A New Lower Bound on the Number of Edges in Colour-Critical Graphs

A.V. Kostochka*
Novosibirsk State University,
630090 Novosibirsk, Russia

M. Stiebitz†
Technische Universität Ilmenau,
D-98684 Ilmenau, Germany

Abstract

A graph G is called k-critical if it has chromatic number k, but every proper subgraph of it is $(k-1)$–colourable. We prove that every k–critical graph ($k \geq 6$) on $n \geq k+2$ vertices has at least $\frac{1}{2}(k-1+\frac{k-3}{(k-c)(k-1)+k-3})n$ edges where $c = (k-5)(\frac{1}{2}-\frac{1}{(k-1)(k-2)})$. This improves earlier bounds established by Gallai [9] and, more recently, by Krivelevich [17].

1 Introduction

A graph G is k–*critical* for some integer $k \geq 1$ if G is not $(k-1)$–colourable but every proper subgraph of G is $(k-1)$–colourable. Then every k–critical graph has chromatic number k and every k–chromatic graph contains a k–critical subgraph. The importance of the concept of criticality consists in the fact that problems for k–chromatic graphs may often be reduced to problems for k–critical graphs, and that the class of k–critical graphs is a narrow subclass of the class of k–chromatic graphs. Critical graphs were first defined and used by Dirac [5] in 1951. In the present paper a new lower bound for the number of edges in a k–critical graph on n vertices is established.

*Research supported in part by the grant 96-01-01614 of the Russian Foundation for Fundamental Research

†This manuscript was finished while this author was visiting the University of Odense

The complete graph K_k is an example of a k-critical graph and for $k = 1, 2$ it is the only one. The only 3-critical graphs are the odd circuits, so for the remainder of this paper we shall restrict our attention to the case $k \geq 4$. Then there are k-critical graphs on n vertices for all $n \geq k$ except for $n = k + 1$. For $n \geq k + 2$, let $f_k(n)$ denote the minimum number of edges possible in a k-critical graph on n vertices. Since every k–critical graph has minimum degree at least $k-1$, we have $2f_k(n) \geq (k-1)n$. Brooks' theorem [3] implies

$$2f_k(n) \geq (k-1)n + 1,$$

and Dirac [6] proved

$$2f_k(n) \geq (k-1)n + k - 3.$$

In [7], he also gave a complete description of the extremal cases. Dirac's proof was rather long. Shorter and more elegant proofs were found by Kronk and Mitchem [18], Weinstein [25] and, for the result in [7], by Deuber, Kostochka and Sachs [4]. In a recent paper [15], the authors proved

$$2f_k(n) \geq (k-1)n + 2(k-3)$$

provided that $n \neq 2k - 1$. For a given constant $c \geq 0$, let

$$g_k(n, c) = (k - 1 + \frac{k-3}{(k-c)(k-1) + k - 3})n.$$

In his fundamental paper [9] Gallai characterized the class of graphs that are subgraphs of some k–critical graph G induced by the set of vertices having degree $k-1$ in G. Based on this result, he proved $2f_k(n) \geq g_k(n, 0)$. Recently, this lower bound was improved by Krivelevich [17] to $2f_k(n) \geq g_k(n, 2)$. In what follows, let

$$\alpha_k = \frac{1}{2} - \frac{1}{(k-1)(k-2)}.$$

The following theorem is one of the main results of this paper.

Theorem 1.1. *If $k \geq 6$ and $n \geq k + 2$, then $2f_k(n) \geq g_k(n, (k-5)\alpha_k)$.* □

For further information about $f_k(n)$ the reader is referred to [12] and [17] (see also section 5).

1.1 Terminology

Concepts and notation not defined in this paper will be used as in standard textbooks.

The graphs considered are finite, undirected and without loops and multiple edges. The set of vertices and the set of edges of a graph G are denoted by $V(G)$ and $E(G)$, respectively. An edge of G joining the distinct vertices $x, y \in V(G)$ is denoted by xy or yx, and the vertices x and y are said to be *adjacent* in G. For $x \in V(G)$, let $N_G(x)$ denote the set of all vertices in G that are adjacent to x in G. The *degree* of x with respect to G is $d_G(x) = |N_G(x)|$.

For $M \subseteq E(G)$, let $G - M = (V(G), E(G) - M)$. Let $X \subseteq V(G)$. The subgraph of G induced by X is denoted by $G(X)$, i.e., $V(G(X)) = X$ and $E(G(X)) = \{e \in E(G) \mid e = xy \text{ and } x, y \in X\}$; further, $G - X = G(V(G) - X)$. The set X will be called a *clique* (respectively, an *independent set*) in G if $G(X)$ is a complete graph (respectively, a graph without edges). A clique of G with p vertices is also said to be a *p–clique* of G. As usual, K_n denotes the complete graph on n vertices. For $H \subseteq V(G)$ and $x \in V(G)$, let $N_G(x : H) = N_G(x) \cap H$ and $d_G(x : H) = |N_G(x : H)|$.

1.2 Φ–critical graphs

For the proof of Theorem 1.1 we shall use the list colouring concept. Consider a graph G and assign to each vertex x of G a set $\Phi(x)$ of colours (positive integers). Such an assignment Φ of sets to vertices in G is referred to as a *colour scheme* (or briefly, a *list*) for G. A *Φ-colouring* of G is a mapping φ of $V(G)$ into the set of colours such that $\varphi(x) \in \Phi(x)$ for all $x \in V(G)$ and $\varphi(x) \neq \varphi(y)$ whenever $xy \in E(G)$. If G admits a Φ-colouring, then G is said to be *Φ-colourable*. In case of $\Phi(x) = \{1, \ldots, k\}$ for all $x \in V(G)$, we also use the terms *k-colouring* and *k-colourable*, respectively. The *chromatic number* of G denoted by $\chi(G)$ is the least number k for which G is k-colourable. If $\chi(G) = k$, then G is called k–chromatic. The list colouring concept was introduced, independently, by Vizing [21] and Erdös, Rubin and Taylor [8].

Let G be a graph and let Φ be a list for G. We say that G is *Φ–critical* if G is not Φ–colourable but every proper subgraph of G is Φ–colourable. In case of $\Phi(x) = \{1, \ldots, k\}$ for all $x \in V(G)$, we also use the term *k–critical.* Then G is k–critical if and only if $\chi(G') < \chi(G) = k$ for every proper subgraph of G. We shall prove the following result.

Theorem 1.2.
Assume that $G \neq K_k$ *is a* Φ–critical *graph where* Φ *is a list for* G *satisfying* $|\Phi(x)| = k-1$ *for every* $x \in V(G)$. *Then*

$$2|E(G)| \geq g_k(|V(G)|, c) = (k-1+\frac{k-3}{(k-c)(k-1)+k-3})|V(G)|$$

provided that $k \geq 9$ *and* $c = \frac{1}{3}(k-4)\alpha_k$ *or* $k \geq 6$, $\Phi(x) = \{1, \ldots, k-1\}$ *for every* $x \in V(G)$ *and* $c = (k-5)\alpha_k$ □

Theorem 1.2 implies immediately Theorem 1.1. The proof of Theorem 1.2 is based on the following result.

Theorem 1.3.
Assume that $G \neq K_k$ *is a* Φ*–critical graph where* Φ *is a list for* G *satisfying* $|\Phi(x)| = k-1$ *for every* $x \in V(G)$. *Let* $L = \{x \in V(G) \mid d_G(x) = k-1\}$ *and* $H = \{x \in V(G) \mid d_G(x) \geq k\}$. *Furthermore, let*

$$\sigma = (k-2+\frac{2}{k-1})|L| - 2|E(G(L))|$$

and

$$\tau_c = 2|E(G(H))| + (k-c-\frac{2}{k-1})\sum_{y \in H}(d_G(y)-k)$$

where c *is a constant. Then* $\sigma + \tau_c \geq c|H|$ *provided that* $k \geq 9$ *and* $c = \frac{1}{3}(k-4)\alpha_k$ *or* $k \geq 6$, $\Phi(x) = \{1, \ldots, k-1\}$ *for every* $x \in V(G)$ *and* $c = (k-5)\alpha_k$ □

If G is Φ–critical where Φ is a list for G, then $|\Phi(x)| \leq d_G(x)$ for every vertex x of G, since otherwise we can extend a Φ–colouring of $G - x$ to a Φ–colouring of G, a contradiction. Therefore, Theorem 1.2 is a consequence of Theorem 1.3 and the following result.

Lemma 1.4. *Let* G *be a graph,* $L = \{x \in V(G) \mid d_G(x) = k-1\}$, $H = \{x \in V(G) \mid d_G(x) \geq k\}$ $(k \geq 4)$, *and let* σ *and* τ_c *be defined as in Theorem* 1.3 *where* $0 \leq c \leq k - \frac{2}{k-1}$ *is a given constant. If* $\sigma + \tau_c \geq c|H|$ *and*

$V(G) = L \cup H$, then $2|E(G)| \geq g_k(|V(G)|, c)$.

