Minimax LQG Control a Simply-Supported Flexible Beam

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Abstract
An optimal-robust Minimax LQG control of vibration on a pinned-pinned flexible beam is presented in this paper. Modeling of the beam is based on the experimental data. The first six modes of the beam are included in the measured impulse response. A Minimax LQG controller is designed for the augmented plant. A trade off between robust stability and robust performance is shown with selecting two different uncertainties modeling. The simulation results show that this robust controller increases the damping of the system in its resonance frequencies.

INTRODUCTION
In many industries vibration has many destructive effects and so it must be eliminated [1]. Passive methods (like using isolators) are not considered in low frequency vibrations because the volume and weight of the structure will be increased and also they can not adapt themselves with environmental changes [1]. To overcome these drawbacks active control of vibrations is focused as the main solution in recent years. In these methods, an appropriate signal is produced to eliminate or increases the damping of undesirable vibrations. The use of classical controllers (like PID or inverse PID controllers) for compensation of the vibratory systems has been investigated [2]. These controllers show limited performance when a small perturbation is occurred in the nominal model, and they make good results only when one mode is to be damped. Another class of controllers uses adaptive algorithms. Iterative algorithms are used to minimize a performance index by reducing the residual vibrations at some specific points on the system. These on-line methods have much computational burden and they will have good performance when the aim is to eliminate narrowband disturbances [3,4].
In recent years, robust controllers have been developed for active control of vibrations [5]. In [6,7], $H_2$ and $H_\infty$ controllers are used to control of vibration spatially in an experimental beam. For this purpose an appropriate cost functional, which is an indicator of the vibration energy on the structure, is selected and by putting some constraints on the control effort, an optimal robust controller is designed. Recently, Minimax LQG control is used as a robust controller for active vibration control applications [8,10]. In this method a cost functional similar to the LQG problem is introduced and then by defining a constraint on the magnitude of the model uncertainties and the maximum level of noise uncertainty, a robust controller is designed. This controller is designed in worst-case and can withstand two kinds of uncertainties: modeling uncertainties as well as measurement noise. Unlike $H_2$ and $H_\infty$ controllers which requires a non-convex optimization, in Minimax LQG control, controllers can be obtained by solving two algebraic Riccati equations in a steady state condition [9].

In [11] $\mu$-synthesis technique which has robust performance and robust stability properties is also used for robust control of vibration in a flexible structure.

Design of robust controller like many other controllers needs an appropriate model of the structure. Analytical modeling is one way usually used to have a suitable model [12,13]. In this method the equations of the structure are solved with respect to the boundary conditions and the transfer function between two arbitrary points of the structure is achieved. This method will be effective only if the governing equations of the physical plant are known in details (like Euler -Bernoulli equation for the beam). Finite element method is a good choice for non-geometrical shapes where analytical equations are hard to achieve. Finite element methods will result in the mass, stiffness and damping matrices as a good approximation of the true system [14,15]. Since these types of modeling result in higher order models and for the purpose of controller design, usually limited bandwidth of the plant is required, it is necessary to truncate the analytical model, and so out of bandwidth modes will cause uncertainty in the modeling.

Another efficient way in modeling of a structure is to use identification techniques. In these methods, the transfer function of the system is calculated based on time or frequency domain data, gathered from an experimental setup. An important point in design of a robust controller is the assumed bound of uncertainty. One way to cope with this problem is to choose a suitable uncertainty bound with respect to the difference between identified and measured frequency response of the structure [8,10]. This model as will be described in subsequent sections in more details can be used to design a robust controller.

In Section 2, the experimental beam for which a controller is to be designed is introduced and the model of beam as well as its uncertainty bound is identified. In Section 3, the Minimax LQG algorithm is introduced briefly. In Section 4, an optimization is performed to achieve good (reasonable) performance and stability and then a Minimax LQG controller is designed for the flexible pinned-pinned beam. Then, robust stability and performance of the controller with two different uncertainty
bounds are compared. Based on the simulation results, in Section 5, conclusions and all the problems which a designer may encounter in this design method are discussed.

