Last name, first name: ... **Major: ...** **Mat.-Nr.: ...**

Ilmenau University of Technology
Communications Research Laboratory
Prof. Dr.-Ing. Martin Haardt

Adaptive and Array Signal Processing

09.02.2011

Total time: 120 min Total points: 48

NB: Those tasks highlighted **bold-faced** can be solved independently from the previous ones. Label the axis of all your graphs properly.

1. Consider the following matrix $\boldsymbol{X} \in \mathbb{R}^{4\times 2}$: (7 pt)

$$\boldsymbol{X} = \frac{1}{4} \cdot \begin{bmatrix} 3 & 1 \\ 3 & 1 \\ 1 & 3 \\ 1 & 3 \end{bmatrix}.$$

Its singular value decomposition $\boldsymbol{X} = \boldsymbol{U} \cdot \boldsymbol{\Sigma} \cdot \boldsymbol{V}^{\mathrm{H}}$ is given by

$$\boldsymbol{U} = \frac{1}{2} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \\ 1 & -1 & 1 & -1 \end{bmatrix}, \quad \boldsymbol{\Sigma} = \begin{bmatrix} \sqrt{2} & 0 \\ 0 & \sqrt{2}/2 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad \boldsymbol{V} = \frac{1}{\sqrt{2}} \cdot \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}.$$

(a) What are the dimensions of the column space, the null space, the row space, and the left null space, respectively?

(b) Let $\tilde{\boldsymbol{X}}$ be the best rank-one approximation of $\boldsymbol{X}$ in the Frobenius norm sense. How can we find $\tilde{\boldsymbol{X}}$ from the SVD of $\boldsymbol{X}$?

(c) Compute $\tilde{\boldsymbol{X}}$.

(d) Provide a basis for the column space of $\tilde{\boldsymbol{X}}$ and a basis for the null space of $\tilde{\boldsymbol{X}}$.

(e) Compute the projection matrix $\boldsymbol{P}$ onto the column space of $\tilde{\boldsymbol{X}}$.

2. Let a tensor $\boldsymbol{\mathcal{X}}$ be given by (11 pt)

$$\boldsymbol{\mathcal{X}} = \boldsymbol{\mathcal{I}}_2 \times_1 \boldsymbol{A} \times_2 \boldsymbol{B} \times_3 \boldsymbol{C}$$

where $\boldsymbol{\mathcal{I}}_2$ is the $2 \times 2 \times 2$ identity tensor (with elements $[\boldsymbol{\mathcal{I}}_2]_{(1,1,1)} = [\boldsymbol{\mathcal{I}}_2]_{(2,2,2)} = 1$ and all other elements zero) and $\boldsymbol{A}$, $\boldsymbol{B}$, $\boldsymbol{C}$ are the loading matrices given by

$$\boldsymbol{A} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \quad \boldsymbol{B} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \quad \boldsymbol{C} = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}$$

(a) What is the (tensor) rank $r = \mathrm{rank}\{\boldsymbol{\mathcal{X}}\}$?

(b) Demonstrate a) by explicitly computing r rank-one tensors $\boldsymbol{\mathcal{X}}_1 \ldots \boldsymbol{\mathcal{X}}_r$ such that $\boldsymbol{\mathcal{X}} = \sum_{i=1}^{r} \boldsymbol{\mathcal{X}}_r$.

(c) Compute the unfoldings $[\boldsymbol{\mathcal{X}}]_{(1)}$, $[\boldsymbol{\mathcal{X}}]_{(2)}$, $[\boldsymbol{\mathcal{X}}]_{(3)}$ in forward (MATLAB) column ordering.

(d) What are the n-ranks of $\boldsymbol{\mathcal{X}}$ for $n = 1, 2, 3$?

(e) Find a basis for the space spanned by the one-mode vectors of $\boldsymbol{\mathcal{X}}$, i.e., the column space of $[\boldsymbol{\mathcal{X}}]_{(1)}$.

(f) Show that for an arbitrary tensor $\boldsymbol{\mathcal{X}} \in \mathbb{C}^{M_1 \times M_2 \times M_3}$ and matrices $\boldsymbol{A} \in \mathbb{C}^{N_1 \times M_1}$, $\boldsymbol{B} \in \mathbb{C}^{P_1 \times N_1}$, the following identity holds: $\boldsymbol{\mathcal{X}} \times_1 \boldsymbol{A} \times_1 \boldsymbol{B} = \boldsymbol{\mathcal{X}} \times_1 (\boldsymbol{B} \cdot \boldsymbol{A})$. Hint: Use a suitable n-mode unfolding.