Proof. Let $m = |E(G)|$, $m_L = |E(G(L))|$, $m_H = |E(G(H))|$ and $n = |V(G)|$. Then

$$\sigma = (k - 2 + \frac{2}{k-1})|L| - 2m_L \text{ and } \tau_c = 2m_H + (k - c - \frac{2}{k-1})\gamma$$

where $\gamma = \sum_{y \in H}(d_G(y) - k)$. On the one hand, since $d_G(x) = k - 1$ for every $x \in L$ and $\sigma + \tau_c \geq c|H|$, we have

$$\begin{aligned} 2m &= 2m_H + 2(k-1)|L| - 2m_L = 2m_H + \sigma + |L|(k - \frac{2}{k-1}) \\ &\geq c|H| - \gamma(k - c - \frac{2}{k-1}) + |L|(k - \frac{2}{k-1}) \\ &= cn - \gamma(k - c - \frac{2}{k-1}) + |L|(k - c - \frac{2}{k-1}). \end{aligned}$$

On the other hand, we have

$$2m = (k-1)n + |H| + \gamma = kn - |L| + \gamma.$$

Therefore, we obtain

$$2m(1 + k - c - \frac{2}{k-1}) \geq (c + k(k - c - \frac{2}{k-1}))n.$$

This implies, by an easy calculation, that

$$2m \geq (k - 1 + \frac{k-3}{(k-c)(k-1) + k - 3})n = g_k(n, c).$$

This proves Lemma 1.4. □

1.3 Gallai trees and bad pairs

Let G be a graph. A vertex x of G is called a *separating vertex* of G if $G - x$ has more components than G. By a *block* of G we mean a maximal connected subgraph B of G such that no vertex of B is a separating vertex of B. Any two blocks of G have at most one vertex in common and, obviously, a vertex of G is a separating vertex of G iff it is contained in more than one block of G. An *end-block* of G is a block that contains at most one separating vertex of G.

A connected graph G all of whose blocks are complete graphs and/or odd circuits is called a *Gallai tree*; a *Gallai forest* is a graph all of whose components are Gallai trees.

By a *bad pair* we mean a pair (G,Φ) consisting of a connected graph G with $n \geq 1$ vertices and a list Φ of G such that $|\Phi(x)| \geq d_G(x)$ for all $x \in V(G)$ and G is not Φ-colourable.

Lemma 1.5 *If (G,Φ) is a bad pair, then the following statements hold.*

(a) $|\Phi(x)| = d_G(x)$ *for all* $x \in V(G)$.

(b) *If G has no separating vertex, then $\Phi(x)$ is the same for all $x \in V(G)$.*

(b) *G is a Gallai tree.* □

Lemma 1.5 was proved independently by Borodin [1, 2] and Erdös, Rubin and Taylor [8]. Proofs of statements (a) and (b) in the graph version based on a sequential colouring argument were given by Vizing [21] and Lovász [19]. For a short proof of Lemma 1.5 based on the following simple reduction idea the reader is referred to [13].

Remark 1.6. Let G be a graph, Φ a list for G, $Y \subseteq V(G)$, and let φ be a Φ–colouring of $G(Y)$. For the graph $G' = G - Y$, we define a list Φ' by

$$\Phi'(x) = \Phi(x) - \{\varphi(y) \;|y \in Y \;\&\; xy \in E(G)\}$$

for every $x \in V(G')$. In what follows, we denote Φ' by $\Phi(Y, \varphi)$ and in case of $Y = \{y\}$ and $\varphi(y) = a$ also by $\Phi(y, a)$. Then it is straightforward to show that the following statements hold.
(a) If G' is Φ'–colourable, then G is Φ–colourable.
(b) If $|\Phi(x)| = d_G(x) + p$ for some $x \in V(G')$, then $|\Phi'(x)| \geq d_{G'}(x) + p$.
(c) If (G, Φ) is a bad pair and G' is connected, then (G', Φ') is a bad pair. □

Consequently, Lemma 1.5 implies immeditely the following result. Note that the only regular Gallai trees are the complete graphs and the odd curcuits.

Lemma 1.7. *Let G be a Φ–critical graph where Φ is a given list for G, $H = \{y \in V(G) \mid d_G(y) > |\Phi(y)|\}$ and $L = V(G) - H$. Then $G(L) = G - H$ is a Gallai forest and $d_G(x) = |\Phi(x)|$ for every $x \in L$. Furthermore, if $|\Phi(x)| = k-1$ for every $x \in V(G)$, then H is non-empty or G is a K_k or $k = 3$ and G is an odd circuit.* □

Let G be an arbitrary Gallai tree. The set of all blocks of G is denoted by $\mathcal{B}(G)$. If $B \in \mathcal{B}(G)$, then B is a complete graph or an odd circuit and we say that B is of *type* b if B is regular of degree $b-1$. Two blocks which have a vertex in common (they cannot have more than one vertex in common) are called *adjacent*.

Let $\mathcal{U}(G)$ denote the set of all mappings u that assign to every block $B \in \mathcal{B}(G)$ of type b a set $u(B)$ of $b-1$ colours such that $u(B) \cap u(B') = \emptyset$ for any two adjacent blocks $B, B' \in \mathcal{B}(G)$. For a given mapping $u \in \mathcal{U}(G)$ define a list $\Phi = \Phi_u$ for the Gallai tree G by $\Phi(x) = \bigcup u(B)$ where B runs through all blocks of G containing the vertex $x \in V(G)$. The following result was proved by Borodin [2].

Lemma 1.8. *Let (G, Φ) be a bad pair. Then $\Phi = \Phi_u$ for some $u \in \mathcal{U}(G)$. This implies, in particular, that $\Phi(x) = \Phi(y)$ provided that x, y are two non-separating vertices of G contained in the same block of G.* □

2 Lower bounds for σ and ϵ_k–graphs

Let $k \geq 4$ be a given integer, and let $r_k = k - 2 + \frac{2}{k-1}$. For an arbitrary graph F and $x \in V(F)$, let $\sigma(x : F) = r_k - d_F(x)$ and $\sigma(F) = \sum_{x \in V(F)} \sigma(x : F)$. Then $\sigma(F) = |V(F)|r_k - 2|E(F)|$. Let $\mathcal{T}_k$ denote the set of all Gallai trees distinct from K_k and with maximum degree at most $k-1$. For $T \in \mathcal{T}_k$ and some end-block B of T, let $T_B = T - (V(B) - x)$ where x is the only separating vertex of T contained in B (if there is no such vertex, then $T = B$ and an arbitrary vertex of B may be taken). The proof of the next Lemma is left to the reader.

Lemma 2.1.

(a) *If B is a complete graph of order $b \leq k-1$, then $\sigma(B) = b(r_k - b + 1)$.*

(b) *If B is an end-block of $T \in \mathcal{T}_k$, then $\sigma(T) = \sigma(T_B) + \sigma(B) - r_k$.* □

Let $T \in \mathcal{T}_k$. For a vertex $x \in V(T)$, let $B_1, \ldots, B_l$ be the blocks of T containing x where B_i is of type b_i $(i = 1, \ldots, l)$. Then x is said to be of *type* $(b_1, \ldots, b_l)$ in T. For an integer $b \geq 1$, let

$$t(b) = \begin{cases} 1 - \alpha_k & \text{if } b = 2 \\ 2 - \frac{2}{b} & \text{otherwise.} \end{cases}$$

For a vertex $x \in V(T)$ of type $(b_1, \ldots, b_l)$ in T, we now define $\sigma'(x : T) = \sigma(x : T) + \sum_{i=1}^{l} t(b_i) - 2$ if x is not contained in a K_{k-1} (that is $b_i \neq k-1$ for all i) and $\sigma'(x : T) = 0$ otherwise. Furthermore, let $\sigma'(T) = \sum_{x \in V(T)} \sigma'(x : T)$.

Lemma 2.2.

(a) *If B is a complete graph of order $b \leq k-1$ or an odd circuit, then $\sigma(B) \geq \sigma'(B) + 2$.*

(b) *Let $T \in \mathcal{T}_k$ be a Gallai tree with at least two blocks. Let B be an end-block of T and x the only separating vertex of T contained in B. If x is not contained in a K_{k-1}, then $\sigma'(T) = \sigma'(T_B) + \sigma'(B) - r_k + 2$.*

Proof. For the proof of (a), let us first consider the case that B is a complete graph of order $b \leq k-1$. If $b = k-1$, then $\sigma'(B) = 0$ and, by Lemma 2.1 (a), $\sigma(B) = (k-1)(r_k - k + 2) = 2 = \sigma'(B) + 2$. If $1 \leq b \leq k-2$, then $\sigma'(B) = \sigma(B) + (t(b) - 2)b$, and, therefore, $\sigma'(B) = \sigma(B) - 2$ if $b \neq 2$ and $\sigma'(B) = \sigma(B) - 2 - 2\alpha_k \leq \sigma(B) - 2$ if $b = 2$. Now assume that B is an odd circuit. Then $\sigma'(B) = \sigma(B) + (t(3) - 2)|B| = \sigma(B) - (2/3)|B| \leq \sigma(B) - 2$. This proves (a).

