Differential-algebraic equations. Control and Numerics V

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Outline

Introduction

- Stability analysis for linear differential-algebraic equations DAEs of the form
$$E(t)\dot{x} = A(t)x + f,$$
with variable coefficients on the half-line $\mathbb{I} = [0, \infty)$.
- They arise as linearization of nonlinear systems
$$F(t, x, \dot{x}) = 0$$
arround reference solutions.

References:

- P. Kunkel and V.M., *Stability properties of differential-algebraic equations and spin-stabilized discretizations.* ELECTRONIC TRANSACTIONS ON NUMERICAL ANALYSIS. Vol. 26, 385–420, 2007.
- V.H. Linh and V.M. *Lyapunov, Bohl and Sacker-Sell Spectral Intervals for Differential-Algebraic Equations.* JOURNAL ON DYNAMICS AND DIFFERENTIAL EQUATIONS, Vol. 21, 153–194, 2009.
- V.H. Linh, V.M., and E. Van Vleck, *QR methods and Error Analysis for Computing Lyapunov and Sacker-Sell Spectral Intervals for Linear Differential-Algebraic Equations*, PREPRINT 676, MATHEON, DFG Research Center *Mathematics for key technologies* in Berlin `url:` `http://www.matheon.de/` . To appear in ADVANCES IN COMPUTATIONAL MATHEMATICS, 2010.

Outline

Spectral theory for ODEs

Classical spectral theory for ODEs

$$\dot{x} = f(t, x), \quad t \in \mathbb{I},\ x(0) = x^0,$$

with $x \in C^1(\mathbb{I}, \mathbb{R}^n)$. By shifting the solution, we may assume that $x(t) \equiv 0$.

Definition

A constant coefficient system $\dot{x} = Ax$ with $A \in \mathbb{R}^{n,n}$ is asymptotically stable if all eigenvalues of A have negative real part.

Asympt. stab. of var. coefficient ODEs

Even if for all $t \in \mathbb{R}$, the matrix $A(t)$ has all eigenvalues in the left half plane, the system $\dot{x} = A(t)x$ may be unstable.

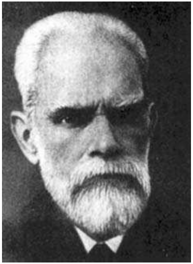
Example For all $t \in \mathbb{R}$

$$A(t) = \begin{bmatrix} \cos^2(3t) - 5\sin^2(3t) & -6\cos(3t)\sin(3t) + 3 \\ -6\cos(3t) sin(3t) + 3 & \sin^2(3t) - 5\cos^2(3t) \end{bmatrix}$$

has a double eigenvalue -2 but the solution of $\dot{x} = A(t)x$, $x(0) = \begin{bmatrix} c_1 \\ 0 \end{bmatrix}$ is

$$x(t) = \begin{bmatrix} c_1 e^t \cos(3t) \\ -c_1 e^t \cos(3t) \end{bmatrix}.$$

A. Lyapunov 1857 - 1918



Lyapunov exponents

For the linear ODE $\dot{x} = A(t)x$ with bounded coefficient function $A(t)$ and nontrivial solution x we define the *upper and lower Lyapunov exponents*,

$$\lambda^u(x) = \limsup_{t\to\infty} \frac{1}{t} \ln \|x(t)\| , \quad \lambda^l(x) = \liminf_{t\to\infty} \frac{1}{t} \ln \|x(t)\| .$$

Since A is bounded, the Lyapunov exponents are finite.

Theorem (Lyapunov 1907)

If the greatest bound of upper Lyapunov exponents for all solutions $\dot{x} = A(t)x$ is negative, then the system is asymptotically stable.

Lyapunov exp. of fundamental sol'n

For the fundamental solution of $\dot{X} = A(t)X$, the Lyapunov exponents for the i-th column of X are

$$\lambda^u(Xe_i), \quad \text{and} \quad \lambda^\ell(Xe_i), \quad i = 1, 2, ..., n, \tag{1}$$

where e_i denotes the i-th unit vector. W.l.o.g. we assume that the columns of X are ordered such that the upper Lyapunov exponents satisfy

$$\lambda^u(Xe_1) \geq \lambda^u(Xe_2) \geq ... \geq \lambda^u(Xe_n).$$

When $\sum_{i=1}^n \lambda^u(Xe_i)$ is minimized with respect to all possible fundamental solution matrices, then the columns of the corresponding fundamental solution matrix are said to form a *normal basis*. It is always possible to construct a normal basis from an arbitrary fundamental solution matrix.

Lagrange identity

Let $\{-\mu_i^u\}_{i=1}^n$ be the upper Lyapunov exponents (ordered increasingly) of the *adjoint equation*

$$\dot{y} = -A^T(t)y,$$

with associated fundamental solution matrices $Y(t)$. Then the fundamental solution matrices satisfy the *Lagrange identity*

$$Y^T(t)X(t) = Y^T(0)X(0), \quad \text{for all } t \geq 0.$$

Furthermore,

$$\lambda_i^\ell = -\mu_i^u, \quad i = 1, 2, ..., n.$$

Lyapunov spectrum

Definition

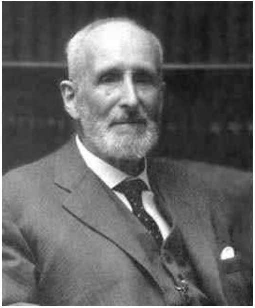
Lyapunov 1907, Perron 1930, Daleckii/Krein 1974 The *Lyapunov spectrum* Σ_L of $\dot{x} = A(t)x$ is the union of *Lyapunov spectral intervals*

$$\Sigma_L := \bigcup_{i=1}^{n} [\lambda_i^\ell, \lambda_i^u].$$

If each of the Lyapunov spectral intervals shrinks to a point, i.e., if $\lambda_i^\ell = \lambda_i^u \quad \forall i = 1, 2, ..., n$, then the system is called *Lyapunov-regular*.

If a system is Lyapunov-regular, then we simply write λ_i for the Lyapunov exponents.

Oskar Perron 1880-1975



Kinematic similarity

Definition

A change of variables $z = T^{-1}x$ with an invertible matrix function $T \in C^1(\mathbb{I}, \mathbb{R}^{n \times n})$ is called a *kinematic similarity transformation* if T and T^{-1} are bounded. If $\dot{T}$ is bounded as well, then it is called a *Lyapunov transformation*.

Theorem (Perron 1930)

For every linear ODE $\dot{x} = A(t)x$, there exists a Lyapunov transformation to upper triangular form, and this transformation can be chosen to be pointwise orthogonal.

Characterization of Lyapunov regularity

Theorem (Lyapunov 1907)

Let $B = [b_{i,j}] \in C(\mathbb{I}, \mathbb{R}^{n\times n})$ *be bounded and upper-triangular. Then the system*

$$\dot{R} = B(t)R,$$

is Lyapunov-regular if and only if all the limits

$$\lim_{t\to\infty} \frac{1}{t}\int_0^t b_{i,i}(s)ds, \quad i = 1, 2, ..., n,$$

exist and these limits coincide with the Lyapunov exponents $\lambda_i,\ i = 1, 2, ..., n$.

Stability of Lyapunov exponents

Definition

Consider $\dot{x} = A(t)x$ with upper Lyapunov exponents λ_i^u and a perturbed system $\dot{x} = [A(t) + \Delta A(t)]x$ with upper Lyapunov exponents ν_i^u, both decreasingly ordered.

