Sum-Rate Maximization in Two-Way AF MIMO Relaying: Polynomial Time Solutions to a Class of DC Programming Problems

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***Abstract*—Sum-rate maximization in two-way amplify-and-forward (AF) multiple-input multiple-output (MIMO) relaying belongs to the class of difference-of-convex functions (DC) programming problems. DC programming problems occur also in other signal processing applications and are typically solved using different modifications of the branch-and-bound method which, however, does not have any polynomial time complexity guarantees. In this paper, we develop two efficient polynomial time algorithms for the sum-rate maximization in two-way AF MIMO relaying. The first algorithm guarantees to find at least a Karush-Kuhn-Tucker (KKT) solution. There is a strong evidence, however, that such a solution is actually globally optimal. The second algorithm that is based on the generalized eigenvectors shows the same performance as the first one with reduced computational complexity.**

The objective function of the problem is represented as a product of quadratic fractional ratios and parameterized so that its convex part (versus the concave part) contains only one (or two) optimization variables. One of the algorithms is called POlynomial Time DC (POTDC) and is based on semi-definite programming (SDP) relaxation, linearization, and an iterative Newton-type search over a single parameter. The other algorithm is called RAte-maximization via Generalized EigenvectorS (RAGES) and is based on the generalized eigenvectors method and an iterative search over two (or one, in its approximate version) optimization variables. We derive an upper-bound for the optimal value of the corresponding optimization problem and show by simulations that this upper-bound is achieved by both algorithms. It provides an evidence that the algorithms find a global optimum. The proposed methods are also superior to other state-of-the-art algorithms.

***Index Terms*—Difference-of-convex functions (DC) programming, non-convex programming, semi-definite programming relaxation, sum-rate maximization, two-way relaying.**

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Color versions of one or more of the figures in this paper are available online at http://ieeexplore.ieee.org.

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I. Introduction

TWO-WAY relaying has recently attracted a significant research interest due to its ability to overcome the drawback of conventional one-way relaying, that is, the factor of 1/2 loss in the rate [1], [2]. Moreover, two-way relaying can be viewed as a certain form of network coding [3] which allows to reduce the number of time slots used for the transmission in one-way relaying by relaxing the requirement of 'orthogonal/non-interfering' transmissions between the terminals and the relay [4]. Specifically, simultaneous transmissions by the terminals to the relay on the same frequencies are allowed in the first time slot, while a combined signal is broadcasted by the relay in the second time slot. In contrast to the one-way relaying, the rate-optimal strategy for two-way relaying is in general unknown [5]. However, some efficient strategies have been developed. Depending on the ability of the relay to regenerate/decode the signals from the terminals, several two-way transmission protocols have been introduced and studied. The regenerative relay adopts the decode-and-forward (DF) protocol and performs the decoding process at the relay [6], while the non-regenerative relay typically adopts a form of amplify-and-forward (AF) protocol and does not perform decoding at the relay, but amplifies and possibly beamforms or precodes the signals to retransmit them back to the terminals [5], [7], [8]. The advantages of the latter are a smaller delay in the transmission and lower hardware complexity of the relay. Most of the research on two-way relaying systems concentrates on studying the corresponding sum-rate, the achievable rate region, and also the bit error probability of different schemes [9]. The tradeoff between the error probability and the achievable rate has been recently studied in [9] using Gallager's random coding error exponent.

In this paper, we consider the AF two-way relaying system with two terminals equipped with a single antenna and one relay with multiple antennas. The task is to find the relay transmit strategy that maximizes the sum-rate of both terminals. This is a basic model which can be extended in many ways. The significant advantage of considering this basic model is that the corresponding capacity region is discussed in the existing literature [4]. It enables us to concentrate on the mathematical issues of the corresponding optimization problem which are of significant and ubiquitous interest.

We show that the optimization problem of finding the relay amplification matrix for the considered AF two-way relaying system is equivalent to finding the maximum of the product of quadratic fractional functions under a quadratic power constraint on the available power at the relay. Such a problem belongs to the class of the so-called difference-of-convex functions (DC) programming problems. It is worth stressing that DC programming problems are very common in signal processing and, in particular, signal processing for communications. For example, the robust adaptive beamforming for the general-rank (distributed source) signal model with a positive semi-definite constraint can be shown to belong to the class of DC programming problems [10], [11]. Specifically, the constraint in the corresponding optimization problem is the difference of two weighted norm functions. The power control for wireless cellular systems is also a DC programming problem when the rate is used as a utility function [12]. Similarly, the dynamic spectrum management for digital subscriber lines [13] as well as the problems of finding the weighted sum-rate point, the proportional-fairness operating point, and the max-min optimal point (egalitarian solution) for the two-user multiple-input single-output (MISO) interference channel [14] are all DC programming problems. The typical approach for solving such problems is the use of various modifications of the branch-and-bound method [14]–[20] that is an efficient global optimization method. The branch-and-bound method is known to work well especially for the case of monotonic functions, i.e., the case which is typically encountered in signal processing and, in particular, signal processing for communications. However, it does not have any worst-case polynomial time complexity guarantees, which significantly limits or even prohibits its applicability in practical communication systems. Thus, methods with guaranteed polynomial time complexity that can find at least a suboptimal solution for different types of DC programming problems are of a great importance.

In the last decade, a significant progress has occurred in the application of optimization theory in signal processing and communications. Some of those results are relevant for the considered problem of maximizing constrained product of quadratic fractional functions [21]–[25]. The worst-case-based robust adaptive beamforming problem is known to belong to the class of second-order cone (SOC) programming problems [21] largely due to the fact that the output signal-plus-interference-to-noise ratio (SINR) of adaptive beamforming is unchanged when the beamforming vector undergoes an arbitrary phase rotation. This allows to simplify the single worst-case distortionless response constraint of the optimization problem into the form of a SOC constraint. The situation is more complicated in the case of multiple constraints of the same type as the constraint in [21] when a single rotation of the beamforming vector is no longer sufficient to satisfy all constraints simultaneously. This situation has been successfully addressed in [22] by considering the semi-definite programming (SDP) relaxation technique. The SDP relaxation technique has been further developed and studied in, for example, [23]–[25] and other works. Interestingly, the work [25] considers the fractional quadratically constrained quadratic programming (QCQP) problem that is closest mathematically to the one addressed in this paper with the significant difference though that the objective in [25] contains only a single quadratic fractional function that simplifies the problem dramatically.

In this paper, we develop two polynomial time algorithms for solving at least sub-optimally the non-convex DC programming problem of maximizing a product of quadratic fractional functions under a quadratic constraint, which precisely corresponds to the sum-rate maximization in two-way AF MIMO relaying.[1] Our algorithms use such parameterizations of the objective function that its convex part (versus the concave part) contains only one (or two) optimization variables. One of the algorithms is named POlynomial Time DC (POTDC) and is based on SDP relaxation, linearization, and an iterative Newton-type search over a single parameter. It is guaranteed that it finds at least a Karush-Kuhn-Tucker (KKT) solution, i.e., a solution which satisfies the KKT optimality conditions. The POTDC algorithm is rigorous and there is great evidence that the KKT solution found by it is also globally optimal. Indeed, the solution given by POTDC coincides with the newly developed upper-bound for the optimal value of the problem. The other algorithm is called RAte-maximization via Generalized EigenvectorS (RAGES) and is based on the generalized eigenvectors method and an iterative search over two (or one, in its approximate version) optimization variables. The RAGES algorithm is somewhat heuristic in its approximate version, but may enjoy a lower complexity. It shows, however, the same performance as the first one.

The rest of the paper is organized as follows. The two-way AF MIMO relaying system model is given in Section II while the sum-rate maximization problem for the corresponding system is formulated in Section III. The POTDC algorithm for the sum-rate maximization is developed in Section IV and an upper-bound for the optimal value of the maximization problem is found in Section V. In Section VI, the RAGES algorithm is developed and investigated. Simulation results are reported in Section VII. Finally, Section VIII presents our conclusions and discussions. This paper is reproducible research [28], and the software needed to generate the simulation results can be obtained from the IEEE Xplore together with the paper.

II. System Model

We consider a two-way relaying system with two single-antenna terminals and an AF relay equipped with M_R antennas. Fig. 1 shows the system we study in the paper. In the first transmission phase, both terminals transmit to the relay. Assuming frequency-flat quasi-static block fading, the received signal at the relay can be expressed as

$$\mathbf{r} = \mathbf{h}_1^{(f)} \cdot x_1 + \mathbf{h}_2^{(f)} \cdot x_2 + \mathbf{n}_R \tag{1}$$

[1]Some preliminary results have been presented in [26] and [27].

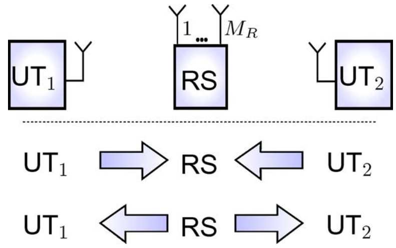

Fig. 1. Two-way relaying system model.

where $\mathbf{h}_i^{(f)} \triangleq [h_{i,1}, \dots, h_{i,M_R}]^T \in \mathbb{C}^{M_R}$ represents the (forward) channel vector between terminal i and the relay, x_i is the transmitted symbol from terminal i, $\mathbf{n}_R \in \mathbb{C}^{M_R}$ denotes the additive noise component at the relay, and $(\cdot)^T$ stands for the transpose of a vector or a matrix. Let $P_{T,i} \triangleq \mathbb{E}\{|x_i|^2\}$ be the average transmit power of terminal i and $\mathbf{R}_{N,R} \triangleq \mathbb{E}\{\mathbf{n}_R \cdot \mathbf{n}_R^H\}$ be the noise covariance matrix at the relay, where $\mathbb{E}\{\cdot\}$ denotes the mathematical expectation, and $(\cdot)^H$ stands for the Hermitian transpose of a vector or a matrix. For the special case of white noise we have $\mathbf{R}_{N,R} = P_{N,R} \cdot \mathbf{I}_{M_R}$ where $P_{N,R} = \frac{\mathrm{tr}\{\mathbf{R}_{N,R}\}}{M_R}$, $\mathbf{I}_{M_R}$ is the identity matrix of size $M_R \times M_R$ and $\mathrm{tr}\{\cdot\}$ denotes the trace operation applied to a square matrix. The relay amplifies the received signal by multiplying it with a relay amplification matrix $\mathbf{G} \in \mathbb{C}^{M_R \times M_R}$, i.e., it transmits the signal

$$\bar{\mathbf{r}} = \mathbf{G} \cdot \mathbf{r}. \tag{2}$$

The transmit power used by the relay can be expressed as

$$\begin{aligned}\mathbb{E}\left\{\|\bar{\mathbf{r}}\|_2^2\right\} &= \mathbb{E}\left\{\mathrm{tr}\{\mathbf{G} \cdot \mathbf{r} \cdot \mathbf{r}^H \cdot \mathbf{G}^H\}\right\} = \mathrm{tr}\{\mathbf{G} \cdot \mathbf{R}_R \cdot \mathbf{G}^H\} \\ &= \mathrm{tr}\{\mathbf{G}^H \cdot \mathbf{G} \cdot \mathbf{R}_R\} \end{aligned} \tag{3}$$

where $\|\cdot\|_2$ denotes the Euclidean norm of a vector and $\mathbf{R}_R \triangleq \mathbb{E}\{\mathbf{r} \cdot \mathbf{r}^H\}$ is the covariance matrix of $\mathbf{r}$ which is given by

$$\mathbf{R}_R = \mathbf{h}_1^{(f)} \cdot \left(\mathbf{h}_1^{(f)}\right)^H \cdot P_{T,1} + \mathbf{h}_2^{(f)} \cdot \left(\mathbf{h}_2^{(f)}\right)^H \cdot P_{T,2} + \mathbf{R}_{N,R}. \tag{4}$$

The covariance matrix $\mathbf{R}_R$ is assumed to be full rank which is true under the common practical assumption that the noise covariance matrix $\mathbf{R}_{N,R}$ is full rank. However, the case of rank deficient $\mathbf{R}_R$ is considered for completeness in Appendix A as well.

Using the equality

$$\mathrm{tr}\{\mathbf{A}^H \cdot \mathbf{B}\} = \mathrm{vec}\{\mathbf{A}\}^H \cdot \mathrm{vec}\{\mathbf{B}\} \tag{5}$$

which holds for any arbitrary square matrices $\mathbf{A}$ and $\mathbf{B}$, the total transmit power of the relay (3) can be equivalently expressed as

$$\mathbb{E}\left\{\|\bar{\mathbf{r}}\|_2^2\right\} = \mathrm{vec}\{\mathbf{G}\}^H \cdot \mathrm{vec}\{\mathbf{G} \cdot \mathbf{R}_R\} \tag{6}$$

where $\mathrm{vec}\{\cdot\}$ stands for the vectorization operation that transforms a matrix into a long vector stacking the columns of the matrix one after another. Finally, using the equality $\mathrm{vec}\{\mathbf{A} \cdot \mathbf{B}\} = (\mathbf{B}^T \otimes \mathbf{I}) \cdot \mathrm{vec}\{\mathbf{A}\}$, which is valid for any arbitrary square matrices $\mathbf{A}$ and $\mathbf{B}$, (6) can be equivalently rewritten as the following quadratic form

$$\mathbb{E}\left\{\|\bar{\mathbf{r}}\|_2^2\right\} = \mathbf{g}^H \cdot \underbrace{\left(\mathbf{R}_R^T \otimes \mathbf{I}_{M_R}\right)}_{\mathbf{Q}} \cdot \mathbf{g} = \mathbf{g}^H \cdot \mathbf{Q} \cdot \mathbf{g} \tag{7}$$

where $\mathbf{g} \triangleq \mathrm{vec}\{\mathbf{G}\}$, $\mathbf{Q} \triangleq \mathbf{R}_R^T \otimes \mathbf{I}_{M_R}$ and $\otimes$ denotes the Kronecker product. Since $\mathbf{Q}$ is the Kronecker product of the two full rank positive definite matrices $\mathbf{R}_R^T$ and $\mathbf{I}_{M_R}$, it is also full rank and positive definite [29].

In the second phase, the terminals receive the relay's transmission via the (backward) channels $\left(\mathbf{h}_1^{(b)}\right)^T$ and $\left(\mathbf{h}_2^{(b)}\right)^T$ (in the special case when reciprocity holds we have $\mathbf{h}_i^{(b)} = \mathbf{h}_i^{(f)}$ for $i = 1, 2$). Consequently, the received signals y_i, $i = 1, 2$ at both terminals can be expressed, respectively, as

$$y_1 = h_{1,1}^{(e)} \cdot x_1 + h_{1,2}^{(e)} \cdot x_2 + \tilde{n}_1 \tag{8}$$

$$y_2 = h_{2,2}^{(e)} \cdot x_2 + h_{2,1}^{(e)} \cdot x_1 + \tilde{n}_2 \tag{9}$$

where $h_{i,j}^{(e)} \triangleq \left(\mathbf{h}_i^{(b)}\right)^T \cdot \mathbf{G} \cdot \mathbf{h}_j^{(f)}$ is the effective channel between terminals i and j for $i, j = 1, 2$ and $\tilde{n}_i \triangleq \left(\mathbf{h}_i^{(b)}\right)^T \cdot \mathbf{G} \cdot \mathbf{n}_R + n_i$ represents the effective noise contribution at terminal i which comprises the terminal's own noise as well as the forwarded relay noise. The first term in the received signal of each terminal represents the self-interference, which can be subtracted by the terminal since its own transmitted signal is known. The required channel knowledge for this step can be easily obtained, for example, via the Least Squares (LS) compound channel estimator of [30].

