www.ietdl.org

Published in IET Signal Processing
Received on 20th April 2013
Revised on 16th September 2013
Accepted on 23rd September 2013
doi: 10.1049/iet-spr.2013.0166

ISSN 1751-9675

Robust adaptive beamforming algorithms using the constrained constant modulus criterion

Lukas Landau[1], *Rodrigo C. de Lamare*[2], *Martin Haardt*[1]

[1]*Communications Research Laboratory, Ilmenau University of Technology, Fachgebiet Nachrichtentechnik, D-98684 Ilmenau, Germany*

[2]*Communications Research Group, Department of Electronics, University of York, North Yorkshire, York Y010 5DD, UK*

E-mail: rcdl500@ohm.york.ac.uk

Abstract: The authors present a robust adaptive beamforming algorithm based on the worst-case (WC) criterion and the constrained constant modulus (CCM) approach, which exploits the constant modulus property of the desired signal. Similar to the existing worst-case beamformer with the minimum variance design, the problem can be reformulated as a second-order cone programme and solved with interior point methods. An analysis of the optimisation problem is carried out and conditions are obtained for enforcing its convexity and for adjusting its parameters. Furthermore, low-complexity robust adaptive beamforming algorithms based on the modified conjugate gradient and an alternating optimisation strategy are proposed. The proposed low-complexity algorithms can compute the existing WC constrained minimum variance and the proposed WC-CCM designs with a quadratic cost in the number of parameters. Simulations show that the proposed WC-CCM algorithm performs better than existing robust beamforming algorithms. Moreover, the numerical results also show that the performances of the proposed low-complexity algorithms are equivalent or better than that of existing robust algorithms, whereas the complexity is more than an order of magnitude lower.

1 Introduction

Beamforming has many applications in wireless communications, radar, sonar, medical imaging, radio astronomy and other areas. One of the most fundamental problems with adaptive beamforming algorithms is the occurrence of mismatches between the presumed and actual signal steering vector [1]. Practical circumstances like local scattering, imperfectly calibrated arrays and imprecisely known wave field propagation conditions are the typical sources of these mismatches and can lead to a performance degradation of the conventional beamforming algorithms [2]. In the past decades, a number of robust approaches have been reported that address this problem [3–13]. These robust methods can be broadly categorised into two main groups: techniques based on previous mismatch assumptions [3–10] and methods that estimate the mismatch or equivalently the actual steering vector [6, 8, 11, 13]. A number of robust designs can be often cast as optimisation problems which end up in the so-called second-order cone (SOC) programme, which can be easily solved with interior point methods. Although those designs for beamformers are based on the minimum variance criterion, it is possible to design them using a constant modulus criterion [14, 15], which can exploit prior knowledge about the desired signal and provide a better performance.

The problem we are interested in solving in this paper is the design of cost-effective adaptive robust beamforming algorithms. In particular, we focus on the design of beamforming algorithms which can exploit prior knowledge about the constant modulus property of a desired signal and that can be implemented in an efficient way with an appropriate modification of adaptive signal processing algorithms. In the first part of this work, the worst-case (WC) optimisation-based beamforming algorithm with the constant modulus criterion (CCM) is developed. To solve the robust constrained constant modulus (CCM), we apply an iterative reformulation of the constant modulus cost function introduced in [16], which is a local second-order approximation. Its derivation is based on the assumption that previous computed weight vectors are close to the solution, which is enforced by the additional constraint.

We reformulate the problem as a SOC programme in a similar fashion to the approach adopted in [5] and devise an adaptive algorithm to adjust the parameters of the beamformer in time-varying scenarios and that can exploit prior knowledge about the constant modulus of the desired signal. An analysis of the optimisation problem is conducted and a condition which ensures convexity is established. In addition, a study about the choice of the parameter ϵ associated with the WC-CCM criterion is carried out. We investigate the performance of the proposed WC-CCM algorithm via simulations. The results show that the proposed WC-CCM algorithm outperforms previously reported methods.

In the second part of this paper, low-complexity robust adaptive beamforming algorithms are developed. Although the robust constraint is similar to that which is known from the WC criterion, the algorithms are based on the modified conjugate gradient (MCG) [17, 18] and an alternating optimisation strategy that performs joint adjustment of the constraint and the parameters of the beamformer. The joint optimisation strategy exploits previous computations and therefore the computational complexity is reduced by more than an order of magnitude from more than cubic $\mathcal{O}(M^{3.5})$ to quadratic $\mathcal{O}(M^2)$ with the number of sensor elements M as compared with the WC optimisation-based approach. A low-complexity approach is also developed for the minimum variance design which is termed the robust constrained minimum variance MCG (robust-CMV (RCMV)-MCG) algorithm. The proposed low-complexity algorithm for the CCM design is termed RCCM-MCG. Although the RCMV-MCG algorithm has a performance equivalent to the WC optimisation-based approach, the RCCM-MCG algorithm which exploits the constant modulus property of the desired signal performs better than existing algorithms. We conduct a simulation study to investigate the performance of the proposed low-complexity algorithms in a number of situations of practical relevance.

This paper is organised as follows. The system model is described in Section 2. Section 3 reviews existing robust adaptive beamforming algorithms. The proposed WC-CCM design is formulated in Section 4. In Section 5, the SOC implementation and the adaptive algorithm are described. An analysis of the optimisation problem is given in Section 6, where a condition is found which ensures convexity and relationships between the parameter ϵ and the signal-to-noise ratio (SNR) are established. In Section 7, the corresponding low-complexity solutions are presented. The simulation results are presented and discussed in Section 8. Section 9 gives the conclusion of this work.