- The upper Lyapunov exponents, $\lambda_1^u \geq ... \geq \lambda_n^u$, are called *stable* if for any $\varepsilon > 0$ there exists a $\delta = \delta(\epsilon) > 0$ such that $\sup_{t \geq 0} \|\Delta A(t)\| < \delta$ implies

$$|\lambda_i^u - \nu_i^u| < \epsilon, \quad i = 1, \ldots, n.$$

- A fundamental solution matrix X is called *integrally separated* if for $i = 1, 2, ..., n-1$, there exist $b > 0$ and $c > 0$ such that

$$\frac{\|X(t)e_i\|}{\|X(s)e_i\|} \cdot \frac{\|X(s)e_{i+1}\|}{\|X(t)e_{i+1}\|} \geq ce^{b(t-s)},$$

Are Lyapunov exponents stable ?

Theorem (see e.g. Dieci/Van Vleck 2006)

i) *Integral separation is invariant under Lyapunov transformations (or kinematic similarity transformations).*

ii) *An integrally separated system has pairwise distinct upper (and pairwise distinct lower) Lyapunov exponents.*

iii) *Distinct upper Lyapunov exponents are stable if and only if there exists an integrally separated fundamental solution matrix.*

iv) *If a system is integrally separated, then so is its adjoint system and thus the lower Lyapunov exponents are stable as well.*

v) *Integral separation is a generic property.*

Vladimir A Steklov, 1864-1926

Can we check integral separation ?

Definition

Consider a scalar continuous function f and suppose that $H > 0$. The *Steklov function* f^H associated with is defined by

$$f^H(t) := \frac{1}{H} \int_t^{t+H} f(\tau) d\tau.$$

Theorem (Adrianova 1995)

Two scalar continuous functions f_1, f_2 are integrally separated if and only if there exists a scalar $H > 0$ such that their Steklov difference is positive, i.e., for H sufficiently large,

$$f_1^H(t) - f_2^H(t) \geq \beta > 0, \quad \forall t \geq 0.$$

Computation of Lyapunov exponents

Theorem (Dieci/Van Vleck 2002)

A system $\dot{x} = B(t)x$ with B bounded, continuous, and triangular, has an integrally separated fundamental solution matrix iff the diagonal elements of B are integrally separated.
If the diagonal of B is integrally separated, then $\Sigma_L = \Sigma_{CL}$, where

$$\Sigma_{CL} := \bigcup_{i=1}^{n} [\lambda_{i,i}^{\ell}, \lambda_{i,i}^{u}],$$

with

$$\lambda_{i,i}^{\ell} := \liminf_{t\to\infty} \frac{1}{t}\int_0^t b_{i,i}(s)ds,\ \ \lambda_{i,i}^{u} := \limsup_{t\to\infty} \frac{1}{t}\int_0^t b_{i,i}(s)ds,\ \ i = 1, 2, \ldots$$

Bohl exponents, Piers Bohl 1865-1921

Definition

Let x be a nontrivial solution of $\dot{x} = A(t)x$. The *(upper) Bohl exponent* $\kappa_B^u(x)$ of this solution is the greatest lower bound of all those numbers ρ for which there exist numbers N_ρ such that

$$\|x(t)\| \leq N_\rho e^{\rho(t-s)} \|x(s)\|$$

for any $t \geq s \geq 0$. If such numbers ρ do not exist, then one sets $\kappa_B^u(x) = +\infty$.

Similarly, the *lower Bohl exponent* $\kappa_B^\ell(x)$ is the least lower bound of all those numbers ρ' for which there exist numbers N'_ρ such that

$$\|x(t)\| \geq N'_\rho e^{\rho'(t-s)} \|x(s)\|, \quad 0 \leq s \leq t.$$

The interval $[\kappa_B^\ell(x), \kappa_B^u(x)]$ is called the *Bohl interval* of the solution.

Mark Grigorievich Krein 1907 - 1989

Bohl exponents

Theorem (Daleckii/Krein 1974)

Bohl and Lyapunov exponents are related via

$$\kappa_B^\ell(x) \le \lambda^\ell(x) \le \lambda^u(x) \le \kappa_B^u(x).$$

The Bohl exponents are given by

$$\begin{aligned} \kappa_B^u(x) &= \limsup_{s,t-s\to\infty} \frac{\ln\|x(t)\| - \ln\|x(s)\|}{t-s}, \\ \kappa_B^\ell(x) &= \liminf_{s,t-s\to\infty} \frac{\ln\|x(t)\| - \ln\|x(s)\|}{t-s}. \end{aligned}$$

If $A(t)$ is integrally bounded, i.e., if $\sup_{t\ge 0}\int_t^{t+1}\|A(s)\|\,ds < \infty$, then the Bohl exponents are finite.

Lyapunov and Bohl exponents

- Bohl exponents characterize the uniform growth rate of solutions, while Lyapunov exponents simply characterize the growth rate of solutions departing from $t = 0$.
- If the greatest bound of upper Lyapunov exponents for all solutions $\dot{x} = A(t)x$ is negative, then the system is asymptotically stable. If the same holds for the greatest upper bound of the upper Bohl exponents then the system is (uniformly) exponentially stable.
- Bohl exponents are stable without any extra assumption.

Sacker-Sell spectra

Definition

The fundamental matrix solution X of $\dot{X} = A(t)X$ is said to admit an *exponential dichotomy* if there exist a projector $P : \mathbb{R}^{n\times n} \to \mathbb{R}^{n\times n}$ and constants $\alpha, \beta > 0$, as well as $K, L \geq 1$, such that

$$\begin{aligned} \left\|X(t)PX^{-1}(s)\right\| &\leq Ke^{-\alpha(t-s)}, \quad t \geq s, \\ \left\|X(t)(I-P)X^{-1}(s)\right\| &\leq Le^{\beta(t-s)}, \quad t \leq s. \end{aligned}$$

The *Sacker-Sell (or exponential-dichotomy) spectrum* Σ_S for is given by those values $\lambda \in \mathbb{R}$ such that the *shifted system*

$$\dot{x}_\lambda = [A(t) - \lambda I]x_\lambda$$

does not have exponential dichotomy. The complement of Σ_S is called the *resolvent set*.

Properties of Sacker-Sell spectra

Theorem (Sacker/Sell 1978)

The property that a system possesses an exponential dichotomy as well as the exponential dichotomy spectrum are preserved under kinematic similarity transformations.

Σ_S *is the union of at most n disjoint closed intervals, and it is stable.*

Furthermore, the Sacker-Sell intervals contain the Lyapunov intervals, i.e.

$$\Sigma_L \subseteq \Sigma_S.$$

Characterization of Sacker-Sell spectra

Theorem (Dieci/Van Vleck 2006)

Consider $\dot{x} = B(t)x$ with B bounded, continuous, and upper triangular. The Sacker-Sell spectrum of this system and that of the corresponding diagonal system

$$\dot{x} = D(t)x, \qquad \text{with } D(t) = \operatorname{diag}(b_{1,1}(t), \ldots, b_{n,n}(t)), \qquad t \geq 0,$$

coincide.

Thus, one can retrieve Σ_S of $\dot{x} = A(t)x$ from the diagonal elements of the triangularized system.