After the cancellation of the self-interference, the two-way relaying system is decoupled into two parallel single-user single-input single-output (SISO) systems. Consequently, the rate r_i of terminal i can be expressed as

$$r_i = \frac{1}{2}\log_2\left(1 + \frac{P_{R,i}}{\tilde{P}_{N,i}}\right) = \frac{1}{2}\log_2\left(\frac{\tilde{P}_{R,i}}{\tilde{P}_{N,i}}\right) \tag{10}$$

where $P_{R,i}$ and $\tilde{P}_{N,i}$ are the powers of the desired signal and the effective noise term at terminal i, respectively, and $\tilde{P}_{R,i} \triangleq P_{R,i} + \tilde{P}_{N,i}$. Specifically, $P_{R,1} \triangleq \mathbb{E}\{|h_{1,2}^{(e)} \cdot x_2|^2\}$, $P_{R,2} \triangleq \mathbb{E}\{|h_{2,1}^{(e)} \cdot x_1|^2\}$, and $\tilde{P}_{N,i} \triangleq \mathbb{E}\{|\tilde{n}_i|^2\}$ for $i = 1, 2$. Note that the factor 1/2 results from the two time slots needed for the bidirectional transmission. The powers of the desired signal and the effective noise term at terminal i can be equivalently expressed as

$$P_{R,1} = P_{T,2}\left|\left(\mathbf{h}_1^{(b)}\right)^T \cdot \mathbf{G} \cdot \mathbf{h}_2^{(f)}\right|^2 \tag{11}$$

$$P_{R,2} = P_{T,1}\left|\left(\mathbf{h}_2^{(b)}\right)^T \cdot \mathbf{G} \cdot \mathbf{h}_1^{(f)}\right|^2 \tag{12}$$

$$\begin{aligned}\tilde{P}_{N,i} &= \mathbb{E}\left\{\left|\left(\mathbf{h}_i^{(b)}\right)^T \cdot \mathbf{G} \cdot \mathbf{n}_R + n_i\right|^2\right\} \\ &= \left(\mathbf{h}_i^{(b)}\right)^T \cdot \mathbf{G} \cdot \mathbf{R}_{N,R} \cdot \mathbf{G}^H \left(\mathbf{h}_i^{(b)}\right)^* + P_{N,i} \end{aligned} \tag{13}$$

where the expectation is taken with respect to the transmit signals and also the additional noise terms, $P_{N,i}$ denotes the variance of the additive noise at terminal i, i.e., n_i and $(\cdot)^*$ stands for the conjugation. Moreover, these powers can be further expressed as quadratic forms in $\mathbf{g}$. For this goal, first note that by using the following equality

$$\mathrm{vec}\{\mathbf{A} \cdot \mathbf{B} \cdot \mathbf{C}\} = (\mathbf{C}^T \otimes \mathbf{A}) \cdot \mathrm{vec}\{\mathbf{B}\} \tag{14}$$

which is valid for any arbitrary matrices $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$ of compatible dimensions, the term $(\mathbf{h}_i^{(b)})^T \cdot \mathbf{G} \cdot \mathbf{h}_j^{(f)}$ can be modified as follows

$$\left(\mathbf{h}_i^{(b)}\right)^T \cdot \mathbf{G} \cdot \mathbf{h}_j^{(f)} = \mathrm{vec}\left\{\left(\mathbf{h}_i^{(b)}\right)^T \cdot \mathbf{G} \cdot \mathbf{h}_j^{(f)}\right\}$$
$$= \left(\left(\mathbf{h}_j^{(f)}\right)^T \otimes \left(\mathbf{h}_i^{(b)}\right)^T\right) \cdot \mathrm{vec}\{\mathbf{G}\}. \quad (15)$$

Using (15), the power of the desired signal at the first terminal can be expressed as

$$P_{R,1} = \mathbf{g}^H \cdot \left(\left(\mathbf{h}_2^{(f)}\right)^T \otimes \left(\mathbf{h}_1^{(b)}\right)^T\right)^H \cdot \left(\left(\mathbf{h}_2^{(f)}\right)^T \otimes \left(\mathbf{h}_1^{(b)}\right)^T\right) \cdot \mathbf{g} \cdot P_{T,2}. \quad (16)$$

Applying also the equality $(\mathbf{A} \otimes \mathbf{B}) \cdot (\mathbf{C} \otimes \mathbf{D}) = (\mathbf{A} \cdot \mathbf{C}) \otimes (\mathbf{B} \cdot \mathbf{D})$ to (16) which is valid for any arbitrary matrices $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$ and $\mathbf{D}$ of compatible dimensions, $P_{R,1}$ can be expressed as the following quadratic form

$$P_{R,1} = \mathbf{g}^H \cdot \left[\left(\mathbf{h}_2^{(f)} \cdot \left(\mathbf{h}_2^{(f)}\right)^H\right) \otimes \left(\mathbf{h}_1^{(b)} \cdot \left(\mathbf{h}_1^{(b)}\right)^H\right)\right]^T \cdot \mathbf{g} \cdot P_{T,2}. \quad (17)$$

The power of the desired signal at the second terminal, i.e., $P_{R,2}$, can be obtained similarly. By defining the matrices $\mathbf{K}_{2,1}$ and $\mathbf{K}_{1,2}$ as follows

$$\mathbf{K}_{2,1} \triangleq \left[\left(\mathbf{h}_2^{(f)} \cdot \left(\mathbf{h}_2^{(f)}\right)^H\right) \otimes \left(\mathbf{h}_1^{(b)} \cdot \left(\mathbf{h}_1^{(b)}\right)^H\right)\right]^T \quad (18)$$
$$\mathbf{K}_{1,2} \triangleq \left[\left(\mathbf{h}_1^{(f)} \cdot \left(\mathbf{h}_1^{(f)}\right)^H\right) \otimes \left(\mathbf{h}_2^{(b)} \cdot \left(\mathbf{h}_2^{(b)}\right)^H\right)\right]^T \quad (19)$$

the powers of the desired signal can be expressed as

$$P_{R,1} = \mathbf{g}^H \cdot \mathbf{K}_{2,1} \cdot \mathbf{g} \cdot P_{T,2} \quad (20)$$
$$P_{R,2} = \mathbf{g}^H \cdot \mathbf{K}_{1,2} \cdot \mathbf{g} \cdot P_{T,1}. \quad (21)$$

Since the matrices $\mathbf{h}_i^{(f)} \cdot (\mathbf{h}_i^{(f)})^H, i = 1, 2$ and $\mathbf{h}_i^{(b)} \cdot (\mathbf{h}_i^{(b)})^H, i = 1, 2$ are all positive semi-definite and the Kronecker product of positive semi-definite matrices is a positive semi-definite matrix [29], the matrices $\mathbf{K}_{2,1}$ and $\mathbf{K}_{1,2}$ are also positive semi-definite. As the last step, the effective noise $\tilde{P}_{N,i}$ can be converted into a quadratic form of $\mathbf{g}$ through the following train of equalities

$$\tilde{P}_{N,i} = \mathbb{E}\left\{\left|\left(\mathbf{h}_i^{(b)}\right)^T \cdot \mathbf{G} \cdot \mathbf{n}_R + n_i\right|^2\right\}$$
$$= \left(\mathbf{h}_i^{(b)}\right)^T \cdot \mathbf{G} \cdot \mathbf{R}_{N,R} \cdot \mathbf{G}^H \left(\mathbf{h}_i^{(b)}\right)^* + P_{N,i} \quad (22)$$
$$= \mathrm{tr}\left\{\mathbf{G}^H \cdot \left(\mathbf{h}_i^{(b)}\right)^* \cdot \left(\mathbf{h}_i^{(b)}\right)^T \cdot \mathbf{G} \cdot \mathbf{R}_{N,R}\right\} + P_{N,i} \quad (23)$$
$$= \mathrm{vec}\{\mathbf{G}\}^H \cdot \mathrm{vec}\left\{\left(\mathbf{h}_i^{(b)}\right)^* \cdot \left(\mathbf{h}_i^{(b)}\right)^T \cdot \mathbf{G} \cdot \mathbf{R}_{N,R}\right\} + P_{N,i} \quad (24)$$
$$= \mathrm{vec}\{\mathbf{G}\}^H \cdot \left[\mathbf{R}_{N,R} \otimes \left(\mathbf{h}_i^{(b)} \cdot \left(\mathbf{h}_i^{(b)}\right)^H\right)\right]^T \cdot \mathrm{vec}\{\mathbf{G}\} + P_{N,i} \quad (25)$$
$$= \mathbf{g}^H \cdot \mathbf{J}_i \cdot \mathbf{g} + P_{N,i} \quad (26)$$

where (24) is obtained from (23) by applying the equality (5), (25) is obtained from (24) by applying the equality (14), and the matrix $\mathbf{J}_i$ in (26) is defined as

$$\mathbf{J}_i \triangleq \left[\mathbf{R}_{N,R} \otimes \left(\mathbf{h}_i^{(b)} \cdot \left(\mathbf{h}_i^{(b)}\right)^H\right)\right]^T. \quad (27)$$

Note that, $\mathbf{J}_i$, $i = 1, 2$ are positive semi-definite matrices because $\mathbf{R}_{N,R}$ and $\mathbf{h}_i^{(b)} \cdot (\mathbf{h}_i^{(b)})^H$, $i = 1, 2$ are positive semi-definite.

III. Problem Statement

Our goal is to find the relay amplification matrix $\mathbf{G}$ which maximizes the sum-rate $r_1 + r_2$ subject to a power constraint at the relay. For convenience we express the objective function and its solution in terms of $\mathbf{g}$. Then the power constrained sum-rate maximization problem can be expressed as

$$\mathbf{g}_{\mathrm{opt}} = \underset{\mathbf{g} | \mathbf{g}^H \cdot \mathbf{Q} \cdot \mathbf{g} \le P_{T,R}}{\arg\max} \; (r_1 + r_2) \quad (28)$$

where $P_{T,R}$ is the total available transmit power at the relay. Using the definitions from the previous section, this optimization problem can be rewritten as

$$\mathbf{g}_{\mathrm{opt}} = \underset{\mathbf{g} | \mathbf{g}^H \cdot \mathbf{Q} \cdot \mathbf{g} \le P_{T,R}}{\arg\max} \frac{1}{2}\log_2\left[\left(1 + \frac{P_{R,1}}{\tilde{P}_{N,1}}\right) \cdot \left(1 + \frac{P_{R,2}}{\tilde{P}_{N,2}}\right)\right]$$
$$= \underset{\mathbf{g} | \mathbf{g}^H \cdot \mathbf{Q} \cdot \mathbf{g} \le P_{T,R}}{\arg\max} \left(1 + \frac{P_{R,1}}{\tilde{P}_{N,1}}\right) \cdot \left(1 + \frac{P_{R,2}}{\tilde{P}_{N,2}}\right) \quad (29)$$
$$= \underset{\mathbf{g} | \mathbf{g}^H \cdot \mathbf{Q} \cdot \mathbf{g} \le P_{T,R}}{\arg\max} \frac{\tilde{P}_{R,1}}{\tilde{P}_{N,1}} \cdot \frac{\tilde{P}_{R,2}}{\tilde{P}_{N,2}} \quad (30)$$

where we have used the fact that $0.5 \cdot \log_2(x)$ is a monotonic function in $x \in \mathbb{R}^+$, and $\tilde{P}_{R,i}, i = 1, 2$ are defined after (10).

It is worth noting that the inequality constraint in the optimization problem (30) has to be active at the optimal point. This can be easily shown by contradiction. Assume that $\mathbf{g}_{\mathrm{opt}}$ satisfies $\mathbf{g}_{\mathrm{opt}}^H \cdot \mathbf{Q} \cdot \mathbf{g}_{\mathrm{opt}} < P_{T,R}$. Then we can find a constant $c > 1$ such that $\bar{\mathbf{g}}_{\mathrm{opt}} = c \cdot \mathbf{g}_{\mathrm{opt}}$ satisfies $\bar{\mathbf{g}}_{\mathrm{opt}}^H \cdot \mathbf{Q} \cdot \bar{\mathbf{g}}_{\mathrm{opt}} = P_{T,R}$. The latter follows from the fact that $\mathbf{Q}$ is positive definite and, therefore, $\mathbf{g}_{\mathrm{opt}}^H \cdot \mathbf{Q} \cdot \mathbf{g}_{\mathrm{opt}}$ is positive. However, inserting $\bar{\mathbf{g}}_{\mathrm{opt}}$ in the objective function of (29), we obtain

$$\left(1 + \frac{c^2 \cdot \mathbf{g}_{\mathrm{opt}}^H \mathbf{K}_{2,1} \mathbf{g}_{\mathrm{opt}} P_{T,2}}{c^2 \cdot \mathbf{g}_{\mathrm{opt}}^H \mathbf{J}_1 \mathbf{g}_{\mathrm{opt}} + P_{N,1}}\right) \cdot \left(1 + \frac{c^2 \cdot \mathbf{g}_{\mathrm{opt}}^H \mathbf{K}_{1,2} \mathbf{g}_{\mathrm{opt}} P_{T,1}}{c^2 \cdot \mathbf{g}_{\mathrm{opt}}^H \mathbf{J}_2 \mathbf{g}_{\mathrm{opt}} + P_{N,2}}\right)$$
$$= \left(1 + \frac{\mathbf{g}_{\mathrm{opt}}^H \mathbf{K}_{2,1} \mathbf{g}_{\mathrm{opt}} P_{T,2}}{\mathbf{g}_{\mathrm{opt}}^H \mathbf{J}_1 \mathbf{g}_{\mathrm{opt}} + \frac{P_{N,1}}{c^2}}\right) \cdot \left(1 + \frac{\mathbf{g}_{\mathrm{opt}}^H \mathbf{K}_{1,2} \mathbf{g}_{\mathrm{opt}} P_{T,1}}{\mathbf{g}_{\mathrm{opt}}^H \mathbf{J}_2 \mathbf{g}_{\mathrm{opt}} + \frac{P_{N,2}}{c^2}}\right) \quad (31)$$

which is monotonically increasing in c. Since we have $c > 1$, the vector $\bar{\mathbf{g}}_{\mathrm{opt}}$ provides a larger value of the objective functions than $\mathbf{g}_{\mathrm{opt}}$ which contradicts the assumption that $\mathbf{g}_{\mathrm{opt}}$ was optimal.

As a result, we have shown that the optimal vector $\mathbf{g}_{\mathrm{opt}}$ must satisfy the total power constraint of the problem (30) with equality, i.e., $\mathbf{g}_{\mathrm{opt}}^H \cdot \mathbf{Q} \cdot \mathbf{g}_{\mathrm{opt}} = P_{T,R}$. Using this fact, the inequality constraint in the problem (30) can be replaced by the constraint $\mathbf{g}^H \cdot \mathbf{Q} \cdot \mathbf{g} = P_{T,R}$. This enables us to substitute the constant term $P_{N,i}$, which appears in the effective noise power at terminal i (26), with the quadratic term of $\left(\frac{P_{N,i}}{P_{T,R}}\right) \cdot \mathbf{g}_{\mathrm{opt}}^H \cdot \mathbf{Q} \cdot \mathbf{g}_{\mathrm{opt}}$. This leads to an equivalent homogeneous expression for the ratio of $\frac{\tilde{P}_{R,i}}{\tilde{P}_{N,i}}$, $i = 1, 2$. Thus, by using such substitution, $\tilde{P}_{N,i}$, $i = 1, 2$ from (26) can be equivalently written as

$$\tilde{P}_{N,i} = \mathbf{g}^H \cdot \mathbf{B}_i \cdot \mathbf{g}, \quad i = 1, 2 \tag{32}$$

where $\mathbf{B}_i$ is defined as

$$\mathbf{B}_i \triangleq \mathbf{J}_i + \frac{P_{N,i}}{P_{T,R}} \cdot \mathbf{Q}. \tag{33}$$

Inserting (20), (21), and (33) into (30), the optimization problem becomes

$$\mathbf{g}_{\text{opt}} = \underset{\mathbf{g}|\mathbf{g}^H\cdot\mathbf{Q}\cdot\mathbf{g}=P_{T,R}}{\arg\max} \frac{\mathbf{g}^H \cdot \mathbf{A}_1 \cdot \mathbf{g}}{\mathbf{g}^H \cdot \mathbf{B}_1 \cdot \mathbf{g}} \cdot \frac{\mathbf{g}^H \cdot \mathbf{A}_2 \cdot \mathbf{g}}{\mathbf{g}^H \cdot \mathbf{B}_2 \cdot \mathbf{g}} \tag{34}$$

where we have defined the new matrices $\mathbf{A}_1 \triangleq \mathbf{K}_{2,1} \cdot P_{T,2} + \mathbf{B}_1$ and $\mathbf{A}_2 \triangleq \mathbf{K}_{1,2} \cdot P_{T,1} + \mathbf{B}_2$. Since the matrices $\mathbf{J}_i$, $i = 1, 2$, $\mathbf{K}_{1,2}$, and $\mathbf{K}_{2,1}$ are positive semi-definite and $\mathbf{Q}$ is a full rank positive definite matrix, the matrices $\mathbf{A}_1$, $\mathbf{A}_2$, $\mathbf{B}_1$, and $\mathbf{B}_2$ are all full rank positive definite matrices and hence invertible. Moreover, $\mathbf{A}_1$, $\mathbf{A}_2$, $\mathbf{B}_1$, and $\mathbf{B}_2$ are all $M_R^2 \times M_R^2$ matrices.

