Last name, first name: … **Major: …** **Mat.-Nr.: …**

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Adaptive and Array Signal Processing

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Total time: 120 min Total points: 49+4

NB: Those tasks highlighted **bold-faced** can be solved independently from the previous ones. Label the axes of all your graphs properly.

1. Consider a linear FIR filter with weight vector $\boldsymbol{w} \in \mathbb{C}^M$ operating on a zero-mean input signal $\boldsymbol{x}[n] \in \mathbb{C}^M$ such that the filter output is given by $y[n] = \boldsymbol{w}^{\mathrm{H}} \cdot \boldsymbol{x}[n]$. The mean output power is given by (12 pt)

$$P_{\mathrm{out}}(\boldsymbol{w}) = \mathbb{E}\left\{|y[n]|^2\right\} = \boldsymbol{w}^{\mathrm{H}} \cdot \mathbb{E}\left\{\boldsymbol{x}[n] \cdot \boldsymbol{x}^{\mathrm{H}}[n]\right\} \cdot \boldsymbol{w} = \boldsymbol{w}^{\mathrm{H}} \cdot \boldsymbol{R}_{xx} \cdot \boldsymbol{w},$$

where $\boldsymbol{R}_{xx} \in \mathbb{C}^{M \times M}$ is the positive definite covariance matrix of the input signal, i.e., it is of full rank.

(a) Find the *unconstrained* optimum, i.e., the weight vector $\boldsymbol{w}$ which minimizes $P_{\mathrm{out}}(\boldsymbol{w})$ if no further constraints are made.

Now we add constraints. Let $M = 3$ such that $\boldsymbol{w} = [w_1, w_2, w_3]^{\mathrm{T}}$ and let two constraints be given by

$$\begin{aligned} w_1 + w_3 &= 1 \\ w_1 - w_2 &= 0 \end{aligned}$$

(b) Express the constraints in the form $\boldsymbol{c}(\boldsymbol{w}) = \boldsymbol{C}^{\mathrm{H}} \cdot \boldsymbol{w} - \boldsymbol{g} \stackrel{!}{=} \boldsymbol{0}$, i.e., find $\boldsymbol{C}$ and $\boldsymbol{g}$.

(c) Find a basis for the column space and the left null space of $\boldsymbol{C}$.

We now want to solve the constrained minimization problem

$$\min_{\boldsymbol{w}} P_{\mathrm{out}}(\boldsymbol{w}) \quad \text{subject to } \boldsymbol{c}(\boldsymbol{w}) = \boldsymbol{0}.$$

(d) Specify the Lagrangian $\mathcal{L}(\boldsymbol{w}, \boldsymbol{\lambda})$ for this *constrained* optimization problem.

(e) What is the dimension of the vector of Lagrangian multipliers $\boldsymbol{\lambda}$?

(f) Is $\mathcal{L}(\boldsymbol{w}, \boldsymbol{\lambda})$ an analytic (holomorphic) function?

An alternative implementation of the constrained optimization is given by the Generalized Sidelobe Canceller, where we decompose $\boldsymbol{w}$ into $\boldsymbol{w} = \boldsymbol{w}_{\mathrm{q}} - \boldsymbol{C}_{\mathrm{a}} \cdot \boldsymbol{w}_{\mathrm{a}}$ such that $\boldsymbol{C}^{\mathrm{H}} \cdot \boldsymbol{C}_{\mathrm{a}}$ is a matrix of zeros.

(g) Compute a vector $\boldsymbol{w}_{\mathrm{q}}$ such that $\boldsymbol{C}^{\mathrm{H}} \cdot \boldsymbol{w} = \boldsymbol{g}$.

(h) How is $\boldsymbol{C}_{\mathrm{a}}$ related to the four fundamental subspaces of $\boldsymbol{C}$?

(i) What is the size of $\boldsymbol{C}_{\mathrm{a}}$ and $\boldsymbol{w}_{\mathrm{a}}$?

(j) Determine the matrix $\boldsymbol{C}_{\mathrm{a}}$.

(k) Find $\boldsymbol{w}_{\mathrm{a}}$ which minimizes the output power $P_{\mathrm{out}}(\boldsymbol{w})$ for $\boldsymbol{R}_{xx} = \boldsymbol{I}_3$.