2 System model

Let us consider a linear array of M sensors that receives signals from D narrowband sources. The vector of array observations $\boldsymbol{x}(i) \in \mathbb{C}^{M\times 1}$ at time instant i can be modelled as

$$\boldsymbol{x}(i) = \boldsymbol{A}(\theta)\boldsymbol{s}(i) + \boldsymbol{n}(i) \tag{1}$$

where $\theta = [\theta_1, \ldots, \theta_D]^{\mathrm{T}} \in \mathbb{R}^{D\times 1}$ is the vector with the directions of arrival (DoA) and $(.)^{\mathrm{T}}$ stands for transpose, $\boldsymbol{A}(\theta) = \left[\boldsymbol{a}_1(\theta_1), \ldots, \boldsymbol{a}_D(\theta_D)\right] \in \mathbb{C}^{M\times D}$ is the matrix containing the array steering vectors $\boldsymbol{a}_m(\theta_m) \in \mathbb{C}^{M\times 1}$, for $m = 1, \ldots, D$. In the following, θ_1 is the direction related to the desired user which is roughly known by the system. The vector $\boldsymbol{s}(i) \in \mathbb{C}^{D\times 1}$ represents the uncorrelated sources. The vector $\boldsymbol{n}(i) \in \mathbb{C}^{M\times 1}$ is the sensor noise, which is assumed as zero-mean complex Gaussian. The true array steering vector is assumed as $\boldsymbol{a}_1(\theta_1) = \boldsymbol{a}(\theta_1) + \boldsymbol{e}$, where $\boldsymbol{e}$ is the mismatch vector and $\boldsymbol{a}(\theta_1)$ is the presumed array steering vector which is known by the system. In what follows, we will use $\boldsymbol{a} = \boldsymbol{a}(\theta_1)$. The output of the beamformer is defined as

$$y(i) = \boldsymbol{w}^{\mathrm{H}}\boldsymbol{x}(i) \tag{2}$$

where $\boldsymbol{w} \in \mathbb{C}^{M\times 1}$ is the complex vector of beamforming weights. The notation $(.)^{\mathrm{H}}$ stands for Hermitian transpose. The signal-to-interference-plus-noise ratio (SINR) is defined as

$$\mathrm{SINR} = \frac{\boldsymbol{w}^{\mathrm{H}}\boldsymbol{R}_{\mathrm{s}}\boldsymbol{w}}{\boldsymbol{w}^{\mathrm{H}}\boldsymbol{R}_{\mathrm{i+n}}\boldsymbol{w}} \tag{3}$$

where $\boldsymbol{R}_{\mathrm{s}}$ is the signal covariance matrix corresponding to the desired user and $\boldsymbol{R}_{\mathrm{i+n}}$ is the interference-plus-noise covariance matrix.

3 Robust adaptive beamforming: a review

We review a few notable approaches to the design of robust adaptive beamforming algorithms. The most common robust approach is the so-called loaded sample matrix inversion (loaded-SMI) beamformer [3], which includes an additional diagonal loading to the signal covariance matrix. The main problem is how to obtain the optimal diagonal loading factor. Typically it is chosen as $10\sigma_{\mathrm{n}}^2$, where σ_{n}^2 is the noise power [3]. Another robust approach is given by the eigen-based beamformer [19]. Here, the presumed array steering vector is replaced by its projection onto the signal-plus-interferer subspace. The approach implies that the noise subspace can be identified, which leads to a limitation in high SNR. A similar method is given by the reduced-rank beamforming approach [20], which avoids an eigen-decomposition and exploits the low rank of the signal-plus-interferer subspace. A different robust beamforming strategy is considered by techniques based on diagonal loading [5, 7, 9, 10], which are more advanced compared with [3]. In these techniques, the algorithms determine a diagonal loading parameter which aims to compensate for the mismatch by adding a suitable factor to the diagonal of the covariance matrix of the input signal.

One of these methods is given by the popular WC performance optimisation-based beamformer [5], which is based on the constraint that the absolute value of the array response is always greater than or equal to a constant for all vectors that belong to a predefined set of vectors in the neighbourhood of the presumed vector. In [5], the set of vectors is a sphere $A = \left\{\mathbf{a} + \mathbf{e}, ||\mathbf{e}||_2 \le \epsilon\right\}$, where the norm of $\boldsymbol{e}$ is upper-bounded by ϵ. The corresponding optimisation problem is given by

$$\begin{aligned} &\min_{\boldsymbol{w}} \boldsymbol{w}^{\mathrm{H}}\boldsymbol{R}_{xx}\boldsymbol{w} \\ &\text{s.t.} \left|\boldsymbol{w}^{\mathrm{H}}(\boldsymbol{a}+\boldsymbol{e})\right| \ge \delta, \ \text{ for all } (\boldsymbol{a}+\boldsymbol{e}) \in \mathcal{A}(\epsilon) \end{aligned} \tag{4}$$

where $\boldsymbol{R}_{xx} = E\{\boldsymbol{x}\boldsymbol{x}^{\mathrm{H}}\}$ is the covariance matrix of the input signal. The problem can be transformed into the following convex SOC problem

$$\begin{aligned} &\min_{\boldsymbol{w}} \boldsymbol{w}^{\mathrm{H}}\boldsymbol{R}_{xx}\boldsymbol{w} \quad \text{s.t.} \mathrm{Re}\left\{\boldsymbol{w}^{\mathrm{H}}\mathbf{a}\right\} - \delta \ge \epsilon||\boldsymbol{w}||_2 \\ &\mathrm{Im}\left\{\boldsymbol{w}^{\mathrm{H}}\boldsymbol{a}\right\} = 0 \end{aligned} \tag{5}$$

where the operator $\mathrm{Re}\{\cdot\}$ retains the real part of the argument and the operator $\mathrm{Im}\{\cdot\}$ retains the imaginary part of the argument. It has been shown that this kind of beamforming technique is related to the class of diagonal loading. In [9], the set of vectors in the neighbourhood can be ellipsoidal as well.