As a final simplifying step we observe that the objective function of (34) is homogeneous in $\mathbf{g}$, meaning that an arbitrary rescaling of $\mathbf{g}$ has no effect on the value of the objective function. Consequently, the equality constraint can be dropped since any solution to the unconstrained problem can be rescaled to meet the equality constraint without any loss in terms of the objective function. Therefore, the final form of our problem statement is given by

$$\mathbf{g}_{\text{opt}} = \arg\max_{\mathbf{g}} \frac{\mathbf{g}^H \cdot \mathbf{A}_1 \cdot \mathbf{g}}{\mathbf{g}^H \cdot \mathbf{B}_1 \cdot \mathbf{g}} \cdot \frac{\mathbf{g}^H \cdot \mathbf{A}_2 \cdot \mathbf{g}}{\mathbf{g}^H \cdot \mathbf{B}_2 \cdot \mathbf{g}}. \tag{35}$$

The optimization problem (35) can be interpreted as the product of two Rayleigh quotients (quadratic fractional functions). Moreover, it can be expressed as a DC programming problem. Indeed, as we will show later the objective function of the problem (35) can be written as a summation of two concave functions with positive sign and one concave function with negative sign. Thus, the objective of the equivalent problem is, in fact, the difference of convex functions which is in general non-convex. The available in the literature algorithms for solving such DC programming problems are based on the so-called branch-and-bound method that does not have any polynomial time computational complexity guarantees [14]–[20]. However, as we show next, at least a KKT solution of the problem (35) can be found in polynomial time with a great evidence that such a solution is also globally optimal.

IV. Polynomial Time Algorithm for the Sum-Rate Maximization Problem in Two-Way AF MIMO Relaying

Since the problem (35) is homogeneous, without loss of generality, we can fix the quadratic term $\mathbf{g}^H \cdot \mathbf{B}_1 \cdot \mathbf{g}$ to be equal to one at the optimal point. By doing so and also by defining the additional variables τ and β, the problem (35) can be equivalently recast as

$$\begin{aligned} &\max_{\mathbf{g},\tau,\beta} \mathbf{g}^H \cdot \mathbf{A}_1 \cdot \mathbf{g} \cdot \frac{\tau}{\beta} \\ &\text{subject to } \mathbf{g}^H \cdot \mathbf{B}_1 \cdot \mathbf{g} = 1,\ \mathbf{g}^H \cdot \mathbf{A}_2 \cdot \mathbf{g} = \tau \\ &\qquad\qquad \mathbf{g}^H \cdot \mathbf{B}_2 \cdot \mathbf{g} = \beta. \end{aligned} \tag{36}$$

For future reference, we need the range of the variable β. Due to the fact that the quadratic function $\mathbf{g}^H \cdot \mathbf{B}_1 \cdot \mathbf{g}$ is set to one, this range can be easily obtained. Specifically, the smallest value of β for which the problem (36) is still feasible can be obtained by solving the following problem

$$\min_{\mathbf{g}} \mathbf{g}^H \cdot \mathbf{B}_2 \cdot \mathbf{g} \quad \text{subject to} \quad \mathbf{g}^H \cdot \mathbf{B}_1 \cdot \mathbf{g} = 1. \tag{37}$$

Note that τ does not impose any restriction on the smallest possible value of β, because if $\mathbf{g}_{\min,\beta}$ denotes the optimal solution of the problem (37), then τ can be chosen as $\tau = g_{\min,\beta}^H \cdot \mathbf{A}_2 \cdot g_{\min,\beta}$. Since the matrix $\mathbf{B}_1$ is positive definite, it can be decomposed as $\mathbf{B}_1 = \mathbf{B}_1^{\frac{1}{2}} \cdot \left(\mathbf{B}_1^{\frac{1}{2}}\right)^H$ where $\mathbf{B}_1^{\frac{1}{2}}$ is a square root of $\mathbf{B}_1$ and it is invertible due to the properties in [31]. By defining the new vector $\mathbf{y} \triangleq \left(\mathbf{B}_1^{\frac{1}{2}}\right)^H \cdot \mathbf{g}$, i.e., $\mathbf{g} = \left(\mathbf{B}_1^{-\frac{1}{2}}\right)^H \cdot \mathbf{y}$ the problem (37) is equivalent to

$$\min_{\mathbf{y}} \mathbf{y}^H \cdot \mathbf{B}_1^{-\frac{1}{2}} \cdot \mathbf{B}_2 \cdot (\mathbf{B}_1^{-\frac{1}{2}})^H \cdot \mathbf{y} \quad \text{subject to} \quad \mathbf{y}^H \cdot \mathbf{y} = 1. \tag{38}$$

It is well known that according to the minimax theorem [32], the optimal value of (38) is the smallest eigenvalue of the matrix $\mathbf{B}_1^{-\frac{1}{2}} \cdot \mathbf{B}_2 \cdot \left(\mathbf{B}_1^{-\frac{1}{2}}\right)^H$. Using the fact that for any arbitrary square matrices $\mathbf{Z}_1$ and $\mathbf{Z}_2$, the eigenvalues of the matrix products $\mathbf{Z}_1 \cdot \mathbf{Z}_2$ and $\mathbf{Z}_2 \cdot \mathbf{Z}_1$ are the same [33], it can be concluded that the smallest eigenvalue of $\mathbf{B}_1^{-\frac{1}{2}} \cdot \mathbf{B}_2 \cdot \left(\mathbf{B}_1^{-\frac{1}{2}}\right)^H$ is the same as the smallest eigenvalue of $\left(\mathbf{B}_1^{-\frac{1}{2}}\right)^H \cdot \mathbf{B}_1^{-\frac{1}{2}} \cdot \mathbf{B}_2$ or, equivalently, $\mathbf{B}_1^{-1} \cdot \mathbf{B}_2$.

The largest value of β for which the problem (36) is still feasible can be obtained in a similar way, and it is equal to the largest eigenvalue of the matrix $\mathbf{B}_1^{-1} \cdot \mathbf{B}_2$. As a result, the range of β is $[\lambda_{\min}(\mathbf{B}_1^{-1} \cdot \mathbf{B}_2), \lambda_{\max}(\mathbf{B}_1^{-1} \cdot \mathbf{B}_2)]$ where $\lambda_{\min}(\cdot)$ and $\lambda_{\max}(\cdot)$ denote the smallest and the largest eigenvalue operators, respectively. Note that, since the matrices $\mathbf{B}_1$ and $\mathbf{B}_2$ are positive definite (hence $\mathbf{B}_1^{-1}$ is also positive definite), the eigenvalues of the product $\mathbf{B}_1^{-1} \cdot \mathbf{B}_2$ are all positive due to the properties in [31] including $\lambda_{\min}(\mathbf{B}_1^{-1} \cdot \mathbf{B}_2)$.

For future reference, we also define the following function of τ and β

$$\begin{aligned} g(\tau, \beta) \triangleq \Big\{ &\max_{\mathbf{g}} \mathbf{g}^H \cdot \mathbf{A}_1 \cdot \mathbf{g} | \mathbf{g}^H \cdot \mathbf{B}_1 \cdot \mathbf{g} = 1, \\ &\mathbf{g}^H \cdot \mathbf{A}_2 \cdot \mathbf{g} = \tau, \mathbf{g}^H \cdot \mathbf{B}_2 \cdot \mathbf{g} = \beta \Big\}, \quad (\tau, \beta) \in \mathcal{D} \end{aligned} \tag{39}$$

where $\mathcal{D} \subset \mathbb{R}^2$ is the set of all pairs (τ, β) such that the corresponding optimization problem obtained from $g(\tau, \beta)$ for fixed τ and β is feasible. The function $g(\tau, \beta)$ is called an *optimal value function*. Therefore, using the optimal value function $g(\tau, \beta)$, the original optimization problem (36) can be equivalently recast as

$$\max_{\tau,\beta} \quad g(\tau, \beta) \cdot \frac{\tau}{\beta}. \tag{40}$$

Introducing the matrix $\mathbf{X} \triangleq \mathbf{g} \cdot \mathbf{g}^H$ and observing that for any arbitrary matrix $\mathbf{Y}$, the relationship $\mathbf{g}^H \cdot \mathbf{Y} \cdot \mathbf{g} = \text{tr}\{\mathbf{Y} \cdot \mathbf{g} \cdot \mathbf{g}^H\}$

holds, the optimal value function $g(\tau,\beta)$ can be equivalently recast as [34]

$$g(\tau,\beta)=\Big\{\max_{\mathbf{X}}\mathrm{tr}\{\mathbf{A}_1\cdot\mathbf{X}\}|\mathrm{tr}\{\mathbf{B}_1\cdot\mathbf{X}\}=1,\mathrm{tr}\{\mathbf{A}_2\cdot\mathbf{X}\}=\tau,\\ \mathrm{tr}\{\mathbf{B}_2\cdot\mathbf{X}\}=\beta,\mathrm{rank}\{\mathbf{X}\}=1,\mathbf{X}\succeq\mathbf{0}\Big\},(\tau,\beta)\in\mathcal{D}. \quad (41)$$

where $\mathrm{rank}\{\cdot\}$ stands for the rank of a matrix. In the optimization problem obtained from the optimal value function $g(\tau,\beta)$ (41) by fixing τ and β, the rank-one constraint $\mathrm{rank}\{\mathbf{X}\}=1$ is the only non-convex constraint with respect to the new optimization variable $\mathbf{X}$. Using the SDP relaxation, the corresponding optimization problem can be relaxed by dropping the rank-one constraint, and the following new optimal value function $h(\tau,\beta)$ can be defined

$$h(\tau,\beta)\triangleq\Big\{\max_{\mathbf{X}}\mathrm{tr}\{\mathbf{A}_1\cdot\mathbf{X}\}|\mathrm{tr}\{\mathbf{B}_1\cdot\mathbf{X}\}=1,\\ \mathrm{tr}\{\mathbf{A}_2\cdot\mathbf{X}\}=\tau,\mathrm{tr}\{\mathbf{B}_2\cdot\mathbf{X}\}=\beta,\mathbf{X}\succeq\mathbf{0}\Big\},(\tau,\beta)\in\mathcal{D}' \quad (42)$$

where $\mathcal{D}'\subset\mathbb{R}^2$ is the set of all pairs (τ,β) such that the optimization problem corresponding to $h(\tau,\beta)$ for fixed τ and β is feasible. For brevity, we will refer to the optimization problems corresponding to the functions $g(\tau,\beta)$ and $h(\tau,\beta)$ when τ and β are fixed simply as the optimization problems of $g(\tau,\beta)$ and $h(\tau,\beta)$, respectively. The following lemma finds the relationship between the domains of the functions $g(\tau,\beta)$ and $h(\tau,\beta)$.

Lemma 1: The domains of the functions $g(\tau,\beta)$ and $h(\tau,\beta)$ are the same, i.e., $\mathcal{D}=\mathcal{D}'$.

Proof: First, we prove that if $(\tau,\beta)\in\mathcal{D}$ then $(\tau,\beta)\in\mathcal{D}'$. Let $(\tau,\beta)\in\mathcal{D}$. It implies that there exists a vector $\mathbf{g}_0$ such that the constraints $\mathbf{g}_0^H\cdot\mathbf{B}_1\cdot\mathbf{g}_0=1$, $\mathbf{g}_0^H\cdot\mathbf{A}_2\cdot\mathbf{g}_0=\tau$, and $\mathbf{g}_0^H\cdot\mathbf{B}_2\cdot\mathbf{g}_0=\beta$ are satisfied. Defining the new matrix $\mathbf{X}_0\triangleq\mathbf{g}_0\cdot\mathbf{g}_0^H$, it is easy to verify that $\mathbf{X}_0$ satisfies the constraints in the optimization problem of $h(\tau,\beta)$ and, therefore, $(\tau,\beta)\in\mathcal{D}'$. Now, let us assume that $(\tau,\beta)\in\mathcal{D}'$. Therefore, there exists a positive semi-definite matrix $\mathbf{X}_0=\mathbf{V}_{M_R\times r}\cdot\mathbf{V}_{M_R\times r}^H$ with rank equal to r and $\mathbf{V}_{M_R\times r}$ being a full rank matrix such that $\mathrm{tr}\{\mathbf{B}_1\cdot\mathbf{X}_0\}=\mathrm{tr}\{\mathbf{V}^H\cdot\mathbf{B}_1\cdot\mathbf{V}\}=1$, $\mathrm{tr}\{\mathbf{A}_2\cdot\mathbf{X}_0\}=\mathrm{tr}\{\mathbf{V}^H\cdot\mathbf{A}_2\cdot\mathbf{V}\}=\tau$, and $\mathrm{tr}\{\mathbf{B}_2\cdot\mathbf{X}_0\}=\mathrm{tr}\{\mathbf{V}^H\cdot\mathbf{B}_2\cdot\mathbf{V}\}=\beta$. If the rank of $\mathbf{X}_0$ is one, then $\mathbf{g}_0=\mathbf{V}_{M_R\times 1}$ satisfies the constraints in the optimization problem of $g(\tau,\beta)$ and trivially $(\tau,\beta)\in\mathcal{D}$. Thus, we assume that r is greater than one and aim to show that based on $\mathbf{X}_0$, another rank-one feasible point for the optimization problem of $h(\tau,\beta)$ can be constructed by following similar lines as in [35]. To this end, let us consider the following set of equations

$$\begin{aligned}&\mathrm{tr}\{\mathbf{V}^H\cdot\mathbf{B}_1\cdot\mathbf{V}\cdot\boldsymbol{\Gamma}\}=0\\&\mathrm{tr}\{\mathbf{V}^H\cdot\mathbf{A}_2\cdot\mathbf{V}\cdot\boldsymbol{\Gamma}\}=0\\&\mathrm{tr}\{\mathbf{V}^H\cdot\mathbf{B}_2\cdot\mathbf{V}\cdot\boldsymbol{\Gamma}\}=0\end{aligned} \quad (43)$$