3. We consider an M-tap linear FIR filter with weight vector $\boldsymbol{w} \in \mathbb{C}^{M\times 1}$ that operates on a received signal $\boldsymbol{x}[n] \in \mathbb{C}^{M\times 1} = [x[n], x[n-1], \dots, x[n-M+1]]^{\mathrm{T}}$ and produces a scalar output $y[n] = \boldsymbol{w}^{\mathrm{H}} \cdot \boldsymbol{x}[n] = \sum_{m=1}^{M} w_m^* \cdot x[n-m+1]$. The goal is to design the filter such that $y[n]$ follows a desired signal $d[n]$, i.e., the squared error signal $|e[n]|^2 = |y[n]-d[n]|^2$ is minimized. (7 pt)

Moreover, the following quantities are given:

$$\boldsymbol{R} = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}, \quad \boldsymbol{R}^{-1} = \frac{1}{4}\begin{bmatrix} 3 & -1 & -1 \\ -1 & 3 & -1 \\ -1 & -1 & 3 \end{bmatrix}, \quad \boldsymbol{p} = \begin{bmatrix} 4 \\ -4 \\ 8 \end{bmatrix} \tag{1}$$

where $\boldsymbol{R} = \mathbb{E}\left\{\boldsymbol{x}[n] \cdot \boldsymbol{x}[n]^{\mathrm{H}}\right\}$ is the autocovariance matrix of the zero mean random process $\boldsymbol{x}[n]$ and $\boldsymbol{p} = \mathbb{E}\left\{\boldsymbol{x}[n] \cdot d[n]^*\right\}$ is the cross-correlation vector between $\boldsymbol{x}[n]$ and the desired signal $d[n]$.

The eigenvalues of $\boldsymbol{R}$ are given by $\lambda_1 = 4$, $\lambda_2 = 1$, and $\lambda_3 = 1$. The variance of the desired signal $d[n]$ is given by $\mathbb{E}\left\{|d(n)|^2\right\} = 100$.

(a) Compute the filter weight vector $\boldsymbol{w}_{\mathrm{opt}}$ which minimizes the mean square error $J(\boldsymbol{w}) = \mathbb{E}\left\{|e(n)|^2\right\}$.

(b) Determine the resulting mean square error $J_{\min} = J(\boldsymbol{w}_{\mathrm{opt}})$.

Alternatively, we can compute $\boldsymbol{w}$ iteratively. We examine the method of steepest descent where we take small steps in the direction of the negative gradient, starting from an initial weight vector $\boldsymbol{w}_0$.

(c) Provide an explicit expression for the gradient of the cost function $\boldsymbol{\gamma}(\boldsymbol{w}) = \frac{\partial J(\boldsymbol{w})}{\partial \boldsymbol{w}^*}$.

(d) For the given covariance matrix, what is the maximum step size $\mu_{\max}$ that still allows the algorithm to converge?

(e) Starting with the initial weight vector $\boldsymbol{w}_0 = \boldsymbol{0}_{M\times 1}$, i.e., the zero vector, perform two steps of the method of steepest descent using the step size $\mu = 0.25$, i.e.,

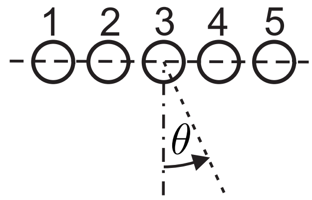
i. Compute the gradient $\boldsymbol{\gamma}(\boldsymbol{w}_0)$.

ii. Perform one step of steepest descent to compute $\boldsymbol{w}_1$.

iii. Compute the new gradient using $\boldsymbol{\gamma}(\boldsymbol{w}_1)$.

iv. Perform a second step of steepest descent to compute $\boldsymbol{w}_2$.

4. We consider a uniform linear array (ULA) with $M = 5$ elements and $\lambda/2$ inter-element spacing. (13 pt)



Its array steering vector $\boldsymbol{a}(\theta) = [a_1(\theta), a_2(\theta), a_3(\theta), a_4(\theta), a_5(\theta)]^{\mathrm{T}}$ satisfies $a_3(\theta) = 1\,\forall\theta$, i.e., the phase reference of the array is chosen in the middle.

(a) Provide expressions for $a_1(\theta), a_2(\theta), a_4(\theta)$, and $a_5(\theta)$, where θ is the azimuth angle.

Now assume that d wavefronts are impinging from distinct directions $\theta_1, \theta_2, \ldots, \theta_d$.

(b) Show that the array steering matrix $\boldsymbol{A} \in \mathbb{C}^{M \times d}$ for d impinging wavefronts is left-$\boldsymbol{\Pi}$-real.

(c) We would like to estimate the directions of arrival θ_i via ESPRIT. If all wavefronts are non-coherent and enough snapshots are available ($N > M$), what is the maximum number of wavefronts $d_{\max}$ that can be resolved with this array?

(d) If all wavefronts are coherent, what is the maximum number of wavefronts $d_{\max}$ that can be resolved via ESPRIT by applying

i. Forward-backward averaging

ii. Spatial smoothing

iii. Both forward-backward averaging and spatial smoothing

as preprocessing steps?

