Last name, first name: … **Major: …** **Mat.-Nr.: …**

Ilmenau University of Technology
Communications Research Laboratory
Prof. Dr.-Ing. Martin Haardt

Mobile Communications

23.03.2011

Total time: 120 min — Total points: 43+3 Bonus

NB: Those tasks highlighted **bold-faced** can be solved independently from the previous ones. Label the axis of all your graphs properly.

1. We consider a 2×2 flat fading MIMO channel, the squared singular values of the channel matrix $\boldsymbol{H}$ are given by $\lambda_1 = \sigma_1^2 = 10$, $\lambda_2 = \sigma_2^2 = 1$. (8+2 pt)

   (a) Compute the open-loop capacity of the MIMO channel assuming that the signal-to-noise ratio (SNR) $\rho = P/\sigma_n^2$ is equal to 0.2, i.e., $P/(M_{\rm T}\sigma_n^2) = 0.1$. You can leave the result as $\log_2(X)$ once you have found the scalar X.

   (b) What is the optimal power distribution $\gamma_1^{\rm opt}$ and $\gamma_2^{\rm opt}$ across the two eigenmodes for SNR $\rightarrow \infty$ and for SNR $\rightarrow -\infty$?

   (c) Compute the optimal power distribution $\gamma_1^{\rm opt}$ and $\gamma_2^{\rm opt}$ via the water pouring algorithm for the same SNR as in (a).

   (d) Compute the closed-loop capacity of the MIMO channel for the same SNR as in (a).

   (e) (**Bonus** 2p): Find the critical SNR $\rho_{\rm crit}$ above which water pouring activates the second stream, i.e., $\gamma_2^{\rm opt} = 0\ \forall \rho < \rho_{\rm crit}$ and $\gamma_2^{\rm opt} > 0\ \forall \rho > \rho_{\rm crit}$.

2. We consider a wireless link where the power received at distance d from the transmitter can be modeled via a path loss model with path loss exponent n, i.e., the received power $P_{\rm R}(d)$ is proportional to d^{-n}. (8+1 pt)

   We have performed two measurements, one at $d_1 = 150$ m and one at $d_2 = 300$ m. The received powers were $P_{\rm R}(d_1) = 1.6$ µW and $P_{\rm R}(d_2) = 0.2$ µW.

   (a) Express the measured $P_{\rm R}(d_1)$ and $P_{\rm R}(d_2)$ in dBW and in dBm.

   (b) Find the path loss exponent n.

   (c) Compute the power we receive at $d_3 = 600$ m and express $P_{\rm R}(d_3)$ in Watt (linear scale).

   To transmit data we consider a transmission scheme that has a diversity order D and achieves the target bit error rate of 10^{-3} at an SNR of $\rho = P_{\rm R}/\sigma_{\rm n}^2 = 10 \triangleq 10$ dB. The noise level is $\sigma_{\rm n}^2 = 0.16$ nW $= 1.6 \cdot 10^{-10}$ W.

   (d) Find the maximum distance $d_{\max}$ so that the bit error rate is below the target bit error rate of 10^{-3}.

   (e) Which BER is achieved at distance $d_{\max}$ if the transmitter increases its transmit power by a factor of 10 for (i) $D = 1$ and (ii) $D = 2$?

   (f) (**Bonus** 1p): The increased transmit power (by a factor of 10) also leads to an increased range where the bit error rate is below 10^{-3}. For $D = 3$, how is the new $d_{\max}$ where the target bit error rate is achieved?

   Hints: $\log_{10}(1.6) \approx 0.2$, $\log_{10}(2) \approx 0.3$, $2^2 = 4, 2^3 = 8, 2^4 = 16$.

3. We consider a 2×2 MIMO system ($M_\mathrm{R} = M_\mathrm{T} = 2$) and assume frequency-flat fading. The transmitter has no channel state information, whereas the receiver knows the channel perfectly. We therefore decide to transmit the information via an Orthogonal Space-Time Block coding scheme. (12 pt)

The input-output relation of the MIMO system for one codeword is expressed as

$$\boldsymbol{Y} = \sqrt{\frac{P}{M_\mathrm{T}}} \cdot \boldsymbol{H} \cdot \boldsymbol{X} + \boldsymbol{N}, \tag{1}$$

where $\boldsymbol{H} \in \mathbb{C}^{M_\mathrm{R} \times M_\mathrm{T}}$ contains the MIMO channel coefficients, $\boldsymbol{X} \in \mathbb{C}^{M_\mathrm{T} \times T}$ represents the Space-Time codeword matrix, $\boldsymbol{N} \in \mathbb{C}^{M_\mathrm{R} \times T}$ contains the samples of the additive white Gaussian noise process, and $\boldsymbol{Y} \in \mathbb{C}^{M_\mathrm{R} \times T}$ is the received signal at the M_R antennas at T subsequent time instants. Moreover, P denotes the total transmit power.

Since $M_\mathrm{T} = 2$ we construct the codeword $\boldsymbol{X}$ from $K = 2$ symbols s_1 and s_2 via the Alamouti Space-Time Block Coding scheme.

(a) What is the number of time snapshots T needed to transmit the two symbols s_1 and s_2?