Another notable idea is the probability-constrained approach [21]. Here, the constraint satisfies operational

doi: 10.1049/iet-spr.2013.0166

conditions that are more likely to occur

$$\min_{w} w^{\mathrm{H}} R_{xx} w \quad \text{s.t.} \Pr\left\{\left|w^{\mathrm{H}}(a+e)\right| \geq \delta\right\} \geq p \tag{6}$$

where Pr denotes the probability operator and p is the desired probability threshold. Here different assumptions on the statistical characteristics of the mismatch vector $\boldsymbol{e}$ lead to different problem formulations. The solutions for the Gaussian probability density function (pdf) case and the general unknown pdf case have been developed in [21].

Another class of robust methods includes those that estimate the mismatch which have been reported in [6, 8, 11, 13]. The main idea behind these approaches is to compute an estimate of the mismatch and then subsequently use this information to obtain an estimate of the actual steering vector. Recently developed approaches estimate the mismatch vector based on sequential quadratic programming [11] or based on semi-definite relaxation [13].

All these beamformers are based on the minimum variance criterion. We assume that a number of these beamformers can benefit from using the CCM design criterion instead of the minimum variance one. Prior work with the CCM design criterion includes the design of adaptive beamformers [22] and receivers for spread spectrum systems [14, 15, 23]. The results in the literature indicate that the CCM design has a superior performance to those designs based on the minimum variance. In the following, we develop a WC performance optimisation-based beamforming algorithm with the CCM design criterion. In addition, we propose low-complexity robust beamforming algorithms.

4 Proposed WC optimisation based constant modulus design

The proposed robust beamformer is based on the WC approach. In case of the minimum variance design, it can be derived from the following optimisation problem

$$\begin{aligned} &\min_{w} w^{\mathrm{H}} R_{xx} w \quad \text{s.t. } \operatorname{Re}\left\{w^{\mathrm{H}} a\right\} - \delta \geq \epsilon \|w\|_2 \\ &\operatorname{Im}\left\{w^{\mathrm{H}} a\right\} = 0 \end{aligned} \tag{7}$$

where ϵ is the level of steering vector mismatch, which is assumed as known a priori. The proposed beamformer uses the constant modulus criterion, which exploits the constant modulus property of the desired signal instead of the minimum variance design criterion. To this end, we will assume that the signals processed have a constant modulus property during the observation time and the proposed algorithms are designed to exploit this property. The constant modulus cost function is defined by

$$\begin{aligned} J &= E\left\{\left(|y(i)|^2 - \gamma\right)^2\right\} \\ &= E\left\{\left(w^{\mathrm{H}}(i) x(i) x^{\mathrm{H}}(i) w(i) - \gamma\right)^2\right\} \end{aligned} \tag{8}$$

where $\gamma \geq 0$ which is a parameter related to and should be chosen according to the energy of the signal. If the parameter gamma is different, then we need to choose the parameter delta of the constraint to satisfy (30). By considering the approximation strategy in [16], that is, replacing in (8) $w^{\mathrm{H}}(i)x(i)$ by $w^{\mathrm{H}}(i-1)x(i)$, we obtain a modified cost function which is a second-order local approximation

$$\begin{aligned} \tilde{J} = \mathrm{E}\{&\left(w^{\mathrm{H}}(i) x(i) x(i)^{\mathrm{H}} w(i-1) - \gamma\right) \\ &\left(w^{\mathrm{H}}(i-1) x(i) x(i)^{\mathrm{H}} w(i) - \gamma\right)\} \end{aligned} \tag{9}$$

This is a special case of the established general constant modulus reformulation suggested in [16] whose validity has been confirmed via computer experiments. Furthermore, it should also be mentioned that the underlying assumption that the previous weight vector is close to the solution is additionally enforced by the direction constraint. Besides this strategy, there are similar second-order approximation strategies in the literature that are based on Taylor series expansion [24] approaches. By discarding the constant term, the objective function is given by

$$\begin{aligned} \hat{J} = \mathbf{w}^{\mathrm{H}} E&\left\{|y(i)|^2 x(i) x^{\mathrm{H}}(i)\right\} w - 2\gamma \mathrm{Re} \\ &\times \left\{w^{\mathrm{H}} E\left\{y^*(i) x(i)\right\}\right\} \end{aligned} \tag{10}$$

where $y(i) = w^{\mathrm{H}}(i-1)x(i)$ denotes the output which assumes small variations of the beamformer that allows the approximation $w^{\mathrm{H}}(i)x(i) \simeq w^{\mathrm{H}}(i-1)x(i)$. In combination with the WC constraint, the proposed WC-CCM design can be cast as the following optimisation problem

$$\begin{aligned} \min_{w} \quad & w^{\mathrm{H}} R_a w - 2\gamma \mathrm{Re}\left\{d^{\mathrm{H}} w\right\} \\ \text{s.t.} \quad & w^{\mathrm{H}} a - \delta \geq \epsilon \|w\|_2 \quad \text{and} \quad \operatorname{Im}\left\{w^{\mathrm{H}} a\right\} = 0 \end{aligned} \tag{11}$$

where $R_a = E\{|y(i)|^2 x(i) x^{\mathrm{H}}(i)\}$ and $d = E\{y^*(i)x(i)\}$ are estimated from the previous snapshots which will be explained in the next section.