For the proof of (b) assume that B is a block of type b and x is of type $(b_1, \ldots, b_l)$ in T where $b_l = b$. Note that $l \geq 2$ and x is of type $(b_1, \ldots, b_{l-1})$ in $T' = T_B$. Then, since x is not contained in a K_{k-1},

$$\sigma'(x : T') = r_k - d_{T'}(x) + \sum_{i=1}^{l-1} t(b_i) - 2,$$

$$\sigma'(x : B) = r_k - d_B(x) + t(b_l) - 2,$$

and,

$$\sigma'(x : T) = r_k - d_T(x) + \sum_{i=1}^{l} t(b_i) - 2.$$

Since $d_T(x) = d_{T'}(x) + d_B(x)$, we conclude that

$$\begin{aligned} \sigma'(T) &= \sigma'(T') + \sigma'(B) + \sigma'(x:T) - \sigma'(x:T') - \sigma'(x:B) \\ &= \sigma'(T') + \sigma'(B) - r_k + 2. \end{aligned}$$

This proves (b). □

For a graph F and an integer $p \geq 2$, let $W^p(F)$ denote the set of all vertices of F that belong to some $(p-1)$–clique of F. If $F \in \mathcal{T}_k$, then $W^{k+1}(F) = \emptyset$ and $F(X)$ is a block for every $(k-1)$–clique X of F. We call F an ϵ_k*–graph* if $F \in \mathcal{T}_k$ and $W^k(F) = V(F)$. Clearly, a graph $F \in \mathcal{T}_k$ is an ϵ_k–graph iff every separating vertex of F is of type $(k-1, 2)$ and every non-separating vertex of F is of type $k-1$.

If a component F' of $F(W^k(F))$ is an ϵ_k–graph, then F' is said to be an ϵ_k*–subcomponent* of F. Obviously, if $T \in \mathcal{T}_k$, then every component of $T(W^k(T))$ is an ϵ_k–graph and, therefore, an ϵ_k–subcomponent of T. The number of all ϵ_k–subcomponents of $T \in \mathcal{T}_k$ is denoted by $s(T)$.

Lemma 2.3. *If $T \in \mathcal{T}_k$ and $k \geq 6$, then*

(a) $\sigma(T) \geq \sigma'(T) + s(T)\alpha_k + 2 - \alpha_k$ *and*

(b) $\sigma'(x:T) \geq \alpha_k(k-1-d_T(x))$ *for every $x \in V(T) - W^k(T)$.*

Proof. We prove statement (a) by induction on the number of blocks of T. If T consists of exactly one block, then T is a complete graph of order $b \leq k-1$ or an odd circuit and statement (a) follows from Lemma 2.2 (a).

Now, assume that T has at least two blocks. Let B be some end-block of T and x the only separating vertex of T contained in B. First, consider the case that $x \in V(T) - W^k(T)$. Then no block of T containing x is a K_{k-1}. Therefore, $s(T_B) = s(T)$ and, by the induction hypothesis, $\sigma(T_B) \geq \sigma'(T_B) + s(T)\alpha_k + 2 - \alpha_k$. Using Lemma 2.1 (b) and Lemma 2.2, we conclude that

$$\begin{aligned} \sigma(T) &= \sigma(T_B) + \sigma(B) - r_k \\ &\geq \sigma'(T_B) + s(T)\alpha_k + 2 - \alpha_k + \sigma'(B) + 2 - r_k \\ &= \sigma'(T) + s(T)\alpha_k + 2 - \alpha_k. \end{aligned}$$

Now, consider the case that $x \in W^k(T)$. Then, since the maximum degree of T is at most $k-1$, $B = K_{k-1}$ or $B = K_2$. If $B = K_2$, then $x \in W^k(T_B) = W^k(T)$ and $s(T_B) = s(T)$. Let y be the vertex of B distinct from x. Then $T_B = T - y$ and, by Lemma 2.1, $\sigma(T) = \sigma(T_B) + \sigma(B) - r_k = \sigma(T_B) - 2(r_k - 1) - r_k = \sigma(T_B) + r_k - 2$. Furthermore, since $\sigma'(x : T_B) = \sigma'(x : T) = 0$, we conclude that $\sigma'(T) = \sigma'(T_B) + \sigma'(y : T) = \sigma'(T_B) + r_k - 1 + 1 - \alpha_k - 2 = \sigma'(T_B) + r_k - 2 - \alpha_k$. By the induction hypothesis, it then follows that

$$\begin{aligned}\sigma(T) &= \sigma(T_B) + r_k - 2 \\ &\geq \sigma'(T_B) + s(T)\alpha_k + 2 - \alpha_k + r_k - 2 \\ &\geq \sigma'(T_B) + r_k - 2 - \alpha_k + s(T)\alpha_k + 2 - \alpha_k \\ &= \sigma'(T) + s(T)\alpha_k + 2 - \alpha_k.\end{aligned}$$

If $B = K_{k-1}$, then we argue as follows. Since the maximum degree of T is at most $k-1$, the vertex x is contained in exactly one further block $B' = K_2$. Let $T' = (T_B)_{B'}$ and let y be the vertex of B' distinct from x. Note that $T' = T - V(B)$. From Lemma 2.1 it then follows that $\sigma(T) = \sigma(T') + \sigma(B) + \sigma(B') - 2r_k = \sigma(T') + 2 + 2(r_k - 1) - 2r_k = \sigma(T')$. If y is contained in a K_{k-1}, then s(T)=s(T') and $\sigma'(T) = \sigma'(T')$. Therefore, by the induction hypothesis, we obtain $\sigma(T) = \sigma(T') \geq \sigma'(T') + s(T)\alpha_k + 2 - \alpha_k = \sigma'(T) + s(T)\alpha_k + 2 - \alpha_k$. If y is not contained in a K_{k-1}, then $s(T) = s(T') + 1$ and $\sigma'(T) = \sigma'(T') + \sigma'(y : T) - \sigma'(y : T') = \sigma'(T') + t(2) - 1 = \sigma'(T') - \alpha_k$. Hence, by the induction hypothesis, we have

$$\begin{aligned}\sigma(T) &= \sigma(T') \\ &\geq \sigma'(T') + (s(T) - 1)\alpha_k + 2 - \alpha_k \\ &= \sigma'(T) + s(T)\alpha_k + 2 - \alpha_k.\end{aligned}$$

Thus statement (a) is proved.

For the proof of (b), let us consider an arbitrary vertex $x \in V(T) - W^k(T)$. Assume that x is of type $(b_1, \ldots, b_l)$ in T where $b_i \neq 2$ for $1 \leq i \leq m$ and $b_{m+1} = \ldots = b_l = 2$. Then $b_i \leq k - 2$ for $i = 1, \ldots, m$. Moreover, $d_T(x) = \sum_{i=1}^{m}(b_i - 1) + l - m \leq k - 1$ and we have to show that

$$\sigma'(x : T) = r_k - d_T(x) + \sum_{i=1}^{l} t(b_i) - 2 \geq \alpha_k(k - 1 - d_T(x)). \tag{1}$$

Let

$$M = (1-\alpha_k)(k-1-\sum_{i=1}^{m}(b_i-1)) + \sum_{i=1}^{m}(2-\frac{2}{b_i})$$

By an easy calculation, it then follows that (1) is equivalent to $M \geq 3-\frac{2}{k-1}$.

If $m=0$, then $M=(1-\alpha_k)(k-1)$ and, for $k \geq 7$, this gives $M \geq 3$ and, for $k=6$, this gives $M=(0.5+1/20)5 = 2.75 \geq 3-2/5$. If $m=1$, then $M=(1-\alpha_k)(k-b_1)+2-2/b_1$. Since $b_1 \leq k-2$,

$$\begin{aligned} M &\geq (1-\alpha_k)(k-(k-2))+2-2/(k-2) \\ &= 1+\frac{2}{(k-1)(k-2)} = 1+\frac{2}{(k-1)(k-2)}+2-2/(k-2) = 3-\frac{2}{k-1}. \end{aligned}$$

If $m=2$, then $3 \leq b_1, b_2$ and $b_1+b_2 \leq k+1$. For $b_1+b_2 \leq k$, we then have $M=(1-\alpha_k)+(2-2/b_1)+(2-2/b_2) \geq 1/2+8/3 \geq 3$ and, for $b_1+b_2=k+1$, we have

$$M = (2-2/b_1)+(2-2/(k+1-b_1)) \geq (2-2/(k-2))+(2-2/3) > 3-2/(k-1).$$

Finally, if $m \geq 3$, then

$$M \geq \sum_{i=1}^{3}(2-\frac{2}{b_i}) \geq 3(2-\frac{2}{3}) = 4.$$

This proves (b). □

3 Bipartite graphs

In this section we prove some auxiliary results concerning bipartite graphs needed for the proof of Theorem 1.3. By $F=F(A,B)$ we denote a bipartite graph with bipartition $\{A,B\}$, i.e., $V(F)=A\cup B$, $A\cap B=\emptyset$ and $E(F)\subseteq\{xy|\ x\in A \text{ and } y\in B\}$.

Lemma 3.1. *Let $F=F(A,B)$ be a bipartite graph, let $r\geq 1$ be an integer, and let B_r be the set of all vertices of B having degree at least r in F. Then there is a subgraph F' of F such that*

(a) $d_{F'}(x) \leq \lceil \frac{d_F(x)}{r} \rceil$ *for every $x\in A$, and*

(b) $d_{F'}(y) = 1$ *for every* $y \in B_r$.