**MODELING OF THE BEAM AND ITS UNCERTAINTIES**

The first step in designing a robust controller is to have a suitable model of the beam. The flexible experimental beam which is considered is made from aluminum with dimensions: 500×51×2 mm. The beam is fixed on a very heavy table with two clamps in whole of the beam’s width in order to isolate it from other undesirable vibrations which may affect it during the experiment.

Fig. 1 shows the beam and the experimental setup. The frequency response of the beam is measured between two collocated points on its surface. In identification procedure, a special hammer is used to excite the beam with an impulse signal and the vibration is measured using a piezoelectric sensor (DJB 120/A/V). Both of input and output are connected to a signal analyzer and the frequency response of the beam is calculated using impulse response techniques. In order to have a more accurate model of the beam and to reduce the effect of measurement noises this procedure is repeated for 10 times and the average of obtained frequency responses which is shown in Fig. 2 is chosen as the actual model of the beam between two points.

As it can be seen only the first six modes of the beam are considered. Since it is assumed that these modes have dominant effect in the response of the beam the required bandwidth for modeling and control is selected up to 800Hz. Since the degree of the controller depends on the degree of the model in order to obtain a lower order controller only 3 modes of the beam are considered for control design purpose. A model of the beam is calculated by fitting a transfer function to second, third and forth mode of the beam and so other modes are considered as undermodeling uncertainties. Fig. 3 shows fitted transfer function for these three modes using least-square method. Because of some priori knowledge about the analytical model of the beam, it is known that in the transfer function obtained from analytical point of view the denominator will have two degree more than the numerator. So the degree of
numerator and denominator of the identified model are selected as 4 and 6 respectively. It is important to note that increasing the degree of the denominator will introduce unstable poles to the system which is undesirable. The calculated transfer function for the 3 mentioned modes is:

\[
P(s) = \frac{-2.555s^4 - 0.0615s^3 - 1.628s^2 + 0.00352s - 0.01232}{s^6 + 0.056s^5 + 3.013s^4 + 0.106s^3 + 2.077s^2 + 0.0350s + 0.267}
\]

This transfer function, which is the nominal plant, is called \( P(s) \) in the remainder of the paper.

The general scheme of the open-loop system is shown in Fig. 4.

\( u \) is the control input, \( y \) is the measured output, \( z \) is the uncertainty output and \( \xi \) is the input which arises from the uncertainty model. Since the uncertainty of the transfer function is modeled by multiplicative uncertainty the real model of the plant can be written as follows:

\[
P_{\Delta} (s) = P (s) \left( 1 + W(s) \Delta (s) \right)
\]

Figure 2. Measured frequency response of the beam

It is obvious that \( p_{\Delta} \) represents the measured (actual) frequency response of the beam and \( \Delta(s) \) is an uncertainty transfer function resulted from under modeling of some modes.
In order the robust stability of the controller will be guaranteed, the below condition on the infinity norm of the uncertainty must be satisfied:

\[ \| \Delta(s) \|_{\infty} \leq 1 \]  

(3)

A smaller norm will result in more stability margin but the performance may be decreased. With these two relations it can be written:

\[ |W(jw)| \geq \left| \frac{P_{\Delta}(jw) - P(jw)}{P(jw)} \right| \]  

(4)

With this assumption, the weighting function \( W(s) \) is obtained. Since the control algorithm used is very sensitive to the specified plant, this part of design procedure is very important in achieving a desired robust controller.

