Pseudospectra and Nonnormal Dynamical Systems

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Overview of the Course

*These lectures describe modern tools for the spectral analysis of dynamical systems. We shall cover a mix of theory, computation, and applications.*

Lecture 1: Introduction to Nonnormality and Pseudospectra
Lecture 2: Functions of Matrices
Lecture 3: Toeplitz Matrices and Model Reduction
Lecture 4: Model Reduction, Numerical Algorithms, Differential Operators
Lecture 5: Discretization, Extensions, Applications
Lecture 3: Functions of Matrices and Model Reduction

- Toeplitz matrices
- Model Reduction: Balanced Truncation
- Nonnormality and Lyapunov Equations
- Model Reduction: Moment Matching
3(a) Toeplitz Matrices
Recall the example that began our investigation of pseudospectra yesterday.

**Example**

*Compute eigenvalues of three similar $100 \times 100$ matrices using MATLAB's `eig`.*

\[
\begin{bmatrix}
0 & 1 \\
1 & 0 \\
\vdots & \vdots \\
1 & 0
\end{bmatrix} \quad \begin{bmatrix}
0 & 1/2 \\
2 & 0 \\
\vdots & \vdots \\
2 & 0
\end{bmatrix} \quad \begin{bmatrix}
0 & 1/3 \\
3 & 0 \\
\vdots & \vdots \\
3 & 0
\end{bmatrix}
\]
Consider the pseudospectra of the $100 \times 100$ matrix in the middle of the last slide, $A = \text{tridiag}(2, 0, 1/2)$.

$A$ is diagonalizable (it has distinct eigenvalues), but Bauer–Fike is useless here: $\kappa(V) = 2^{99} \approx 6 \times 10^{29}$. 
We’ve already analyzed pseudospectra of Jordan blocks near $\lambda$ for small $\varepsilon > 0$. Here we want to investigate the entire pseudospectrum for larger $\varepsilon$.

Near the eigenvalue, the resolvent norm grows with dimension $n$; outside the unit disk, the resolvent norm does not seem to get big. We would like to prove this.
We’ve already analyzed pseudospectra of Jordan blocks near $\lambda$ for small $\varepsilon > 0$. Here we want to investigate the entire pseudospectrum for larger $\varepsilon$.

$S_n = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & \cdots & 0 & 0 \end{bmatrix}$

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Near the eigenvalue, the resolvent norm grows with dimension $n$; outside the unit disk, the resolvent norm does not seem to get big. We would like to prove this.
Consider the generalization of the Jordan block to the domain

\[ \ell^2(\mathbb{N}) = \{(x_1, x_2, \ldots) : \sum_{j=1}^{\infty} |x_j|^2 < \infty \}. \]

The shift operator \( S \) on \( \ell^2(\mathbb{N}) \) is defined as

\[ S(x_1, x_2, \ldots) = (x_2, x_3, \ldots). \]
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In particular,

\[ S(1, z, z^2, \ldots) = (z, z^2, z^3, \ldots) = z(1, z, z^2, \ldots) . \]

So if \((1, z, z^2, \ldots) \in \ell^2(\mathbb{N})\), then \( z \in \sigma(S) \).
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So if \( (1, z, z^2, \ldots) \in \ell^2(\mathbb{N}) \), then \( z \in \sigma(S) \).

If \( |z| < 1 \), then

\[ \sum_{j=1}^{\infty} |z^{j-1}|^2 = \frac{1}{1 - |z|^2} < \infty. \]

So,

\[ \{z \in \mathbb{C} : |z| < 1\} \subseteq \sigma(S). \]
Jordan Blocks/Shift Operator

\[ S(x_1, x_2, \ldots) = (x_2, x_3, \ldots). \]

We have seen that \( \{z \in \mathbb{C} : |z| < 1\} \subseteq \sigma(S) \).

Observe that
\[ \|S\| = \sup_{\|x\|=1} \|Sx\| = 1, \]
and so
\[ \sigma(S) \subseteq \{z \in \mathbb{C} : |z| \leq 1\}. \]

The spectrum is closed, so
\[ \sigma(A) = \{z \in \mathbb{C} : |z| \leq 1\}. \]

For any finite dimensional \( n \times n \) Jordan block \( S_n \),
\[ \sigma(S_n) = \{0\}. \]
\[ S(x_1, x_2, \ldots) = (x_2, x_3, \ldots). \]

We have seen that
\[ \{ z \in \mathbb{C} : |z| < 1 \} \subseteq \sigma(S). \]

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For any finite dimensional \( n \times n \) Jordan block \( S_n \),
\[ \sigma(S_n) = \{ 0 \}. \]

So the \( S_n \to S \) strongly, but there is a discontinuity in the spectrum:
\[ \sigma(S_n) \not\to \sigma(S). \]
Pseudospectra resolve this unpleasant discontinuity.

Recall the eigenvectors \((1, z, z^2, \ldots)\) for \(S\).