Computation of Sacker-Sell spectra I

The Sacker-Sell spectrum of the diagonal system can be approximated as follows. For $i = 1, 2, ..., n$, and for $\lambda \in \mathbb{R}$, one introduces the two diagonal systems

$$\dot{y}_i = \begin{bmatrix} \lambda & 0 \\ 0 & b_{i,i}(t) \end{bmatrix} y_i \quad \text{and} \quad \dot{y}_i = \begin{bmatrix} b_{i,i}(t) & 0 \\ 0 & \lambda \end{bmatrix} y_i .$$

Considering the sets
$\Lambda_i := \{\lambda \in \mathbb{R} : \text{ the systems are not integrally separated}\}$,
$i = 1, 2, ..., n$ one defines the *integral separation spectrum* Σ_I for the diagonal system as

$$\Sigma_I := \bigcup_{i=1}^{n} \Lambda_i .$$

Computation of Sacker-Sell spectra II

For $H > 0$ and $i = 1, 2, ..., n$ one defines

$$\alpha_i^H := \inf_{t \geq 0} \frac{1}{H} \int_t^{t+H} b_{i,i}(s) ds \quad \text{and} \quad \beta_i^H := \sup_{t \geq 0} \frac{1}{H} \int_t^{t+H} b_{i,i}(s) ds.$$

Theorem (Dieci/Van Vleck 2002)

Consider the diagonal system and let Λ_i, $i = 1, 2, ..., n$, be the i-th interval in Σ_S for this system. Then, for $H > 0$ sufficiently large,

$$\Lambda_i = [\alpha_i^H, \beta_i^H], \quad \text{for all } i = 1, 2, ..., n.$$

Outline

Stability Theory for DAEs

- Lyapunov theory for regular constant coeff. DAEs Stykel 2002
- Index 1 systems Ascher/Petzold 1993,
- Systems of tractability index ≤ 2, Tischendorf 1994, Hanke/Macana/März 1998,
- Systems with properly stated leading term, Higueras/März/Tischendorf 2003, März 1998, März/Riazza 2002, Riazza 2002, Riazza/Tischendorf 2004.
- Lyapunov exponents and regularity, Cong/Nam 2003/2004.
- Exponential dichotomy in bound. val. problems, Lentini/März 1990.
- Exponential stability and Bohl exponents, Du/Linh 2006, 2007.
- General theory for linear DAEs Linh/M. 2008
- Perturbation theory Linh/M./Van Vleck 2009

Fundamental solution matrices

Definition

A matrix function $X \in C^1(\mathbb{I}, \mathbb{R}^{n\times k}), \quad d \leq k \leq n$, is called *fundamental solution matrix of* $E(t)\dot{X} = A(t)X$ if each of its columns is a solution to $E(t)\dot{x} = A(t)x$ and $\operatorname{rank} X(t) = d$, for all $t \geq 0$.

A fundamental solution matrix is said to be *maximal* if $k = n$ and *minimal* if $k = d$, respectively.

A maximal fundamental matrix solution, denoted by $X(t, s)$, is called *normalized* if it satisfies the *projected initial condition* $E(0)(X(0, 0) - I) = 0$..

Every fundamental solution matrix has exactly d linearly independent columns and a minimal fundamental matrix solution can be easily made maximal by adding $n - d$ zero columns.

Lyapunov exponents for DAEs

Definition

For a fundamental solution matrix X of a strangeness-free DAE system $E(t)\dot{x} = A(t)x$, and for $d \leq k \leq n$, we introduce

$$\lambda_i^u = \limsup_{t\to\infty} \frac{1}{t} \ln \|X(t)e_i\| \text{ and } \lambda_i^\ell = \liminf_{t\to\infty} \frac{1}{t} \ln \|X(t)e_i\|, \quad i = 1, 2, ..$$

The columns of a minimal fundamental solution matrix form a *normal basis* if $\Sigma_{i=1}^{d} \lambda_i^u$ is minimal. The $\lambda_i^u, i = 1, 2, ..., n$, belonging to a normal basis are called *(upper) Lyapunov exponents* and the intervals $[\lambda_i^\ell, \lambda_i^u], i = 1, 2, ..., d$, are called *Lyapunov spectral intervals*.

The DAE system is said to be *Lyapunov-regular* if $\lambda_i^l = \lambda_i^u, i = 1, 2, ..., d$.

Kinematic equivalence

Definition

Suppose that $W \in C(\mathbb{I}, \mathbb{R}^{n\times n})$ and $T \in C^1(\mathbb{I}, \mathbb{R}^{n\times n})$ are pointwise nonsingular matrix functions such that T and T^{-1} are bounded. Then the transformed DAE system

$$\tilde{E}(t)\dot{\tilde{x}} = \tilde{A}(t)\tilde{x},$$

with $\tilde{E} = WET$, $\tilde{A} = WAT - WE\dot{T}$ and $x = T\tilde{x}$ is called *globally kinematically equivalent* to $E(t)\dot{x} = A(t)x$. If, furthermore, also W and W^{-1} are bounded then we call this a *strong global kinematical equivalence transformation.*

Transformation to triangular form

Lemma

Consider a strangeness-free DAE $E(t)\dot{x} = A(t)x$ *with continuous coefficients and a minimal fundamental solution matrix* X*. Then there exist pointwise orthogonal matrix functions* $U \in C(\mathbb{I}, \mathbb{R}^{n\times n})$ *and* $V \in C^1(\mathbb{I}, \mathbb{R}^{n\times n})$ *such that in* $E\dot{X} = AX$ *the change of variables* $X = VR$*, with* $R = \begin{bmatrix} R_1 \\ 0 \end{bmatrix}$ *and* $R_1 \in C^1(\mathbb{I}, \mathbb{R}^{d\times d})$ *and the multiplication from the left with* U^T *leads to the system*

$$E_d \dot{R}_1 = A_d R_1,$$

where $E_d := U_1^T E V_1$ *is pointwise nonsingular and* $A_d := U_1^T A V_1 - U_1^T E \dot{V}_1$*. Here* U_1, V_1 *are the matrix functions consisting of the first* d *columns of* U, V*, respectively.*

Normal basis

Theorem

Let Z be a minimal fundamental solution matrix for the strangeness-free DAE $E(t)\dot{x} = A(t)x$ such that the upper Lyapunov exponents of its columns are ordered decreasingly. Then there exists a nonsingular upper triangular matrix $C \in \mathbb{R}^{d\times d}$ such that the columns of $X(t) = Z(t)C$ form a normal basis.

Adjoint equation

Definition

The DAE system

$$\frac{d}{dt}(E^T y) = -A^T y, \quad \text{or} \quad E^T(t)\dot{y} = -[A^T(t) + \dot{E}^T(t)]y, \quad t \geq 0,$$

is called the *adjoint system* associated to $E(t)\dot{x} = A(t)x$.

Lemma

Fundamental solution matrices X, Y of a strangeness-free DAE and its adjoint equation satisfy the Lagrange identity

$$Y^T(t)E(t)X(t) = Y^T(0)E(0)X(0), \; t \geq 0.$$

Transformation to block form

Theorem

Consider a reduced strangeness-free DAE system. If the coefficient matrices are sufficiently smooth, then there exists an orthogonal matrix function $\hat{Q} \in C^1(\mathbb{I}, \mathbb{R}^{n \times n})$ *such that with* $\hat{x} = \hat{Q}^T x$, *the submatrix* E_1 *is compressed, i.e., the transformed system has the form*

$$\begin{bmatrix} \hat{E}_{11} & 0 \\ 0 & 0 \end{bmatrix} \dot{\hat{x}} = \begin{bmatrix} \hat{A}_{11} & \hat{A}_{12} \\ \hat{A}_{21} & \hat{A}_{22} \end{bmatrix} \hat{x}, \quad t \geq 0,$$

Furthermore, this system is still strangeness-free and thus $\hat{E}_{11}$ *and* $\hat{A}_{22}$ *are pointwise nonsingular.*

The *associated underlying (implicit) ODE* is,

$$\hat{E}_{11} \dot{\hat{x}}_1 = \hat{A}_s \hat{x}_1,$$

Relation between Lyapunov exponents

Theorem

Let $\lambda^u(\hat{A}_{22}^{-1}\hat{A}_{21})$ *be the upper Lyapunov exponent of the matrix function* $\hat{A}_{22}^{-1}\hat{A}_{21}$*. If the* **boundedness condition**

$$\lambda^u(\hat{A}_{22}^{-1}\hat{A}_{21}) \leq 0$$

holds, then the upper Lyapunov exponents of the transformed DAE and its underlying ODE coincide if they are both ordered decreasingly.