where an $r\times r$ Hermitian matrix $\boldsymbol{\Gamma}$ is an unknown variable. Due to the fact that $\mathrm{tr}\{\mathbf{V}^H\cdot\mathbf{B}_1\cdot\mathbf{V}\cdot\boldsymbol{\Gamma}\}$, $\mathrm{tr}\{\mathbf{V}^H\cdot\mathbf{A}_2\cdot\mathbf{V}\cdot\boldsymbol{\Gamma}\}$, and $\mathrm{tr}\{\mathbf{V}^H\cdot\mathbf{B}_2\cdot\mathbf{V}\cdot\boldsymbol{\Gamma}\}$ are all real valued functions of $\boldsymbol{\Gamma}$, the set of (43) is a linear set of 3 equations with r^2 variables, that is, the total number of real and imaginary variables in the matrix $\boldsymbol{\Gamma}$. Since the number of variables r^2, $(r\geq 2)$ is greater than the number of equations, there exists a nonzero solution denoted as $\boldsymbol{\Gamma}_0$ for the linear set of equations (43). Let δ_0 denote the eigenvalue of the matrix $\boldsymbol{\Gamma}_0$ which has the largest absolute value. Without loss of generality, we can assume that $\delta_0>0$, which is due to the fact that both $\boldsymbol{\Gamma}_0$ and $-\boldsymbol{\Gamma}_0$ are solutions of (43). Using $\mathbf{X}_0$ and $\boldsymbol{\Gamma}_0$, we can construct a new matrix $\mathbf{X}_0^{new}=\mathbf{V}\cdot\left(\mathbf{I}_r-\frac{\boldsymbol{\Gamma}_0}{\delta_0}\right)\cdot\mathbf{V}^H$. It is then easy to verify that the expressions $\mathrm{tr}\{\mathbf{B}_1\cdot\mathbf{X}_0^{new}\}=1$, $\mathrm{tr}\{\mathbf{A}_2\cdot\mathbf{X}_0^{new}\}=\tau$, $\mathrm{tr}\{\mathbf{B}_2\cdot\mathbf{X}_0^{new}\}=\beta$, and $\mathbf{X}_0^{new}\succeq\mathbf{0}$ are valid and the rank of $\mathbf{X}_0^{new}$ is less than or equal to $r-1$. It is because the rank of the matrix $\left(\mathbf{I}_r-\frac{\boldsymbol{\Gamma}_0}{\delta_0}\right)$ is less than or equal to $r-1$ and the fact that rank of any matrix product is less than or equal to the rank of each of the matrices. It means that $\mathbf{X}_0^{new}$ is another feasible point of the optimization problem of $h(\tau,\beta)$ and its rank has reduced at least by one. This process can be repeated until $r^2\leq 3$ or, equivalently, a rank-one feasible point is found. After a rank one feasible point $\mathbf{v}_{M_R\times 1}\cdot\mathbf{v}_{M_R\times 1}^H$ is constructed, $\mathbf{g}_0=\mathbf{v}_{M_R\times 1}$ is also a feasible point for the optimization problem of $g(\tau,\beta)$. Thus, $(\tau,\beta)\in\mathcal{D}$ which completes the proof. ■

So far, we have shown that both optimal value functions $g(\tau,\beta)$ and $h(\tau,\beta)$ have the same domain. Since the feasible set of the optimization problem of $g(\tau,\beta)$ is a subset of the feasible set of the optimization problem of $h(\tau,\beta)$, we expect that $g(\tau,\beta)$ is less than or equal to $h(\tau,\beta)$ at every feasible point. However, due to the specific structure of the function $g(\tau,\beta)$, these two optimal value functions are equivalent as it is shown in the following Theorem.

Theorem 1: The optimal value functions $g(\tau,\beta)$ and $h(\tau,\beta)$ are equivalent, i.e., $g(\tau,\beta)=h(\tau,\beta)$, $(\tau,\beta)\in\mathcal{D}$.

Proof: In order to show that these optimal value functions are equal, we use the dual problem of the optimization problems of $g(\tau,\beta)$ and $h(\tau,\beta)$. It is easy to verify that the optimization problems of $g(\tau,\beta)$ and $h(\tau,\beta)$ have the same following dual problem

$$\begin{aligned}&\min_{\delta_1,\delta_2,\delta_3}\ \delta_1+\delta_2\cdot\tau+\delta_3\cdot\beta\\&\text{subject to }-\mathbf{A}_1+\mathbf{B}_1\cdot\delta_1+\mathbf{A}_2\cdot\delta_2+\mathbf{B}_2\cdot\delta_3\succeq\mathbf{0}.\end{aligned} \quad (44)$$

The following optimal value function can be defined based on (44)

$$k(\tau,\beta)\triangleq\Big\{\min_{\delta_1,\delta_2,\delta_3}\ \delta_1+\delta_2\cdot\tau+\delta_3\cdot\beta|-\mathbf{A}_1+\mathbf{B}_1\cdot\delta_1+\mathbf{A}_2\cdot\delta_2\\ +\mathbf{B}_2\cdot\delta_3\succeq\mathbf{0}\Big\},\quad(\tau,\beta)\in\mathcal{D} \quad (45)$$

Since the dual problem (44) gives an upper-bound for the optimization problems of $g(\tau,\beta)$ and $h(\tau,\beta)$, consequently, the function (45) is greater than or equal to $g(\tau,\beta)$ and $h(\tau,\beta)$ for every $(\tau,\beta)\in\mathcal{D}$. The optimization problem of $h(\tau,\beta)$ is convex and satisfies the Slater's condition [34] because for every $(\tau,\beta)\in\mathcal{D}$, there exists a strictly feasible point for its dual problem (44). Specifically, the point ($\delta_1=2\cdot\lambda_{\max}\{\mathbf{B}_1^{-1}\cdot\mathbf{A}_1\}$, $\delta_2=0$, $\delta_3=0$) is a strictly feasible point for the dual problem (44) as it can be easily verified that the matrix $-\mathbf{A}_1+2\cdot\lambda_{\max}\{\mathbf{B}_1^{-1}\cdot\mathbf{A}_1\}\cdot\mathbf{B}_1$, or equivalently, $\mathbf{B}_1\cdot(-\mathbf{B}_1^{-1}\cdot\mathbf{A}_1+2\cdot\lambda_{\max}\{\mathbf{B}_1^{-1}\cdot\mathbf{A}_1\}\cdot\mathbf{I})$ is positive definite. Therefore, the duality gap between the optimization problem of $h(\tau,\beta)$ and its dual problem (44) is zero [34] which implies that for every $(\tau,\beta)\in\mathcal{D}$, $h(\tau,\beta)=k(\tau,\beta)$.

Regarding the optimization problem of $g(\tau,\beta)$ which is a QCQP problem [35]–[37], it is known that the duality gap between a QCQP problem with three or less constraints and its dual problem is zero [35]. Specifically, Corollary 3.3 of [35, Section 3] implies that the duality gap between the optimization problem of $g(\tau,\beta)$ and its dual optimization problem (44) is zero and hence, $g(\tau,\beta)=k(\tau,\beta)$. Since both of the optimization problems of $g(\tau,\beta)$ and $h(\tau,\beta)$ have zero duality gap with their dual problem (44), it can be concluded that in addition to having the same domain, the functions $g(\tau,\beta)$ and $h(\tau,\beta)$ have the same optimal values, i.e., $g(\tau,\beta)=h(\tau,\beta)$ for every feasible (τ,β). ■

For any feasible point of the optimization problem of $g(\tau,\beta)$ denoted as $\mathbf{g}_0$, the matrix $\mathbf{g}_0\cdot\mathbf{g}_0^H$ is also a feasible point of the optimization problem of $h(\tau,\beta)$ and their corresponding objective values are the same. Based on the latter fact and also the fact that the functions $g(\tau,\beta)$ and $h(\tau,\beta)$ have the same optimal values, it can be concluded that if $g^{\rm opt}_{(\tau,\beta)}$ denotes the optimal solution of the optimization problem of $g(\tau,\beta)$, then $g^{\rm opt}_{(\tau,\beta)}\cdot(g^{\rm opt}_{(\tau,\beta)})^H$ is also the optimal solution of the optimization problem of $h(\tau,\beta)$. Therefore, for every $(\tau,\beta)\in\mathcal{D}$, there exists a rank-one solution for the optimization problem of $h(\tau,\beta)$. The algorithm for constructing such a rank-one solution from a general-rank solution of the optimization problem of $h(\tau,\beta)$ has been explained in [35].

Although the optimal value functions $g(\tau,\beta)$ and $h(\tau,\beta)$ are equal, however, compared to the optimization problem of $g(\tau,\beta)$ which is non-convex, the optimization problem of $h(\tau,\beta)$ is convex. Using this fact and replacing $g(\tau,\beta)$ by $h(\tau,\beta)$ in the original optimization problem (40), the problem (36) can be simplified as

$$\begin{aligned}&\max_{\mathbf{X},\tau,\beta}\ \mathrm{tr}\{\mathbf{A}_1\cdot\mathbf{X}\}\cdot\frac{\tau}{\beta}\\ &\text{subject to } \mathrm{tr}\{\mathbf{B}_1\cdot\mathbf{X}\}=1,\quad \mathrm{tr}\{\mathbf{A}_2\cdot\mathbf{X}\}=\tau\\ &\qquad\qquad \mathrm{tr}\{\mathbf{B}_2\cdot\mathbf{X}\}=\beta,\quad \mathbf{X}\succeq\mathbf{0}.\end{aligned}\tag{46}$$

Therefore, instead of the original optimization problem (36), we can solve the simplified problem (46). Based on the optimal solution of the simplified problem, denoted as $\mathbf{X}_{\rm opt}$, $\tau_{\rm opt}$, and $\beta_{\rm opt}$, the optimal solution of the original problem can be found. The optimal values of τ and β are equal to the corresponding optimal values of the simplified problem, while the optimal $\mathbf{g}$ can be constructed based on $\mathbf{X}_{\rm opt}$ using rank-reduction techniques [35] mentioned above.

It is worth stressing that for every feasible point of the optimization problem (46) denoted as $\mathbf{X}$, the terms $\mathrm{tr}\{\mathbf{A}_1\cdot\mathbf{X}\}$, $\mathrm{tr}\{\mathbf{A}_2\cdot\mathbf{X}\}$, and $\mathrm{tr}\{\mathbf{B}_2\cdot\mathbf{X}\}$ are positive, and therefore, the corresponding objective value is positive as well. The latter can be easily verified by applying Lemma 1 of [38, Section 2] which states that for every Hermitian matrix $\mathbf{A}$ and Hermitian positive semi-definite matrix $\mathbf{B}$, $\mathrm{tr}\{\mathbf{A}\cdot\mathbf{B}\}$ is greater than or equal to $\lambda_{\min}\{\mathbf{A}\}\cdot\mathrm{tr}\{\mathbf{B}\}$. Applying this lemma, it can be found that

$$\begin{aligned}\mathrm{tr}\{\mathbf{A}_1\cdot\mathbf{X}\}&=\mathrm{tr}\{\mathbf{X}\cdot\mathbf{A}_1\}\\ &=\mathrm{tr}\left\{\mathbf{X}\cdot\mathbf{B}_1^{\frac{1}{2}}\cdot\mathbf{B}_1^{-\frac{1}{2}}\cdot\mathbf{A}_1\cdot\left(\mathbf{B}_1^{\frac{1}{2}}\cdot\mathbf{B}_1^{-\frac{1}{2}}\right)^H\right\}\\ &=\mathrm{tr}\left\{\left(\mathbf{B}_1^{\frac{1}{2}}\right)^H\cdot\mathbf{X}\cdot\mathbf{B}_1^{\frac{1}{2}}\cdot\mathbf{B}_1^{-\frac{1}{2}}\cdot\mathbf{A}_1\cdot\left(\mathbf{B}_1^{-\frac{1}{2}}\right)^H\right\}\\ &\geq\mathrm{tr}\{\mathbf{B}_1\cdot\mathbf{X}\}\cdot\lambda_{\min}\left\{\mathbf{B}_1^{-1}\cdot\mathbf{A}_1\right\}\\ &=\lambda_{\min}\left\{\mathbf{B}_1^{-1}\cdot\mathbf{A}_1\right\}\end{aligned}\tag{47}$$

where Lemma 1 of [38, Section 2] has been applied in the third line of (47) and the last equality follows from the fact that $\mathrm{tr}\{\mathbf{B}_1\cdot\mathbf{X}\}=1$ as $\mathbf{X}$ is a feasible point. Since $\mathbf{B}_1^{-1}$ and $\mathbf{A}_1$ are positive definite, all the eigenvalues of the product $\mathbf{B}_1^{-1}\cdot\mathbf{A}_1$ are positive [31], and therefore, $\lambda_{\min}\{\mathbf{A}_1\cdot\mathbf{B}_1^{-1}\}$ is positive. In a similar way, it can be proved that $\mathrm{tr}\{\mathbf{A}_2\cdot\mathbf{X}\}$ and $\mathrm{tr}\{\mathbf{B}_2\cdot\mathbf{X}\}$ are necessarily positive, and therefore, the variables τ and β are also positive. Thus, the task of maximizing the objective function in the problem (46) is equivalent to maximizing the logarithm of this objective function because $\ln(x)$ is a strictly increasing function and the objective function in (46) is positive. Then, the optimization problem (46) can be equivalently rewritten as

$$\begin{aligned}&\max_{\mathbf{X},\tau,\beta}\quad \ln\left(\mathrm{tr}\{\mathbf{A}_1\cdot\mathbf{X}\}\right)+\ln(\tau)-\ln(\beta)\\ &\text{subject to}\quad \mathrm{tr}\{\mathbf{B}_1\cdot\mathbf{X}\}=1,\quad \mathrm{tr}\{\mathbf{A}_2\cdot\mathbf{X}\}=\tau\\ &\qquad\qquad\quad \mathrm{tr}\{\mathbf{B}_2\cdot\mathbf{X}\}=\beta,\quad \mathbf{X}\succeq\mathbf{0}.\end{aligned}\tag{48}$$

In summary, by replacing $g(\tau,\beta)$ by $h(\tau,\beta)$, we are able to write our optimization problem as a DC programming problem, where the fact that $\ln(\mathrm{tr}\{\mathbf{A}_1\cdot\mathbf{X}\})$ in the objective of (48) is a concave function is also considered. Although the problem (48) boils down to the known family of DC programming problems, still there exists no solution for such DC programming problems with guaranteed polynomial time complexity. However, the problem (48) has a very particular structure, such as, all the constraints are convex and the terms $\ln(\mathrm{tr}\{\mathbf{A}_1\cdot\mathbf{X}\})$ and $\ln(\tau)$ in the objective are concave. Thus, the only term that makes the problem overall non-convex is the term $-\ln(\beta)$ in the objective. If $-\ln(\beta)$ is piece-wise linearized over a finite number of intervals,[2] then the objective function becomes concave on these intervals and the whole problem (48) becomes convex. The resulting convex problems over different linearization intervals for $-\ln(\beta)$ can be solved efficiently in polynomial time, and then, the sub-optimal solution of the problem (48) can be found. The fact that such a solution is sub-optimal follows from the linearization, which has a finite accuracy. The smaller the intervals are, the more accurate the solution of (48) becomes. However, such a solution procedure is not the most efficient in terms of computational complexity. Thus, we develop another method (the POTDC algorithm) which makes it possible to find at least a KKT solution for the problem (48) (with a great evidence that such a solution is globally optimal) in a more efficient way.