Truncate this vector to length \(n\), and apply it to \(S_n\):

\[
\begin{bmatrix}
0 & 1 \\
0 & \ddots \\
\vdots & \ddots & 1 \\
0 & \ldots & 0 & 1 \\
&\phantom{0} & \phantom{0} & \phantom{0} & \phantom{0}
\end{bmatrix}
\begin{bmatrix}
1 \\
z \\
\vdots \\
z^{n-2} \\
z^{n-1}
\end{bmatrix} =
\begin{bmatrix}
z \\
z^2 \\
\vdots \\
z^{n-2} \\
z^{n-1}
\end{bmatrix}
\]
Pseudospectra of Jordan Blocks

Pseudospectra resolve this unpleasant discontinuity.

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Truncate this vector to length \(n\), and apply it to \(S_n\):

\[
\begin{bmatrix}
0 & 1 \\
0 & \ddots \\
\vdots & & \ddots & 1 \\
0 & & \ddots & 0 \\
z & & \ddots & z^n \\
z^n & & \ddots & z^n-1 \\
\end{bmatrix}
\begin{bmatrix}
1 \\
z \\
\vdots \\
z^n-1 \\
z^n \\
\end{bmatrix}
= \begin{bmatrix}
z \\
z^2 \\
\vdots \\
z^{n-1} \\
z^n \\
\end{bmatrix}
= z
\begin{bmatrix}
1 \\
z \\
\vdots \\
z^{n-2} \\
z^n-1 \\
\end{bmatrix}
- \begin{bmatrix}
0 \\
0 \\
\vdots \\
z^n \\
z^n \\
\end{bmatrix}
\]
Pseudospectra of Jordan Blocks

Pseudospectra resolve this unpleasant discontinuity.

Recall the eigenvectors \((1, z, z^2, \ldots)\) for \(S\).

Truncate this vector to length \(n\), and apply it to \(S_n\):

\[
\begin{bmatrix}
0 & 1 & \cdots & \cdots & \cdots \\
0 & \ddots & \ddots & & \\
\vdots & \ddots & \ddots & \ddots & \\
\vdots & & \ddots & \ddots & \ddots \\
0 & \cdots & \cdots & 0 & 1
\end{bmatrix}
\begin{bmatrix}
1 \\
z \\
\vdots \\
\vdots \\
z^{n-2}
\end{bmatrix}
= 
\begin{bmatrix}
z \\
z^2 \\
\vdots \\
\vdots \\
z^{n-1}
\end{bmatrix}
= z
\begin{bmatrix}
1 \\
z \\
\vdots \\
\vdots \\
z^{n-2}
\end{bmatrix}
- 
\begin{bmatrix}
0 \\
0 \\
\vdots \\
\vdots \\
z^n
\end{bmatrix}.
\]

Hence, \(\|S_n x - zx\| = |z|^n\), so for all \(\varepsilon > \frac{|z|^n}{\|x\|} = |z|^n \frac{\sqrt{1 - |z|^{2n}}}{\sqrt{1 - |z|^2}},\)

\[z \in \sigma_\varepsilon(S_n).\]

We conclude that for fixed \(|z| < 1\), the resolvent norm \(\|(z - S_n)^{-1}\|\) grows exponentially with \(n\).
Upper Triangular Toeplitz Matrices

Consider an upper triangular Toeplitz matrix giving the matrix with constant diagonals containing the Laurent coefficients:

\[
A_n = \begin{bmatrix}
a_0 & a_1 & a_2 & \cdots \\
a_0 & \ddots & \ddots & \ddots \\
\vdots & \ddots & \ddots & \ddots \\
& \cdots & a_1 & a_2 \\
& & a_0 & a_1 \\
& & & a_0
\end{bmatrix} \in \mathbb{C}^{n \times n}.
\]

Definition (Symbol, Symbol Curve)

Toeplitz matrices are described by their symbol \( a \) with Taylor expansion

\[
a(z) = \sum_{k=0}^{\infty} a_k z^k.
\]

Call the image of the unit circle \( T \) under \( a \) the symbol curve, \( a(T) \).
Apply the same approximate eigenvector we used for the Jordan block:

\[
\begin{bmatrix}
    a_0 & a_1 & a_2 & \cdots & a_{n-1} \\
    a_0 & \ddots & & & \vdots \\
    \vdots & & \ddots & \ddots & \vdots \\
    \vdots & & & a_1 & a_2 \\
    a_0 & a_1 & a_0 & & a_0 \\
\end{bmatrix}
\begin{bmatrix}
    1 \\
    z \\
    \vdots \\
    z^{n-2} \\
    z^{n-1} \\
\end{bmatrix} =
\begin{bmatrix}
    \sum_{k=0}^{n-1} a_k z^k \\
    \sum_{k=0}^{n-1} a_k z^{k+1} \\
    \vdots \\
    a_0 z^{n-2} + a_1 z^{n-1} \\
    a_0 z^{n-1} \\
\end{bmatrix}.
\]