Proof. Split each vertex $x \in A$ into $m_x = \lceil \frac{d_F(x)}{r} \rceil$ vertices of degree at most r. This results in a bipartite graph $H = H(A', B)$ such that $d_H(x') \leq r$ for every $x' \in A'$ and $d_H(y) = d_F(y)$ for every $y \in B$. Then, clearly, for each $S \subseteq B_r$, we have $|N_H(S)| \geq |S|$ where $N_H(S) := \bigcup_{y \in S} N_H(y)$. Now, Hall's theorem implies that there is a matching M in H that covers all vertices in B_r. From the graph $(V(H), M)$ we then obtain a subgraph F' of F such that $d_{F'}(x) \leq m_x$ for every $x \in A$ and $d_{F'}(y) = 1$ for every $y \in B_r$. This proves Lemma 3.1. □

Lemma 3.2. *Let* $F = F(A, B)$ *be a bipartite graph and, for an integer* $r \geq 1$, *let* B_r *be the set of all vertices of* B *having degree at least* r *in* F. *Assume that* $d_F(x) \geq 4$ *for every* $x \in A$. *Then there is a subgraph* F' *of* F *such that*

(a) $d_{F'}(x) = 2$ *for every* $x \in A$,

(b) $d_{F'}(y) \leq d_F(y) - 2$ *for every* $y \in B_4$, *and*

(c) $d_{F'}(y) \leq d_F(y) - 1$ *for every* $y \in B_3$.

Proof. Because of Lemma 3.1, there is a subgraph H of F such that $d_H(x) \leq \lceil \frac{d_F(x)}{4} \rceil$ for every $x \in A$ and $d_H(y) = 1$ for every $y \in B_4$. Let $\tilde{F} = F - E(H)$ and let $\tilde{B}_3$ be the set of all vertices of B having degree at least 3 in $\tilde{F}$. Obviously, $B_4 \subseteq \tilde{B}_3$. Then Lemma 3.1 implies that there is a subgraph $\tilde{H}$ of $\tilde{F}$ such that $d_{\tilde{H}}(x) \leq \lceil \frac{d_{\tilde{F}}(x)}{3} \rceil$ for every $x \in A$ and $d_{\tilde{H}}(y) = 1$ for every $y \in \tilde{B}_3$.

Let $G = \tilde{F} - E(\tilde{H}) = F - E(H) - E(\tilde{H})$. For $y \in B_4$, we have $d_H(y) = d_{\tilde{H}}(y) = 1$ and, therefore, $d_G(y) = d_F(y) - d_H(y) - d_{\tilde{H}}(y) = d_F(y) - 2$. For $y \in B_3$, we have $d_H(y) \geq 1$ or $y \in \tilde{B}_3$ and, therefore, $d_{\tilde{H}}(y) = 1$ implying that $d_G(y) = d_F(y) - d_H(y) - d_{\tilde{H}}(y) \leq d_F(y) - 1$. Let $x \in A$. Since $d_F(x) \geq 4$, we have $d_{\tilde{F}}(x) = d_F(x) - d_H(x) \geq d_F(x) - \lceil \frac{d_F(x)}{4} \rceil \geq 3$ and, therefore, $d_G(x) = d_{\tilde{F}}(x) - d_{\tilde{H}}(x) \geq d_{\tilde{F}}(x) - \lceil \frac{d_{\tilde{F}}(x)}{3} \rceil \geq 2$. Consequently, there is a subgraph F' of G such that $d_{F'}(x) = 2$ for every $x \in A$, $d_{F'}(y) \leq d_G(y) \leq d_F(y) - 2$ for every $y \in B_4$, and $d_{F'}(y) \leq d_G(y) \leq d_F(y) - 1$ for every $y \in B_3$. This proves Lemma 3.2. □

Lemma 3.3. *Let $F = F(A, B)$ be a bipartite graph and let $\mathcal{P}$ be a mapping that assigns to every vertex $x \in A$ a partition $\mathcal{P}(x)$ of $N_F(x)$. Assume that $d_F(x) \geq |\mathcal{P}(x)| + 2^{r-3}$ for every $x \in A$ where $r \geq 3$ is a given integer. Then there is a subgraph F' of F such that the following statements hold.*

(a) *If $x \in A$, then $d_{F'}(x) = 2$ and $N_{F'}(x) \subseteq N$ for some $N \in \mathcal{P}(x)$.*

(b) *If $y \in B$ and $d_F(y) \geq s$* where $3 \leq s \leq r$, *then $d_{F'}(y) \leq d_F(y) - s + 3$.*

Proof (by induction on r and $|E(F)|$). A subgraph F' of F satisfying the conditions (a) and (b) of Lemma 3.3 is called a *good subgraph* of F with respect to $\mathcal{P}$ and r. Let $F_1 = F_1(A, B)$ be a subgraph of F and define $\mathcal{P}_1$ by

$$\mathcal{P}_1(x) = \{N \cap N_{F_1}(x) | N \in \mathcal{P}(x) \ \& \ N \cap N_{F_1}(x) \neq \emptyset\}$$

for every $x \in A$. In this case we write $\mathcal{P}_1 = \mathcal{P}|F_1$. It is easy to check that if F' is a good subgraph of F_1 with respect to $\mathcal{P}_1 = \mathcal{P}|F_1$ and r, then F' is a good subgraph of F with respcet to $\mathcal{P}$ and r.

We have to show that there is a good subgraph of F with respect to $\mathcal{P}$ and r providing that $d_F(x) \geq |\mathcal{P}(x)| + 2^{r-3}$ for every $x \in A$. For $r = 3$ this is evident. Now, assume $r \geq 4$.

First, consider the case that, for some $x \in A$, there is a set $N \in \mathcal{P}(x)$ such that $N = \{y\}$. Let $F_1 = F - \{xy\}$ and $\mathcal{P}_1 = P|F_1$. Then $d_{F_1}(x) = d_F(x) - 1 \geq |\mathcal{P}(x)| + 2^{r-3} - 1 = |\mathcal{P}_1(x)| + 2^{r-3}$ and, by the induction hypothesis, there is a good subgraph F' of F_1 with respect to $\mathcal{P}_1$ and r. Then F' is a good subgraph of F with respect to $\mathcal{P}$ and r.

Now, consider the case that $|N| \geq 2$ for every $N \in \mathcal{P}(x)$ and every $x \in A$. If $d_F(x) > |\mathcal{P}(x)| + 2^{r-3}$ for some $x \in A$, then let $F_1 = F - \{xy\}$ and $\mathcal{P}_1 = \mathcal{P}|F_1$ where $y \in N_F(x)$. Since $d_{F_1}(x) \geq |\mathcal{P}(x)| + 2^{r-3} = |\mathcal{P}_1(x)| + 2^{r-3}$, it then follows from the induction hypothesis that there is a good subgraph F' of F_1 with respect to $\mathcal{P}_1$ and r. Then F' is a good subgraph of F with respect to $\mathcal{P}$ and r. If $d_F(x) = |\mathcal{P}(x)| + 2^{r-3}$ for every $x \in A$, then we argue as follows. Since every set of $\mathcal{P}(x)$ has at least two elements, $|\mathcal{P}(x)| \leq 2^{r-3}$ and, therefore, $d_F(x) \leq 2^{r-2}$ for every $x \in A$. By Lemma 3.2, there is a subgraph H of F such that $d_H(x) \leq \lceil \frac{d_F(x)}{r} \rceil \leq \lceil \frac{2^{r-2}}{4} \rceil = 2^{r-4}$ for every $x \in A$ and $d_H(y) = 1$ for every $y \in B$ with $d_F(y) \geq r$. Let $\tilde{F} = F - E(H)$ and $\tilde{\mathcal{P}} = \mathcal{P}|\tilde{F}$. Then, for every $x \in A$, $d_{\tilde{F}}(x) = d_F(x) - d_H(x) \geq |\mathcal{P}(x)| + 2^{r-3} - 2^{r-4} = |\mathcal{P}(x)| + 2^{r-4} \geq |\tilde{\mathcal{P}}(x)| + 2^{r-4}$. Therefore, by the induction hypothesis, there is a good subgraph F' of $\tilde{F}$ with respect to $\tilde{\mathcal{P}}$ and $r-1$. Then F' is a good subgraph of F with respect to $\mathcal{P}$ and $r-1$. If $y \in B$ and $d_F(y) = r$, then $d_H(y) = 1$ and, therefore, $d_{\tilde{F}}(y) = r-1$ implying that $d_{F'}(y) \leq d_{\tilde{F}}(y) - (r-1) + 3 = d_F(y) - r + 3$. Consequently, F' is a good subgraph of F with respect to $\mathcal{P}$ and r. Thus Lemma 3.3 is proved. $\square$

Remark. Lemma 3.3 remains valid if the condition $d_F(x) \geq |\mathcal{P}(x)| + 2^{r-3}$ is replaced by $d_F(x) \geq |\mathcal{P}(x)| + m_r$ where $m_3, m_4, ...$ is a sequence of integers satisfying $m_3 = 1$ and $m_r - \lceil \frac{2m_r}{r} \rceil \geq m_{r-1}$ for $r \geq 4$. For $r = 5$, the case we are interested in, this gives $m_5 = 4$.

Lemma 3.4. *Let $F = F(A, B)$ be a bipartite graph, let R, d be integers with $R \geq d \geq 1$ and, for every $x \in A$,* let *$a(x) \geq 1$ be an integer. Assume that $d_F(y) \geq R$ for every $y \in B$. Then*

$$(R-d)|B| \leq \sum_{x \in A} a(x)$$

or there are non-empty subsets $A' \subseteq A$ and $B' \subseteq B$ such that for the bipartite graph $F' = F(A' \cup B')$ we have $d_{F'}(x) > a(x)$ for every $x \in A'$ and $d_{F'}(y) > d$ for every $y \in B'$.