Let us introduce a new additional variable t and then express the problem (48) equivalently as

$$\begin{aligned}&\max_{\mathbf{X},\tau,\beta,t}\quad \ln\left(\mathrm{tr}\{\mathbf{A}_1\cdot\mathbf{X}\}\right)+\ln(\tau)-t\\ &\text{subject to}\quad \mathrm{tr}\{\mathbf{B}_1\cdot\mathbf{X}\}=1,\quad \mathrm{tr}\{\mathbf{A}_2\cdot\mathbf{X}\}=\tau\\ &\qquad\qquad\quad \mathrm{tr}\{\mathbf{B}_2\cdot\mathbf{X}\}=\beta,\quad \ln(\beta)\leq t,\quad \mathbf{X}\succeq\mathbf{0}.\end{aligned}\tag{49}$$

The objective function of the optimization problem (49) is concave and all the constraints except the constraint $\ln(\beta)\leq t$ are convex. Thus, we can develop an iterative method that is different from the aforementioned piece-wise linearization-based method, and is based on linearizing the non-convex term $\ln(\beta)$ in the constraint $\ln(\beta)\leq t$ around a suitably selected point in each iteration. More specifically, the linearizing point in each iteration is selected so that the objective function increases in every iteration of the iterative algorithm. Roughly speaking, the main idea of this iterative method is similar to the gradient based Newton-type methods. In the first iteration, we start with an arbitrary point selected in the interval $[\lambda_{\min}\{\mathbf{B}_1^{-1}\cdot\mathbf{B}_2\},\lambda_{\max}\{\mathbf{B}_1^{-1}\cdot\mathbf{B}_2\}]$ and denoted as β_c. Then

[2] As explained before, the parameter β can take values only in a finite interval. Thus, a finite number of linearization intervals for $-\ln(\beta)$ is needed.

the non-convex function $\ln(\beta)$ can be replaced by its linear approximation around this point β_c, that is,

$$\ln(\beta) \approx \ln(\beta_c) + \frac{1}{\beta_c}\cdot(\beta-\beta_c) \tag{50}$$

which results in the following convex optimization problem

$$\begin{aligned}\max_{\mathbf{X},\tau,\beta,t}\quad & \ln\left(\mathrm{tr}\{\mathbf{A}_1\cdot\mathbf{X}\}\right)+\ln(\tau)-t\\ \text{subject to}\quad & \mathrm{tr}\{\mathbf{B}_1\cdot\mathbf{X}\}=1,\ \mathrm{tr}\{\mathbf{A}_2\cdot\mathbf{X}\}=\tau,\ \mathrm{tr}\{\mathbf{B}_2\cdot\mathbf{X}\}=\beta\\ & \ln(\beta_c)+\frac{1}{\beta_c}\cdot(\beta-\beta_c)\le t,\quad \mathbf{X}\succeq\mathbf{0}.\end{aligned} \tag{51}$$

The problem (51) can be efficiently solved using the interior-point-based numerical methods with the worst-case complexity of $\mathcal{O}((M_R^4+3)^{3.5})$ where M_R^4+3 is the total number of optimization variables in the problem (51) including the real and imaginary parts of the elements of $\mathbf{X}$ as well as τ, β, and t. Once the optimal solution of this problem, denoted in the first iteration as $\mathbf{X}_{\rm opt}^{(1)}$, $\tau_{\rm opt}^{(1)}$, $\beta_{\rm opt}^{(1)}$, and $t_{\rm opt}^{(1)}$, is found, the algorithm proceeds to the second iteration by replacing the function $\ln(\beta)$ by its linear approximation around $\beta_{\rm opt}^{(1)}$ found from the previous (first) iteration. Fig. 2 shows how $\ln(\beta)$ is replaced by its linear approximation around β_c where $\beta_{\rm opt}$ is the optimal value of β obtained through solving (51) using such a linear approximation. In the second iteration, the resulting optimization problem has the same structure as the problem (51) in which β_c has to be set to $\beta_{\rm opt}^{(1)}$ obtained from the first iteration. This process continues, and kth iteration is obtained by replacing $\ln(\beta)$ by its linearization of type (50) around $\beta_{\rm opt}^{(k-1)}$ found at the iteration $k-1$. The POTDC algorithm for solving the problem (49) is summarized in Algorithm 1.

Algorithm 1: The POTDC algorithm for solving the optimization problem (49)

Initialize: Select an arbitrary β_c from the interval $[\lambda_{\min}\{\mathbf{B}_1^{-1}\cdot\mathbf{B}_2\},\lambda_{\max}\{\mathbf{B}_1^{-1}\cdot\mathbf{B}_2\}]$, set the counter k to be equal to 1 and choose an accuracy parameter ϵ.

while The difference between the values of the objective function in two consecutive iterations is larger than ϵ. **do**

Use the linearization of type (50) and solve the optimization problem (51) to obtain $\mathbf{X}_{\rm opt}^{(k)}$, $\tau_{\rm opt}^{(k)}$, $\beta_{\rm opt}^{(k)}$ and $t_{\rm opt}^{(k)}$.

Set $\mathbf{X}_{\rm opt} := \mathbf{X}_{\rm opt}^{(k)}$, and $\beta_c := \beta_{\rm opt}^{(k)}$.

$k := k+1$.

end while

Output: $\mathbf{X}_{\rm opt}$.

The following two lemmas regarding the proposed POTDC algorithm are of interest. First, the termination condition in the POTDC algorithm is guaranteed to be satisfied due to the following lemma which states that by choosing β_c in the above proposed manner, the optimal values of the objective function of (51) for $\mathbf{X}_{\rm opt}^{(k)}$, $\tau_{\rm opt}^{(k)}$, $\beta_{\rm opt}^{(k)}$ and $t_{\rm opt}^{(k)}$ are non-decreasing.

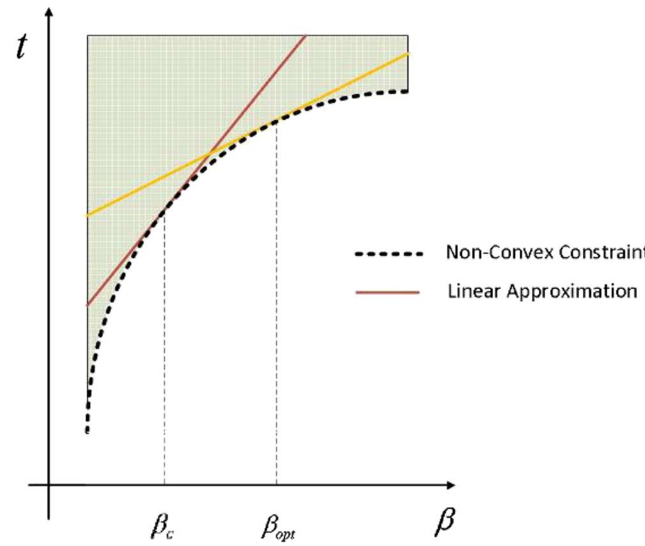

Fig. 2. Linear approximation of $\ln(\beta)$ around β_c. The region above the dashed curve is non-convex.

Lemma 2: The optimal values of the objective function of the optimization problem (51) obtained over the iterations of the POTDC algorithm are non-decreasing.

Proof: Considering the linearized problem (51) at the iteration $k+1$. It is easy to verify that $\mathbf{X}_{\rm opt}^{(k)}$, $\tau_{\rm opt}^{(k)}$, $\beta_{\rm opt}^{(k)}$, and $t_{\rm opt}^{(k)}$ give a feasible point for this problem. Therefore, the optimal value at the iteration $k+1$ must be greater than or equal to the optimal value at the iteration k which completes the proof. ■

Second, the following lemma regarding the solution found by the POTDC algorithm is of interest.

Lemma 3: The solution obtained using the POTDC algorithm satisfies the KKT conditions.

Proof: This lemma follows straightforwardly from Proposition 3.2 of [39, Section 3]. ■

As soon as the solution of the relaxed problem (49) is found, the solution of the original problem (36), which is equivalent to the solution of the sum-rate maximization problem (35), can be found using one of the existing methods for extracting a rank-one solution [35]–[37]. The rank reduction-based technique of [35] and the algebraic technique of [37] have been mentioned earlier, while the method based on solving the dual problem (see [36]) exploits the fact that a QCQP problem with only two constraints has zero duality gap. Note that the computational complexity of the POTDC algorithm is equivalent to that of solving an SDP problem, i.e., $\mathcal{O}((M_R^4+3)^{3.5})$, times the number of the iterations. Although the POTDC finds a KKT solution for the considered sum-rate maximization problem, we also aim at showing the evidence that such a solution is globally optimal. Toward this end, we will need an upper-bound for the optimal value.

V. An Upper-Bound for the Optimal Value

Through extensive simulations we have observed that regardless of the initial value chosen for β_c in the first iteration of the POTDC algorithm, the proposed iterative method always converges to the global optimum of the problem (49). However, since the original problem is not convex, this can not be easily verified analytically. A comparison between the optimal value obtained by using the proposed iterative method and also the global optimal value can be, however, done by developing a tight upper-bound for the optimal value of the problem and comparing the solution to such an upper-bound. Thus, in this section, we find such an upper-bound for the optimal value of the optimization problem (49). For this goal, we first consider the following lemma which gives an upper-bound for the optimal value of the variable β in the problem (49). This lemma will further be used for obtaining the desired upper-bound for our problem.

Lemma 4: The optimal value of the variable β in (49) or equivalently (48), denoted as $\beta_{\rm opt}$, is upper-bounded by $e^{(q^\star - p^\star)}$, where $p^\star$ is the value of the objective function in the problem (49) or equivalently (48) corresponding to any arbitrary feasible point and $q^\star$ is the optimal value of the following convex optimization problem[3]

$$\begin{aligned} q^\star = \max_{\mathbf{X},\tau,\beta} \quad & \ln\left(\mathrm{tr}\{\mathbf{A}_1 \cdot \mathbf{X}\}\right) + \ln\{\tau\} \\ \text{subject to} \quad & \mathrm{tr}\{\mathbf{B}_1 \cdot \mathbf{X}\} = 1,\ \mathrm{tr}\{\mathbf{A}_2 \cdot \mathbf{X}\} = \tau,\ \mathbf{X} \succeq \mathbf{0}. \end{aligned} \quad (52)$$

Proof: Let $\mathbf{X}_{\rm opt}$, $\tau_{\rm opt}$, and $\beta_{\rm opt}$ denote the optimal solution of the optimization problem (48). We define another auxiliary optimization problem based on the problem (48) by fixing the variable β to $\beta_{\rm opt}$ as

$$\begin{aligned} \max_{\mathbf{X},\tau} \quad & \ln\left(\mathrm{tr}\{\mathbf{A}_1 \cdot \mathbf{X}\}\right) + \ln(\tau) - \ln(\beta_{\rm opt}) \\ \text{subject to} \quad & \mathrm{tr}\{\mathbf{B}_1 \cdot \mathbf{X}\} = 1, \quad \mathrm{tr}\{\mathbf{A}_2 \cdot \mathbf{X}\} = \tau \\ & \mathrm{tr}\{\mathbf{B}_2 \cdot \mathbf{X}\} = \beta_{\rm opt}, \quad \mathbf{X} \succeq \mathbf{0}. \end{aligned} \quad (53)$$

For every feasible point of the problem (53), denoted as $(\mathbf{X}, \tau)$, it is easy to verify that $(\mathbf{X}, \tau, \beta_{\rm opt})$ is also a feasible point of the problem (48). Based on this fact, it can be concluded that the optimal value of the problem (53) is less than or equal to the optimal value of the problem (48). However, since $(\mathbf{X}_{\rm opt}, \tau_{\rm opt})$ is a feasible point of the problem (53) and the value of the objective function at this feasible point is equal to the optimal value of the problem (48), that is, $\ln(\mathrm{tr}\{\mathbf{A}_1 \cdot \mathbf{X}_{\rm opt}\}) + \ln\{\tau_{\rm opt}\} - \ln\{\beta_{\rm opt}\}$, we find that both optimization problems (53) and (48) have the same optimal value. To find an upper-bound for $\beta_{\rm opt}$, we make the feasible set of the problem (53) independent of $\beta_{\rm opt}$ which can be done by dropping the constraint $\mathrm{tr}\{\mathbf{B}_2 \cdot \mathbf{X}\} = \beta_{\rm opt}$ in the problem (53). Then the following problem is obtained:

$$\begin{aligned} \max_{\mathbf{X},\tau,\beta} \quad & \ln\left(\mathrm{tr}\{\mathbf{A}_1 \cdot \mathbf{X}\}\right) + \ln(\tau) - \ln(\beta_{\rm opt}) \\ \text{subject to} \quad & \mathrm{tr}\{\mathbf{B}_1 \cdot \mathbf{X}\} = 1,\ \mathrm{tr}\{\mathbf{A}_2 \cdot \mathbf{X}\} = \tau,\ \mathbf{X} \succeq \mathbf{0}. \end{aligned} \quad (54)$$

Noticing that the feasible set of the optimization problem (53) is a subset of the feasible set of the problem (54), it is straightforward to conclude that the optimal value of the problem (54) is equal to or greater than the optimal value of the problem (53) and, thus, also the optimal value of the problem (48). On the other hand, $p^\star$ is the value of the objective function of the problem (48) which corresponds to an arbitrary feasible point and as a result is less than or equal to the optimal value of the problem (48). Since the optimal value of the problem (54) is greater than or equal to the optimal value of the problem (48) and the optimal value of the problem (48) is greater than or equal to $p^\star$, the optimal value of the problem (54), denoted as $q^\star - \ln(\beta_{\rm opt})$, is greater than or equal to $p^\star$, and therefore, $\beta_{\rm opt} \leq e^{(q^\star - p^\star)}$ which completes the proof. ■

[3]Note that this optimization problem can be solved efficiently using numerical methods, for example, interior point methods.

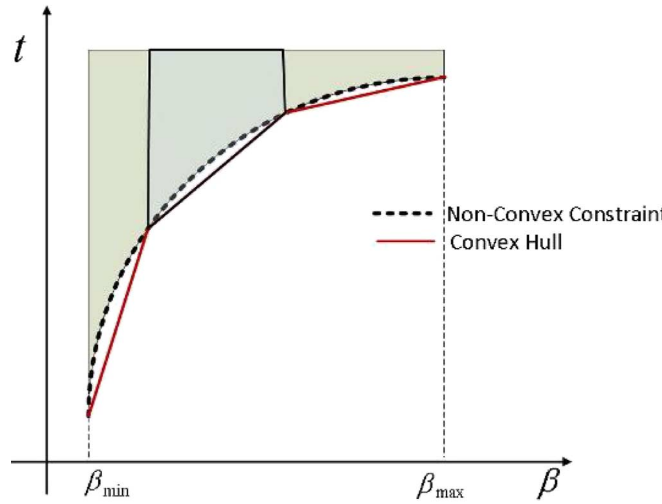

Fig. 3. Feasible region of the constraint $\ln(\beta) \leq t$ and the convex hull in each sub-division.

Note that as mentioned earlier, $p^\star$ is the objective value of the problem (48) that corresponds to an arbitrary feasible point. In order to obtain the tightest possible upper-bound for $\beta_{\rm opt}$, we choose $p^\star$ to be the largest possible value that we already know. A suitable choice for $p^\star$ is then the one which is obtained using the POTDC algorithm. In other words, we choose $p^\star$ as the corresponding objective value of the problem (48) at the optimal point which is resulted from the POTDC algorithm. Thus, we have obtained an upper-bound for $\beta_{\rm opt}$ which makes it further possible to develop an upper-bound for the optimal value of the optimization problem (48). To this end, we consider the only non-convex constraint of this problem, i.e., $\ln(\beta) \leq t$. Fig. 3 illustrates a subset of the feasible region corresponding to the non-convex constraint $\ln(\beta) \leq t$ where $\beta_{\min}$ equals $\lambda_{\min}\{\mathbf{B}_1^{-1} \cdot \mathbf{B}_2\}$, i.e., the smallest value of β for which the problem (49) is feasible, and $\beta_{\max}$ is the upper-bound for the optimal value $\beta_{\rm opt}$ given by Lemma 4 ($\beta_{\max}$ is equal to $\lambda_{\max}\{\mathbf{B}_1^{-1} \cdot \mathbf{B}_2\}$ if it is smaller than the upper-bound of $\beta_{\rm opt}$ obtained using Lemma 4). For obtaining an upper-bound for the optimal value of the problem (49), we divide the interval $[\beta_{\min}, \beta_{\max}]$ into N sections as it is shown in Fig. 3. Then, each section is considered separately. In each such section, the corresponding non-convex feasible set is replaced by its convex-hull and each corresponding optimization problem is solved separately as well. The maximum optimal value of such N convex optimization problems is then the upper-bound. Indeed, solving the resulting N convex optimization problems and choosing the maximum optimum value among them is equivalent to replacing the constraint $\ln(\beta) \leq t$ with the feasible set which is described by the region above the solid line in Fig. 3. The upper-bound becomes more and more accurate when the number of the intervals, i.e., N increases.