5 Proposed SOC implementation and adaptive algorithm

In the first part of this section, we show how to implement the SOC programme and in the second part we devise an adaptive algorithm to adjust the weights of the beamformer according to the WC-CCM design.

5.1 SOC implementation

In this subsection, inspired by the approach in [5], we present a SOC implementation of the proposed WC-CCM design. Introducing a scalar variable τ, an equivalent problem to (10) can be formulated

$$\begin{aligned} &\min_{\tau, w} \quad \tau \quad \text{s.t.} \quad w^{\mathrm{H}} R_{\mathrm{ac}}^{\mathrm{H}} R_{\mathrm{ac}} w - 2\gamma \mathrm{Re}\left\{d^{\mathrm{H}} w\right\} \leq \tau \\ &\operatorname{Re}\left\{w^{\mathrm{H}} a\right\} - \delta \geq \epsilon \|w\|_2 \\ &\operatorname{Im}\left\{w^{\mathrm{H}} a\right\} = 0 \end{aligned} \tag{12}$$

where $R_{\mathrm{ac}}^{\mathrm{H}} R_{\mathrm{ac}} = R_a$ is the Cholesky factorisation. Introducing the real-valued matrix and the real-valued vectors given by

$$\breve{\boldsymbol{R}}_{\mathrm{ac}} = \begin{bmatrix} \mathrm{Re}\{\boldsymbol{R}_{\mathrm{ac}}\} & -\mathrm{Im}\{\boldsymbol{R}_{\mathrm{ac}}\} \\ \mathrm{Im}\{\boldsymbol{R}_{\mathrm{ac}}\} & \mathrm{Re}\{\boldsymbol{R}_{\mathrm{ac}}\} \end{bmatrix} \in \mathbb{R}^{(2M)\times(2M)}$$

$$\breve{\boldsymbol{d}} = [\mathrm{Re}\{\boldsymbol{d}\}^{\mathrm{T}}, \ Im\{\boldsymbol{d}\}^{\mathrm{T}}]^{\mathrm{T}} \in \mathbb{R}^{(2M)\times 1},$$

$$\breve{\boldsymbol{a}} = [\mathrm{Re}\{\boldsymbol{a}\}, \ Im\{\boldsymbol{a}\}]^{\mathrm{T}} \in \mathbb{R}^{(2M)\times 1}$$

$$\bar{\boldsymbol{a}} = [\mathrm{Im}\{\boldsymbol{a}\}^{\mathrm{T}}, \ -\mathrm{Re}\{\boldsymbol{a}\}^{\mathrm{T}}]^{\mathrm{T}} \in \mathbb{R}^{(2M)\times 1},$$

$$\breve{\boldsymbol{w}} = [\mathrm{Re}\{\boldsymbol{w}\}^{\mathrm{T}}, \ Im\{\boldsymbol{w}\}]^{\mathrm{T}} \in \mathbb{R}^{(2M)\times 1}$$

The problem can be rewritten as

$$\begin{aligned} &\min_{\tau, \boldsymbol{w}} \ \tau \text{ s.t} \\ &\breve{\boldsymbol{w}}^{\mathrm{T}}\breve{\boldsymbol{R}}_{\mathrm{ac}}^{\mathrm{T}}\breve{\boldsymbol{R}}_{\mathrm{ac}}\breve{\boldsymbol{w}} - 2\gamma\breve{\boldsymbol{d}}^{\mathrm{T}}\breve{\boldsymbol{w}} \le \tau\breve{\boldsymbol{w}}^{\mathrm{T}}\boldsymbol{a} - \delta \ge \epsilon\left\|\breve{\boldsymbol{w}}\right\|_2 \\ &\breve{\boldsymbol{w}}^{\mathrm{T}}\bar{\boldsymbol{a}} = 0 \end{aligned} \tag{13}$$

The quadratic constraint can be converted into an equivalent SOC constraint, because the convexity of the optimisation problem can be enforced as will be shown in the next section. This leads to the following optimisation problem

$$\min_{\tau, \boldsymbol{w}} \ \tau \text{ s.t.} \ \ \frac{1}{2} + \gamma\breve{\boldsymbol{d}}^{\mathrm{T}}\breve{\boldsymbol{w}} + \frac{\tau}{2} \ge$$
$$\left\| \left[\frac{1}{2} - \gamma\breve{\boldsymbol{d}}^{\mathrm{T}}\breve{\boldsymbol{w}} - \frac{\tau}{2}(14)\breve{\boldsymbol{R}}_{\mathrm{ac}}\breve{\boldsymbol{w}} \right] \right\|_2 \breve{\boldsymbol{w}}^{\mathrm{T}}\boldsymbol{a} - \delta \ge \epsilon||\breve{\boldsymbol{w}}||_2\breve{\boldsymbol{w}}^{\mathrm{T}}\bar{\boldsymbol{a}} = 0 \tag{14}$$