If the matrix has fixed bandwidth \( b \ll n \), (i.e., \( a_k = 0 \) for \( k > b \)), then

\[
\begin{bmatrix}
    a_0 & \cdots & a_b & \cdots \\
    a_0 & \ddots & & \vdots \\
    \vdots & & \ddots & \ddots \\
    \vdots & & & a_b \\
    a_0 & \vdots & a_0 & \vdots \\
\end{bmatrix}
\begin{bmatrix}
    1 \\
    z \\
    \vdots \\
    z^{n-2} \\
    z^{n-1} \\
\end{bmatrix} =
\begin{bmatrix}
    \sum_{k=0}^{b} a_k z^k \\
    \sum_{k=0}^{b} a_k z^{k+1} \\
    \vdots \\
    \sum_{k=0}^{b-1} a_k z^{k+n-b} \\
    \vdots \\
    a_0 z^{n-1} \\
\end{bmatrix}.
\]
Pseudospectra of Upper Triangular Toeplitz Matrices

\[
\begin{pmatrix}
a_0 & \cdots & a_b \\
a_0 & \ddots & \ddots \\
\vdots & \ddots & \ddots & a_b \\
a_0 & \ddots & \ddots & \ddots \\
a_0 & \ddots & \ddots & \ddots & \ddots \\
\end{pmatrix}
\begin{pmatrix}
1 \\
z \\
\vdots \\
z^{n-2} \\
z^{n-1} \\
\end{pmatrix}
= 
\begin{pmatrix}
\sum_{k=0}^{b} a_k z^k \\
\sum_{k=0}^{b} a_k z^{k+1} \\
\vdots \\
\sum_{k=0}^{b-1} a_k z^{k+n-b} \\
\sum_{k=0}^{b} a_k z^{n-1} \\
\end{pmatrix}
\begin{pmatrix}
0 \\
\vdots \\
0 \\
\sum_{k=1}^{b} a_k z^{n+k-1} \\
\end{pmatrix}
\]

Hence \( A_n \) gets very small in size as \( n \to \infty \).
Hence \( A_n x - a(z)x \) gets very small in size as \( n \to \infty \).
In fact, this reveals the spectrum of the infinite Toeplitz operator on \( \ell^2(\mathbb{N}) \ldots \).
Spectrum of Toeplitz Operators on $\ell^2(\mathbb{Z})$

For the Toeplitz operator (semi-infinite matrix) $A_\infty$ with the same symbol:

$$
\begin{bmatrix}
a_0 & \cdots & a_b \\
a_0 & \ddots & \ddots \\
\vdots & \ddots & \ddots \\
a_0 & \cdots & \\
\end{bmatrix}
\begin{bmatrix}
1 \\
z \\
z^2 \\
\vdots \\
\end{bmatrix}
= 
\begin{bmatrix}
\sum_{k=0}^{b} a_k z^k \\
\sum_{k=0}^{b} a_k z^{k+1} \\
\sum_{k=0}^{b} a_k z^{k+2} \\
\vdots \\
\end{bmatrix}
= a(z)
\begin{bmatrix}
1 \\
z \\
z^2 \\
\vdots \\
\end{bmatrix}
$$

Thus $a(z) \in \sigma(A_\infty)$ for all $|z| < 1$. In fact, one can show that for this banded upper triangular symbol, $\sigma(A_\infty) = \{ a(z) : |z| \leq 1 \}$.

The calculation on the last slide guarantees that for any $\epsilon > 0$, there exists $N > 0$ such that if $n > N$, $\sigma(A_\infty) \subseteq \sigma_\epsilon(A_n)$.
Spectrum of Toeplitz Operators on $\ell^2(\mathbb{Z})$

For the Toeplitz operator (semi-infinite matrix) $A_\infty$ with the same symbol:

$$
\begin{bmatrix}
a_0 & \cdots & a_b \\
a_0 & \cdots & \cdots & \cdots & a_b \\
\vdots & \ddots & \ddots & \ddots & \ddots \\
a_0 & \cdots & \cdots & \cdots & \cdots
\end{bmatrix}
\begin{bmatrix}1 \\ z \\ z^2 \\ \vdots \\ \vdots \end{bmatrix}
= \begin{bmatrix}
\sum_{k=0}^b a_k z^k \\
\sum_{k=0}^b a_k z^{k+1} \\
\sum_{k=0}^b a_k z^{k+2} \\
\vdots \\
\vdots
\end{bmatrix}
= a(z)
\begin{bmatrix}1 \\ z \\ z^2 \\ \vdots \end{bmatrix}
$$

Thus $a(z) \in \sigma(A)$ for all $|z| < 1$. In fact, one can show that for this banded upper triangular symbol,

$$\sigma(A_\infty) = \{ a(z) : |z| \leq 1 \}.$$
Spectrum of Toeplitz Operators on $\ell^2(\mathbb{Z})$