VI. Semi-Algebraic Solution via Generalized Eigenvectors (RAGES)

In this section we present RAGES as an alternative solution to the sum-rate maximization problem (35) which is based on generalized eigenvectors. It requires a different parameterization than the one used in the POTDC algorithm and in some cases it is more computationally efficient.

A. Basic Approach: Generalized Eigenvectors

To derive the link between (35) and generalized eigenvectors we start with the necessary condition for optimality that the gradient of (35) vanishes. Therefore, if we find all vectors $\mathbf{g}$ for which the gradient of the objective functions is zero, the global optimum must be one of them. By using the product rule and the chain rule of differentiation, the condition of zero gradient can be expressed as [27]

$$\frac{\tilde{P}_{R,2}}{\tilde{P}_{N,1}\cdot\tilde{P}_{N,2}}\cdot\mathbf{A}_1\cdot\mathbf{g}+\frac{\tilde{P}_{R,1}}{\tilde{P}_{N,1}\cdot\tilde{P}_{N,2}}\cdot\mathbf{A}_2\cdot\mathbf{g} = \frac{\tilde{P}_{R,1}\cdot\tilde{P}_{R,2}}{\tilde{P}_{N,1}^2\cdot\tilde{P}_{N,2}}\cdot\mathbf{B}_1\cdot\mathbf{g}+\frac{\tilde{P}_{R,1}\cdot\tilde{P}_{R,2}}{\tilde{P}_{N,1}\cdot\tilde{P}_{N,2}^2}\cdot\mathbf{B}_2\cdot\mathbf{g}. \quad (55)$$

Rearranging (55) we obtain

$$(\mathbf{A}_1+\rho_{\mathrm{sig}}\cdot\mathbf{A}_2)\cdot\mathbf{g}=\frac{\tilde{P}_{R,1}}{\tilde{P}_{N,1}}\cdot(\mathbf{B}_1+\rho_{\mathrm{noi}}\cdot\mathbf{B}_2)\cdot\mathbf{g} \quad (56)$$

where ρ_{sig} and ρ_{noi} are defined via

$$\rho_{\mathrm{sig}}=\frac{\tilde{P}_{R,1}}{\tilde{P}_{R,2}} \quad \text{and} \quad \rho_{\mathrm{noi}}=\frac{\tilde{P}_{N,1}}{\tilde{P}_{N,2}}. \quad (57)$$

It follows from (56) that the optimal $\mathbf{g}$ must be a generalized eigenvector of the pair of matrices $(\mathbf{A}_1+\rho_{\mathrm{sig}}\cdot\mathbf{A}_2)$ and $(\mathbf{B}_1+\rho_{\mathrm{noi}}\cdot\mathbf{B}_2)$. Moreover, the corresponding generalized eigenvalue is given by $\frac{\tilde{P}_{R,1}}{\tilde{P}_{N,1}}$ which is logarithmically proportional to the rate of the terminal one r_1. Unfortunately, the matrices $(\mathbf{A}_1+\rho_{\mathrm{sig}}\cdot\mathbf{A}_2)$ and $(\mathbf{B}_1+\rho_{\mathrm{noi}}\cdot\mathbf{B}_2)$ contain the parameters ρ_{sig} and ρ_{noi} which also depend on $\mathbf{g}$ and are hence not known in advance. Therefore, we still need to optimize over these two parameters. However, compared to the original problem of finding a complex-valued $M_R\times M_R$ matrix, optimizing over the two real-valued scalar parameters is significantly simpler. The following subsections show how to simplify this 2-D search even further.

B. Bounds on the Parameters ρ_{sig} and ρ_{noi}

Since both parameters ρ_{sig} and ρ_{noi} have a physical interpretation, the lower and upper-bounds for them can be easily found. Such bounds are useful since they limit the search space that has to be tested. For instance, ρ_{noi} can be expanded into

$$\rho_{\mathrm{noi}}=\frac{\tilde{P}_{N,1}}{\tilde{P}_{N,2}}=\frac{\mathbf{g}^H\cdot\mathbf{J}_1\cdot\mathbf{g}+P_{N,1}}{\mathbf{g}^H\cdot\mathbf{J}_2\cdot\mathbf{g}+P_{N,2}}. \quad (58)$$

The quadratic forms can be bounded by using the fact that for any Hermitian matrix $\mathbf{R}$ we have

$$\lambda_{\min}\{\mathbf{R}\}\cdot\gamma^2\le\mathbf{g}^H\cdot\mathbf{R}\cdot\mathbf{g}\le\lambda_{\max}\{\mathbf{R}\}\cdot\gamma^2 \quad (59)$$

where $\lambda_{\min}\{\mathbf{R}\}$ and $\lambda_{\max}\{\mathbf{R}\}$ are the smallest and the largest eigenvalues of $\mathbf{R}$, respectively, and $\gamma^2\triangleq\|\mathbf{g}\|_2^2$. It follows from (27) that

$$\lambda_{\min}\{\mathbf{J}_1\}=0 \text{ and } \lambda_{\max}\{\mathbf{J}_1\}=\lambda_{\max}\{\mathbf{R}_{N,R}\}\cdot\left(\alpha_i^{(b)}\right)^2 \quad (60)$$

where $\alpha_i^{(b)}$ is a short hand notation for $\|\mathbf{h}_i^{(b)}\|_2$. Furthermore, in general the following inequality holds

$$\lambda_{\max}\{\mathbf{R}_{N,R}\}\le P_{N,R}\cdot M_R \quad (61)$$

which for the case of white noise at the relay boils down to the following tighter condition $\lambda_{\max}\{\mathbf{R}_{N,R}\}=P_{N,R}$.

The relations (60) and (61) can be used to bound (58). Specifically, an upper-bound for ρ_{noi} can be found by upper-bounding the enumerator and lower-bounding the denominator, while the lower-bound can be found by lower-bounding the enumerator and upper-bounding the denominator. This yields

$$\rho_{\mathrm{noi}}\le\frac{P_{N,R}}{P_{N,2}}\cdot M_R\cdot\left(\alpha_1^{(b)}\right)^2\cdot\gamma^2+\frac{P_{N,1}}{P_{N,2}} \quad (62)$$

$$\rho_{\mathrm{noi}}\ge\left(\frac{P_{N,R}}{P_{N,1}}\cdot M_R\cdot\left(\alpha_2^{(b)}\right)^2\cdot\gamma^2+\frac{P_{N,2}}{P_{N,1}}\right)^{-1} \quad (63)$$

where γ^2 and M_R can be dropped if the noise at the relay is white. However, an upper-bound for γ^2 is till needed. Due to the relay power constraint we have $\mathbf{g}^H\cdot\mathbf{Q}\cdot\mathbf{g}=P_{T,R}$. Using the latter condition, the following bound can be derived $\gamma^2\le\frac{P_{T,R}}{\lambda_{\min}\{\mathbf{Q}\}}$. However, it is easy to check that this bound is very loose since for white noise at the relay we have $\lambda_{\min}\{\mathbf{Q}\}=P_{N,R}$ and for arbitrary relay noise covariance matrices no lower-bound exists (the infimum over $\lambda_{\min}$ is zero). This bound is so loose because it is extremely pessimistic: it measures the norm of $\mathbf{g}$ in the case when only noise is amplified and no power is put on the eigenvalues related to the signals of interest. However, such a case is practically irrelevant since it corresponds to a sum-rate equal to zero. Therefore, we propose to replace γ^2 in (62) and (63) by[4]

$$\gamma^2=\frac{P_{T,R}}{\lambda_2(\mathbf{R}_R)} \quad (64)$$

where $\lambda_2(\mathbf{R}_R)$ is the second largest eigenvalue of the matrix $\mathbf{R}_R$.

In a similar manner, ρ_{sig} can be bounded. In this case, the enumerator and the denominator have the additional terms $P_{T,2}\cdot\mathbf{g}^H\cdot\mathbf{K}_{2,1}\cdot\mathbf{g}$ and $P_{T,1}\cdot\mathbf{g}^H\cdot\mathbf{K}_{1,2}\cdot\mathbf{g}$, respectively. A pessimistic (loose) bound is obtained by bounding these two terms independently, i.e., $0\le\mathbf{g}^H\cdot\mathbf{K}_{2,1}\cdot\mathbf{g}\le\gamma^2\cdot(\alpha_2^{(f)})^2\cdot(\alpha_1^{(b)})^2$ and $0\le\mathbf{g}^H\cdot\mathbf{K}_{1,2}\cdot\mathbf{g}\le\gamma^2\cdot(\alpha_1^{(f)})^2\cdot(\alpha_2^{(b)})^2$. This yields

$$\rho_{\mathrm{sig}}\le\frac{P_{T,2}}{P_{N,2}}\cdot\left(\alpha_2^{(f)}\right)^2\cdot\left(\alpha_1^{(b)}\right)^2\cdot\gamma^2+\frac{P_{N,R}}{P_{N,2}}\cdot M_R\cdot\left(\alpha_1^{(b)}\right)^2\cdot\gamma^2+\frac{P_{N,1}}{P_{N,2}} \quad (65)$$

$$\rho_{\mathrm{sig}}\ge\left(\frac{P_{T,1}}{P_{N,1}}\cdot\left(\alpha_1^{(f)}\right)^2\cdot\left(\alpha_2^{(b)}\right)^2\cdot\gamma^2+\frac{P_{N,R}}{P_{N,1}}\cdot M_R\cdot\left(\alpha_2^{(b)}\right)^2\cdot\gamma^2+\frac{P_{N,2}}{P_{N,1}}\right)^{-1}. \quad (66)$$

Again, these bounds are pessimistic since they assume that there exists an optimal relay strategy for which $P_{R,1}=P_{T,2}\cdot(\alpha_2^{(f)})^2\cdot(\alpha_1^{(b)})^2\cdot\gamma^2$ but $P_{R,2}=0$, i.e., the rate of the second terminal is equal to zero. However, it is typically sum-rate optimal to have significantly more balanced rates between the two users. In fact, for the "symmetric" scenario when $P_{T,1}=P_{T,2}$, $\mathbf{h}_i^{(f)}=\mathbf{h}_i^{(b)}$, $i=1,2$, and $\alpha_1^{(f)}=\alpha_2^{(f)}$, we always have $P_{R,1}=P_{R,2}$ at optimality. Therefore, these bounds can be further tightened if a priori knowledge about the specific scenario is available.

C. Efficient 2-D and 1-D Search

Once the search space for ρ_{sig} and ρ_{noi} has been fixed, we can find the maximum via optimization over these two param-

[4]We have observed in all our simulations that this value poses indeed an upper-bound on the norm of the optimal solution $\mathbf{g}_{\mathrm{opt}}$.

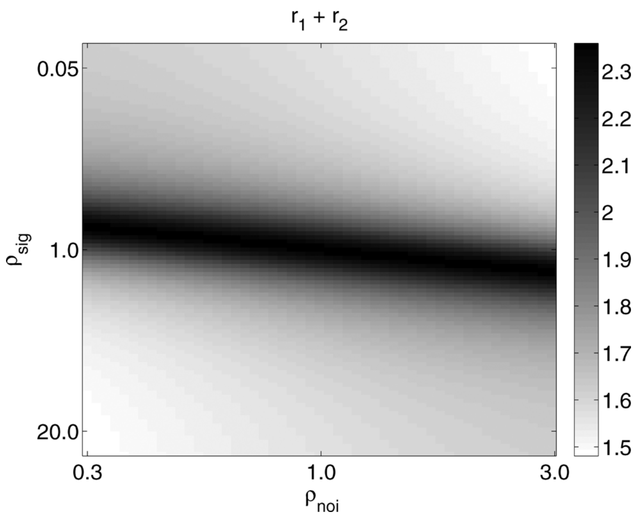

Fig. 4. Sum-rate $r_1 + r_2$ versus $\rho_{\rm sig}$ and $\rho_{\rm noi}$ for $M_R = 6$, $P_{T,1} = P_{T,2} = P_{T,R} = 1$, $P_{N,1} = P_{N,2} = P_{N,R} = 0.1$.

eters using a 2-D search. In general, a 2-D exhaustive search can be computationally demanding, i.e., the computational complexity will be higher than that of the POTDC algorithm. However, as we show in the sequel, for the problem at hand, this search can be implemented efficiently. These efficient implementations are, however, heuristic since they rely on properties of the objective functions that are apparent by visual inspection. As we will see in simulations, the performance of the resulting RAGES algorithm coincides with that of the rigorous POTDC algorithm.

Fig. 4 demonstrates a typical example of the sum-rate $r_1 + r_2$ as a function of $\rho_{\rm sig}$ and $\rho_{\rm noi}$. For this example we have chosen $M_R = 6$, $P_{T,1} = P_{T,2} = P_{T,R} = 1$, $P_{N,1} = P_{N,2} = P_{N,R} = 0.1$ and we have drawn the channel vectors from an uncorrelated Rayleigh fading distribution assuming reciprocity. By visual inspection, this sample objective function shows two interesting properties. First, it is a quasi-convex function with respect to the parameters $\rho_{\rm noi}$ and $\rho_{\rm sig}$ which allows for efficient (quasi-convex) optimization tools for finding its maximum. Albeit this property is only demonstrated for one example here, it has been always present in our numerical evaluations even when largely varying all system parameters. Second, for every value of $\rho_{\rm noi}$ the corresponding maximization over $\rho_{\rm sig}$ yields one maximal value which depends on $\rho_{\rm noi}$ only very weakly. This is illustrated by Fig. 5 which displays the relative change of the objective function $r_1 + r_2$ for different choices of $\rho_{\rm noi}$, each time optimizing it over $\rho_{\rm sig}$. The displayed values represent the relative decrease of the objective functions compared to the global optimum, i.e., for the worst choice of $\rho_{\rm noi}$, the achieved sum-rate is about $2 \cdot 10^{-5} = 0.002\%$ lower than for the best choice of $\rho_{\rm noi}$. Consequently, the 2-D search over $\rho_{\rm sig}$ and $\rho_{\rm noi}$ can be replaced essentially without any loss by a 1-D search over $\rho_{\rm sig}$ only for one fixed value of $\rho_{\rm noi}$ (e.g., the geometric mean of the upper and the lower-bound).

In addition, instead of performing the search directly over the original objective function $r_1 + r_2$, we can find an even simpler objective functions by using the physical meaning of our two search parameters. To this end, let us introduce a new parameter $\hat{\rho}_{\rm sig}$ as a function of $\mathbf{g}$ as follows

$$\hat{\rho}_{\rm sig}(\mathbf{g}) = \frac{\mathbf{g}^H \cdot \mathbf{A}_1 \cdot \mathbf{g}}{\mathbf{g}^H \cdot \mathbf{A}_2 \cdot \mathbf{g}}. \tag{67}$$

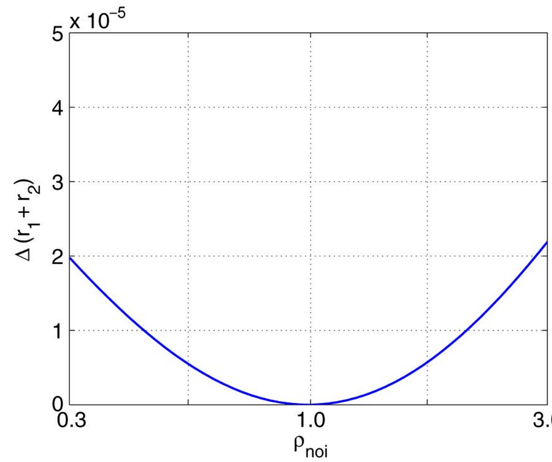

Fig. 5. Relative change in sum-rate $r_1 + r_2$ versus $\rho_{\rm noi}$: optimizing over $\rho_{\rm sig}$ for every choice of $\rho_{\rm noi}$. The same data set as in Fig. 4 is used.