Let us define

$$\boldsymbol{p} = [1, \ 0^{\mathrm{T}}]^{\mathrm{T}} \in \mathbb{R}^{(2M+1)\times 1}$$

$$\boldsymbol{u} = [\tau, \ \breve{\boldsymbol{w}}^{\mathrm{T}}]^{\mathrm{T}} \in \mathbb{R}^{(2M+1)\times 1}$$

$$\boldsymbol{f} = [1/2, \ 1/2, \ 0^{\mathrm{T}}, \ -\delta, \ 0^{\mathrm{T}}, \ 0]^{\mathrm{T}} \in \mathbb{R}^{(4M+4)\times 1}$$

$$\boldsymbol{F}^{\mathrm{T}} = \begin{bmatrix} \frac{1}{2} & \gamma\breve{\boldsymbol{d}}^{\mathrm{T}} \\ -\frac{1}{2} & -\gamma\breve{\boldsymbol{d}}^{\mathrm{T}} \\ 0 & \breve{\boldsymbol{R}}_{\mathrm{ac}} \\ 0 & \breve{\boldsymbol{a}} \\ 0 & \epsilon\boldsymbol{I} \\ 0 & \bar{\boldsymbol{a}} \end{bmatrix} \in \mathbb{R}^{(4M+4)\times(2M+1)}$$

where $\boldsymbol{I}$ is a $2M \times 2M$ identity matrix and 0 is a vector of zeros of compatible dimensions. Finally, the problem can be formulated as the dual form of the SOC problem (equivalent to (8) in [25])

$$\begin{aligned} &\min_{\boldsymbol{u}} \boldsymbol{p}^{\mathrm{T}}\boldsymbol{u} \quad \text{s.t.} \\ &\boldsymbol{f} + \boldsymbol{F}^{\mathrm{T}}\boldsymbol{u} \in \mathrm{SOC}_1^{2M+2} \times \mathrm{SOC}_2^{2M+1} \times \{0\} \end{aligned} \tag{15}$$

where $\boldsymbol{f} + \boldsymbol{F}^{\mathrm{T}}\boldsymbol{u}$ describes a SOC with a dimension $2M+2$, a SOC with a dimension $2M+1$ and $\{0\}$ is the so-called zero cone that determines the hyperplane because of the equality constraint $\boldsymbol{w}^{\mathrm{T}}\bar{\boldsymbol{a}} = 0$. Finally, the weight vector of the beamformer $\boldsymbol{w}$ can be retrieved in the form $\boldsymbol{w} = [\boldsymbol{u}_2, \ldots, \boldsymbol{u}_{M+1}]^{\mathrm{T}} + j[\boldsymbol{u}_{M+2}, \ldots, \boldsymbol{u}_{2M+1}]^{\mathrm{T}}$. Alternatively, (10) can be solved by using [10], which transforms it automatically into an appropriate form.

Table 1 Proposed WC-CCM algorithm

initialisation: $\hat{\boldsymbol{R}}_a(0) = \sigma_n^2\boldsymbol{I}$; $\hat{\boldsymbol{d}}(0) = 0$; $w(0) = \frac{\mathbf{a}}{M}$
update for each time instant $i = 1, \ldots, \boldsymbol{N}$
$y(i) = w^{\mathrm{H}}(i-1)x(i)$
$\hat{\boldsymbol{R}}_a(i) = \mu\hat{\boldsymbol{R}}_a(i-1) + |y(i)|^2 x(i)x^{\mathrm{H}}(i)$
$\boldsymbol{R}_{\mathrm{ac}}(i) = \mathrm{chol}\left(\hat{\boldsymbol{R}}_a(i)\right)$
$\hat{d}(i) = \mu\hat{d}(i-1) + x(i)y^*(i)$
$\boldsymbol{R}_{\mathrm{acr}}(i) = \begin{bmatrix} \mathrm{Re}\{\boldsymbol{R}_{\mathrm{ac}}(i)\} & -Im\{\boldsymbol{R}_{\mathrm{ac}}(i)\} \\ Im\{\boldsymbol{R}_{\mathrm{ac}}(i)\} & \mathrm{Re}\{\boldsymbol{R}_{\mathrm{ac}}(i)\} \end{bmatrix}$
$\boldsymbol{d}_{\mathrm{r}}(i) = \left[\mathrm{Re}\left\{\hat{\boldsymbol{d}}(i)\right\}^{\mathrm{T}}, \ Im\left\{\hat{\boldsymbol{d}}(i)\right\}^{\mathrm{T}}\right]^{\mathrm{T}}$
$p = [1, 0^{\mathrm{T}}]^{\mathrm{T}}$
$f = [1/2, 1/2, 0^{\mathrm{T}}, -\delta, 0^{\mathrm{T}}, 0]^{\mathrm{T}}$
$\boldsymbol{F}^{\mathrm{T}} = \begin{bmatrix} \frac{1}{2} & \gamma d_{\mathrm{r}}^{\mathrm{T}}(i) \\ -\frac{1}{2} & -\gamma d_{\mathrm{r}}^{\mathrm{T}}(i) \\ 0 & \boldsymbol{R}_{\mathrm{acr}}(i) \\ 0 & a \\ 0 & \epsilon\boldsymbol{I} \\ 0 & \bar{a} \end{bmatrix}$
$\min_u p^{\mathrm{T}}u$ s.t.
$f + \boldsymbol{F}^{\mathrm{T}}u \in \mathrm{SOC}_1^{2M+2} \times \mathrm{SOC}_2^{2M+1} \times \{0\}$
$w(i) = [u_2, \ldots, u_{M+1}]^{\mathrm{T}} + j[u_{M+2}, \ldots, u_{2M+1}]^{\mathrm{T}}$