For the Toeplitz operator (semi-infinite matrix) $A_{\infty}$ with the same symbol:

$$
\begin{bmatrix}
  a_0 & \cdots & a_b \\
  & a_0 & \cdots & \cdots \\
  & & \ddots & \vdots \\
  & & & a_b \\
  & & & & a_0 \\
  & & & & & \vdots
\end{bmatrix}
\begin{bmatrix}
  1 \\
  z \\
  z^2 \\
  \vdots
\end{bmatrix}
= 
\begin{bmatrix}
  \sum_{k=0}^b a_k z^k \\
  \sum_{k=0}^b a_k z^{k+1} \\
  \sum_{k=0}^b a_k z^{k+2} \\
  \vdots
\end{bmatrix}
= a(z)
\begin{bmatrix}
  1 \\
  z \\
  z^2 \\
  \vdots
\end{bmatrix}
$$

Thus $a(z) \in \sigma(A)$ for all $|z| < 1$. In fact, one can show that for this banded upper triangular symbol,

$$
\sigma(A_{\infty}) = \{a(z) : |z| \leq 1\}.
$$

The calculation on the last slide guarantees that for any $\varepsilon > 0$, there exists $N > 0$ such that if $n > N$,

$$
\sigma(A_{\infty}) \subseteq \sigma_\varepsilon(A_n).
$$
Symbols of Upper Triangular Toeplitz Matrices

Symbol curves for banded upper triangular Toeplitz matrices.

- $a(z) = z + 2z^2$
- $a(z) = z + 2z^2 + z^3 + z^5$
- $a(z) = z + z^3 - z^5$
- $a(z) = z^2 + 3z^5$
- $a(z) = z^2 + z^7$
More generally, the dense Toeplitz matrix

$$A_n = \begin{bmatrix}
    a_0 & a_1 & a_2 & \cdots \\
    a_{-1} & a_0 & \ddots & \vdots \\
    a_{-2} & \ddots & \ddots & a_{1} & a_2 \\
    \vdots & \ddots & a_{-1} & a_0 & a_1 \\
    \cdots & \ddots & a_{-2} & a_{-1} & a_0 \\
\end{bmatrix} \in \mathbb{C}^{n \times n}$$

follows the same terminology.

**Definition (Symbol, Symbol Curve)**

Toeplitz matrices are described by their *symbol* $a$ with Laurent expansion

$$a(z) = \sum_{k=-\infty}^{\infty} a_k z^k.$$ 

Call the image of the unit circle $T$ under $a$ the *symbol curve*, $a(T)$. 
Theorem (Spectrum of a Toeplitz Operator)

Suppose the Toeplitz operator \( A_\infty : \ell^2(N) \rightarrow \ell^2(N) \) has a symbol that is continuous. Then

\[
\sigma(A_\infty) = a(T) \cup \{\text{all points } a(T) \text{ encloses with nonzero winding number}\}.
\]

Due variously to: Wintner; Gohberg; Krein; Calderón, Spitzer, & Widom.

See Albrecht Böttcher and colleagues for many more details.
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Suppose the Toeplitz operator $A_\infty : \ell^2(N) \to \ell^2(N)$ has a symbol that is continuous. Then

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Due variously to: Wintner; Gohberg; Krein; Calderón, Spitzer, & Widom.

See Albrecht Böttcher and colleagues for many more details.
What can be said of the eigenvalues of a finite-dimensional Toeplitz matrix?

**Theorem (Limiting Spectrum of Finite Toeplitz Matrices)**

Consider the family of banded Toeplitz matrices \( \{A_n\}_{n \in \mathbb{N}} \) with upper bandwidth \( b \) and lower bandwidth \( d \). For any fixed \( \lambda \in \mathbb{C} \), label the roots \( \zeta_1, \ldots, \zeta_{b+d} \) of the polynomial \( z^d (a(z) - \lambda) \) by increasing modulus.

If \( |\zeta_d| = |\zeta_{d+1}| \), then \( \lambda \in \lim_{n \to \infty} \sigma(A_n) \).

This result, proved by [Schmidt & Spitzer, 1960], shows that in general:

\[
\lim_{n \to \infty} \sigma(A_n) \neq \sigma(A_\infty).
\]
Spectrum of a General Toeplitz Matrix

What can be said of the eigenvalues of a finite-dimensional Toeplitz matrix?

**Theorem (Limiting Spectrum of Finite Toeplitz Matrices)**

Consider the family of banded Toeplitz matrices \( \{A_n\}_{n \in \mathbb{N}} \) with upper bandwidth \( b \) and lower bandwidth \( d \). For any fixed \( \lambda \in \mathbb{C} \), label the roots \( \zeta_1, \ldots, \zeta_{b+d} \) of the polynomial \( z^d(a(z) - \lambda) \) by increasing modulus.