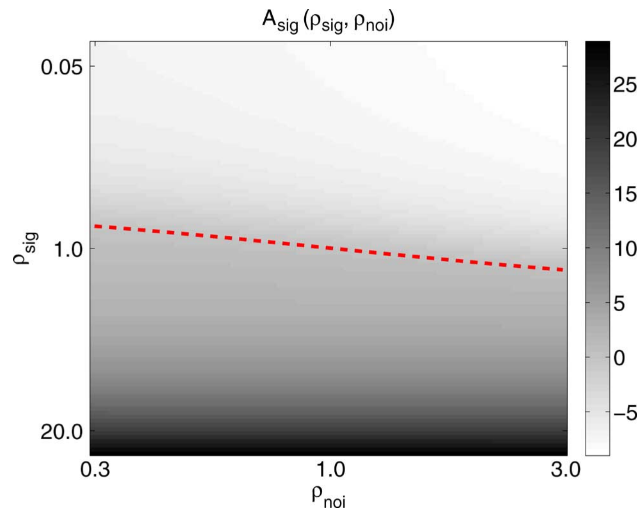

Fig. 6. Objective function $A_{\rm sig}(\rho_{\rm sig}, \rho_{\rm noi})$. The same data set as in Fig. 4 is used. The dashed line indicates the points where $A_{\rm sig}(\rho_{\rm sig}, \rho_{\rm noi}) = 0$.

Here $\mathbf{g}$ is the relay weight vector at the current search point $(\rho_{\rm sig}, \rho_{\rm noi})$. Then we know that in the optimal point $\mathbf{g}_{\rm opt}$, we have $\hat{\rho}_{\rm sig}(\mathbf{g}_{\rm opt}) = \rho_{\rm sig}$. This can be used to construct a new objective function

$$A_{\rm sig}(\rho_{\rm sig}, \rho_{\rm noi}) = \hat{\rho}_{\rm sig}(\mathbf{g}) - \rho_{\rm sig} \tag{68}$$

where $\mathbf{g}$ is the dominant eigenvector of (56) for the current search point $(\hat{\rho}_{\rm sig}, \hat{\rho}_{\rm noi})$.

Using the same data set as before, we display the corresponding shape of $A_{\rm sig}(\rho_{\rm sig}, \rho_{\rm noi})$ in Fig. 6. The dashed line indicates the set of points where $A_{\rm sig}(\rho_{\rm sig}, \rho_{\rm noi}) = 0$. It can be observed that for every value of $\rho_{\rm noi}$, $A_{\rm sig}(\rho_{\rm sig}, \rho_{\rm noi})$ is a monotonic function in $\rho_{\rm sig}$. Therefore, the bisection method can be used to find a zero crossing in $\rho_{\rm sig}$ which coincides with the sum-rate-optimal $\rho_{\rm sig}$ for a given $\rho_{\rm noi}$.

D. Summary

In summary, it can be concluded that the RAGES approach simplifies the optimization over a complex-valued $M_R \times M_R$ matrix into the optimization over two real-valued parameters which both have a physical interpretation. Even more, the 2-D search can be simplified into a 1-D search by fixing one of the parameters. The loss incurred to this step is typically small. In the example provided above, it is only 0.002%, but even varying the system parameters largely and using many random trials we never found a relative difference higher than a few percents.

Moreover, the 1-D search can be efficiently implemented by exploiting the quasi-convexity of r_1+r_2 or the monotonicity of $A_{\rm sig}$ (e.g., using the bisection method). Again, these properties are only demonstrated by examples but we have observed in all our simulations that the resulting algorithm yields a sum-rate very close to the optimum found by the exact solution and its upper-bound described before. This comparison is further illustrated in next section via numerical simulations.

Comparing the POTDC and RAGES approaches, it is noticeable that the POTDC approach is rigorous, while the RAGES approach is at some points heuristic. As it has been mentioned, the computational complexity of solving the proposed sum-rate maximization problem for two-way AF MIMO relaying using the POTDC algorithm is the same as the complexity of solving the SDP problem (49) product the number of iterations, where the number of iterations is significantly smaller than the dimension of the problem. For example, for a relay with 4 antennas (the corresponding dimension of the problem is 16), the number of iterations is only 4–6. Alternatively, the complexity of solving the same problem using the RAGES approach is equivalent to the complexity of finding the dominant generalized eigenvector, which has to be performed for each combination of the parameters $\rho_{\rm sig}$ and $\rho_{\rm noi}$. Since the search over one parameter only is sufficient, the complexity of the RAGES approach is typically lower than that of the POTDC algorithm for the 1-D RAGES.

VII. Simulation Results

In this section, we evaluate the performance of the new proposed methods via numerical simulations. Consider a communication system consisting of two single-antenna terminals and an AF MIMO relay with M_R antenna elements. The communication between the terminals is bidirectional, i.e., it is performed based on the two-way relaying scheme. It is assumed that perfect channel knowledge is available at the terminals and at the relay, while the terminals use only effective channels (scalars), but the relay needs full channel vectors. The relay estimates the corresponding channel coefficients between the relay antenna elements and the terminals based on the pilots which are transmitted from the terminals. Then based on these channel vectors, the relay computes the relay amplification matrix $\mathbf{G}$ and then uses it for forwarding the pilot signals to the terminals. After receiving the forwarded pilot signals from the relay via the effective channels, the terminals can estimate the effective channels using a suitable pilot-based channel estimation scheme, e.g., the LS.

The noise powers of the relay antenna elements and the single-antenna terminals $P_{N,R}$, $P_{N,1}$, and $P_{N,2}$ are assumed to be equal to σ^2. Uncorrelated Rayleigh fading channels are considered and it is assumed that reciprocity holds, i.e., $\mathbf{h}_i^{(f)}=\mathbf{h}_i^{(b)}$ for $i=1,2$. The relay is assumed to be located over a line of unit length which connects the terminals to each other and the variances of the channel coefficients between terminal i and the relay antenna elements are all assumed to be proportional to $\frac{1}{d_i^{\nu}}$, where $d_i \in (0,1)$ is the distance between the relay and the terminal i and ν is the path-loss exponent which is assumed to be equal to 3 throughout the simulations.[5] For obtaining each point, 100 independent simulation runs are used unless otherwise is specified.

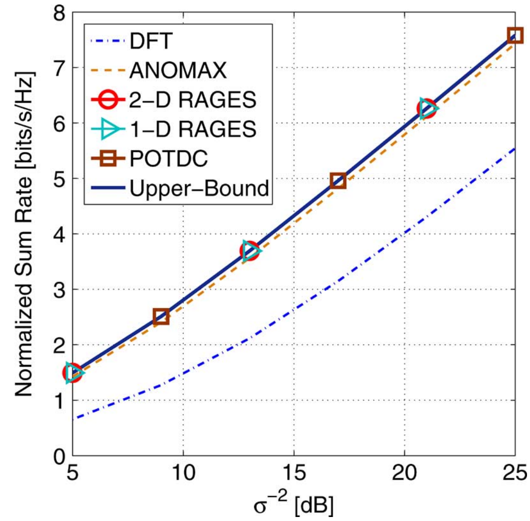

Fig. 7. Sum-rate r_1+r_2 versus σ^{-2} for $M_R=3$ antennas. Example 1: the case of symmetric channel conditions.

In order to design the relay amplification matrix $\mathbf{G}$, five different methods are considered including the proposed POTDC, 2-D RAGES and 1-D RAGES algorithms, the algebraic norm-maximizing (ANOMAX) transmit strategy of [40] and the discrete Fourier transform (DFT) method that chooses the relay precoding matrix as a scaled DFT matrix. Note that the ANOMAX strategy provides a closed-form solution for the problem. Also note that for the DFT method no channel knowledge is needed. Thus, the DFT method serves as a benchmark for evaluating the gain achieved by using channel knowledge. The upper-bound is also shown in simulation examples 1 and 2. For obtaining the upper-bound, the interval $[\beta_{\min},\beta_{\max}]$ is divided in 30 segments. In addition, the proposed techniques are compared to the SNR-balancing technique of [41] for the scenario when multiple single-antenna relay nodes are used and the method of [41] is applicable.

A. Example 1: Symmetric Channel Conditions

In our first example, we consider the case when the channels between the relay antenna elements and both terminals have the same channel quality. More specifically, it is assumed that the relay is located in the middle of the connecting line between the terminals and the transmit powers $P_{T,1}$ and $P_{T,2}$ and the total transmit power of the MIMO relay $P_{T,R}$ are all assumed to be equal to 1.

Fig. 7 shows the sum-rate achieved by different aforementioned methods versus σ^{-2} for the case of $M_R=3$. It can be seen in this figure that the performance of the proposed methods coincides with the upper-bound. Thus, the methods perform optimally in terms of providing the maximum sum-rate. The ANOMAX technique performs close to the optimal, while the DFT method gives a significantly lower sum-rate.

[5]It is experimentally found that typically $2\leq\nu\leq 6$ (see [42, p. 46–48] and references therein). However, ν can be smaller than 2 when we have a waveguide effect, i.e., indoors in corridors or in urban scenarios with narrow street canyons.

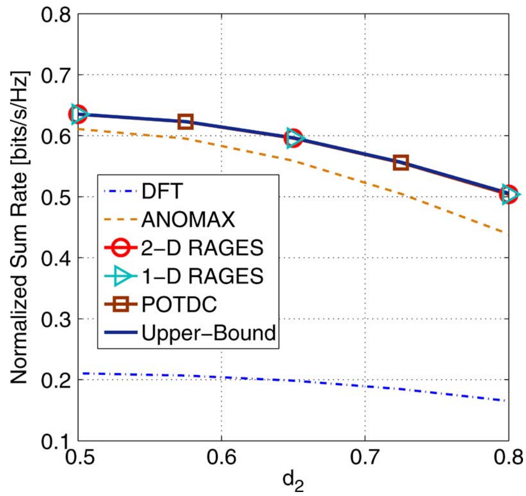

Fig. 8. Sum-rate $r_1 + r_2$ versus the distance between the relay and the second terminal d_2 for $M_R = 3$ antennas. Example 2: the case of asymmetric channel conditions.

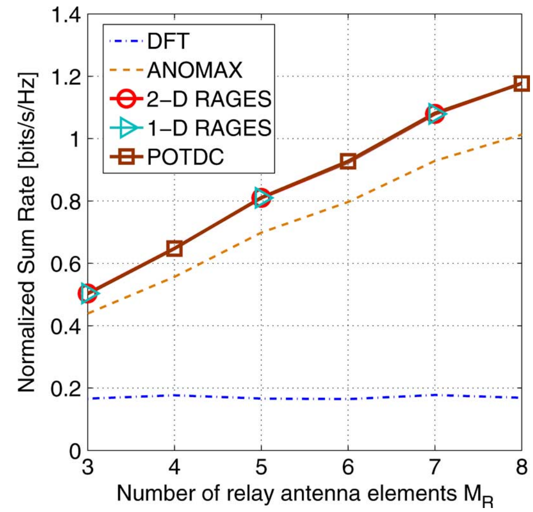

Fig. 9. Sum-rate $r_1 + r_2$ versus the number of the relay antenna elements M_R. Example 3: the case of asymmetric channel conditions.

B. Example 2: Asymmetric Channel Conditions

In the second example, we consider the case when the channels between the relay antenna elements and the second terminal have better channel quality than the channels between the relay antenna elements and the first terminal. Thus, we evaluate the effect of the relay location on the achievable sum-rate. Particularly, we consider the case when the distance between the relay and the second terminal d_2 is less than or equal to the distance between the relay and the first terminal d_1. The total transmit power of the terminals, i.e., $P_{T,1}$ and $P_{T,2}$ and the total transmit power of the MIMO relay $P_{T,R}$ all are assumed to be equal to 1 and the noise powers in the relay antenna elements and the terminals all are assumed to be equal to 1.

Fig. 8 shows the sum-rate achieved in this scenario by different methods tested versus the distance between the relay and the second terminal d_2, for the case of $M_R = 3$. It can be seen in this figure that the proposed methods perform optimally, while the performance (sum-rate) of ANOMAX is slightly worse.

As mentioned earlier, it is guaranteed that the POTDC algorithm converges to at least a KKT solution of the sum-rate maximization problem. However, our extensive simulation results confirm that the POTDC algorithm converges to the global maximum of the problem in all simulation runs. It is approved by the fact that the performance of the POTDC algorithm coincides with the upper-bound. Moreover, the 2-D RAGES and 1-D RAGES are, in fact, globally optimal, too. The ANOMAX and DFT methods, however, do not achieve the maximum sum-rate. The loss in sum-rate related to the DFT method is quite significant while the loss in sum-rate related to the ANOMAX method grows from small in the case of symmetric channel conditions to significant in the case of asymmetric channel conditions. Although ANOMAX enjoys a closed-form solution and it is even applicable in the case when terminals have multiple antennas, it is not a good substitute for the proposed methods because of the significant gap in performance in the asymmetric case.

C. Example 3: Effect of the Number of Relay Antenna Elements

In this example, we consider the effect of the number of relay antenna elements M_R on the achievable sum-rate for the aforementioned methods. The powers assigned to the first and second terminals as well as to the relay are all equal to 1. The relay is assumed to be located at the distance of 1/5 from the second user. Moreover, the noise powers at the terminals and at the relay antenna elements are all assumed to be equal to 1. For obtaining each point in this simulation example, 200 independent runs are used.

Fig. 9 depicts the sum-rates achieved by different methods versus the number of relay antenna elements M_R. As it is expected, by increasing M_R (thus, increasing the number of degrees of freedom), the sum-rate increases. For the DFT method, the sum-rate does not increase with the increase of M_R because of the lack of channel knowledge for this method. The proposed methods achieve higher sum-rate compared to ANOMAX.

D. Example 4: Performance Comparison for the Scenario of Two-Way Relaying via Multiple Single-Antenna Relays

In our last example, we compare the proposed methods with the SNR balancing-based approach of [41]. The method of [41] is developed for two-way relaying systems which consist of two single-antenna terminals and multiple single-antenna relay nodes. Subject to the constraint on the total transmit power of the relay nodes and the terminals, the method of [41] designs a beamforming vector for the relay nodes and the transmit powers of the terminals to maximize the minimum received SNR at the terminals. In order to make a fair comparison, we consider a diagonal structure for the relay amplification matrix $\mathbf{G}$ that corresponds to the case of [41] when multiple single-antenna nodes are used for relaying. It is worth mentioning that for imposing such a diagonal structure for the relay amplification matrix $\mathbf{G}$ in POTDC and RAGES, the vector $\mathbf{g}_{M_R^2 \times 1} = \mathrm{vec}\{\mathbf{G}\}$ is replaced with $\mathbf{g}_{M_R \times 1} = \mathrm{diag}\{\mathbf{G}\}$ and the matrices $\mathbf{A}_i$ and $\mathbf{B}_i$, $i = 1, 2$ are replaced with new square matrices $\tilde{\mathbf{A}}_i$ and $\tilde{\mathbf{B}}_i$, $i = 1, 2$ of size $M_R \times M_R$ such that $[\tilde{\mathbf{A}}_i]_{m,n} = [\mathbf{A}_i]_{(m-1)\cdot M_R+m,(n-1)\cdot M_R+n}$ and $[\tilde{\mathbf{B}}_i]_{m,n} = [\mathbf{B}_i]_{(m-1)\cdot M_R+m,(n-1)\cdot M_R+n}$, $m, n = 1, \cdots, M_R$. Moreover, for the proposed methods, we assume fixed transmit powers at the terminals and fixed total transmit power at the relay nodes that are all equal to 1, while for the method of [41], the total transmit power at the relay nodes and the terminals is assumed to be equal to 3. Thus, the overall consumed power by the proposed methods and the method of [41] is the same,

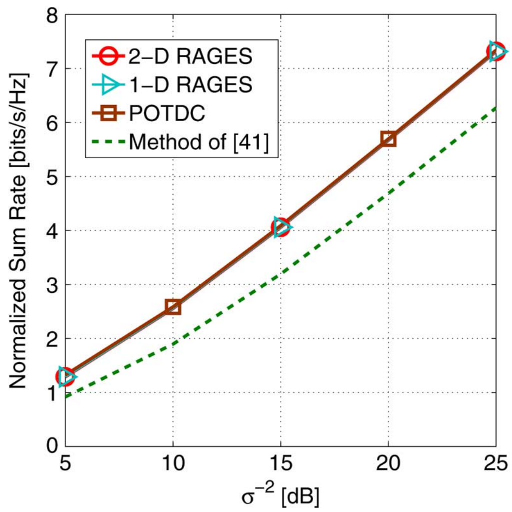

Fig. 10. Sum-rate $r_1 + r_2$ versus σ^{-2} for $M_R = 3$ antennas. Example 4: the case of symmetric channel conditions and distributed single antenna relays.

however, compared to [41], which also optimizes the power usage of the terminals, the transmit powers of the terminals in the proposed methods are fixed. The relay is assumed to lie in the middle in between the terminals. Fig. 10 shows the corresponding performance of the methods tested. From this figure it can be seen that the proposed methods demonstrate a better performance compared to the method of [41] as it may be expected even though they use a fixed transmit power for the terminals.