5.2 Adaptive algorithm

It has already been mentioned that the optimisation problem corresponding to the WC-CCM algorithm design is solved iteratively. As a result, the underlying optimisation problem is to be solved periodically. In this case, the proposed adaptive algorithm solves it at each time instant. For the adaptive implementation, we use an exponentially decayed data window for the estimation of $\boldsymbol{R}_a$ and $\boldsymbol{d}$ given by

$$\hat{\boldsymbol{R}}_a(i) = \mu\hat{\boldsymbol{R}}_a(i-1) + |y(i)|^2\boldsymbol{x}(i)\boldsymbol{x}^{\mathrm{H}}(i) \tag{16}$$

$$\hat{\boldsymbol{d}}(i) = \mu\hat{\boldsymbol{d}}(i-1) + \boldsymbol{x}(i)y^*(i) \tag{17}$$

where $0 < \mu < 1$ is the forgetting factor. Each iteration includes a Cholesky factorisation and also a transformation into a real-valued problem. Finally, the problem is formulated in the dual form of the SOC problem and solved with SeDuMi [25]. The structure of the adaptive algorithm is summarised in Table 1. Compared with the algorithm based on the minimum variance constraint, the proposed algorithm increases the dimension of the first SOC from $2M+1$ to $2M+2$.

6 Analysis of the optimisation problem

In this section, we analyse the optimisation problem associated with the design of the proposed robust WC-CCM beamformer. For the purpose of analysis, we rely on the equality of the robust constraint described in (11). In particular, we derive a sufficient condition for enforcing the convexity of the proposed WC-CCM beamformer design as a function of the power of the desired signal. We also

doi: 10.1049/iet-spr.2013.0166

provide design guidelines for the adjustment of the parameter ϵ in the optimisation problem.

6.1 Convexity of the optimisation problem

The objective function for the constant modulus design criterion is

$$J_{\text{cm}} = \mathrm{E}\left\{\left(|y(i)|^2 - \gamma\right)^2\right\} \tag{18}$$

To ensure that the constraint $\boldsymbol{w}^{\mathrm{H}}\boldsymbol{a} = \delta + \epsilon||\boldsymbol{w}||_2$ is fulfiled, the beamformer $\boldsymbol{w}$ is replaced by

$$\boldsymbol{w} = \frac{\boldsymbol{a}}{M}\left(\delta + \epsilon||\boldsymbol{w}||_2\right) + \boldsymbol{B}z \tag{19}$$

where the columns of $\boldsymbol{B}$ are unitary and span the null space of $\boldsymbol{a}^{\mathrm{H}}$, $z \in \mathbb{C}^{M-1\times 1}$ and $\boldsymbol{a}^{\mathrm{H}}\boldsymbol{a} = M$. To obtain a function which does not depend on $||\boldsymbol{w}||_2$, we compute the squared norm of (19) and obtain the following quadratic equation to be solved

$$||\boldsymbol{w}||_2 = \tau = \sqrt{\frac{1}{M}(\delta + \epsilon\tau)^2 + z^{\mathrm{H}}z} \tag{20}$$

Since the norm is greater than zero the following holds

$$\tau = \frac{\epsilon\delta}{M-\epsilon^2} + \sqrt{\frac{Mz^{\mathrm{H}}z + \delta^2}{M-\epsilon^2} + \left(\frac{\epsilon\delta}{M-\epsilon^2}\right)^2} \tag{21}$$

Therefore, by inserting (20) and (21) in (19), the resulting weight vector $\boldsymbol{w}$ is a function of z as described by

$$\boldsymbol{w}(z) = \frac{\boldsymbol{a}}{M}\left(\delta + \frac{\epsilon^2\delta}{M-\epsilon^2} + \epsilon\sqrt{\frac{Mz^{\mathrm{H}}z + \delta^2}{M-\epsilon^2} + \left(\frac{\epsilon\delta}{M-\epsilon^2}\right)^2}\right) + \boldsymbol{B}z \tag{22}$$

Replacing the $\boldsymbol{w}$ in the objective function leads to an equivalent problem to the original

$$J = E\left\{|y(i)|^2 - \gamma\right\} = E\left\{\left[\boldsymbol{w}^{\mathrm{H}}(z)\boldsymbol{x}\boldsymbol{x}^{\mathrm{H}}\boldsymbol{w}(z) - \gamma\right]^2\right\} \tag{23}$$

The function above is convex if the Hessian $\boldsymbol{H} = (\partial/\partial z^{\mathrm{H}})(\partial J/\partial z)$ is positive semi-definite. The Hessian corresponding to the objective function is given by

$$\begin{aligned}\boldsymbol{H} &= 2\frac{\partial}{\partial z^{\mathrm{H}}}\left(E\left\{|y|^2 - \gamma\right\}\right)\frac{\partial}{\partial z}\left(E\left\{|y|^2 - \gamma\right\}\right)\\ &\quad + 2E\left\{|y|^2 - \gamma\right\}\frac{\partial}{\partial z^{\mathrm{H}}}\frac{\partial}{\partial z}\left(E\left\{|y|^2 - \gamma\right\}\right)\end{aligned} \tag{24}$$