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\[
\lim_{n \to \infty} \sigma(A_n) \neq \sigma(A_\infty).
\]
Spectrum of a General Toeplitz Matrix: Example

\[ a(T) \]

\[ \lim_{n \to \infty} \sigma(A_n) \]

\[ \sigma(A_{50}) \]
Spectrum of a General Toeplitz Matrix: Example

\[ a(T) \lim_{n \to \infty} \sigma(A_n) \]

\[ \sigma(A_{100}) \]
Spectrum of a General Toeplitz Matrix: Example

\[ a(T) \]

\[ \lim_{n \to \infty} \sigma(A_n) \]

\[ \sigma(A_{200}) \]
Spectrum of a General Toeplitz Matrix: Example

\[ a(T) \]

\[ \lim_{n \to \infty} \sigma(A_n) \]

\[ \sigma(A_{400}) \]
Spectrum of a General Toeplitz Matrix: Example

\[ a(T) \]

\[ \lim_{n \to \infty} \sigma(A_n) \]

\[ \sigma(A_{800}) \]
Theorem (Landau; Reichel and Trefethen; Böttcher)

Let \( a(z) = \sum_{k=-M}^{M} a_k z^k \) be the symbol of a banded Toeplitz operator.

- The pseudospectra of \( A_n \) converge to the pseudospectra of the Toeplitz operator on \( \ell^2(N) \) as \( n \to \infty \).

Let \( z \in \mathbb{C} \) have nonzero winding number w.r.t. the symbol curve \( a(T) \).

- \( \| (z - A_n)^{-1} \| \) grows exponentially in \( n \).
- For all \( \varepsilon > 0 \), \( z \in \sigma_\varepsilon(A_n) \) for all \( n \) sufficiently large.
Pseudospectra of Toeplitz Matrices

symbol curve

\[ \sigma_\varepsilon(A_{100}) \]
Pseudospectra of Toeplitz Matrices

symbol curve

$\sigma_\varepsilon(A_{500})$
Hermitian Toeplitz Matrices

We have seen “large” pseudospectra arise for generic Toeplitz matrices.

But what about Hermitian Toeplitz matrices?

For example, the matrix

\[
\begin{bmatrix}
0 & 1 & & \\
1 & 0 & & \\
& & \ddots & \\
& & & 1
\end{bmatrix}
\]

has symbol \( a(z) = z^{-1} + z \), so

\[
a(e^{i\theta}) = e^{-i\theta} + e^{i\theta} = 2\cos(\theta) \in [-2, 2] \subset \mathbb{R}.
\]
Hermitian Toeplitz Matrices

We have seen “large” pseudospectra arise for generic Toeplitz matrices.

But what about Hermitian Toeplitz matrices?

For example, the matrix

\[
\begin{bmatrix}
0 & 1 \\
1 & 0 \\
& \ddots \\
& & \ddots & 1 \\
& & & 1 & 0
\end{bmatrix}
\]

has symbol \( a(z) = z^{-1} + z \), so

\[
a(e^{i\theta}) = e^{-i\theta} + e^{i\theta} = 2 \cos(\theta) \in [-2, 2] \subset \mathbb{R}.
\]

In general, Hermitian symbols have \( a_{-k} = \overline{a_k} \), so

\[
a(e^{i\theta}) = a_0 + \sum_{k=-\infty}^{-1} a_k e^{ik\theta} + \sum_{k=1}^{\infty} a_k e^{ik\theta}
\]

\[
= a_0 + \sum_{k=1}^{\infty} \overline{a_k} e^{-ik\theta} + \sum_{k=1}^{\infty} a_k e^{ik\theta} = a_0 + \sum_{k=1}^{\infty} 2 \text{Re}(a_k e^{ik\theta}) \in \mathbb{R}.
\]

As \( n \to \infty \), eigenvalues of \( A_n \) distribute according to Szegő’s theorem.
Tridiagonal Toeplitz Matrices

Consider the case of a tridiagonal Toeplitz matrix,

\[
A_n = \begin{bmatrix}
\beta & \gamma \\
\alpha & \beta & \ddots \\
& \ddots & \ddots & \gamma \\
& & \alpha & \beta
\end{bmatrix} \in \mathbb{C}^{n \times n}
\]

with symbol

\[
a(z) = \frac{\alpha}{z} + \beta + \gamma z.
\]

The symbol curve is the ellipse

\[
a(T) = \{ z \in \mathbb{T} : \frac{\alpha}{z} + \beta + \gamma z \}.
\]
Eigenvectors of Tridiagonal Toeplitz Matrices

\[ A = \text{tridiag}(1, 0, 1), \ n = 20 \]

Normal matrix: orthogonal eigenvectors
Eigenvectors of Tridiagonal Toeplitz Matrices

\[ A = \text{tridiag}(2, 0, 1/2), \ n = 20 \]

Nonnormal matrix: non-orthogonal eigenvectors
In contrast, consider the *[circulant matrix]*

\[
C_n = \begin{bmatrix}
    a_0 & a_1 & \cdots & a_{-2} & a_{-1} \\
    a_{-1} & a_0 & \ddots & \ddots & a_{-2} \\
    \vdots & \ddots & \ddots & a_1 & \ddots \\
    a_2 & \ddots & a_{-1} & a_0 & a_1 \\
    a_1 & a_2 & \cdots & a_{-1} & a_0
\end{bmatrix} \in \mathbb{C}^{n \times n}
\]
Circulant Matrices and Laurent Operators