(b) Provide the explicit codeword matrix $\boldsymbol{X} \in \mathbb{C}^{M_\mathrm{T} \times T}$ for $s_1 = j$ and $s_2 = 1$.

(c) What is the array gain and the diversity order we can achieve?

The decoding process of the Alamouti Space-Time code can be described by

$$\boldsymbol{z} = \boldsymbol{H}_\mathrm{eff}^\mathrm{H} \cdot \tilde{\boldsymbol{y}}, \tag{2}$$

where $\boldsymbol{z} \in \mathbb{C}^{2 \times 1}$ is the output of the Alamouti receiver, $\tilde{\boldsymbol{y}} \in \mathbb{C}^{M_\mathrm{T} \cdot T \times 1}$ is a column vector constructed from $\boldsymbol{Y}$, and the matrix $\boldsymbol{H}_\mathrm{eff} \in \mathbb{C}^{M_\mathrm{T} \cdot T \times 2}$ is a linear receive filtering matrix, which is constructed from the channel matrix $\boldsymbol{H}$.

(d) Describe how to construct $\tilde{\boldsymbol{y}}$ from $\boldsymbol{Y}$.

(e) Describe how to construct $\boldsymbol{H}_\mathrm{eff}$ from $\boldsymbol{H}$.

(f) What is the general relation between the elements of $\boldsymbol{z}$ and the transmitted symbols s_1 and s_2?

We now consider the following channel realization:

$$\boldsymbol{H} = \begin{bmatrix} 2 & 1 \\ 1 & -1 \end{bmatrix} \tag{3}$$

(g) Compute the received signal matrix $\boldsymbol{Y} \in \mathbb{C}^{M_\mathrm{T} \times T}$ explicitly, ignoring the noise contribution ($\boldsymbol{N} = \boldsymbol{0}$).

(h) Determine the vector $\tilde{\boldsymbol{y}}$ explicitly.

(i) Construct the linear receive filtering matrix $\boldsymbol{H}_\mathrm{eff}$ explicitly.

(j) Compute the vector $\boldsymbol{z}$.

(k) How do we find the symbols s_1 and s_2 from $\boldsymbol{z}$?

Let $\rho = P/\sigma_n^2$ be the signal-to-noise ratio (SNR) of an equivalent SISO system, where σ_n^2 is the variance of each noise sample in the noise matrix $\boldsymbol{N}$. Furthermore, let η be the SNR after the Alamouti receive filtering.

(l) What is the SNR improvement η/ρ for the specific MIMO channel matrix that we consider here?

4. Let X be a random variable with probability density function (PDF) $f_X(x)$ and cumulative density function (CDF) $F_X(x) = \mathrm{P}(X \leq x)$. Furthermore, let Y be another random variable with PDF $f_Y(y)$ and CDF $F_Y(y) = \mathrm{P}(Y \leq y)$. We assume that X and Y are statistically **independent**. (7 pt)

   Then, a new random variable Z is computed via $Z = \max(X, Y)$.

   (a) Express the CDF $F_Z(z) = \mathrm{P}(Z \leq z)$ of Z in terms of the CDFs of X and Y.

   (b) Find an expression for the PDF $f_Z(z)$ in terms of the CDFs/PDFs of X and Y.

   (c) Compute the PDF $f_Z(z)$ of Z assuming that X and Y are uniformly distributed in the interval $[0, 1]$, i.e., $f_X(x) = \mathrm{rect}(x - 1/2)$, and sketch $f_Z(z)$.

   (d) Sketch the CDF $F_Z(z)$ of Z.

   Hint: $\Big(\max(X, Y) \leq z\Big) \Leftrightarrow \Big((X \leq z) \text{ and } (Y \leq z)\Big)$.

5. We investigate a wireless communication system operating at 2 GHz. We want to support moving mobile terminals. The maximum speed where a wireless connection is still guaranteed should be $v = 30$ m/s. (8 pt)

   (a) Calculate the maximum Doppler shift $f_{\mathrm{D,max}}$ that can occur by a transmission from a mobile to the fixed base station.

   (b) What does this mean for the coherence time of the channel $(\Delta t)_{\mathrm{c}}$?

   The signal is received by the base station via three distinct paths. The path delays are $\tau_1 = 300$ ns, $\tau_2 = 200$ ns, $\tau_3 = 500$ ns and the Doppler shifts are $f_{\mathrm{D},1} = -50$ Hz, $f_{\mathrm{D},2} = 0$ Hz, and $f_{\mathrm{D},3} = 100$ Hz. Finally, the corresponding powers of the three paths are given by $\gamma_1 = 2$ W, $\gamma_2 = 6$ W, and $\gamma_3 = 2$ W.

   (c) Provide an explicit expression for the scattering function $\Phi_h(\tau, f_{\mathrm{D}})$ and sketch it (pseudo-3D or view from the top).

   (d) Determine the Doppler power spectrum $\Phi_H(f_{\mathrm{D}})$ and sketch it.

   (e) Compute the average Doppler shift $\overline{f_{\mathrm{D}}}$, the mean square Doppler shift $\overline{f_{\mathrm{D}}^2}$, and the RMS Doppler spread $f_{\mathrm{D,rms}}$. You can leave the result as $\sqrt{X}$ once you have found the scalar X.

   Hints: The speed of light is $c \approx 3 \cdot 10^8$ m/s.