Since it is the product of a vector multiplied with its Hermitian transposed the first term in (3), is positive semi-definite. Although it is assumed that $E\{|y(i)|^2 - \gamma\} \geq 0$, the positive semi-definiteness of $\boldsymbol{H}_2 = (\partial/\partial z^{\mathrm{H}})(\partial/\partial z)(E\{|y|^2 - \gamma\})$ still needs to be shown. It can be expressed as a sum $\boldsymbol{H}_2 = \sum_{k=1}^{4}\boldsymbol{H}_{2,k}$ and is given by

$$\begin{aligned}\boldsymbol{H}_2 = E\Bigg\{&\left(-\frac{\epsilon}{2}\left(\frac{1}{\sqrt{\alpha}}\right)\right)^3\left(\frac{M}{M-\epsilon^2}\right)^2\mathrm{Re}\{\xi\}zz^{\mathrm{H}}\\ &+ \epsilon\frac{1}{\sqrt{\alpha}}\left(\frac{M}{M-\epsilon^2}\right)\mathrm{Re}\{\xi\}\boldsymbol{I}_{M-1}\\ &+ \frac{\epsilon}{2}\frac{1}{\sqrt{\alpha}}\left(\frac{M}{M-\epsilon^2}\right)\frac{\boldsymbol{a}^{\mathrm{H}}}{M}\boldsymbol{x}\boldsymbol{x}^{\mathrm{H}}\frac{\boldsymbol{a}}{M}\\ &\frac{\epsilon}{2}\frac{1}{\sqrt{\alpha}}\left(\frac{M}{M-\epsilon^2}\right)zz^{\mathrm{H}}\\ &+\left(\frac{\epsilon}{2}\frac{1}{\sqrt{\alpha}}\left(\frac{M}{M-\epsilon^2}\right)z\frac{\boldsymbol{a}^{\mathrm{H}}}{M} + \boldsymbol{B}^{\mathrm{H}}\right)\boldsymbol{x}\boldsymbol{x}^{\mathrm{H}}\\ &\times\left(\frac{\boldsymbol{a}}{M}\frac{\epsilon}{2}\frac{1}{\sqrt{\alpha}}\left(\frac{M}{M-\epsilon^2}\right)z^{\mathrm{H}} + \boldsymbol{B}\right)\Bigg\}\end{aligned} \tag{25}$$

where

$$\alpha = \left(\tfrac{Mz^{\mathrm{H}}z+\delta^2}{M-\epsilon^2} + \left(\tfrac{\epsilon\delta}{M-\epsilon^2}\right)^2\right),$$

$$\beta = \left(\delta + \tfrac{\epsilon^2\delta}{M-\epsilon^2} + \epsilon\sqrt{\alpha}\right)$$

$$\text{and} \quad i = \tfrac{\boldsymbol{a}^{\mathrm{H}}}{M}\boldsymbol{x}\boldsymbol{x}^{\mathrm{H}}\left(\beta\tfrac{\boldsymbol{a}}{M} + \boldsymbol{B}z\right)$$

To show that $\boldsymbol{H}_2$ is positive semi-definite the following steps are made. Here, it is assumed that

$$\mathrm{Re}\{\xi\} = \mathrm{Re}\left\{\frac{\boldsymbol{a}^{\mathrm{H}}}{M}\boldsymbol{x}\boldsymbol{x}^{\mathrm{H}}\left(\beta\frac{\boldsymbol{a}}{M} + \boldsymbol{B}z\right)\right\} \geq 0 \tag{26}$$

This assumption is reasonable as far as the term $\boldsymbol{x}^{\mathrm{H}}\boldsymbol{B}z$ is basically the compensating term of the unwanted contribution of $\boldsymbol{x}^{\mathrm{H}}(\beta(\boldsymbol{a}/M))$. Under this condition, all terms in the sum of $\boldsymbol{H}_2$ are positive semi-definite except the first term $\boldsymbol{H}_{2,1}$. The inequality $\boldsymbol{v}^{\mathrm{H}}(\boldsymbol{H}_{2,2})\boldsymbol{v} \geq \boldsymbol{v}^{\mathrm{H}}(-\boldsymbol{H}_{2,1})\boldsymbol{v}\forall\boldsymbol{v}$ is a sufficient condition to ensure positive semi-definiteness which is described as

$$\begin{aligned}&\boldsymbol{v}^{\mathrm{H}}\epsilon\frac{1}{\sqrt{\alpha}}\left(\frac{M}{M-\epsilon^2}\right)\mathrm{Re}\{\xi\}\boldsymbol{I}_{M-1}\boldsymbol{v}\\ &\geq z^{\mathrm{H}}\left(\frac{\epsilon}{2}\left(\frac{1}{\sqrt{\alpha}}\right)\right)^3\left(\frac{M}{M-\epsilon^2}\right)^2\mathrm{Re}\{\xi\}zz^{\mathrm{H}}z\\ &\geq \boldsymbol{v}^{\mathrm{H}}\left(\frac{\epsilon}{2}\left(\frac{1}{\sqrt{\alpha}}\right)\right)^3\left(\frac{M}{M-\epsilon^2}\right)^2\mathrm{Re}\{\xi\}zz^{\mathrm{H}}\boldsymbol{v}\end{aligned} \tag{27}$$

where $\boldsymbol{v}$ is any vector with the same norm of z and $z^{\mathrm{H}}(-\boldsymbol{H}_{2,1})z$ is introduced as the upper bound for $\boldsymbol{v}^{\mathrm{H}}(-\boldsymbol{H}_{2,1})\boldsymbol{v}$. Since $z^{\mathrm{H}}z = \boldsymbol{v}^{\mathrm{H}}\boldsymbol{v}$, the inequality can be reduced to

$$2\alpha \geq \left(\frac{Mz^{\mathrm{H}}z}{M-\epsilon^2}\right) \tag{28}$$

Replacing α gives the proof for positive semi-definiteness

$$2\left(\frac{Mz^{\mathrm{H}}z + \delta}{M-\epsilon^2} + \left(\frac{\epsilon\delta}{M-\epsilon^2}\right)^2\right) \geq \left(\frac{Mz^{\mathrm{H}}z}{M-\epsilon^2}\right) \tag{29}$$

which is always true. To ensure that $E\{|y|^2 - \gamma\} \geq 0$, it can be assumed that $\boldsymbol{w}^{\mathrm{H}}(\boldsymbol{a} + \boldsymbol{e}) \geq \delta$, where $\boldsymbol{e}$ is the array steering vector mismatch. Therefore

$$\gamma \leq \delta E\left\{|s_1|^2\right\} \tag{30}$$

is a sufficient condition to enforce convexity, where $|s_1|^2$ is the power of the desired user. Therefore the parameter gamma should be chosen such that the convexity condition given in (30) is satisfied.