In contrast, consider the *circulant matrix*

\[
C_n = \begin{bmatrix}
a_0 & a_1 & \cdots & a_{-2} & a_{-1} \\
a_{-1} & a_0 & \cdots & \cdots & a_{-2} \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
a_{2} & \vdots & a_{-1} & a_0 & a_{1} \\
a_{1} & a_{2} & \cdots & a_{-1} & a_{0}
\end{bmatrix} \in \mathbb{C}^{n \times n}
\]

- \(C_n\) is *normal* for all symbols.
- \(C_n\) is diagonalized by the Discrete Fourier Transform matrix.
- \(\sigma(C_n) = \{a(z) : z \in T_n\}\), where \(T_n := \{e^{2k\pi i/n}, k = 0, \ldots, n-1\}\): i.e., \(\sigma(C_n)\) comprises the image of the \(n\)th roots of unity under the symbol.
- Infinite dimensional generalization: Laurent operators \(C_{\infty}\) on \(\ell^2(\mathbb{Z})\) (doubly-infinite matrices) with spectrum \(\sigma(C_{\infty}) = a(T)\).
Circulant Matrix and Laurent Operators

\[ \sigma(C_{\infty}) = a(T') \]

\[ \sigma(C_{100}) \]
\[ \sigma(C_{200}) \]
\[ \sigma(C_{400}) \]
It is possible for the symbol

\[ a(z) = \sum_{k=-\infty}^{\infty} a_k z^k \]

to, e.g., have a jump discontinuity. This compromises the exponential growth of the norm of the resolvent [Böttcher, E., Trefethen, 2002].

For example, take \( a(e^{i\theta}) = \theta e^{i\theta} \), a symbol studied by [Basor & Morrison, 1994].
“Twisted” Toeplitz Matrices

A “twisted” Toeplitz matrix is a Toeplitz-like matrix with varying coefficients [Trefethen & Chapman, 2004].

For example, for \( x_j = \frac{2\pi j}{n} \), set

\[
A_n = \begin{bmatrix}
    x_1 & \frac{1}{2}x_1 & & & \\
    & x_2 & \ddots & & \\
    & & \ddots & \frac{1}{2}x_{n-1} & \\
    & & & x_n & \\
\end{bmatrix}.
\]

The “symbol” now depends on two variables: \( a(x, z) = x + \frac{1}{2}xz \).

The pseudospectra resemble those of standard Toeplitz matrices, but the (pseudo)-eigenvectors have an entirely different character.
For $x_j = 2\pi j/n$, set

\[
A_n = \begin{bmatrix}
  x_1 & 1/2 x_1 \\
  x_2 & \ddots & \ddots \\
  \vdots & \ddots & \ddots & \ddots \\
  x_{n-1} & \ddots & \ddots & 1/2 x_{n-1} \\
  x_n & \ddots & \ddots & \ddots & 1/2 x_n
\end{bmatrix}
\]

Pseudospectra of $\sigma_\varepsilon(A_{100})$
Eigenvectors of a Twisted Toeplitz Matrix

Eigenvectors form *wave packets* for twisted Toeplitz matrices. Pseudoeigenvectors for $z \in \sigma_{\varepsilon}(A)$ have a similar form.

Eigenvectors of $A_{20}$ for $a(x, z) = x + \frac{1}{2}xz$
Where do Toeplitz (and twisted Toeplitz) matrices come from?

Numerous applications – be we will highlight just one of them: discretization of differential operators.

Consider the steady–state convection diffusion equation $Lu = f$, where

$$Lu = u'' + cu'$$

posed over for $x \in [0, 1]$ with $u(0) = u(1) = 0$.

- Fix a discretization parameter $n$
- Approximate the problem on a simple grid $\{x_j\}_{j=0}^{n+1}$ with spacing $h = 1/(n + 1)$:
  $$x_j = jh.$$
- Replace the first and second derivatives with second-order accurate formulas on the grid:
  $$u'(x_j) = \frac{u(x_{j+1}) - u(x_j)}{2h} + O(h^2)$$
  $$u''(x_j) = \frac{u(x_{j-1}) - 2u(x_j) + u(x_{j+1})}{h^2} + O(h^2).$$
Finite Difference Discretizations

Now let $u_j \approx u(x_j)$ with $u_0 = u_{n+1} = 0$, so that

$$u'(x_j) \approx \frac{u_{j+1} - u_{j+1}}{2h}$$

$$u''(x_j) \approx \frac{u_{j-1} - 2u_j + u_{j+1}}{h^2}.$$  

Approximate $Lu = f$, with

$$Lu = u'' + cu', \quad u(0) = u(1) = 0,$$

via

$$\begin{bmatrix}
-2 & 1 + ch/2 \\
1 - ch/2 & -2 & \ddots \\
\vdots & \ddots & \ddots & 1 + ch/2 \\
1 - ch/2 & -2 \\
\end{bmatrix} \begin{bmatrix}
u_1 \\
u_2 \\
\vdots \\
u_n \\
\end{bmatrix} = \begin{bmatrix}
f(x_1) \\
f(x_2) \\
\vdots \\
f(x_n) \\
\end{bmatrix}.$$  

Label these components: $A_n u = f$.  