Relation between DAE and Adjoint DAE

Theorem

Consider the transformed DAE system and suppose that

$$\lambda^u(\hat{A}_{22}^{-1}\hat{A}_{21}) \leq 0, \lambda^u(\hat{A}_{12}\hat{A}_{22}^{-1}) \leq 0, \quad \lambda^u(\hat{E}_{11}) \leq 0, \quad \lambda^u(\hat{E}_{11}^{-1}) \leq 0.$$

If λ_i^l are the lower Lyapunov exponents of the DAE and $-\mu_i^u$ are the upper Lyapunov exponents of the adjoint system, both in increasing order, then

$$\lambda_i^l = \mu_i^u, \quad i = 1, 2, ..., d,$$

Furthermore, the DAE is Lyapunov regular if and only if the adjoint DAE is regular, and in this case we have the Perron identity

$$\lambda_i = \mu_i, \quad i = 1, 2, ..., d,$$

Differences between DAEs and ODEs

Example 1 Consider the DAE

$$\begin{array}{rcr} e^{\alpha t}\dot{x}_1 & = & e^{\alpha t}\lambda x_1 + x_2, \\ 0 & = & -e^{\beta t}x_2, \end{array}$$

where $\alpha \le 0, \beta \le 0$ and λ are real. The adjoint system is

$$\begin{array}{rcl} e^{\alpha t}\dot{y}_1 & = & -(e^{\alpha t}\lambda + \alpha e^{\alpha t})y_1, \\ 0 & = & -y_1 + e^{\beta t}y_2. \end{array}$$

Both are Lyapunov regular, the Lyapunov exponent for the DAE is λ, while the Lyapunov exponent for the adjoint is $-\lambda - \alpha - \beta$. So, the Perron identity does not hold if $\alpha + \beta \neq 0$.

Differences between DAEs and ODEs

Example 1, continued: Consider the same DAE

$$\begin{aligned} e^{\alpha t}\dot{x}_1 &= e^{\alpha t}\lambda x_1 + x_2, \\ 0 &= -e^{\beta t}x_2, \end{aligned}$$

where $\alpha \leq 0, \beta \leq 0$ and λ are real. The adjoint system is

$$\begin{aligned} e^{\alpha t}\dot{y}_1 &= -(e^{\alpha t}\lambda + \alpha e^{\alpha t})y_1, \\ 0 &= -y_1 + e^{\beta t}y_2. \end{aligned}$$

If α is positive and $\lambda(t) = \sin(\ln(t+1)) + \cos(\ln(t+1))$, then the Lyapunov spectrum of the DAE is $[-1, 1]$ and that of the adjoint is $[-1-\alpha-\beta, 1-\alpha-\beta]$.
Neither the DAEs nor the underlying ODEs are Lyapunov-regular. However, if $\alpha + \beta = 2$, then the Perron identity holds for the upper Lyapunov exponents.

Differences between DAEs and ODEs

Example 2

- ▷ The underlying ODE of

$$\dot{x}_1 = -x_1,\; 0 = x_1 - e^{-t+t\sin(t)}x_2.$$

  is Lyapunov-regular, but the DAE itself is not.

- ▷

$$\begin{array}{rcl} \dot{x}_1 & = & [\sin(\ln(t+1)) + \cos(\ln(t+1))]x_1, \\ 0 & = & -x_1 + e^{t\sin(\ln(t+1))-t}x_2. \end{array}$$

  is Lyapunov-regular but the underlying ODE is not.

- ▷

$$\dot{x}_1 = -3x_1 + e^{t\sin(t)-t}x_2,\; 0 = x_2,$$

  is Lyapunov-regular. However, its adjoint system

$$\dot{y}_1 = 3y_1,\; 0 = e^{t\sin(t)-t}y_1 + y_2,$$

  is not.

Perturbed DAE systems

Consider perturbed DAEs

$$(E(t) + \Delta E(t))\dot{x} = (A(t) + \Delta A(t))x, \quad t \geq 0,$$

with *special perturbations* $(\Delta E, \Delta A),\ \Delta E, \Delta A \in C(\mathbb{I}, \mathbb{R}^{n \times n})$ that are sufficiently smooth and small enough such that by appropriate orthogonal transformation we obtain

$$\begin{bmatrix} \hat{E}_{11} + \Delta\hat{E}_{11} & 0 \\ 0 & 0 \end{bmatrix} \dot{\hat{x}} = \begin{bmatrix} \hat{A}_{11} + \hat{A}_{11} & \hat{A}_{12} + \Delta\hat{A}_{12} \\ \hat{A}_{21} + \Delta\hat{A}_{21} & \hat{A}_{22} + \Delta\hat{A}_{22} \end{bmatrix} \hat{x}, \quad t \geq 0.$$

If this is the case then we say that the perturbations are *admissible*.

Stability of Lyapunov exponents

Definition

The upper Lyapunov exponents $\lambda_1^u \geq ... \geq \lambda_d^u$ are said to be *stable* if for any $\epsilon > 0$, there exists $\delta > 0$ such that the conditions $\sup_t \left\|\Delta\tilde{E}(t)\right\| < \delta,\ \sup_t \left\|\Delta\tilde{A}(t)\right\| < \delta$ on admissable perturbations imply that the perturbed DAE system is strangeness-free and

$$|\lambda_i^u - \gamma_i^u| < \epsilon, \quad \forall i = 1, 2, ..., d,$$

where the γ_i^u are the ordered upper Lyapunov exponents of the perturbed system.
A DAE system and an admissably perturbed system are called *asymptotically equivalent* if they are strangeness-free and

$$\lim_{t\to\infty} \|\Delta E(t)\| = \lim_{t\to\infty} \|\Delta A(t)\| = 0.$$

Asymptotic equivalence

Theorem

Suppose that the DAE and an admissibly perturbed DAE are asymptotically equivalent. If the Lyapunov exponents are stable then $\lambda_i^u = \gamma_i^u$*, for all* $i = 1, 2, ..., d$*, where again the* γ_i^u *are the ordered upper Lyapunov exponents of the perturbed system.*

Integral separation

Definition

A minimal fundamental solution matrix X for a strangeness-free DAE is called *integrally separated* if for $i = 1, 2, ..., d - 1$ there exist $b > 0$ and $c > 0$ such that

$$\frac{\|X(t)e_i\|}{\|X(s)e_i\|} \cdot \frac{\|X(s)e_{i+1}\|}{\|X(t)e_{i+1}\|} \geq ce^{b(t-s)},$$

for all t, s with $t \geq s \geq 0$.

Distinct Lyapunov exponents

Lemma

Consider a strangeness-free DAE.

1. *If the DAE is integrally separated then the same holds for any strongly globally kinematically equivalent system.*
2. *If the DAE is integrally separated, then it has pairwise distinct upper and pairwise distinct lower Lyapunov exponents.*
3. *If* $\hat{A}_{22}^{-1}\hat{A}_{21}$ *is bounded, then the DAE is integrally separated if and only if and the underlying ODE is integrally separated.*

Stability of exponents

Theorem

Consider a strangeness-free DAE and its transformed system, satisfying that

$$\hat{A}_{22}^{-1}\hat{A}_{21},\ \hat{A}_{12}\hat{A}_{22}^{-1},\ \hat{E}_{11},\ \hat{E}_{11}^{-1}(\hat{A}_{11}-\hat{A}_{12}\hat{A}_{22}^{-1}\hat{A}_{21}),$$

are bounded. The DAE has d pairwise distinct upper and pairwise distinct lower Lyapunov exponents and they are stable iff it is integrally separated.
If $\hat{A}_{22}^{-1}\hat{A}_{21}$, $\hat{A}_{12}\hat{A}_{22}^{-1}$, $\hat{E}_{11}$, and $\hat{E}_{11}^{-1}$ are bounded, then the system has an integrally separated fundamental solution matrix iff its adjoint has.