Proof. We define a sequence $B_0 = \emptyset,\ A_1, B_1, A_2, B_2, ...$ of sets as follows. For $i \geq 1$, let

$$A_i = \{x \in A \mid d_F(x : B - B_{i-1}) \leq a(x)\}$$

and

$$B_i = \{y \in B \mid d_F(y : A_i) \geq R - d\}.$$

Then, for every $i \geq 1$, we have $A_i \subseteq A_{i+1} \subseteq A$ and $B_i \subseteq B_{i+1} \subseteq B$. Let $A' = A - \bigcup A_i$, $B' = B - \bigcup B_i$ and $F' = F(A' \cup B')$. For $x \in A'$, we have $d_F(x : B - B_{i-1}) > a(x)$ for every $i \geq 1$ implying that $d_{F'}(x) = d_F(x : B') > a(x)$ and, hence, $B' \neq \emptyset$. For $y \in B'$, we have $d_F(y : A_i) < R - d$ for every $i \geq 1$ and, therefore, $d_F(y : \bigcup A_i) < R - d$. This implies that $d_{F'}(y) = d_F(y : A') = d_F(y : A) - d_F(y : \bigcup A_i) > R - (R - d) = d$ and, hence, $A' \neq \emptyset$. This proves Lemma 3.4 in case of $A' \neq \emptyset$. If $A' = \emptyset$, then $B' = \emptyset$ and, therefore, $A = \bigcup A_i$ and $B = \bigcup B_i$. Let $E = \{xy \in E(F) | x \in A_i$ and $y \in B - B_{i-1}$ for some $i \geq 1\}$. Then

$$(R-d)|B| \leq |E| \leq \sum_{x \in A} a(x).$$

Thus Lemma 3.4 is proved. □

4 List critical graphs

For the proof of Theorem 1.3 we need the following result.

Lemma 4.1. *Asumme that $G \neq K_k$ is a Φ–critical graph where Φ is a list for G satisfying $|\Phi(x)| = k-1$ for every $x \in V(G)$. Let $L = \{x \in V(G) \mid d_G(x) = k-1\}$, $X \subseteq L$, $Y \subseteq \{y \in V(G) \mid d_G(y) = k\}$ and let $W = W^k(G(X))$. Furthermore, let $\mathcal{C}$ be the set of all components of $G(X)$ and, for $y \in Y$ and $T \in \mathcal{C}$, let*

$$d(y) = |\{T \in \mathcal{C} \ \mid \ N_G(y : V(T) \cap W) \neq \emptyset\}|$$

and

$$d(T) = |\{y \in Y \ \mid \ N_G(y : V(T) \cap W) \neq \emptyset\}|.$$

Then the following statements hold.

(a) *If $k \geq 5$, then $d(y) \geq d_G(y : W) - 1$ for every $y \in Y$.*

(b) *If $\Phi(x) = \{1, \ldots, k-1\}$ for every $x \in V(G)$ and $k \geq 5$, then $d(y) \leq 4$ for some $y \in Y$ or $d(T) \leq s(T) + 3$ for some $T \in \mathcal{C}$.*

(c) *If every member of $\mathcal{C}$ is an ϵ_k–graph and $k \geq 9$, then $d(y) \leq 3$ for some $y \in Y$ or $d(T) \leq 3$ for some $T \in \mathcal{C}$.*

The proof of this result is given in subsection 4.2. First, we show how Lemma 4.1 is used in the proof of Theorem 1.3.

4.1 Proof of Theorem 1.3

In this subsection, let $G \neq K_k$ be a Φ–critical graph where $k \geq 4$ and Φ is a list for G satisfying $|\Phi(x)| = k-1$ for every $x \in V(G)$. Let $L = \{x \in V(G) \mid d_G(x) = k-1\}$, $H = \{x \in V(G) \mid d_G(x) \geq k\}$, $W = W^k(G(L))$ and $L' = L - W$. Moreover, let $\mathcal{C}$ be the set of all components of $G(L)$ and let $\mathcal{D}$ be the set of all components of $G(W)$. By Lemma 1.7, $H \neq \emptyset$, $\mathcal{C} \subseteq \mathcal{T}_k$ and, therefore, every member of $\mathcal{D}$ is an ϵ_k–graph. Let σ and τ_c be defined as in Theorem 1.3 and, for $y \in H$, let

$$\tau_c(y) = d_G(y : H) + (k - c - \frac{2}{k-1})(d_G(y) - k).$$

Then we have

$$\sigma = \sigma(G(L)) = \sum_{T \in \mathcal{C}} \sigma(T) \text{ and } \tau_c = \sum_{y \in H} \tau_c(y).$$

From Lemma 2.3 (a) it then follows that

$$\sigma \geq \sum_{T \in \mathcal{C}} (\sigma'(T) + s(T)\alpha_k + 2 - \alpha_k)$$

provided that $k \geq 6$. If $x \in L$ belongs to some component $T \in \mathcal{C}$, then $d_G(x : H) = k - 1 - d_T(x)$. Therefore, Lemma 2.3 (b) implies that

$$\sigma \geq \sum_{x \in L'} \alpha_k d_G(x : H) + \sum_{T \in \mathcal{C}} (s(T)\alpha_k + 2 - \alpha_k) \tag{2}$$

provided that $k \geq 6$. Let $H' = \{y \in H \mid d_G(y) = k\}$ and let

$$S = \sigma + \sum_{y \in H'} \tau_c(y) = \sigma + \sum_{y \in H'} d_G(y : H).$$

For the proof of Theorem 1.3 we consider two cases.

Case 1: Assume that $\Phi(x) = \{1, \ldots, k-1\}$ for every $x \in V(G)$, $k \geq 6$ and $c = (k-5)\alpha_k$. We have to show that $\sigma + \tau_c \geq c|H|$. Since, for $y \in H - H'$, we have $\tau_c(y) \geq k - c - \frac{2}{k-1} \geq c$, it is sufficient to show that $S \geq c|H'|$. To prove this, we define a bipartite graph $F = F(A, B)$ subject to the following conditions.

(a) $B = H'$ and A is the disjoint union of the sets A_1, A_2 and A_3.

(b) $A_1 = \mathcal{C}$ and, for $T \in \mathcal{C}$, $N_F(T) = \{y \in H' \mid N_G(y : V(T) \cap W) \neq \emptyset\}$.

(c) For each vertex x of L', let $A_2(x)$ be a set of $n_x = d_G(x : H')$ vertices joined in F to the set $N_G(x : H')$ by an matching with n_x edges. Let $A_2 = \bigcup_{x \in L'} A_2(x)$.

(d) For each vertex $y \in B$, let $A_3(y)$ be a set of $d_G(y:H)$ vertices which are all joined to y in F. Let $A_3 = \bigcup_{y \in B} A_3(y)$.

Now, we use Lemma 4.1 with $X = L$ and $Y = H'$. Then, for $y \in B$, we have $d_F(y : A_1) = |\{T \in \mathcal{C} \mid N_G(y : V(T) \cap W) \neq \emptyset\}| \geq d_G(y : W) - 1$ and, therefore,

$$\begin{aligned} d_F(y) &= d_F(y:A_1) + d_F(y:A_2) + d_F(y:A_3) \\ &\geq d_G(y:W) - 1 + d_G(y:L') + d_G(y:H) \\ &= d_G(y:L) + d_G(y:H) - 1 = d_G(y) - 1 = k-1. \end{aligned}$$

From (2) we infer that

$$\begin{aligned} S &\geq \sum_{x \in L'} \alpha_k d_G(x:H) + \sum_{T \in \mathcal{C}} (s(T)\alpha_k + 2 - \alpha_k) + \sum_{y \in H'} d_G(y:H) \\ &\geq \alpha_k |A_2| + \sum_{T \in \mathcal{C}} ((s(T)+3)\alpha_k + |A_3| \\ &\geq \alpha_k(|A_2| + \sum_{T \in A_1} ((s(T)+3) + |A_3|) \end{aligned}$$

Now, we apply Lemma 3.4 to $F = F(A,B)$ where $R = k-1$, $d = 4$ and $a(x) = 1$ if $x \in A_2 \cup A_3$ and $a(x) = s(T)+3$ if $x = T \in A_1$. If $(R-d)|B| \leq \sum_{x \in A} a(x)$, then the above inequality for S implies $S \geq \alpha_k(k-5)|B| = c|H'|$. Otherwise, by Lemma 3.4, there are non-empty subsets $A' \subseteq A$ and $B' \subseteq B = H'$ such that for $F' = F(A' \cup B')$ we have $d_{F'}(x) > a(x)$ for every $x \in A'$ and $d_{F'}(y) > d = 4$ for every $y \in B'$. Since every vertex of $A_2 \cup A_3$ has degree 1 in F, we have $A' \subseteq A_1 = \mathcal{C}$. This gives a contradiction to Lemma 4.1 (b) where $X = \bigcup_{T \in A'} V(T)$ and $Y = B'$.