2. We are given the following $2 \times 2 \times 2$ tensor $\boldsymbol{\mathcal{X}}$: (9 pt)

$$\boldsymbol{\mathcal{X}} = \begin{bmatrix}1\\0\end{bmatrix} \circ \begin{bmatrix}1\\0\end{bmatrix} \circ \begin{bmatrix}2\\0\end{bmatrix} \quad + \quad \begin{bmatrix}1\\1\end{bmatrix} \circ \begin{bmatrix}1\\-1\end{bmatrix} \circ \begin{bmatrix}3\\3\end{bmatrix} \quad + \quad \begin{bmatrix}0\\1\end{bmatrix} \circ \begin{bmatrix}0\\1\end{bmatrix} \circ \begin{bmatrix}0\\4\end{bmatrix}$$

(a) What is the (tensor-) rank of $\boldsymbol{\mathcal{X}}$?

(b) Compute the "three-mode slices" $[\boldsymbol{\mathcal{X}}]_{(:,:,k)}$ of $\boldsymbol{\mathcal{X}}$, i.e., the matrices $[\boldsymbol{\mathcal{X}}]_{(:,:,1)} = [x_{i,j,k}]_{\substack{i=1,2\\j=1,2\\k=1}}$ and $[\boldsymbol{\mathcal{X}}]_{(:,:,2)} = [x_{i,j,k}]_{\substack{i=1,2\\j=1,2\\k=2}}$, where $x_{i,j,k}$ denotes the (i,j,k)-th element of $\boldsymbol{\mathcal{X}}$.

(c) Find three matrices $\boldsymbol{A}$, $\boldsymbol{B}$, and $\boldsymbol{C}$ such that $\boldsymbol{\mathcal{X}} = \boldsymbol{\mathcal{I}} \times_1 \boldsymbol{A} \times_2 \boldsymbol{B} \times_3 \boldsymbol{C}$, where $\boldsymbol{\mathcal{I}}$ is the diagonal 3-D identity tensor of appropriate size.

(d) Find the one-mode, the two-mode, and the three-mode unfoldings of $\boldsymbol{\mathcal{X}}$ in forward (MATLAB) column ordering.

(e) What are the n-ranks of $\boldsymbol{\mathcal{X}}$ for $n = 1, 2, 3$?

3. Let a 2×3 matrix $\boldsymbol{X}$ be given by (9+2 pt)

$$\boldsymbol{X} = \begin{bmatrix} 2-j & 1+3j & -1-2j \\ x & y & z \end{bmatrix}$$

(a) Find $x, y, z \in \mathbb{C}$ such that $\boldsymbol{X}$ is left-$\boldsymbol{\Pi}$-real.

(b) Find $x, y, z \in \mathbb{C}$ such that $\boldsymbol{X}$ is centro-Hermitian.

Now consider a generic 2×2 matrix $\boldsymbol{A}$ given by

$$\boldsymbol{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

(c) Which conditions should $a, b, c, d \in \mathbb{C}$ satisfy such that $\boldsymbol{A}$ is centro-Hermitian?

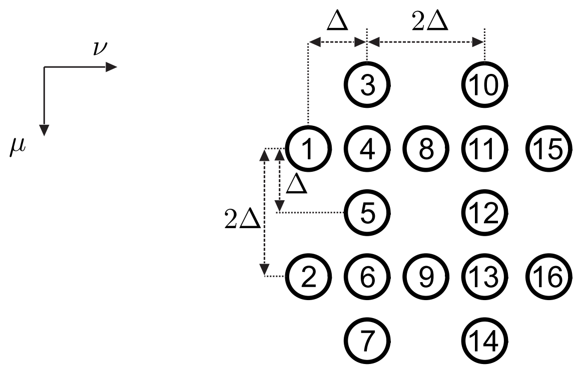
The set of centro-Hermitian matrices $\boldsymbol{A}$ can be mapped onto the set of real-valued matrices via the mapping $\varphi(\boldsymbol{A}) = \boldsymbol{Q}^{-1} \cdot \boldsymbol{A} \cdot \boldsymbol{Q}$, where $\boldsymbol{Q}$ is a left-$\boldsymbol{\Pi}$-real matrix.