Differences between ODEs and DAEs

Example 2 For the DAE system

$$\begin{array}{rclrcl} \dot{x}_1 & = & x_1 + x_2, & 0 & = & x_1 - x_3, \\ \dot{x}_2 & = & x_2, & 0 & = & x_2 - e^{-t} x_4, \end{array}$$

the underlying ODE is not integrally separated, but the Lyapunov exponents are stable and the minimal fundamental solution

$$X(t) = \begin{bmatrix} e^t & te^t \\ 0 & e^t \\ e^t & te^t \\ 0 & e^{2t} \end{bmatrix},$$

is integrally separated. However, the Lyapunov exponents of the DAE ($\{1, 2\}$), are not stable.

Shifted systems

Definition

Consider a strangeness-free DAE. For a scalar $\lambda \in \mathbb{R}$, the DAE system

$$E(t)\dot{x} = [A(t) - \lambda E(t)]x, \quad t \geq 0,$$

is called a *shifted DAE system*.
The shifted DAE transforms as

$$\begin{bmatrix} \hat{E}_{11} & 0 \\ 0 & 0 \end{bmatrix} \dot{\hat{x}} = \begin{bmatrix} \hat{A}_{11} - \lambda \hat{E}_{11} & \hat{A}_{12} \\ \hat{A}_{21} & \hat{A}_{22} \end{bmatrix} \hat{x}, \quad t \geq 0,$$

and clearly, the shifted DAE system inherits the strangeness-free property from the original DAE.

Exponential Dichotomy

Definition

A strangeness-free DAE system is said to have an *exponential dichotomy* if for a maximal fundamental solution $\hat{X}(t)$, there exists a projection matrix $P \in \mathbb{R}^{d\times d}$ and constants $\alpha, \beta > 0$, and $K, L \geq 1$ such that

$$\left\| \hat{X}(t) \begin{bmatrix} P & 0 \\ 0 & 0 \end{bmatrix} \hat{X}^{-}(s) \right\| \leq K e^{-\alpha(t-s)}, \quad t \geq s,$$
$$\left\| \hat{X}(t) \begin{bmatrix} I_d - P & 0 \\ 0 & 0 \end{bmatrix} \hat{X}^{-}(s) \right\| \leq L e^{\beta(t-s)}, \quad t \leq s.$$

Here $X^{-}(s)$ is a reflexive generalized inverse.

Existence of exponential Dichotomy

Theorem

A strangeness-free DAE system has an exponential dichotomy if and only if in the transformed system $\hat{A}_{22}^{-1}\hat{A}_{21}$ *is bounded and the underlying ODE has an exponential dichotomy.*

Sacker-Sell spectra

Definition

- The *Sacker-Sell (or exponential dichotomy) spectrum* of a strangeness-free DAE system is defined by

$$\Sigma_S := \{\lambda \in \mathbb{R},\ \text{the shifted DAE has no exponential dichotomy}\}\,.$$

- The complement of Σ_S is called the *resolvent set*.
- The Sacker-Sell spectrum of a DAE system does not depend on the choice of the orthogonal change of basis that brings it to the transformed system.

Sacker-Sell spectra

Theorem

Consider a DAE and suppose that in the transformed DAE $\hat{A}_{22}^{-1}\hat{A}_{21}$ *is bounded.*

- ▷ *The Sacker-Sell spectrum is exactly the Sacker-Sell spectrum of the underlying ODE. It consists of at most* d *closed intervals*
- ▷ *If the Sacker-Sell spectrum of the DAE system is given by* d *disjoint closed intervals, then there exists a minimal fundamental solution matrix* $\hat{X}$ *with integrally separated columns.*
- ▷ *In this case it is given exactly by the* d *(not necessarily disjoint) Bohl intervals associated with the columns of* $\hat{X}$ *and* $\Sigma_L \subseteq \Sigma_S$*.*

Stability of Sacker-Sell spectra

Theorem

Consider a strangeness-free DAE system

$$\hat{A}_{22}^{-1}\hat{A}_{21},\ \hat{A}_{12}\hat{A}_{22}^{-1},\ \hat{E}_{11},\ \hat{E}_{11}^{-1}(\hat{A}_{11}-\hat{A}_{12}\hat{A}_{22}^{-1}\hat{A}_{21}),$$

are bounded and for $k \leq d$ $\Sigma_S = \bigcup_{i=1}^{k}[a_i, b_i]$.
Consider an admissible perturbation. Let $\varepsilon > 0$ *be sufficiently small such that* $b_i + \varepsilon < a_{i+1} - \varepsilon$ *for some* i*,* $0 \leq i \leq k$*. For* $i = 0$ *and* $i = k$*, set* $b_0 = -\infty$ *and* $a_{k+1} = \infty$*, respectively. Then, there exists* $\delta > 0$ *such that the inequality*

$$\max\{\sup_t \|\Delta E(t)\|, \sup_t \|\Delta A(t)\|\} \leq \delta,$$

implies that the interval $(b_i + \varepsilon, a_{i+1} - \varepsilon)$ *is contained in the resolvent of the perturbed DAE system.*

Stability of Sacker-Sell spectra

Corollary

Let the assumptions of the last Theorem hold and let $\varepsilon > 0$ *be sufficiently small such that* $b_{i-1} + \varepsilon < a_i - \varepsilon < a_i \leq b_i < b_i + \varepsilon < a_{i+1} - \varepsilon$*, for* $0 \leq i \leq k$*. For* $i = 0$ *and* $i = k$*, set* $b_0 = -\infty$ *and* $a_{k+1} = \infty$*, respectively. Then, there exists* $\delta > 0$ *such that the inequality*

$$\max\{\sup_t \left\|\Delta \tilde{E}(t)\right\|, \sup_t \left\|\Delta \tilde{A}(t)\right\|\} \leq \delta,$$

implies that the Sacker-Sell interval $[a_i, b_i]$ *either remains a Sacker-Sell interval under the perturbation or it is possibly split into several new intervals, but the smallest left end-point and the largest right end-point stay in the interval* $[a_i - \varepsilon, b_i + \varepsilon]$*.*

Outline

1. Introduction
2. Spectral theory for ODEs
3. Stability Theory for DAEs
4. **Numerical methods**
5. Summary spectral theory
6. Stability of numerical methods
7. Stability functions/regions
8. Spin-stabilized discretization

Numerical methods

For the numerical computation we have first have to obtain the strangeness-free form of $E(t)$, $A(t)$.

It can be obtained pointwise for every t via the `FORTRAN` code `GELDA` Kunkel/M./Rath/Weickert 1997 or the corresponding `MATLAB` version Kunkel/M./Seidel 2005.

It comes in the form

$$\begin{bmatrix} E_1(t) \\ 0 \end{bmatrix} \dot{x} = \begin{bmatrix} A_1(t) \\ A_2(t) \end{bmatrix} x$$

where A_2 is full row rank.

Smooth QR factorizations

Kunkel/M. 1991, Dieci/Eirola 1999 Suppose that the original DAE has sufficiently smooth coefficients.

- There exists a pointwise nonsingular, upper triangular matrix function $\tilde{A}_{22} \in C^1(\mathbb{I}, \mathbb{R}^{(n-d)\times(n-d)})$ and a pointwise orthogonal matrix function $\tilde{Q} \in C^1(\mathbb{I}, \mathbb{R}^{n\times n})$ such that

$$A_2 = \left[\begin{array}{cc} 0 & \tilde{A}_{22} \end{array}\right] \hat{Q}.$$

- To make the factorization unique and to obtain smoothness, we require the diagonal elements of $\tilde{A}_{22}$ to be positive.
- Alternatively we can derive differential equations for $\tilde{Q}$ (or its Householder factors) and to solve the corresponding initial value problems.