Case 2: Assume that $k \geq 9$ and $c = \frac{1}{3}(k-4)\alpha_k$. We have to show that $\sigma + \tau_c \geq c|H|$. Since, for $y \in H - H'$, we have $\tau_c(y) \geq c$, it is sufficient to show that $S \geq c|H'|$. Let $F^* = F^*(A^*, B)$ be the bipartite graph obtained from $F - A_1$ by adding the set $A_1^* = \mathcal{D}$ where $y \in B$ and $T \in \mathcal{D}$ are joined by an edge in F^* iff $N_G(y : V(T) \cap W) \neq \emptyset$. Since every ϵ_k–graph $T \in \mathcal{D}$ is an ϵ_k–subcomponent of some member in $\mathcal{C}$, we infer from (2) that

$$S \geq \sum_{x \in L'} \alpha_k d_G(x:H) + \sum_{T \in \mathcal{D}} \alpha_k + \sum_{y \in H'} d_G(y:H) \geq \alpha_k |A^*|.$$

We use Lemma 4.1 with $X = W = W^k(G(L))$ and $Y = H'$. As in case 2, we have $d_{F^*}(y) \geq k-1$ for every $y \in B$. Now, we apply Lemma 3.4 to F^* where $R = k-1$, $d = 3$ and $a(x) = 3$ for every $x \in A^*$. If $(R-d)|B| \leq 3|A^*|$, then we obtain $S \geq \alpha_k|A^*| \geq \frac{1}{3}(k-4)\alpha_k|B| = c|H'|$. Otherwise, by Lemma 3.4, there are non-empty subsets $A' \subseteq A^*$ and $B' \subseteq B = H'$ such that for $F' = F^*(A' \cup B')$ we have $d_{F'}(x) > 3$ for every $x \in A'$ and $d_{F'}(y) > 3$ for every $y \in B'$. Since every vertex of $A_2 \cup A_3$ has degree 1 in F^*, we have $A' \subseteq A_1^* = \mathcal{D}$. This gives a contradiction to Lemma 4.1 (c) where $X = \bigcup_{T \in A'} V(T)$ and $Y = B'$. Thus Theorem 1.3 is proved. □

4.2 Proof of Lemma 4.1

Let G be a graph, $z \in V(G)$, and let Φ be a list for G. We call (G, z, Φ, k) a *configuration of type* 1 if
(1) $G - z$ is a Gallai forest, $G \neq K_k$,
(2) $d_G(z) \leq k$, $d_G(x) \leq k-1$ for every $x \in V(G-z)$,
(3) $|\Phi(z)| \geq d_G(z) - 1$, and $|\Phi(x)| \geq d_G(x)$ for every $x \in V(G-z)$.

Lemma 4.2. *Let (G, z, Φ, k) be a configuration of type 1 where $k \geq 5$, let m be the number of components of $G - z$ and let $W = W^k(G-z)$. Assume that $V(T) \cap W \neq \emptyset$ for every component T of $G - z$. If G is not Φ–colourable, then $m \geq |N_G(z : W)| - 1$.*

Proof. Let us consider a possible counterexample (G, z, Φ, k) such that $|V(G)|$ is minimum. Let $T_1, \ldots, T_m$ denote the components of $G - z$. For $i = 1, \ldots, m$, let $d_i = |N_G(z : V(T_i) \cap W)|$. We may assume that $d_1 \geq d_2 \geq \ldots \geq d_m$. Then $d_m \geq 1$ and $m \leq |N_G(z : W)| - 2 = d_1 + \ldots d_m - 2$. We claim that $m = 1$ and $d_1 \geq 3$ or $m = 2$ and $d_1 = d_2 = 2$. Obviously, if $m = 1$, then $d_1 \geq 3$. Now, assume $m \geq 2$. Let $T = T_m$. Then $|\Phi(x)| \geq d_G(x) \geq d_T(x)$ for all $x \in V(T)$. Since z is adjacent in G to some vertex x of T, we have $|\Phi(x)| > d_T(x)$ for this vertex x. Therefore, by Lemma 1.5, there is a Φ–colouring φ of T. Let $G' = G - V(T)$ and $\Phi' = \Phi(V(T), \varphi)$ (see Remark 1.6). Since G is not Φ–colourable, G' is not Φ'-colourable. Moreover, it is easy to check that (G', z, Φ', k) is a configration of type 1 satisfying the assumption of Lemma 4.2. Therefore, $m - 1 \geq |N_{G'}(z : W^k(G' - z))| - 1 = d_1 + \ldots + d_{m-1} - 1$ implying that $m = 2$ and $d_1 = d_2 = 2$. This proves our claim. Now, we consider two cases.

Case 1: $m = 2$ and $d_1 = d_2 = 2$. Let $i \in \{1, 2\}$ and let $G_i = G(V(T_i) \cup \{z\})$. For $x \in V(G_i - z)$, we have $|\Phi(x)| \geq d_G(x) = d_{G_i}(x)$. Since z has at least two neigbours in G_i that belongs to $(k-1)$–cliques of $G_i - z$ and every vertex of $G_i - z$ has degree at most $k - 1$ where $k \geq 5$, we conclude that G_i is not a Gallai tree and $|\Phi(z)| \geq d_G(z) - 1 > d_{G_i}(z)$.

Let M_i be the set of all colours $a \in \Phi(z)$ such that $\varphi(z) \neq a$ for every Φ–colouring φ of G_i. Since G is not Φ–colourable, $M_1 \cup M_2 = \Phi(z)$. From $|\Phi(z)| \geq d_G(z) - 1 = d_{G_1}(z) + d_{G_2}(z) - 1$ we conclude that $|M_i| \geq d_{G_i}(z)$ for some i, say $i = 1$. Now, let Φ' be the list for G_1 with $\Phi'(x) = \Phi(x)$ for $x \in V(G_1 - z)$ and $\Phi'(z) = M_1$. Since G_1 is a connected graph but not a Gallai tree, we infer from Lemma 1.5 that G_1 is Φ'–colourable. This implies that there is a Φ–colouring φ of G_1 with $\varphi(z) \in M_1$, a contradiction.

Case 2: $m = 1$ and $d_1 \geq 3$. Then $T = G - z \in \mathcal{T}_k$. Since G is not Φ–colourable, we may assume that $|\Phi(x)| = d_G(x)$ for all $x \in V(G - z)$. Let B be an arbitrary end-block of T and let X be the set of all non-separating vertices of T that belong to B. Consider a vertex $y \in X$. Since $|\Phi(y)| = d_G(y) \geq 1$, there is a colour $a \in \Phi(y)$. Let $G' = G - y$ and $\Phi' = \Phi(y, a)$. Then G' is not Φ'–colourable and (G', z, Φ', k) is a configuration of type 1. If no vertex of B belongs to $N_G(z : W)$, then $N_G(z : W) = N_{G'}(z : W^k(G'))$ and, therefore, (G', z, Φ', k) is a smaller counterexample, a contradiction. Hence $|V(B) \cap N_G(z : W)| \geq 1$. Since $d_G(x) \leq k - 1$ for all $x \in V(T)$, this implies that B is a K_{k-1}.

Let $y \in V(B) \cap N_G(z : W)$. Since $d_G(y) \leq k - 1$, we have $|\Phi(y)| = d_G(y) = k - 1$ and $y \in X$. We claim that $X \subseteq N_G(z : W)$. Suppose, on the contrary, that there is a vertex $x \in X - N_G(z : W)$. Then $|\Phi(x)| = d_G(x) = k - 2$ and, therefore, there is a colour $a \in \Phi(y) - \Phi(x)$. Since $|\Phi(z)| \geq d_G(z) - 1 \geq 3$, there is a colour $b \in \Phi(z)$ with $b \neq a$. Let $\Phi' = \Phi(z, b)$. Then $T = G - z$ is not Φ'–colourable and $|\Phi'(u)| \geq d_T(u)$ for all $u \in V(T)$. Therefore, (T, Φ') is a bad pair and $\Phi'(x) \neq \Phi'(y)$, a contradiction to Lemma 1.8. This proves our claim, i.e. $X \subseteq N_G(z : W)$. If B is the only block of T, then $X = V(B) = V(T)$ and, therefore $G = K_k$, a contradiction. Hence, there is an end-block $B' \neq B$ of T. For the set X' of all vertices of B' that are non-separating vertices of T, we have $X' \subseteq N_G(z : W)$. This yields $d_G(z) \geq |X| + |X'| \geq 2(k - 2) \geq k + 1$, a contradiction.

Thus Lemma 4.2 is proved. □

Proof of Lemma 4.1 (a). Consider a vertex $y \in Y$. Let $\mathcal{C}'$ be the set of all components $T \in \mathcal{C}$ satisfying $N_G(z : W \cap V(T)) \neq \emptyset$, let $X' = \{z\} \cup \bigcup(V(T))$ where the union is taken over all $T \in \mathcal{C}'$ and let $G' = G(X')$. Then $d(y) = |\mathcal{C}'|$ and $d_G(y : W) = |N_{G'}(z : W^k(G' - z))|$. Since G is Φ–critical, there is a Φ–colouring φ of $G - X'$. Let $\Phi' = \Phi(V(G) - X', \varphi)$. Then G' is not Φ'-colourable and, see Remark 1.6, (G', z, Φ', k) is a configuration of type 1. From Lemma 4.2 we then infer that $d(y) \geq d_G(y : W) - 1$. This proves statement (a) of Lemma 4.2. □

Lemma 4.3. *Assume that (T, Φ) is a bad pair where $T \in \mathcal{T}_k$ and $\Phi(z) \subseteq \{1, \ldots, k-1\}$ for every $z \in V(T)$. Let x, y be two non-separating vertices of T contained in the same ϵ_k–subcomponent of T, then $\Phi(x) = \Phi(y)$.*

Proof. By Lemma 1.8, $\Phi = \Phi_u$ for some mapping $u \in \mathcal{U}(G)$. If x, y are contained in the same block, then the statement is evident. Otherwise, since x, y belong to the same component of $T(W^k(T))$ and $T \in \mathcal{T}_k$, there is a sequence $B_1, B_2, \ldots, B_{2l+1}$ of blocks of T such that $x \in V(B_1)$, $y \in V(B_{2l+1})$, B_{2i+1} is a K_{k-1} for $i = 0, \ldots, l$ and B_{2i} is a K_2 for $i = 1, \ldots, l$ and $V(B_i) \cap V(B_{i+1}) \neq \emptyset$ for $i = 1, \ldots, l-1$. Then $u(B_i) \cap u(B_{i+1}) = \emptyset$ for $i = 1, \ldots, l-1$. Since $u(B_i)$ is a subset of $\{1, \ldots, k-1\}$, $|u(B_{2i+1})| = k-2$ and $|u(B_{2i})| = 1$, we conclude that $u(B_1) = u(B_{2l+1})$ and, therefore $\Phi(x) = \Phi(y)$.