$A_n$ is Toeplitz for this constant-coefficient differential operator.
Finite Difference Discretizations

\( A_n := \frac{1}{h^2} \begin{bmatrix} -2 & 1 + ch/2 \\ 1 - ch/2 & -2 & \ddots \\ \ddots & \ddots & \ddots & 1 + ch/2 \\ 1 - ch/2 & -2 \end{bmatrix} \)

Notice that the symbol of \( A_n \) depends on \( n \) (recall \( h = 1/(n + 1) \)):

\[ a_n(z) = \left( \frac{1}{h^2} - \frac{c}{2h} \right) z^{-1} - \left( \frac{2}{h^2} \right) + \left( \frac{1}{h^2} + \frac{c}{2h} \right) z \]
Finite Difference Discretizations

\[ A_n := \frac{1}{h^2} \begin{bmatrix} -2 & 1 + ch/2 \\ 1 - ch/2 & -2 & \ddots \\ \ddots & \ddots & \ddots & 1 + ch/2 \\ 1 - ch/2 & -2 \end{bmatrix} \]

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- For all \( n \), the symbol curve \( a_n(T) \) is an ellipse.
Finite Difference Discretizations

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- For all \( n \), the symbol curve \( a_n(T) \) is an ellipse.
- If \( c = 0 \) (no convection), then \( a_n(T) \) is a real line segment: \( A_n \) and \( L \) are both self-adjoint, hence normal.
- If \( c \neq 0 \), the eigenvalues of \( A_n \) will be real, but the pseudospectra can be far from these eigenvalues.
Finite Difference Discretizations

\[
A_n := \frac{1}{h^2} \begin{bmatrix}
-2 & 1 + ch/2 \\
1 - ch/2 & -2 & \ddots \\
& \ddots & \ddots & 1 + ch/2 \\
& & 1 - ch/2 & -2
\end{bmatrix}
\]

Notice that the symbol of \( A_n \) depends on \( n \) (recall \( h = 1/(n + 1) \)):

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a_n(z) = \left( \frac{1}{h^2} - \frac{c}{2h} \right) z^{-1} - \left( \frac{2}{h^2} \right) + \left( \frac{1}{h^2} + \frac{c}{2h} \right) z
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- If \( c = 0 \) (no convection), then \( a_n(T) \) is a real line segment: \( A_n \) and \( L \) are both self-adjoint, hence normal.
- If \( c \neq 0 \), the eigenvalues of \( A_n \) will be real, but the pseudospectra can be far from these eigenvalues.
- Rightmost part of \( \sigma_\varepsilon(A_n) \) approximates the corresponding part of \( \sigma_\varepsilon(L) \).
Finite Differences: Symbol Curves

Symbol curves $a_n(T)$ for $n = 16, 32, 64$
Finite Differences: Symbol Curves

Symbol curves $a_n(T)$ for $n = 16, 32, 64$
Rightmost part of $\sigma_{\varepsilon}(A_n)$ for $n = 64$
Rightmost part of $\sigma_{\varepsilon}(A_n)$ for $n = 128$
Rightmost part of $\sigma_\varepsilon(A_n)$ for $n = 256$
Rightmost part of $\sigma_\varepsilon(A_n)$ for $n = 512$
3(b) Balanced Truncation Model Reduction
Consider the single-input, single-output (SISO) linear dynamical system:

$$\dot{x}(t) = Ax(t) + bu(t)$$
$$y(t) = cx(t),$$

$A \in \mathbb{C}^{n \times n}$, $b, c^T \in \mathbb{C}^n$. We assume that $A$ is stable: $\alpha(A) < 0$.

We wish to reduce the dimension of the dynamical system by projecting onto well-chosen subspaces.

**Balanced truncation:** Change basis to match states that are easy to reach and easy to observe, then project onto that prominent subspace.
Controllability and Observability Gramians

To gauge the **observability** of an initial state $x_0 = \hat{x}$, measure the energy in its output (when there is no input, $u = 0$):

$$y(t) = ce^{tA}\hat{x}.$$  

Then

$$\int_{0}^{\infty} |y(t)|^2 \, dt = \int_{0}^{\infty} \hat{x}^* e^{tA^*} c^* ce^{tA}\hat{x} \, dt$$

$$= \hat{x}^* \left( \int_{0}^{\infty} e^{tA^*} c^* ce^{tA} \, dt \right) \hat{x} =: \hat{x}^* Q\hat{x}.$$
Controllability and Observability Gramians

To gauge the *observability* of an initial state $x_0 = \hat{x}$, measure the energy in its output (when there is no input, $u = 0$):

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Then

$$\int_0^\infty |y(t)|^2 \, dt = \int_0^\infty \hat{x}^* e^{tA^*} c^* ce^{tA}\hat{x} \, dt$$

$$= \hat{x}^* \left( \int_0^\infty e^{tA^*} c^* ce^{tA} \, dt \right) \hat{x} =: \hat{x}^* Q\hat{x}.$$  

