each fixed point.

5.3.3 k-(2,1)-heteroclinic orbits

In this section we search for k-(2,1)-heteroclinic orbits for $k \geq 2$. Hence the bifurcation equation for these orbits reads, see also Section 4.2.1, Equation (4.10):

$$
\begin{aligned}
\Xi_1(\boldsymbol{\omega}, \mu) &= \begin{pmatrix} L_{11}(\omega_{1,2}, \mu) + r_{1,1}(\boldsymbol{\omega}, \mu) \\ \mu + L_{12}(\omega_{1,2}, \mu) + r_{2,1}(\boldsymbol{\omega}, \mu) \end{pmatrix} = 0 \\
\Xi_i(\boldsymbol{\omega}, \mu) &= \begin{pmatrix} L_{11}(\omega_{1,i+1}, \mu) - L_{11}(\omega_{2,i}, \mu) + r_{1,i}(\boldsymbol{\omega}, \mu) \\ \mu + L_{12}(\omega_{1,i+1}, \mu) + L_{12}(\omega_{2,i}, \mu) + r_{2,i}(\boldsymbol{\omega}, \mu) \end{pmatrix} = 0 \\
& \qquad i = 2, \ldots, k-1 \\
\Xi_k(\boldsymbol{\omega}, \mu) &= \begin{pmatrix} -L_{11}(\omega_{2,k}, \mu) + r_{1,k}(\boldsymbol{\omega}, \mu) \\ \mu + L_{12}(\omega_{2,k}, \mu) + r_{2,k}(\boldsymbol{\omega}, \mu) \end{pmatrix} = 0.
\end{aligned}
\tag{5.6}
$$

First we search for symmetric k-(2,1)-heteroclinic orbits. From Lemma 5.9, with $i_0 = -2$, the k-periodicity of the corresponding sequence $\boldsymbol{\omega}$, and $\omega_{1,1} = \omega_{2,1}$ we get

$$\omega_{1,i} = \omega_{2,-i+2}, \quad i = 1, \ldots, k. \tag{5.7}$$

Then Corollary 5.10 provides

$$R\,\Xi_i = -\Xi_{-i+1}, \quad i = 1, \ldots, k. \tag{5.8}$$

So the set of bifurcation equations (5.6) reduces down to

$$
\begin{aligned}
\Xi_1(\boldsymbol{\omega}, \mu) &= \begin{pmatrix} L_{11}(\omega_{1,2}, \mu) + r_{1,1}(\boldsymbol{\omega}, \mu) \\ \mu + L_{12}(\omega_{1,2}, \mu) + r_{2,1}(\boldsymbol{\omega}, \mu) \end{pmatrix} = 0 \\
\Xi_i(\boldsymbol{\omega}, \mu) &= \begin{pmatrix} L_{11}(\omega_{1,i+1}, \mu) - L_{11}(\omega_{2,i}, \mu) + r_{1,i}(\boldsymbol{\omega}, \mu) \\ \mu + L_{12}(\omega_{1,i+1}, \mu) + L_{12}(\omega_{2,i}, \mu) + r_{2,i}(\boldsymbol{\omega}, \mu) \end{pmatrix} = 0 \\
& \qquad i = 2, \ldots, \lfloor (k+1)/2 \rfloor,
\end{aligned}
\tag{5.9}
$$

where $\lfloor x \rfloor$ denotes the integer part of x. Note that in these bifurcation equations $\boldsymbol{\omega}$ has the form (5.7). Further, if k is odd, then due to (5.8), $\Xi_{\lfloor (k+1)/2 \rfloor} \in \mathrm{Fix}\,(-R)$.

First we consider

$$
\begin{aligned}
L_{11}(\omega_{1,2}, \mu) &= 0 \\
\mu + L_{12}(\omega_{1,2}, \mu) &= 0\ .
\end{aligned}
$$

The second equation can be solved for $\mu(\omega_{1,2})$ (by applying for sufficiently large $\omega_{1,2}$ the Banach fixed point theorem). Similar to Section 4.2.1 this yields $(0, \mu) = \tilde{L}_1(\omega_{1,2})$, where $\tilde{L}_1$ can be seen as a spiral in the plane centred at $(0,0)$. Intersections of this spiral with the abscissa axis yields parameter values $(\hat{\omega}_{1,2})_m$ and corresponding $\hat{\mu}_m$, $m \in \mathbb{Z}$. Similar to Section 4.2.1 we have $((\hat{\omega}_{1,2})_m, \hat{\mu}_m) \to (\infty, 0)$ as $m \to \infty$. Now we can proceed in the same way as in Section 4.2.1 to solve the full equation (5.9) This proves the existence of a countable set of parameter values for which k-(2,1)-heteroclinic orbits exist. From this construction it follows that $\mu = 0$ is a accumulation point of this set.