Thus Lemma 4.3 is proved. □

Proof of Lemma 4.1 (b). Suppose, on the contrary, that $d(y) \geq 5$ for every $y \in Y$ and $d(T) \geq s(T) + 4$ for every $T \in \mathcal{C}$.

Now, let $F = F(A, B)$ be the bipartite graph with $A = \mathcal{C}$ and $B = Y$ where $N_F(T) = \{y \in Y \mid N_G(y : V(T) \cap W) \neq \emptyset\}$ for every $T \in A$. Then, clearly, $d_F(y) = d(y) \geq 5$ for every $y \in Y$ and $d_F(T) = d(T) \geq s(T) + 4$ for every $T \in \mathcal{C}$. If $N_G(y : V(T) \cap W) \neq \emptyset$ where $y \in Y$ and $T \in \mathcal{C}$, then there is an ϵ_k–subcomponent T' of T such that $N_G(y : V(T') \cap W) \neq \emptyset$. Therefore, for every $T \in A = \mathcal{C}$, there exists a partition $\mathcal{P}(T)$ of $N_F(T)$ such that for every $N \in \mathcal{P}(T)$ there is an ϵ_k–subcomponent T' of T with $N \subseteq \{y \in Y \mid N_G(y : V(T') \cap W) \neq \emptyset\}$. Then $d_F(T) \geq s(T) + 4 \geq |\mathcal{P}(T)| + 4$ for every $T \in A$ and, by Lemma 3.3, there is a subgraph F' of F such that, for every $T \in A$, $d_{F'}(T) = 2$ and $N_{F'}(T) \subseteq N$ for some $N \in \mathcal{P}(T)$ and, for every $y \in B$, $d_{F'}(y) \leq d_F(y) - 2$.

Now, let G' be the graph with $V(G') = V(G) - X$ and $E(G') = E(G - X) \cup \{yy' \mid N_{F'}(T) = \{y, y'\} \,\&\, T \in A\}$. For $y \in Y$, we have $d_G(y) = k$ and, therefore, $d_{G'}(y) \leq d_G(y) - d_F(y) + d_{F'}(y) \leq k-2$. Since G is Φ–critical, there is a Φ–coluring φ of $G - X - Y = G' - Y$. For every $y \in Y$, we have $|\Phi(y)| = k - 1 \geq d_{G'}(y) + 1$. This implies that φ can be extended to some Φ–colouring φ' of G'. Let $\Phi^* = \Phi(V(G'), \varphi')$ and $G^* = G - V(G') = G(X)$. Then G^* is not Φ^*-colourable and $|\Phi^*(x)| \geq d_{G^*}(x)$ for every $x \in X$. Therefore, there is a component T of $G(X)$, such that (T, Φ_1) is a bad pair where Φ_1 is the restrictzion of Φ^* to T. Consider the two vertices y_1, y_2 of $N_{F'}(T)$. Then there is a ϵ_k–subcomponent T' of T and two vertices x_1, x_2 in $V(T')$ such that x_iy_i is an edge of G for $i = 1, 2$. Since every vertex of T has degree $k-1$ in G and T' is an ϵ_k–subcomponent of T it follows that x_1 and x_2 are two distinct non-separating vertices of T and y_i is the only vertex in $V(G')$ adjacent to x_i in G. Therefore, $\Phi_1(x_i) = \Phi(x_i) - \{\varphi'(y_i)\} = \{1, \ldots, k-1\} - \{\varphi'(y_i)\}$ for $i = 1, 2$. Since $\varphi'(y_1) \neq \varphi'(y_2)$, this yields $\Phi_1(x_1) \neq \Phi_1(x_2)$, a contradiction to Lemma 4.3.

Thus Lemma 4.1 (b) is proved. □

Let G be a graph, let F be a subgraph of G, $Y \subseteq V(G)$ and let Φ be a list for G. Then we call (G, F, Y, Φ) a *configuration of type* 2 if the following conditions hold.

(1) $G - Y$ is a Gallai forest.
(2) $|\Phi(x)| \geq d_G(x)$ for every $x \in V(G) - Y$.
(3) $|\Phi(y)| \geq d_G(y : Y) + d_F(y) + 1$ for every $y \in Y$.
(4) For every component T of $G - Y$ there are two edges $x_1y_1, x_2y_2 \in E(F)$ such that x_1, x_2 are two distinct non-separating vertices of T, $y_1 \neq y_2$ and, for $i = 1, 2$, $N_G(x_i : Y) = \{y_i\}$. Moreover, if B_i $(i = 1, 2)$ is the only block of T containing x_i, then $B_1 = B_2$ or , for some $i \in \{1, 2\}$, there is a non-separating vertex x of T such that $x \in B_i$ and $N_G(x : Y) = \emptyset$.

Lemma 4.4. *If* (G, F, Y, Φ) *is a configuration of type* 2, *then* G *is* Φ–*colourable.*

Proof (by induction on $m = |V(G) - Y|$). If $m = 0$, then $G = G(Y)$ and $|\Phi(y)| \geq d_G(y) + 1$ for every $y \in V(G)$ implying that G is Φ-colourable. Now, assume $m \geq 1$. Then there is a component T of $G - Y$ and two edges $x_1y_1, x_2y_2 \in E(F)$ subject to condition (4). For $i = 1, 2$, let B_i be the only block of T containing x_i. Let $G' = G - V(T)$ and $F' = F - V(T)$. We consider two cases.

Case 1: $B_1 = B_2$. First, assume $\Phi(x_1) = \Phi(x_2)$. Let G^* be the graph obtained from G' by adding the edge y_1y_2. Then (G^*, F', Y, Φ) is a configuration of type 2 and, by the induction hypothesis, there is a Φ–colouring φ of G^*. Consider the list $\Phi' = \Phi(V(G^*), \varphi)$ for $T = G - V(G^*)$. If T is not Φ'–colourable, then (T, Φ') is a bad pair and, since y_i is the only vertex in Y adjacent to x_i in G and $\varphi(y_1) \neq \varphi(y_2)$, $\Phi'(x_1) \neq \Phi'(x_2)$, a contradiction to Lemma 1.8. Hence, T is Φ'-colurable implying that G is Φ–colourable.

Now, assume $\Phi(x_1) \neq \Phi(x_2)$, say $a \in \Phi(x_1) - \Phi(x_2)$. Let Φ' be the list obtained from Φ by removing colour a from $\Phi(y_1)$. Then (G', F', Y, Φ') is a configuration of type 2 and, by the induction hypothesis, there is a Φ'–colouring φ of G'. Consider the list $\Phi_1 = \Phi(V(G'), \varphi)$ for $T = G - V(G')$. Then $\Phi_1(x_1) \neq \Phi_2(x_2)$ and, by Lemma 1.8, it follows that T is Φ_1-colourable implying that G is Φ–colourable.

Case 2: $B_1 \neq B_2$. Then, because of condition (4), in one of these two blocks, say in B_1, there is a non-separating vertex x of T such that $N_G(x : Y) = \emptyset$. We may assume that $|\Phi(x)| = d_G(x)$. Then $|\Phi(x_1)| \geq d_G(x_1) > d_G(x)$ impliying that there is a colour $a \in \Phi(x_1) - \Phi(x)$. Let Φ' be the list obtained from Φ by removing colour a from $\Phi(y_1)$. Then (G', F', Y, Φ') is a configuration of type 2 and, by the induction hypothesis, there is a Φ'–colouring φ of G'. Consider the list $\Phi_1 = \Phi(V(G'), \varphi)$ for $T = G - V(G')$. Then $\Phi_1(x_1) \neq \Phi_2(x)$ and, by Lemma 1.8, it follows that T is Φ_1-colurable implying that G is Φ–colourable.

Thus Lemma 4.4 is proved. □

Proof of Lemma 4.1 (c). Suppose, on the contrary, that $d(y) \geq 4$ for every $y \in Y$ and $d(T) \geq 4$ for every $T \in \mathcal{C}$. Let $G_1 = G(X \cup Y)$. Since G is Φ–critical there is a Φ–colouring φ of $G' = G - X - Y$. Let $\Phi_1 = \Phi(V(G'), \varphi)$ be the list for G_1. To arrive at a contradiction, we shall show that that there is a Φ_1–colouring of G_1 and, therefore, a Φ–colouring of G.

For $Z \subseteq X$, let $N(Z) = \{y \in Y \mid N_G(y : Z) \neq \emptyset\}$, and, for a set of blocks $\mathcal{B}$ of $G(X)$, let $X(\mathcal{B})$ be the set of all vertices contained in some block of $\mathcal{B}$.