Transformed DAE

The transformation $\tilde{x} = \tilde{Q}^T x$ leads to

$$\begin{bmatrix} \tilde{E}_{11} & \tilde{E}_{12} \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} E_1 \\ 0 \end{bmatrix} \tilde{Q}, \quad \begin{bmatrix} \tilde{A}_{11} & \tilde{A}_{12} \\ 0 & \tilde{A}_{22} \end{bmatrix} = \begin{bmatrix} A_1 \\ A_2 \end{bmatrix} \tilde{Q} - \begin{bmatrix} E_1 \\ 0 \end{bmatrix} \dot{\tilde{Q}}.$$

To evaluate $\dot{\tilde{Q}}$ at time t, we may use either a finite difference formula or the smooth QR method derived in Kunkel/M. 1991. The solution component $\tilde{x}_2$ associated with the algebraic equations is simply 0, thus we only have to deal with $\tilde{x}_1$.

Basic Idea for Computing Exponents

- Determine for every point t

$$\mathcal{E} = [e_{ij}] = P^T \tilde{E}_{11} Q, \ \mathcal{A} = [a_{ij}] = P^T \tilde{A}_{11} Q - P^T \tilde{E}_{11} \dot{Q}$$

  such that $\mathcal{E}$, $\mathcal{A}$ are upper triangular.
- Determine strictly lower triangular part of the skew symmetric $S(Q) = Q^T \dot{Q}$ by corresponding part of $\mathcal{E}^{-1} P^T \tilde{A}_{11} Q$ and the remaining part by skew-symmetry.
- Determine P and $\mathcal{E}$ via a smooth QR-factorization $\tilde{E}_{11} Q = P\mathcal{E}$.
- Keep orthogonality via orthogonal integrators Hairer/Lubich/Wanner 2002 or projected ODE integrators Dieci/Van Vleck 2003.
- Compute the spectral intervals from

$$\mathcal{E}_1(t)\dot{R}_1 = \mathcal{A}_1(t)R_1, \quad t \in \mathbb{I},$$

  where R_1 is the fundamental solution matrix of the triangularized underlying implicit ODE.

Continuous QR algorithm

- Compute a smooth QR factorization of A_2

$$\begin{bmatrix} \mathcal{E}_1 & U_1^T \tilde{E}_{12} \\ 0 & 0 \end{bmatrix} \dot{z} = \begin{bmatrix} \mathcal{A}_1 & U_1^T \tilde{A}_{12} \\ 0 & \tilde{A}_{22} \end{bmatrix} z,$$

  with $\mathcal{E}_1$, $\mathcal{A}_1$, $\tilde{A}_{22}$ upper triangular.
- Apply ODE methods of Dieci/Van Vleck to

$$\mathcal{E}_1(t)\dot{R}_1 = \mathcal{A}_1(t)R_1, \quad t \in \mathbb{I},$$

- Compute

$$\lambda_i(t_j) = \frac{1}{t_j} \ln[R_1(t_j)]_{i,i} = \frac{1}{t_j} \ln \prod_{\ell=1}^{j} [\Theta_\ell]_{i,i} = \frac{1}{t_j} \sum_{\ell=1}^{j} \ln[\Theta_\ell]_{i,i}, \quad i = 1, 2, ..$$

- Solve optimization problems $\inf_{\tau \le t \le T} \lambda_i(t)$ and $\sup_{\tau \le t \le T} \lambda_i(t)$, $i = 1, 2, \ldots, d$ for a given $\tau \in (0, T)$.

Discrete QR algorithm

- Take a mesh $0 = t_0 < t_1 < \ldots < t_{N-1} < t_N = T$.
- Compute the fundamental solution $X^{[j]}$ on $[t_{j-1}, t_j]$ by solving

$$E\dot{X}^{[j]} = AX^{[j]},\ t_{j-1} \leq t \leq t_j,\ X^{[j]}(t_{j-1}) = \chi_{j-1},$$

  using the DAE integrator `GELDA`.
- Determine QR factorizations

$$\tilde{Q}(t_j)^T X^{[j]}(t_j) = \begin{bmatrix} Z_j \\ 0 \end{bmatrix}, Z_j = Q_j \Theta_j, \quad j = 1, 2, \ldots, N$$

- Compute

$$\lambda_i(t_j) = \frac{1}{t_j} \ln[R_1(t_j)]_{i,i} = \frac{1}{t_j} \ln \prod_{\ell=1}^{j} [\Theta_\ell]_{i,i} = \frac{1}{t_j} \sum_{\ell=1}^{j} \ln[\Theta_\ell]_{i,i}, \quad i = 1, 2, \ldots$$

- Solve the optimization problems $\inf_{\tau \leq t \leq T} \lambda_i(t)$ and $\sup_{\tau \leq t \leq T} \lambda_i(t)$, $i = 1, 2, \ldots, d$ for a given $\tau \in (0, T)$.

Lyapunov exp. via cont. QR-Euler method

Lyapunov regular 2 $\times$ 2 DAE $\lambda_1 = 5, \lambda_2 = 0$.

T	h	λ_1	λ_2	$CPU(s)$
500	0.1	4.9341	-0.0043	2.55
500	0.05	4.9337	-0.0038	5.01
500	0.01	4.9337	-0.0037	24.89
1000	0.1	4.9632	-0.0006	5.01
1000	0.05	4.9628	-0.0001	10.01
1000	0.01	4.9627	-0.0001	49.84
2000	0.1	4.9799	-0.0010	10.17
2000	0.05	4.9794	-0.0005	20.02
10000	0.1	4.9956	-0.0009	49.91
10000	0.05	4.9951	-0.0003	99.71

Lyapunov exp. via disc. QR-Euler method

Lyapunov regular 2 × 2 DAE $\lambda_1 = 5, \lambda_2 = 0$.

T	h	λ_1	λ_2	$CPU(s)$
500	0.1	5.0324	-0.0137	9.87
500	0.05	4.9818	-0.0087	19.59
500	0.01	4.9431	-0.0047	97.31
1000	0.1	5.0625	-0.0100	19.63
1000	0.05	5.0114	-0.0050	38.87
2000	0.1	5.0799	-0.0104	39.20
2000	0.05	5.0284	-0.0053	78.15
10000	0.1	5.0963	-0.0102	194.89
10000	0.05	5.0443	-0.0052	389.64

Lyapunov exp. via cont. QR-Euler method

Lyapunov non regular 2×2 DAE with Lyapunov intervals $[-1, 1]$, $[-6, -4]$.

T	t_0	h	$[\lambda_1^l, \lambda_1^u]$	$[\lambda_2^l, \lambda_2^u]$	$CPU(s)$
1000	100	0.1	$[-1.0018, 0.5865]$	$[-6.0006, -4.8928]$	5.3
5000	100	0.1	$[-1.0018, 1.0004]$	$[-6.0006, -4.3846]$	26.0
10000	100	0.1	$[-1.0018, 1.0004]$	$[-6.0006, -4.0235]$	51.5
10000	500	0.1	$[-0.0647, 1.0004]$	$[-6.0006, -4.0235]$	51.6
10000	100	0.05	$[-1.0028, 1.0000]$	$[-6.0001, -4.0229]$	103.4
20000	100	0.1	$[-1.0018, 1.0004]$	$[-6.0006, -4.0007]$	103.5
20000	500	0.1	$[-0.4598, 1.0004]$	$[-6.0006, -4.0007]$	103.3
20000	100	0.05	$[-1.0028, 1.0000]$	$[-6.0001, -4.0001]$	211.0
50000	100	0.05	$[-1.0028, 1.0000]$	$[-6.0001, -4.0001]$	519.5
50000	500	0.05	$[-0.9844, 1.0000]$	$[-6.0001, -4.0001]$	518.2
100000	100	0.05	$[-1.0028, 1.0000]$	$[-6.0001, -4.0001]$	1044.9
100000	500	0.05	$[-0.9998, 1.0000]$	$[-6.0001, -4.0001]$	1050.4

Sacker-Sell interv. via cont. QR-Euler

Sacker-Sell intervals $[-\sqrt{2}, \sqrt{2}]$ and $[-5-\sqrt{2}, -5+\sqrt{2}]$.