The multiplicative uncertainty of the plant is modeled with two filters as shown in Fig. 5 and 6. The first filter is of Chebychev type I of order (11) and the second one is a \( p^\text{th} \) norm filter. The first filter is achieved by subtracting bode plots of two high pass analog filters with different cut-off frequencies which are adjusted to make a hole in middle frequencies. The characteristics of second filters like cut off frequency order and pass band ripples are adjusted after some trial and error using FDA toolbox of
MATLAB. The stop band filter is converted to an analog filter using Tustin method. The resultant analog filter will set an upper bound for the uncertainties. As it is clear from Fig. 5 the fit has not deliberately tighten to unmodeled dynamics. For example, for the first mode which isn’t concluded in model, there is no uncertainty bound. In Fig. 6 a $p^{th}$ norm filter of order (10) is fitted, but a tighter fit is selected. It will be shown that this loose or tight fitting will result in better robust performance and robust stability respectively. The ripple of the pass band is selected small (2 dB). Besides, increasing the degree of the filter in spite of good fitting may result in an unstable controller. It should be noticed that in Minimax LQG, this weighting transfer function has important effect on the performance of the closed-loop system.

Fig 4. General scheme of the open-loop plant
Figure 5. Multiplicative uncertain bound with Chebychev filter

Figure 6. Multiplicative uncertain bound with P-th norm filter
A significant idea in robust control design is to solve the problem with minmax optimization procedure. This is a game-type problem such that the design of a controller is performed in a worst case situation and in the presence of uncertainties. In this game type problem, the designer may be considered as a minimizing player who endeavors to find an optimal control strategy to maintain a certain level in robust performance of the closed loop system when it is faced with some degrees of uncertainty. In contrast, the uncertainty in the underlying plant may impair the performance of the closed-loop system (or even unstabilizes it). Thus, one may think of the uncertainty as a maximizing player in this game. The advantage of this method is that it may allow one to convert the issue of robustness into a mathematically tractable game-type minmax optimization framework. This may called a robust LQG or Minimax problem. This problem will be more complicated when the noise which is affected the system is not a white Gaussian noise. Like the LQG control, it is assumed that a white Gaussian noise is input to the plant. It is proved that if the real input noise to the plant is not Gaussian, this algorithm can result in robust stability of the closed-loop system by appropriate tuning of some parameters [8]. However, the system may show poor robust performance.

As it is shown in Fig. 4 the disturbance $W$ is applied at the output of the system (In the case of the disturbance is applied at the input of the plant the algorithm can be implemented in the same way [8]). It should be noticed that the general plant includes the states of the main system ($P(s)$) as well as its uncertainty weight ($W(s)$). Assuming the state-space realization of the main system as $(A, B, C, D)$ and for uncertainty weighting as $(A_b, B_b, C_b, D_b)$ the augmented plant can be written as follows:

\[
\begin{align*}
\xi &= \begin{bmatrix} A & 0 \\ 0 & A_b \end{bmatrix} x + \begin{bmatrix} B_x \\ B_b \end{bmatrix} u + \begin{bmatrix} B_x \\ 0 \end{bmatrix} \xi + \begin{bmatrix} 0 \\ 0 \end{bmatrix} W \\
z &= \begin{bmatrix} 0 & C_x \end{bmatrix} x + D_b u \\
y &= C_x x + del \times W
\end{align*}
\]

In the above equation, $\rho$ is a tuning parameter. As it is clear from Fig. 4 $\rho$ is a weighting parameter for the white Gaussian noise where applied at the plant output. $d$ is a parameter which may restrict the size of total disturbance acting on the system and can be written as follows (An Integral Quadratic Constraint):

\[
\lim_{T \to \infty} \frac{1}{T} E \left[ \int_0^T \| \xi(t) \|^2 dt - \int_0^T \| z(t) \|^2 dt \right] \leq d
\]

(6)
Since choosing \( d \) small enough means a larger infinity norm of the \( \Delta \) we will have a better robust performance. Instead, if an uncertainty exists in the disturbance signal (like non-Gaussian noises) a larger \( d \) will result in good robust stability \([8]\).