VIII. Conclusions and Discussions

We have shown that the sum-rate maximization problem in two-way AF MIMO relaying belongs to the class of DC programming problems. Although DC programming problems can be solved by the branch-and-bound method, this method does not have any polynomial time guarantees for its worst-case complexity. In this paper, we have developed the so-called POTDC algorithm for finding a KKT solution of the aforementioned problem with polynomial time worst-case complexity. There is, however, a great evidence that the global optimal solution is also achieved. The other method called RAGES shows the same performance as the POTDC algorithm. The POTDC algorithm is based on a specific parameterization of the objective function, that is, the product of quadratic fractional functions; and then application of SDP relaxation, linearization and iterative search over a single parameter. Its design is rigorous and is based on the recent advances in convex optimization. The RAGES algorithm is based on a different parameterization of the objective function and the generalized eigenvectors method. It may enjoy a lower computational complexity that makes it a valid alternative to the POTDC algorithm especially if 1-D search is used. The upper-bound for the solution of the problem is developed and it is demonstrated by simulations that both proposed methods achieve the upper-bound and are, thus, globally optimal.

The proposed POTDC algorithm represents a general optimization technique applicable for solving a wide class of DC programming problems. Essentially, the optimization problems consisting of the maximization/minimization of a product of quadratic fractional functions can be handled using the proposed POTDC approach. Moreover, the POTDC algorithm can be used for solving optimization problems with constraints represented as a difference of two quadratic forms. More general problems can be also addressed by POTDC approach by applying some relatively straightforward modifications. For example, if the problem is to optimize a product of more than two quadratic fractional functions under a single quadratic (power) constraint, the number of constraints in the corresponding DC programming problem will be more than three. In this case, randomization procedures have to be adopted to recover a rank-one solution from the solution of the relaxed problem. Such solutions obviously may not be exact.

Other signal processing problems that can be addressed using the proposed POTDC approach are the general-rank robust adaptive beamformer with a positive semi-definite constraint, the dynamic spectrum management for digital subscriber lines, the problems of finding the weighted sum-rate point, the proportional-fairness operating point, and the max-min optimal point for the MISO interference channel, the problem of robust beamforming design for AF relay networks with multiple relay nodes, the proportional fairness as well as the max-min rate problem for two-way relaying, and so on. The extensions of the POTDC approach to some of the aforementioned problem is an issue of future research.

Appendix A

The matrix $\mathbf{R}_R$ is rank deficient only if the noise is spatially correlated. Let the rank of $\mathbf{R}_R$ be denoted as N, $N < M_R$. Since the matrix $\mathbf{Q} \triangleq \mathbf{R}_R \otimes \mathbf{I}_{M_R}$, its rank is equal to the rank of the matrix $\mathbf{R}_R$ multiplied by the rank of the matrix $\mathbf{I}_{M_R}$, i.e., $\mathrm{rank}\{\mathbf{Q}\} = M_R \cdot N$. Then if $\mathbf{R}_R$ is rank deficient, $\mathbf{Q}$ is also rank deficient.

For convenience, we restate the sum-rate maximization problem (29) as

$$\mathbf{g}_{\mathrm{opt}} = \underset{\mathbf{g}|\mathbf{g}^H\cdot\mathbf{Q}\cdot\mathbf{g}\le P_{T,R}}{\arg\max} \; s(\mathbf{g}) \tag{69}$$

where

$$s(\mathbf{g}) \triangleq \frac{1}{2}\log_2\left(\left(1+\frac{\mathbf{g}^H\cdot\mathbf{K}_{2,1}\cdot\mathbf{g}\cdot P_{T,2}}{\mathbf{g}^H\cdot\mathbf{J}_1\cdot\mathbf{g}+P_{N,1}}\right)\cdot\left(1+\frac{\mathbf{g}^H\cdot\mathbf{K}_{1,2}\cdot\mathbf{g}\cdot P_{T,1}}{\mathbf{g}^H\cdot\mathbf{J}_2\cdot\mathbf{g}+P_{N,2}}\right)\right). \tag{70}$$

If $\mathbf{Q}$ is rank deficient, then for every $\mathbf{g} \in Null(\mathbf{Q})$ where $Null(\mathbf{Q})$ is the null space of $\mathbf{Q}$, the total transmit power from the relay is zero, i.e., $\mathbf{g}^H \cdot \mathbf{Q} \cdot \mathbf{g} = 0$. Moreover, the corresponding sum-rate $s(\mathbf{g})$ for any vector $\mathbf{g} \in Null(\mathbf{Q})$ is equal to zero. To show this, let us consider $\mathbf{g}_0 = \mathrm{vec}\{\mathbf{G}_0\} \in Null(\mathbf{Q})$ that straightforwardly implies that

$$\mathbf{g}_0^H\cdot\mathbf{Q}\cdot\mathbf{g}_0 = \mathrm{tr}\left\{\mathbf{G_0}\cdot\mathbf{R}_R\cdot\mathbf{G_0}^H\right\} = 0. \tag{71}$$

Note that the first equality in (71) follows from (3). Substituting (4) in (71), we obtain

$$\mathrm{tr}\left\{\left(\mathbf{G}_0\cdot\mathbf{h}_1^{(f)}\right)\cdot\left(\mathbf{G}_0\cdot\mathbf{h}_1^{(f)}\right)^H\cdot P_{T,1}+\left(\mathbf{G}_0\cdot\mathbf{h}_2^{(f)}\right)\cdot\left(\mathbf{G}_0\cdot\mathbf{h}_2^{(f)}\right)^H\cdot P_{T,2}+\left(\mathbf{G}_0\cdot\mathbf{R}_{N,R}^{\frac{1}{2}}\right)\cdot\left(\mathbf{G}_0\cdot\mathbf{R}_{N,R}^{\frac{1}{2}}\right)^H\right\}=0. \tag{72}$$

Since the matrices $(\mathbf{G}_0 \cdot \mathbf{h}_1^{(f)}) \cdot (\mathbf{G}_0 \cdot \mathbf{h}_1^{(f)})^H$, $(\mathbf{G}_0 \cdot \mathbf{h}_2^{(f)}) \cdot (\mathbf{G}_0 \cdot \mathbf{h}_2^{(f)})^H$, and $\left(\mathbf{G}_0\cdot\mathbf{R}_{N,R}^{\frac{1}{2}}\right)\cdot\left(\mathbf{G}_0\cdot\mathbf{R}_{N,R}^{\frac{1}{2}}\right)^H$ in (72) are all positive semi-definite and the powers $P_{T,1}$ and $P_{T,2}$ are strictly positive, (72) is satisfied only if $\mathbf{G}_0 \cdot \mathbf{h}_1^{(f)} = \mathbf{0}$, $\mathbf{G}_0 \cdot \mathbf{h}_2^{(f)} = \mathbf{0}$, and $\mathbf{G}_0 \cdot \mathbf{R}_{N,R}^{\frac{1}{2}} = \mathbf{0}$. Therefore, the following equations are in order

$$\mathbf{g_0}^H \cdot \mathbf{K}_{2,1} \cdot \mathbf{g_0} = \left| \left(\mathbf{h}_1^{(b)}\right)^T \cdot \mathbf{G}_0 \cdot \mathbf{h}_2^{(f)} \right|^2 = 0 \tag{73}$$

$$\mathbf{g_0}^H \cdot \mathbf{K}_{1,2} \cdot \mathbf{g_0} = \left| \left(\mathbf{h}_2^{(b)}\right)^T \cdot \mathbf{G}_0 \cdot \mathbf{h}_1^{(f)} \right|^2 = 0 \tag{74}$$

$$\mathbf{g}_0^H \cdot \mathbf{J}_i \cdot \mathbf{g}_0 = \left(\mathbf{h}_i^{(b)}\right)^T \cdot \mathbf{G}_0 \cdot \mathbf{R}_{N,R} \cdot \mathbf{G}_0^H \left(\mathbf{h}_i^{(b)}\right)^* = 0, \quad i = 1, 2. \tag{75}$$

Substituting (73)–(75) in (70), we conclude that the sum-rate is indeed zero for any $\mathbf{g}_0 \in Null(\mathbf{Q})$. Furthermore, in a similar way, it can be shown that, for any $\mathbf{g} = \mathbf{g}_0 + \mathbf{g}_1$ such that $\mathbf{g}_0 \in Null(\mathbf{Q})$, $s(\mathbf{g}) = s(\mathbf{g}_1)$, and $(\mathbf{g}_0 + \mathbf{g}_1)^H \cdot \mathbf{Q} \cdot (\mathbf{g}_0 + \mathbf{g}_1) = \mathbf{g}_1^H \cdot \mathbf{Q} \cdot \mathbf{g}_1$, which means that $\mathbf{g}_0$ does not have any contribution in the transmit power as well as the sum-rate.

Using the above observations, it is easy to see that if $\mathbf{R}_R$ is rank deficient, the only thing required to do is reformulating the rate function (70) in the following manner. Denote the eigenvalue decomposition of the matrix $\mathbf{Q}$ as $\mathbf{Q} = \mathbf{U} \cdot \mathbf{\Lambda} \cdot \mathbf{U}^H$ where $\mathbf{U}_{M_R^2 \times M_R^2}$ and $\mathbf{\Lambda}_{M_R^2 \times M_R^2}$ are unitary and diagonal matrices of eigenvectors and eigenvalues, respectively. The ith eigenvector and the ith eigenvalue of $\mathbf{Q}$ denoted as $\mathbf{u}_i$ and λ_i, respectively, constitutes the column i of $\mathbf{U}$ and ith diagonal element of $\mathbf{\Lambda}$. It is assumed without loss of generality that the eigenvalues λ_i, $i = 1, \cdots, M_R^2$ are ordered in the descending order, i.e., $\lambda_i \geq \lambda_{i+1}$, $i = 1, \cdots, M_R^2 - 1$. Since in the case of rank deficient $\mathbf{R}_R$, the rank of $\mathbf{Q}$ is equal to $M_R \cdot N$, the last $M_R \cdot (M_R - N)$ eigenvalues of $\mathbf{Q}$ are zero. By splitting $\mathbf{U}$ to the $M_R^2 \times (M_R \cdot N)$ matrix $\mathbf{U}_1$ and the $M_R^2 \times M_R \cdot (M_R - N)$ matrix $\mathbf{U}_2$ as $\mathbf{U} = [\mathbf{U}_1\ \mathbf{U}_2]$, the matrix $\mathbf{Q}$ can be decomposed as $\mathbf{Q} = \mathbf{U}_1 \cdot \mathbf{\Lambda}_1 \cdot \mathbf{U}_1^H + \mathbf{U}_2 \cdot \mathbf{\Lambda}_2 \cdot \mathbf{U}_2^H$ where the $M_R \cdot N \times M_R \cdot N$ diagonal matrix $\mathbf{\Lambda}_1$ contains the $M_R \cdot N$ dominant eigenvalues, while the other $M_R \cdot (M_R - N) \times M_R \cdot (M_R - N)$ diagonal matrix $\mathbf{\Lambda}_2$ contains the $M_R \cdot (M_R - N)$ zero eigenvalues. Since $\mathbf{U}$ is unitary, any arbitrary vector $\mathbf{g}$ can be expressed as $\mathbf{g} = \mathbf{U}_1 \cdot \boldsymbol{\alpha} + \mathbf{U}_2 \cdot \boldsymbol{\beta}$ where $\boldsymbol{\alpha}_{M_R \times N}$ and $\boldsymbol{\beta}_{M_R \times (M_R - N)}$ are the coefficient vectors. It is easy to verify that the component $\mathbf{U}_2 \cdot \boldsymbol{\beta}$ lies inside $Null(\mathbf{Q})$ and as a result $\mathbf{s}(\mathbf{U}_1 \cdot \boldsymbol{\alpha}_1 + \mathbf{U}_2 \cdot \boldsymbol{\alpha}_2) = s(\mathbf{U}_1 \cdot \boldsymbol{\alpha}_1)$ and $(\mathbf{U}_1 \cdot \boldsymbol{\alpha}_1 + \mathbf{U}_2 \cdot \boldsymbol{\alpha}_2)^H \cdot \mathbf{Q} \cdot (\mathbf{U}_1 \cdot \boldsymbol{\alpha}_1 + \mathbf{U}_2 \cdot \boldsymbol{\alpha}_2) = (\mathbf{U}_1 \cdot \boldsymbol{\alpha}_1)^H \cdot \mathbf{Q} \cdot (\mathbf{U}_1 \cdot \boldsymbol{\alpha}_1)$. Therefore, $\boldsymbol{\beta}$ can be any arbitrary vector and we only need to find the optimal $\boldsymbol{\alpha}$. Substituting $\mathbf{U}_1 \cdot \boldsymbol{\alpha}$ in (70), the sum-rate can be expressed only as a function of $\boldsymbol{\alpha}$ as follows

$$s(\mathbf{U}_1 \cdot \boldsymbol{\alpha}) = \frac{1}{2} \log_2 \left(\left(1 + \frac{\boldsymbol{\alpha}^H \cdot (\mathbf{U}_1^H \cdot \mathbf{K}_{2,1} \cdot \mathbf{U}_1) \cdot \boldsymbol{\alpha} \cdot P_{T,2}}{\boldsymbol{\alpha}^H \cdot (\mathbf{U}_1^H \cdot \mathbf{J}_1 \cdot \mathbf{U}_1) \cdot \boldsymbol{\alpha} + P_{N,1}}\right) \cdot \left(1 + \frac{\boldsymbol{\alpha}^H \cdot (\mathbf{U}_1^H \cdot \mathbf{K}_{1,2} \cdot \mathbf{U}_1) \cdot \boldsymbol{\alpha} \cdot P_{T,1}}{\boldsymbol{\alpha}^H \cdot (\mathbf{U}_1^H \cdot \mathbf{J}_2 \cdot \mathbf{U}_1) \cdot \boldsymbol{\alpha} + P_{N,2}}\right) \right). \tag{76}$$

Then the optimization problem (69) is equivalent to the maximization of (76) under the constraint

$$\boldsymbol{\alpha}^H \cdot \mathbf{\Lambda}_1 \cdot \boldsymbol{\alpha} \leq P_{T,R}. \tag{77}$$

Since the matrix $\mathbf{\Lambda}_1$ is full rank, the corresponding optimization problem can be solved by the methods that we develop in the paper.

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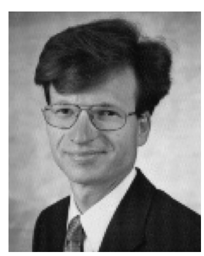

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