(d) Compute the transformed real-valued matrix $\varphi(\boldsymbol{A})$ using the generic centro-Hermitian 2×2 matrix from above.
Hint: A unitary left-$\boldsymbol{\Pi}$-real matrix of size $2n \times 2n$ is given by $\boldsymbol{Q}_{2n}^{(s)} = \frac{1}{\sqrt{2}} \cdot \begin{bmatrix} \boldsymbol{I}_n & j \cdot \boldsymbol{I}_n \\ \boldsymbol{\Pi}_n & -j \cdot \boldsymbol{\Pi}_n \end{bmatrix}$.

(e) Show that the matrix above is in fact real-valued.
Hint: $a + a^* = 2 \cdot \mathrm{Re}\{a\}$ and $a - a^* = 2 \cdot j \cdot \mathrm{Im}\{a\}$.

(f) (**Bonus +2p**): In class we have shown that for an arbitrary centro-Hermitian matrix $\boldsymbol{A}$, the transformed matrix $\varphi(\boldsymbol{A})$ is real. Now show that the converse is true, i.e., for an arbitrary real-valued matrix $\boldsymbol{Z}$, show that $\varphi^{-1}(\boldsymbol{Z})$ is a centro-Hermitian matrix. Here $\varphi^{-1}(\boldsymbol{Z})$ is the inverse mapping which retrieves $\boldsymbol{A}$ from $\varphi(\boldsymbol{A})$.

4. We would like to apply 2-D ESPRIT on the following sensor array: (10 pt)



(a) For $\Delta = \lambda/2$, find the sensor elements belonging to the largest possible subarrays in vertical (μ) and horizontal (ν) directions, respectively. Note that the displacement between the corresponding elements of the first and the second subarray must be less than or equal to $\lambda/2$.

(b) Find the selection matrices for the vertical (μ) direction $\boldsymbol{J}_{\mu,1}$ and $\boldsymbol{J}_{\mu,2}$.

(c) What is the maximum number of incoherent wavefronts $d_{\max}$ which can be resolved?

(d) Repeat task (a) for $\Delta = \lambda/4$.

(e) How does $d_{\max}$ change for $\Delta = \lambda/4$?

5. Consider a uniform rectangular array of $M = M_x \times M_y$ sensors. Without loss of generality, we assume $M_x \leq M_y$ (for $M_x > M_y$ we can simply flip the dimensions). (9+2 pt)

(a) You are given a total of $M = 48$ sensors. Find all arrangements of the M sensors in form of an $M_x \times M_y$ URA where $1 < M_x \leq M_y$.

(b) For each arrangement find the maximum number of *incoherent* wavefronts $d_{\max}$ we can have for 2-D ESPRIT. Note that in both directions x and y, the number of sensors per subarray must be greater than or equal to d.

If sources are *coherent*, we need to apply proper preprocessing to decorrelate them. Since the array is centro-symmetric we use forward-backward averaging. Additionally, we apply 2-D spatial smoothing by dividing the array into $L_x \times L_y$ subarrays (L_x in x-direction and L_y in y-direction) where $1 \leq L_x < M_x$ and $1 \leq L_y < M_y$. In total, this decorrelates $2 \cdot L = 2 \cdot L_x \cdot L_y$ sources.

(c) What is the size of the array after 2-D spatial smoothing has been applied?

(d) How many coherent sources can be estimated via 2-D Unitary ESPRIT for a given choice of L_x and L_y?
Hint: Use the same reasoning as in (b).

(e) Find the *maximum* possible number of coherent sources (i.e., choose the best possible L_x and L_y) for $M_x = 3$ and $M_y = 4$.
Hint: Try out all combinations of L_x and L_y for $1 \leq L_x < M_x$ and $1 \leq L_y < M_y$.

(f) (**Bonus +2p**) Consider a URA with $M = M_x \times M_y$ elements where M is a square number, i.e., $M = m^2$ for an integer m. Show that in this case, for *incoherent* wavefronts the best distribution of M sensors in form of a URA which maximizes $d_{\max}$ is given by setting $M_x = \sqrt{M} = m$.