Similarly, we measure the *controllability* by the total input energy required to steer $x_0 = 0$ to a target state $\hat{x}$ as $t \to \infty$. The special form of $u$ that drives the system to $\hat{x}$ with minimal energy satisfies

$$\int_0^\infty |u(t)|^2 \, dt = \hat{x}^* \left( \int_0^\infty e^{tA} bb^* e^{tA^*} \, dt \right)^{-1} \hat{x} =: \hat{x}^* P^{-1}\hat{x}.$$
Controllability and Observability Gramians

To gauge the *observability* of an initial state \( x_0 = \hat{x} \), measure the energy in its output (when there is no input, \( u = 0 \)):

\[
y(t) = ce^{tA}\hat{x}.
\]

Then

\[
\int_0^\infty |y(t)|^2 \, dt = \int_0^\infty \hat{x}^* e^{tA^*} c^* ce^{tA}\hat{x} \, dt
\]

\[
= \hat{x}^* \left( \int_0^\infty e^{tA^*} c^* ce^{tA} \, dt \right) \hat{x} =: \hat{x}^* Q\hat{x}.
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Similarly, we measure the *controllability* by the total input energy required to steer \( x_0 = 0 \) to a target state \( \hat{x} \) as \( t \to \infty \). The special form of \( u \) that drives the system to \( \hat{x} \) with minimal energy satisfies

\[
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\]

Thus we have the infinite controllability and observability gramians \( P \) and \( Q \):

\[
P := \int_0^\infty e^{tA} bb^* e^{tA^*} \, dt, \quad Q := \int_0^\infty e^{tA^*} c^* c e^{tA} \, dt.
\]

See, e.g., [Antoulas, 2005].
Balanced Truncation Model Reduction

The gramians

\[ P := \int_0^\infty e^{tA}bb^*e^{tA^*} \, dt, \quad Q := \int_0^\infty e^{tA^*}c^*ce^{tA} \, dt \]

(Hermitian positive definite, for a controllable and observable stable system) can be determined by solving the Lyapunov equations – see the next lecture.

If \( x_0 = 0 \), the minimum energy of \( u \) required to drive \( x \) to state \( \hat{x} \) is

\[ \hat{x}^*P^{-1}\hat{x}. \]

Starting from \( x_0 = \hat{x} \) with \( u(t) \equiv 0 \), the energy of output \( y \) is

\[ \hat{x}^*Q\hat{x}. \]

\( \hat{x}^*P^{-1}\hat{x} \): \( \hat{x} \) is hard to reach if it is rich in the lowest modes of \( P \).

\( \hat{x}^*Q\hat{x} \): \( \hat{x} \) is hard to observe if it is rich in the lowest modes of \( Q \).

Balanced truncation transforms the state space coordinate system to make these two gramians the same, then it truncates the lowest modes.
Consider a generic coordinate transformation, for $S$ invertible:

$$(Sx)'(t) = (SAS^{-1})(Sx(t)) + (Sb)u(t)$$

$$y(t) = (cS^{-1})(Sx(t)) + du(t), \quad (Sx)(0) = Sx_0.$$ 

With this transformation, the controllability and observability gramians are

$$\hat{P} = SPS^*, \quad \hat{Q} = S^{-*}QS^{-1}.$$ 

For balancing, we seek $S$ so that $\hat{P} = \hat{Q}$ are diagonal.
Consider a generic coordinate transformation, for $S$ invertible:
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For balancing, we seek $S$ so that $\hat{P} = \hat{Q}$ are diagonal.

**Observation (How does nonnormality affect balancing?)**

- $\sigma_{\varepsilon/\kappa}(S)(SAS^{-1}) \subseteq \sigma_{\varepsilon}(A) \subseteq \sigma_{\varepsilon\kappa}(S)(SAS^{-1})$.
- The choice of internal coordinates will affect $P$, $Q$, ...
Balanced Truncation Model Reduction

Consider a generic coordinate transformation, for $S$ invertible:

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For balancing, we seek $S$ so that $\hat{P} = \hat{Q}$ are diagonal.

Observation (How does nonnormality affect balancing?)

- $\sigma_{\varepsilon/\kappa}(S)(SAS^{-1}) \subseteq \sigma_{\varepsilon}(A) \subseteq \sigma_{\varepsilon\kappa}(S)(SAS^{-1})$.
- *The choice of internal coordinates will affect $P$, $Q$, ...*
- *but not the Hankel singular values: $\hat{P}\hat{Q} = SPQS^{-1}$,*
- *and not the transfer function:*
  \[
  (cS^{-1})(z - SAS^{-1})^{-1}(Sb) = d + c(z - A)^{-1}b,
  \]
- *and not the system moments:*
  \[
  (cS^{-1})(Sb) = cb, \quad (cS^{-1})(SAS^{-1})(Sb) = cAb, \quad ....
  \]