In Minimax LQG cost functional which is given in (7) is like an LQG problem, but here there are additional constraints on uncertainties:

\[
J = \lim_{T \to \infty} \frac{1}{2T} E \int_0^T (x(t)' R(x(t) + u(t)' G u(t))) \, dt \tag{7}
\]

where \( R \) the state weighting matrix is equal to \( C_2' C_2 \). In this case, (7) indicates the total energy of the output of the plant. \( Q \) is input weighting matrix which limits the control effort, but it is usually selected small enough to prevent input saturation. The solution of the Minimax LQG problem can be obtained as described in \([9]\) by solving two algebraic Riccati equations in steady-state mode as follows:

\[
(A - B_2 D_2' (D_2 D_2')^{-1} C_2) Y_\infty + Y_\infty (A - B_2 D_2' (D_2 D_2')^{-1} C_2)' \nonumber \\
- Y_\infty (C_2' (D_2 D_2')^{-1} C_2 - \frac{1}{\tau} R_\tau) Y_\infty \\
+ B_2 (I - D_2' (D_2 D_2')^{-1} D_2) B_2' = 0
\]

\[
(A - B_1 G_\tau^{-1} \Pi_\tau') X_\infty + X_\infty (A - B_1 G_\tau^{-1} \Pi_\tau') \\
- X_\infty (B_1 G_\tau^{-1} B_1' - \frac{1}{\tau} B_2 B_2') X_\infty \\
+ (R_\tau - \Pi_\tau G_\tau^{-1} \Pi_\tau') = 0 \tag{8}
\]

which the new matrices are defined as below:

\[
R_\tau = R + \tau C_1' C_i
\]
\[
G_\tau = G + \tau D_1' D_i
\]
\[
\Pi_\tau = \tau C_i' D_i
\tag{9}
\]

where the solutions are required to satisfy the conditions:

\[
Y_\infty > 0, \\
X_\infty > 0, \\
I - \frac{1}{\tau} Y_\infty X_\infty > 0 \tag{10}
\]
As it is clear from above equations, the parameter $\tau$ has a dominant effect in design of the controller. In order to be able to choose the value of $\tau$ appropriately, a new cost functional derived from (7) can be written as follows [8]:

$$W_{\tau} = \frac{1}{2} \text{trace} \left[ Y_{\omega} R_{\tau} + (Y_{\omega} C_{2} + B_{2} D_{2}) (D_{2} D_{2}^{'})^{-1} \right] \times (C_{2} Y_{\omega} + D_{2} B_{2}) X_{\omega} (I - \frac{1}{\tau} Y_{\omega} X_{\omega})^{-1} + \tau d$$  \hspace{1cm} (11)

Since the parameter $\tau$ does not depend on $X_{\infty}$ and $Y_{\infty}$ linearly, minimizing the resulted cost functional is not a straightforward task. The key problem in designing a good Minimax LQG controller is to choose a suitable value for $\tau$. It can be shown [8] that this cost functional has an infimum at $\tau > 0$ which must be found by trial and error. Then the Minimax LQG controller is obtained as follows in state space representation:

$$A_{\tau} = A + \frac{Y_{\omega} R_{\tau}}{\tau} - (Y_{\omega} C_{2} + B_{2} D_{2}) (D_{2} D_{2}^{'})^{-1} C_{2} - (B_{1} + \frac{Y_{\omega}}{\tau} \Pi_{\tau}) G_{\tau}^{-1}$$

$$\times (B_{1} X_{\omega} + \Pi_{\tau}) (I - \frac{Y_{\omega} X_{\omega}}{\tau})^{-1}$$

$$B_{\tau} = (Y_{\omega} C_{2} + B_{2} D_{2}) (D_{2} D_{2}^{'})^{-1}$$

$$C_{\tau} = G_{\tau}^{-1} (B_{1} X_{\omega} + \Pi_{\tau}) (I - \frac{Y_{\omega} X_{\omega}}{\tau})^{-1}$$

$$D_{\tau} = 0$$  \hspace{1cm} (12)

In this algorithm, there are several parameters to be tuned properly to have good closed-loop performance. In tune of these parameters the performance of the system must be checked in every step.