T	H	h	$[\kappa_1^l, \kappa_1^u]$	$[\kappa_2^l, \kappa_2^u]$	$CPU(s)$
1000	100	0.1	$[-1.2042, 1.3811]$	$[-6.4049, -4.8927]$	6.2
5000	100	0.1	$[-1.2042, 1.4131]$	$[-6.4049, -3.5990]$	30.8
10000	100	0.1	$[-1.2042, 1.4131]$	$[-6.4049, -3.5867]$	61.9
10000	500	0.1	$[-0.7327, 1.4030]$	$[-6.2142, -3.5872]$	94.8
10000	100	0.05	$[-1.2049, 1.4127]$	$[-6.4046, -3.5860]$	147.2
20000	100	0.1	$[-1.3461, 1.4131]$	$[-6.4049, -3.5867]$	123.6
20000	500	0.1	$[-1.3416, 1.4030]$	$[-6.2142, -3.5872]$	201.3
20000	100	0.05	$[-1.3468, 1.4127]$	$[-6.4046, -3.5860]$	283.1
50000	100	0.1	$[-1.4132, 1.4131]$	$[-6.4049, -3.5867]$	310.4
50000	500	0.1	$[-1.4132, 1.4030]$	$[-6.2142, -3.5872]$	506.7
100000	100	0.1	$[-1.4132, 1.4131]$	$[-6.4049, -3.5867]$	646.2
100000	500	0.1	$[-1.4132, 1.4030]$	$[-6.3633, -3.5872]$	976.3
200000	500	0.1	$[-1.4132, 1.4030]$	$[-6.4147, -3.5872]$	1973.4

Outline

Summary spectral theory

- The classical Theory of Lyapunov/Bohl/Sacker-Sell has been extended to linear DAEs with variable coefficients.
- Boundedness conditions and strangeness-free formulations are the key tools.
- In principle we can compute spectral intervals.
- Numerical methods for computing the Sacker-Sell spectra of DAEs extending work of Dieci/Van Vleck. They are **expensive**.
- Perturbation theory of Dieci/Van Vleck for ODEs has been extended and the methods have been modified to deal only with partial exponents (e.g. the largest Sacker-Sell interval) Linh/M./Van Vleck Dec. 2009
- SVD based methods for more accurate spectra.
- Implementation Diploma thesis, Jens Möckel.

Outline

An example

Example März/Rodriguez-Santiesteban 2002

$$\begin{bmatrix} \delta - 1 & \delta t \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -\eta(\delta - 1) & -\eta\delta t \\ \delta - 1 & \delta t - 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix},$$

with real parameters η and $\delta \neq 1$.
This system has differentiation index 1 and the solution

$$x_1(t) = (\delta - 1)^{-1}(1 - \delta t)x_2(t), \quad x_2(t) = e^{(\delta-\eta)t}x_2(0).$$

The system is asymptotically stable, i.e. $x(t) \to 0$ as $t \to \infty$ independently of $x_2(0)$ for $\delta < \eta$.

Implicit Euler method

Using a constant stepsize h, the implicit Euler method

$$x_{i+1} = [E(t_{i+1} - hA(t_{i+1})]^{-1}(E(t_{i+1})x_i + hf(t_{i+1})),\ x_0 = x^0$$

for this example yields numerical approximations

$$x_{i,1} = (\delta - 1)^{-1}(1 - \delta t_i)x_{i,2}, \quad x_{i,2} = \frac{1 + h\delta}{1 + h\eta}x_{i-1,2}.$$

Here $x_i \to 0$ as $i \to \infty$ independently of $x_{0,2}$ if and only if $|1 + h\delta| < |1 + h\eta|$.
There exist parameter values (δ, η) for which the exact solution asymptotically goes to 0, while the numerical solution grows unboundedly.

What is the problem, what to do ?

- The instability is caused by the time dependence of kernel($E(t)$).
- This does not occur for ODEs.
- The classical test equation

$$\dot{x} = \lambda x, \quad \lambda \in \mathbb{C}$$

  for ODEs is not sufficient to analyze this instability.
- **We need a new test equation.**
- **We need new discretization techniques that avoid these instabilities.**

A test equation for DAEs

$$\begin{bmatrix} 1 & -\omega t \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} \lambda & \omega(1-\lambda t) \\ -1 & 1+\omega t \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

With initial data $x_1(0) = 1$, $x_2(0) = 1$, the solution is

$$x(t) = \begin{bmatrix} 1 & \omega t \\ 0 & 1 \end{bmatrix} \begin{bmatrix} e^{\lambda t} \\ e^{\lambda t} \end{bmatrix} = \begin{bmatrix} (1+\omega t)e^{\lambda t} \\ e^{\lambda t} \end{bmatrix},$$

- ▷ Solution is asymptotically stable for $\mathrm{Re}(\lambda) < 0$ and ω arbitrary, i.e. asymptotic stability does not depend on ω.
- ▷ All transformations of x such that the transforming matrix function and its pointwise inverse are polynomially bounded for $t \to \infty$ preserve the asymptotic stability of the solution.

Normal form of test equation

With

$$Q(t) = \frac{1}{\sqrt{1+\omega^2 t^2}} \begin{bmatrix} 1 & \omega t \\ -\omega t & 1 \end{bmatrix}, \quad \dot{Q}(t) = \frac{\omega}{(1+\omega^2 t^2)^{3/2}} \begin{bmatrix} -\omega t & 1 \\ -1 & -\omega t \end{bmatrix}$$

we have that

$$\begin{aligned} EQ &= \frac{1}{\sqrt{1+\omega^2 t^2}} \begin{bmatrix} 1+\omega^2 t^2 & 0 \\ 0 & 0 \end{bmatrix}, \\ AQ - E\dot{Q} &= \frac{1}{\sqrt{1+\omega^2 t^2}} \begin{bmatrix} \lambda - \omega^2 t + \lambda \omega^2 t^2 & 0 \\ -1 - \omega t - \omega^2 t^2 & 1 \end{bmatrix}, \end{aligned}$$

i.e., the DAE is equivalent to the pair in normal form

$$(\tilde{E}, \tilde{A}) = \left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} \lambda - \frac{\omega^2 t}{1+\omega^2 t^2} & 0 \\ 1 + \omega t + \omega^2 t^2 & -1 \end{bmatrix} \right),$$

New vs classical test equation

- Since the kernel(E) is changing in t, the DAE is not autonomous. Any discretization of the test equation will explicitly include time positions.
- The transformation that separates dynamic and algebraic part is not pointwise orthogonal. An orthogonal variant would be

  $$\frac{1}{\sqrt{1+\omega^2 t^2}} \begin{bmatrix} 1 & \omega t \\ -\omega t & 1 \end{bmatrix}$$

  or

  $$\begin{bmatrix} \sin(\omega t) & \cos(\omega t) \\ -\sin(\omega t) & \cos(\omega t) \end{bmatrix}.$$

  but this would make the analysis very technical.
- Numerical tests show that there is no essential difference in the corresponding stability regions.

Outline

Stability functions/regions

Definition A function

$$\mathcal{R}(h\lambda, h\omega) = \mathcal{R}(z, w)$$

with $z = h\lambda$, $w = h\omega$ is called **DAE-stability function** of a numerical discretization method if it can be interpreted as the numerical solution after one step for the DAE test equation. The set

$$\mathcal{S} = \{(z, w) \in \mathbb{C} \times \mathbb{R}, \quad |R(z, \omega)| \leq 1\}$$

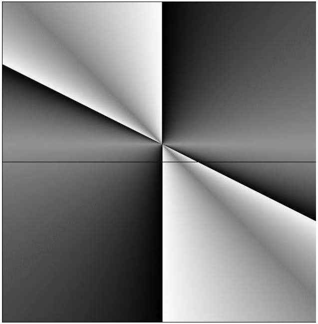
is called **DAE-stability region** for this method.