The leading term of the right-hand side of $\mu = -L_{12}(\omega_{1,2}(\mu), \mu) - r_{2,1}(\boldsymbol{\omega}(\mu), \mu)$ has the form

$$c(\mu)e^{-\left(\frac{\rho(\mu)}{\phi(\mu)}(n\pi-\varphi_1)\right)} \sin(n\pi + \varphi_2 - \varphi_1) = \tilde{c}(\mu)e^{-\left(\frac{\rho(\mu)}{\phi(\mu)}(n\pi-\varphi_1)\right)}(-1)^n,$$

see Lemma 4.3. This proves the existence of sequences (μ_n^+) and (μ_n^-) as stated in the theorem.

Finally, we comment on the non-existence of non-symmetric k-(2,1)-heteroclinic orbits. For that we emphasise that the bifurcation equations for detecting those orbits do not depend on the dimension of the phase space. In other words the existence or non-existence of non-symmetric k-(2,1)-heteroclinic orbits does not depend on the dimension of the phase space. In $\mathbb{R}^3$ however there are no such non-symmetric heteroclinic orbits for a simple geometrical reason: in this case the unstable manifold of p_2 is one-dimensional – and non-symmetric orbits arise in pairs.

5.3.4 k-(1,2)-heteroclinic orbits

Next we search for k-(1,2)-heteroclinic orbits for $k \geq 2$. The bifurcation equation for these orbits reads, see also Section 4.2.2:

$$\Xi_i(\boldsymbol{\omega}, \mu) = \begin{pmatrix} L_{11}(\omega_{1,i+1}, \mu) - L_{11}(\omega_{2,i}, \mu) + r_{1,i}(\boldsymbol{\omega}, \mu) \\ \mu + L_{12}(\omega_{1,i+1}, \mu) + L_{12}(\omega_{2,i}, \mu) + r_{2,i}(\boldsymbol{\omega}, \mu) \end{pmatrix} = 0 \qquad (5.10)$$
$$i = 1, \ldots, k-1.$$

In order to solve these equations we can proceed similar as in Section 4.2.2. We simply choose $\hat{\omega} \in \Omega_{n_0}$, defined by (5.4).

In this way we find at $\mu = 0$ countably many solutions of (5.10). Therefore there are countably many k-(1,2)-heteroclinic orbits (for each k) at $\mu = 0$.

By construction $q_1(0)$ is an accumulation point of the set of intersections of these orbits with Σ_1 because $\{(n\pi - \varphi_2)/\phi(0), n \geq n_0\}$ is unbounded.

Let, for fixed $\hat{\boldsymbol{\omega}}$, $\mathcal{O}_k$ be the k-(1,2)-heteroclinic orbit which corresponds to the solution $(\boldsymbol{\omega}_{\hat{\boldsymbol{\omega}}}(0), 0)$ of (5.10). Since $\boldsymbol{\omega}_{\hat{\boldsymbol{\omega}}}(\cdot)$ is also defined for μ close to 0, the orbit $\mathcal{O}_k$ can be continued for those μ.

Again as in Section 4.2.2 we find that for fixed $\mu \neq 0$ equation $\Xi_i(\boldsymbol{\omega}, \mu) = 0$ has only finitely many solutions, see also Section 5.3.2 and Figure 6. So there are only finitely many k-(1,2)-heteroclinic orbits for those μ.

5.3.5 Homoclinic Orbits

It is sufficient to consider k-homoclinic orbits to p_1 because their R-images are just the k-homoclinic orbits to p_2.

The bifurcation equation for those orbits reads, see also (4.14):

$$\begin{aligned}
\Xi_i(\boldsymbol{\omega},\mu) = & \quad \begin{pmatrix} L_{11}(\omega_{1,i+1},\mu) - L_{11}(\omega_{2,i},\mu) + r_{1,i}(\boldsymbol{\omega},\mu) \\ \mu + L_{12}(\omega_{1,i+1},\mu) + L_{12}(\omega_{2,i},\mu) + r_{2,i}(\boldsymbol{\omega},\mu) \end{pmatrix} = 0 \\
& \qquad\qquad\qquad i = 1,\ldots,k-1 \\
\Xi_k(\boldsymbol{\omega},\mu) = & \qquad\qquad \begin{pmatrix} -L_{11}(\omega_{2,k},\mu) + r_{1,k}(\boldsymbol{\omega},\mu) \\ \mu + L_{12}(\omega_{2,k},\mu) + r_{2,k}(\boldsymbol{\omega},\mu) \end{pmatrix} = 0.
\end{aligned} \tag{5.11}$$

These equations have the same structure as (5.9); they differ only in the number of equations and variables. So the discussion of (5.11) runs along the same lines as the discussion in Section 5.3.3.

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