Consider an arbitrary component $T \in \mathcal{C}$ of $G(X)$. Since T is an ϵ_k–graph, $V(T) \subseteq W = W^k(G(X)) = X$ and, therefore, $|N(V(T))| \geq 4$. Let S denote the set of all vertices of T that are in G adjacent to a vertex of Y and let R denote the set of all non-separating vertices of T. Since every vertex of T has degree $k-1$ in G, $S \subseteq R$. From Lemma 4.1 (a) it follows that every vertex of Y has in G at most two neighbours belonging to T. This implies, in particular, that $|N(Z)| \geq 4$ provided that $|Z \cap S| \geq 7$.

Let $\mathcal{B}_1$ denote the set of all blocks B of T such that $V(B) \cap (R-S) \neq \emptyset$. We claim that there is a set $\mathcal{B} = \mathcal{B}_T$ of blocks of T such that all but at most one block of $\mathcal{B}$ belongs to $\mathcal{B}_1$ and $|N(X(\mathcal{B}))| \geq 4$. If some end-block B of T is not in $\mathcal{B}_1$, then $V(B) \cap R \subseteq S$ and, since B is a K_{k-1}, $|V(B) \cap S| = |V(B) \cap R| = k-2 \geq 7$ implying that the claim is satisfied for $\mathcal{B} = \{B\}$. Now, assume that every end-block of T belonge to $\mathcal{B}_1$ and $|N(\mathcal{B}_1)| \leq 3$. Since $|N(V(T))| \geq 4$, there is a block B of T not contained in $\mathcal{B}_1$. Let $\mathcal{B} = \mathcal{B}_1 \cup \{B\}$. Since $\emptyset \neq V(B) \cap R \subseteq S$ and T has at least $|V(B) - R|$ end-blocks, we conclude that B is a K_{k-1} and $|X(\mathcal{B}) \cap S| \geq |V(B)| = k-1 \geq 8$ and, therefore, $|N(X(\mathcal{B}))| \geq 4$. This proves our claim.

Now, let $F = F(\mathcal{C}, Y)$ be the bipartite graph where $N_F(T) = N(X(\mathcal{B}_T))$. Then $d_F(T) \geq 4$ for every $T \in \mathcal{C}$ and, by Lemma 3.2, there is a subgraph F' of F such that $d_{F'}(T) = 2$ for every $T \in \mathcal{C}$ and $d_{F'}(y) \leq d_F(y) - 2$ for every $y \in Y$ with $d_F(y) \geq 4$ and $d_{F'}(y) \leq 2$ for every $y \in Y$ with $d_F(y) \geq 3$.

For every component $T \in \mathcal{C}$, $N_{F'}(T)$ consists of two distinct vertices $y_1(T), y_2(T)$ and, moreover, there are two distinct vertices $x_1(T), x_2(T) \in X(\mathcal{B}_T)$ such that $x_i(T)$ is in G adjacent to $y_i(T)$ for $i = 1, 2$. Let F_1 be the subgraph of G_1 with the same vertex set as G_1 and with $E(F_1) = \{x_i(T)y_i(T) \mid T \in \mathcal{C} \;\&\; i = 1, 2\}$. Then (G_1, F_1, Y, Φ_1) is a configuration of type 2 and, therefore, G_1 is Φ_1–colourable. This contradiction proves Lemma 4.1 (c). □

5 Concluding remarks

The main result of this paper is that $2f_k(n) \geq g_k(n, c)$ where $c = (k-5)\alpha_k$ and $k \geq 6$. Our method of proof yields two upper bounds for the possible values of the constant c, namely $c \leq k - 2/(k-1)$ (see Lemma 1.4) and $c \leq \frac{1}{2}(k - 2/(k-1))$ (see the proof of Theorem 1.3, the part where we show that $\tau_c(y) \geq c$ provided that $d_G(y) > k$). For integers p, k satisfying $k \geq 4$

and $2 \leq p \leq k$, let

$$c_{k,p} = f_k(k+p) - \frac{1}{2}g_k(k+p, k - \frac{2}{k-1})$$

be a constant depending on k and p, and let

$$h_{k,p}(n) = \frac{1}{2}g_k(n, k - \frac{2}{k-1}) + c_{k,p} = \frac{1}{2}(k - 1 + \frac{k-3}{k-1})n + c_{k,p}.$$

We claim that if $n \geq k+2$ and $n \equiv p-1 \mod (k-1)$ where $2 \leq p \leq k$, then there is a k–critical graph with n vertices and $h_{k,p}(n)$ edges implying that

$$2f_k(n) \leq 2h_{k,p}(n) = g_k(n, k - \frac{2}{k-1}) + 2c_{k,p}. \tag{3}$$

For $n = k+p$, we have $h_{k,p}(n) = f_k(n)$ and the claim is evidently true. Now, assume $n \equiv p-1 \mod (k-1)$. If G is a k–critical graph with n vertices and $h_{k,p}(n)$ edges, then we apply the Hajós construction (see [11] or [12]) to G and K_k. This results in a k–critical graph with $n+k-1$ vertices and $m = |E(G)| + \binom{k}{2} - 1$ edges. By an easy calculation, we then obtain $m = h_{k,p}(n) + \binom{k}{2} - 1 = h_{k,p}(n+k-1)$. This proves our claim. By a conjecture of Ore [23] (see also problem 5.3 in [12]) it follows that we have equality in (3). In [10] Gallai proved that $2f_k(k+p) = (k-1)(k+p) + p(k-p)$ provided that $2 \leq p \leq k-1$ and in [15] it was proved that $f_k(2k) = k^2 - 3$.

References

[1] O.V. Borodin, Criterion of chromaticity of a degree prescription (in Russian), in: Abstracts of IV All-Union Conf. on Theoretical Cybernetics (Novosibirsk) 1977, 127-128.

[2] O.V. Borodin, Problems of colouring and of covering the vertex set of a graph by induced subgraphs (in Russian), Ph.D.Thesis, Novosibirsk State University, Novosibirsk, 1979.

[3] R.L. Brooks, On colouring the nodes of a network, Proc. Cambridge Phil. Soc. 37 (1941) 194-197.

[4] W.A. Deuber, A.V. Kostochka and H. Sachs, A short proof of Dirac's theorem on the number of edges in chromatically critical graphs. Diskretnyi analiz i issledovanie operacii (in Russian), 3 (1996), No. 4, 28-34.

[5] G.A. Dirac, Note on the colouring of graphs. Math. Z. 54 (1951), 347-353.

[6] G.A. Dirac, A theorem of R.L. Brooks and a conjecture of H. Hadwiger, Proc. London Math. Soc. (3) 7 (1957), 161-195.

[7] G.A. Dirac, The number of edges in critical graphs, J. Reine Angew. Math. 268/269 (1974), 150-164.

[8] P. Erdös, A.L. Rubin and H. Taylor, Choosability in graphs, Proc. West Coast Conf. on Combinatorics, Graph Theory and Computing, Congressus Numerantium XXVI (1979), 125-157.

[9] T. Gallai, Kritische Graphen I, Publ. Math. Inst. Hungar. Acad. Sci. 8 (1963), 165-192.

[10] T. Gallai, Kritische Graphen II, Publ. Math. Inst. Hungar. Acad. Sci. 8 (1963), 373-395.

[11] G. Hajós, Über eine Konstruktion nicht n–färbbarer Graphen, Wiss. Z. Martin-Luther-Univ. Halle-Wittenberg Math.-Naturw. Reihe 10 (1961), 116-117.

[12] T.R. Jensen and B. Toft, Graph Coloring Problems, Wiley-Interscience Series in Discrete Mathematics and Optimization, John Wiley & Sons, New York, 1995.

[13] A.V. Kostochka, M. Stiebitz and B. Wirth, The colour theorems of Brooks and Gallai extended. Discrete Math. 162 (1996), 299-303.

[14] A.V. Kostochka and M. Stiebitz, Colour-critical graphs with few edges, to appear in Discrete Math..

[15] A.V. Kostochka and M. Stiebitz, Excess in colour-critical graphs, submitted.

[16] M. Krivelevich, An improved bound on the minimal number of edges in color-critical graphs, to appear in Combinatorica.

[17] M. Krivelevich, On the minimal number of edges in color-critical graphs, to appear in Combinatorica.

[18] H. V. Kronk and J. Mitchem, On Dirac's generalization of Brooks' theorem, Canad. J. Math., 24(1972), 805-807.

[19] L. Lovász, Combinatorial Problems and Exercises, Akad. Kiadó, Budapest, 1979.

[20] C.Thomassen, Color-critical graphs on a fixed surface, Report, Technical University of Denmark, Lyngby, 1995.

[21] V.G. Vizing, Colouring the vertices of a graph with prescribed colours (in Russian), Metody Diskretnogo Analiza v Teorii Kodov i Skhem No 29 (1976) 3-10.

[22] J. Mitchem, A new proof of a theorem of Dirac on the number of edges in critical graphs, J. Reine u. Angew. Math. 299/300 (1978), 84-91.

[23] O. Ore, The Four Colour Problem, Academic Press, New York, 1967.

[24] B. Toft, Colour-critical graphs and hypergraphs, J. Combin. Th.(B), 16(1974), 145-161.

[25] J. Weinstein, Excess in critical graphs, J. Combin. Th.(B), 18(1975), 24-31.