For more details on Minimax LQG problem and its solution you can refer to [9,10,16].

**CONTROLLER DESIGN FOR FLEXIBLE PINNED-PINNED BEAM**

In this section the identified model in section 2 is used to design a robust controller by Minimax LQG technique described in section 3.

As mentioned before, selecting a large $d$ may result in good robust stability in the presence of noise uncertainties but it may deteriorate the performance of the closed-loop system, besides the increment of it can increase the cost functional linearly. After some trial and error, $d$ is chosen as $10^{-10}$. Since we have a SISO plant $Q$ will be a scalar selected as $10^{-4}$, $\rho$ which relates noise to the output and has great effect on the value of the cost functional is chosen as $10^{-7}$.

By plotting the performance index $W_{\tau}$ for different values of $\tau$, the interval where the global infimum of the cost functional may lie is obtained. Then with a more exact search using nonlinear optimization algorithm the fine tune of $\tau$ will be possible. It
should be noted that a small change in $\tau$ may impair the performance of the system. This search has been done for two type of uncertainty filters designed in section 2, and so two different values for $\tau$ is obtained. Having these values the robust controller can be designed using relation (12).

$$\tau_1 = 1972 \quad W_{1c} = 1888377$$
$$\tau_2 = 9316 \quad W_{2c} = 4202641$$

The frequency responses of two controllers are plotted in Fig. 7 and 8 distinctively. As it can be inferred from these figures the robust controller has the same nature as the plant and the resonant peaks of the controllers occurs near the sonant peaks of the plant. As stated before, the aim of these two controllers is to decrease the resonant peaks of the frequency response of the beam, while holding the robust stability condition respect to unmodeled dynamics.

Open-loop and closed-loop frequency responses of the plant for uncertainties modeling W1 and W2 are shown in Fig. 9 and 10 respectively. As it can be seen, for the first controller the quantity of damping is larger since a tighter bound for uncertainties is choosed.

![Controller Bode Plot for Chebychev Uncertainty Weighting](image-url)

*Figure 7. Controller magnitude bode plot for Chebychev uncertainty weighting*
Figure 8. Controller magnitude bode plot for P-th norm uncertainty weighting

Figure 9. Simulated controlled and uncontrolled frequency response of the beam with Chebychev uncertainty weighting
5. CONCLUSION
Modeling and robust control of vibration for a flexible pinned-pinned beam is investigated. By fitting a continuous transfer function to second, third and fourth modes of the beam using experimental data, three other modes are assumed as model uncertainty. Two different weighting functions by loose and tight fitting for uncertainty frequency response are selected to show a trade-off between robust stability and robust performance. Multiplicative type for uncertainty is used in modeling.

After modeling procedure, a Minimax LQG controller is designed for the augmented plant. The aim is to design a controller which must have good robust stability and performance against model and noise uncertainties. The controller is obtained in a worst-case situation and an optimal problem is solved to achieve good performance for the closed-loop system. Because uncertainties have undesired effects on stability of the system, selecting a good weighting function to model them may lead to robust stability. However, designer should make a good trade-off between stability margin and performance of the closed-loop system.

Since the LQG controller design is based on white Gaussian noise disturbance, if there are other uncertainties in noise like non-Gaussian or Gaussian noise with varying variance, etc. the performance of the closed-loop system may be impaired. However, the robust stability against this type of uncertainty is attained. Because
there are many parameters which can affect the performance of the closed-loop system, it must be remembered that choosing some suitable parameters, desirable performance can be achieved. As it is shown applying Minimax LQG controller may not eliminate the vibrations perfectly, but since the controller decreases the resonance peaks of the beam frequency response in three controlled modes the compensated structure has more damping.

6. REFERENCES