Now we consider the special case $d = 2$ and ignore the contribution of the noise for clarity. Consequently, the covariance matrix of the received signal $\boldsymbol{x}[n]$ is given by $\boldsymbol{R}_{\mathrm{xx}} = \mathbb{E}\left\{\boldsymbol{x}[n] \cdot \boldsymbol{x}[n]^{\mathrm{H}}\right\} = \boldsymbol{A} \cdot \boldsymbol{R}_{\mathrm{ss}} \cdot \boldsymbol{A}^{\mathrm{H}}$, where $\boldsymbol{A} \in \mathbb{C}^{M \times 2}$ and the source covariance matrix $\boldsymbol{R}_{\mathrm{ss}} \in \mathbb{C}^{2 \times 2}$ can be expressed as

$$\boldsymbol{R}_{\mathrm{ss}} = \mathbb{E}\left\{\boldsymbol{s}[n] \cdot \boldsymbol{s}[n]^{\mathrm{H}}\right\} = \begin{bmatrix} 1 & \rho \\ \rho^* & 1 \end{bmatrix}. \tag{2}$$

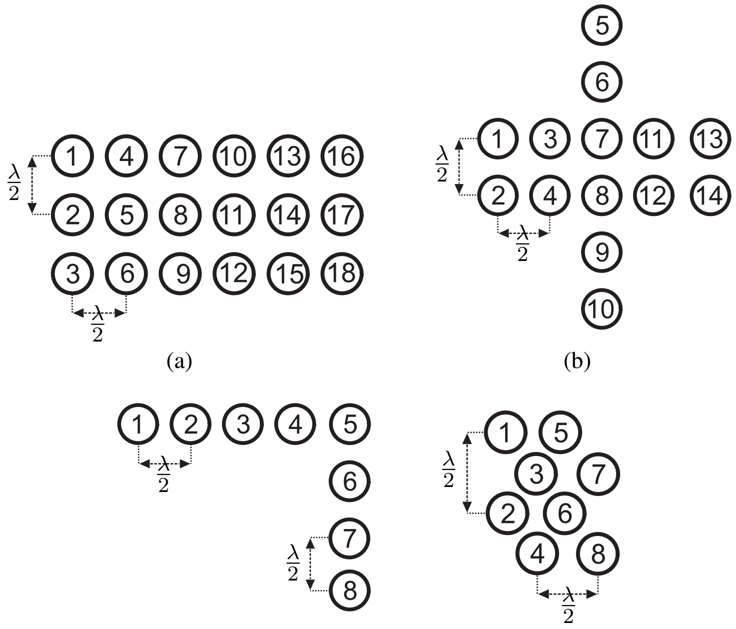
Here $\rho = |\rho| e^{\jmath \varphi_\rho} \in \mathbb{C}$ denotes the correlation coefficient.

(e) Show that $\operatorname{rank}\{\boldsymbol{R}_{\mathrm{ss}}\} = 1$ if $|\rho| = 1$.

(f) Prove that after applying forward-backward averaging, the corresponding covariance matrix $\boldsymbol{R}_{\mathrm{xx}}^{(\mathrm{fba})}$ can be written as $\boldsymbol{R}_{\mathrm{xx}}^{(\mathrm{fba})} = \boldsymbol{A} \cdot \tilde{\boldsymbol{R}}_{\mathrm{ss}} \cdot \boldsymbol{A}^{\mathrm{H}}$, where $\tilde{\boldsymbol{R}}_{\mathrm{ss}} = \boldsymbol{R}_{\mathrm{ss}} + \boldsymbol{R}_{\mathrm{ss}}^*$.

(g) Show under which conditions the rank of $\boldsymbol{R}_{\mathrm{xx}}^{(\mathrm{fba})}$ is 2.

5. Consider the following four 2-D arrays: (10 pt)



(a) (b) (c) (d)

Each of these arrays has a double shift-invariance structure, i.e., it is shift-invariant in horizontal (x) and in vertical (y) direction.

(a) For each array find the largest possible subarrays (i.e., with the most number of sensors M_{sub} in x- and y-direction). Document your findings in a table like this

Array	Subarray 1, x	Subarray 2, x	Subarray 1, y	Subarray 2, y
(a)	...	...	...	...
(b)	...	...	...	...
(c)	...	...	...	...
(d)	...	...	...	...

by filling in the indices of the sensors that belong to the respective subarrays. Note that the displacement between the two subarrays must not exceed $\lambda/2$.

(b) For each array find the largest possible number of wavefronts $d_{\max}$ that can be resolved via 2-D ESPRIT if all wavefronts are non-coherent and a sufficient number of snapshots is available ($N > M$).

(c) For array (c) provide the selection matrices $\boldsymbol{J}_{\mu,1}, \boldsymbol{J}_{\mu,2}, \boldsymbol{J}_{\nu,1}, \boldsymbol{J}_{\nu,2}$ that are needed for the 2-D shift invariance equations explicitly. Here, μ corresponds to the horizontal and ν corresponds to the vertical direction.

(d) On which of the arrays can 2-D Unitary ESPRIT be applied?