DAE Stability Regions

In the following plots of stability functions the depicted region is given by $\mathrm{Im}(z) = 0$ $(\mathrm{Re}\, z, w) \in [-9, 9]^2$.
Dark regions are those with $|R(z, w)| \leq 1$ and the shading is according to the modulus of $R(z, w)$, i.e. **the darker the more stable**.

For the implicit Euler method applied to the test function we obtain the DAE stability function

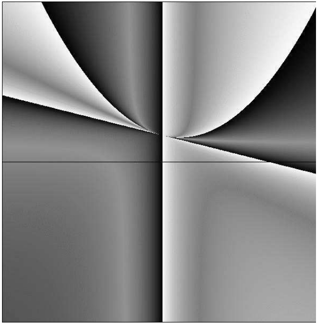
$$R(z,w) = \frac{1-w}{1-z-w}.$$



Radau IIa method with two stages

The stiffly accurate Radau IIa methods are some of the best methods for solving DAEs. The DAE stability function for the 2-stage Radau IIa method is

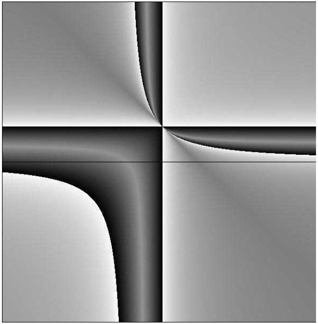
$$R(z,w) = \frac{6 - 4w + 2z - 2zw}{6 - 4z - 4w + z^2 + 2zw}.$$



Implicit projected trapezoidal rule

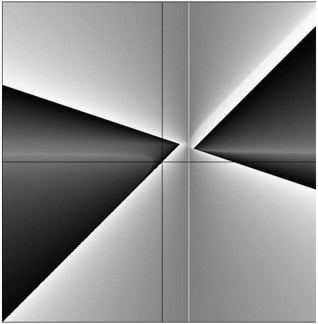
In circuit simulation besides the implicit Euler methods the trapezoidal rule is commonly used. The DAE stability function is

$$R(z, w) = \frac{2 + z - w - zw}{2 - z - w}.$$



BDF Methods

The most commonly used methods for DAEs are BDF methods. DAE stability function for the BDF method with $k = 2$.



DAE stability functions

Method	DAE-stability function $R(z, w)$
Implicit Euler	$\mathcal{R}(z, w) = \frac{1-w}{1-z-w}$
Radau IIa $s = 2$	$\mathcal{R}(z, w) = \frac{6-4w+2z-2zw}{6-4z-4w+z^2+2zw}$
Radau IIa $s = 3$	$\mathcal{R}R(z, w) = \frac{60-36w+24z-18zw+3z^2-3z^2w}{60-36w-36z+18zw+9z^2-z^3-3z^2w}$
Implicit midpoint rule	$\mathcal{R}(z, w) = \frac{2+z-w}{2-z-w}$
Gauß $s = 2$	$\mathcal{R}(z, w) = \frac{12-6w+6z-4zw+z^2}{12-6w-6z+2zw+z^2}$
Gauß-Lobatto $k = 1$	$\mathcal{R}(z, w) = \frac{4+2z-zw}{4-2z-zw}$
Gauß-Lobatto $k = 2$	$\mathcal{R}(z, w) = \frac{24+12z-2zw+2z^2-z^2w}{24-12z-2zw+2z^2+z^2w}$
Implicit trapezoidal rule	$\mathcal{R}(z, w) = \frac{2+z-w-zw}{2-z-w}$

Intermediate summary

- DAE stability regions are often much smaller than ODE stability regions.
- DAE stability regions change drastically with ω.
- It is difficulty to guarantee stability of the discretized equation.
- Classical methods can drastically fail.
- Other methods, Higueras/März/Tischendorf 2003, Voigtmann 2006
- We need methods that can actively react to a changing kernel, i.e. where the transformation Q that transforms to normal form yields a large term $[\, I_d \;\; 0 \,] Q^T \dot{Q}$.

Outline

Spin stabilized discretization

Assume that the spin-effect in the kernel of

$$\hat{E}(t) = \begin{bmatrix} Z_1(t)^T F_{\dot{x}}(t, x(t), \dot{x}(t)) \\ 0 \end{bmatrix}.$$

is covered by the linearization of Q.

Idea for spin stabilization Use a linear approximation

$$\tilde{Q}(t) = \tilde{Q}(\hat{t}) + (t - \hat{t})\dot{\tilde{Q}}, \quad \dot{\tilde{Q}} \in \mathbb{R}^{n,n}, \ \hat{t} \in \mathbb{I} \text{ fixed},$$

such that in the i-th step of a k-step method with stepsize h

$$\tilde{Q}(t) - Q(t) = \mathcal{O}(h^2), \quad \dot{\tilde{Q}} - \dot{Q}(t) = \mathcal{O}(h)$$

holds for all $t \in [t_i, t_{i+k}]$ with small constants in the remainder terms.

Then use this linear approximation to transform the given DAE before carrying out the discretization step.

Smoothness of $\tilde{Q}$

- Note that Q is not unique and thus we usually do not get a smooth representation of Q.
- The selection can be made unique by freezing the pivoting and all other decisions performed during the computation of $Q(t_{i+k})$ say by QR-decomposition, when we determine $Q(t_{i+k-1})$.

Spin stabilization procedure

- Choose a step-size *h*.
- Determine an approximate transformation to normal form.
- Transform the DAE.
- Carry out the discretization step.
- Transform back.

Spin Stabilized Method

$$\mathfrak{X}_{i+1} = \mathfrak{F}(t_i, \mathfrak{X}_i; h),$$

with

$$\mathfrak{F}(t_i, \mathfrak{X}_i; h) = \mathfrak{Q}_{i+1}\tilde{\mathfrak{F}}(t_i, \mathfrak{Q}_i^{-1}\mathfrak{X}_i; h).$$

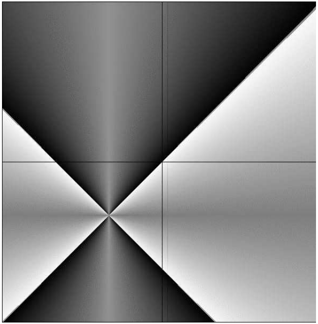
Spin-stabilized Runge-Kutta and BDF.

Theorem A spin-stabilized stiffly accurate Runge-Kutta method based on a stiffly accurate Runge-Kutta method of order p with invertible $\mathcal{A}$ is **convergent of order p**.

Theorem A spin-stabilized BDF method based on a BDF method of order k, $1 \leq k \leq 6$ is **convergent of order k**.

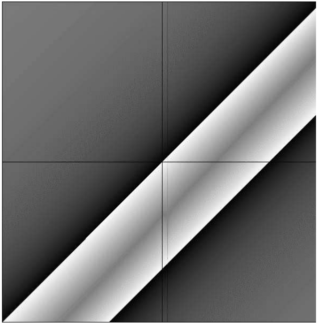
A numerical experiment I

Stability for implicit Euler method applied to the example of März/Rodriguez-Santiesteban for $(\delta, \eta) \in [-3, 3]^2$ and $h = 0.1$
One can recognize the stability restriction $|1 + h\delta| < |1 + h\eta|$.



A numerical experiment II

Spin-stabilized implicit Euler method is stable in the region $\delta < \eta$, where the actual solution is stable.



Take home message

- DAEs are a wonderful modeling tool.
- Methods work safely only for strangeness-free problems.
- Remodeling is necessary and possible in most cases.
- Everything works for strangeness-free problems.
- Input/output maps are nice but a behavior modeling is better.
- There are still lots of things to do.

Thank you very much
for your attention.