6.2 Adjustment of the design parameter ϵ

Let us define the beamforming weight vector as

$$\boldsymbol{w} = c\boldsymbol{a}/M + \boldsymbol{b} \tag{31}$$

where c is a scalar and $\boldsymbol{b}$ is orthogonal to $\boldsymbol{a}$. Using it with the WC constraint leads to

$$c - \delta \geq \epsilon\sqrt{\frac{c^2}{M} + \boldsymbol{b}^{\mathrm{H}}\boldsymbol{b}} \tag{32}$$

From the above inequality, the following relation holds

$$c - \delta \geq \epsilon\sqrt{\frac{c^2}{M} + \boldsymbol{b}^{\mathrm{H}}\boldsymbol{b}} \geq \epsilon\sqrt{\frac{c^2}{M}} \tag{33}$$

Rewriting the relation shows that c tends to infinity when ϵ is close to $\sqrt{M}$

$$c \geq \frac{\delta}{1 - \epsilon/\sqrt{M}} \tag{34}$$

In addition, it is mentioned in [7] that for $||\boldsymbol{a}||_2 \leq \epsilon$ there is no $\boldsymbol{w}$ that satisfies the constraint. By rewriting the inequality in (33), we obtain

$$c \geq \frac{M\delta}{M - \epsilon^2} + \sqrt{\frac{M\epsilon^2\boldsymbol{b}^{\mathrm{H}}\boldsymbol{b} - M\delta}{M - \epsilon^2} + \left(\frac{M\delta}{M - \epsilon^2}\right)^2} \tag{35}$$

Now by assuming that $\epsilon \simeq \sqrt{M}$ and strictly less than $\sqrt{M}$, then we have $c \gg \delta$. In that case, the inequality in (12) can be rewritten as

$$c \geq \epsilon\sqrt{\frac{c^2}{M} + \boldsymbol{b}^{\mathrm{H}}\boldsymbol{b}} \tag{36}$$

or equivalently as

$$\frac{\boldsymbol{b}^{\mathrm{H}}\boldsymbol{b}}{c^2} \leq \frac{1}{\epsilon^2} - \frac{1}{M} \tag{37}$$

As a result of (37), the choice of ϵ affects the ratio between the components of the weight vector defined by (31), which can become negligible. This corresponds to $\boldsymbol{w} \simeq c\boldsymbol{a}/M$ and an equivalent diagonal loading which is above the level of the interference. Hence, the diagonal loading can be chosen by an appropriate procedure if ϵ is chosen in the allowed interval $\left[0, \sqrt{M}\right]$, where the constraint can be enforced. Obviously, in the case of ϵ being close to $\sqrt{M}$ the ratio $(\boldsymbol{b}^{\mathrm{H}}\boldsymbol{b}/c^2)$ tends to a small value, which can lead to a performance degradation. The ratio is small for low SNR values, and this is caused by the assumption that the additional noise appears as a diagonal loading in the signal covariance matrix and this eventually decreases $||\boldsymbol{b}||$. This means that the relation in (37) and its penalty has a more significant impact on the performance for higher SNR values. As a consequence, our suggestion is to choose ϵ with respect to the SNR as well as with respect to the assumed mismatch level. This will be investigated in the simulations (see Fig. 3).

7 Low-complexity algorithms using the modified conjugate gradient

The existing algorithms which use the WC optimisation-based constraint do not take advantage of previous computations as the conventional SMI beamforming algorithm solved by the modified conjugate method (MCG) algorithm or the recursive-least-squares algorithm in the so-called 'online' mode. For this reason, the existing robust beamforming algorithms are not suitable for low-complexity implementations and are unable to track time-varying signals.

In this section, a robust constraint is shown which is just slightly different compared with the WC optimisation-based approaches. As a result, the corresponding optimisation problem is a quadratically constrained quadratic programme instead of a SOC programme. It is shown how to solve the problem with a joint optimisation strategy. The method includes a system of equations which is solved efficiently with a MCG algorithm and an alternating optimisation strategy [26]. As a result, the computational complexity is reduced from more than cubic $O(M^{3.5})$ to quadratic $O(M^2)$ with the number of sensor elements, while the SINR performance is equivalent to the WC optimisation-based approach. The proposed method is presented in the robust CMV design using the MCG method (RCMV-MCG) and in the RCCM design using the MCG method (RCCM-MCG), which exploits the constant modulus property of the desired signal.

7.1 Proposed design and joint optimisation approach

In this part, we detail the main steps of the proposed design and the low-complexity algorithms as well as the joint optimisation approach that is employed to compute the parameters of the adaptive robust beamformer and the diagonal loading. Specifically, the proposed algorithms are based on an alternating optimisation strategy [26] that updates the beamformer $\boldsymbol{w}(i)$, whereas the diagonal loading $\lambda(i)$ is fixed and then updates $\lambda(i)$ while $\boldsymbol{w}(i)$ is held fixed. The algorithm is illustrated in Fig. 1.

Since the joint optimisation of the parameters $\boldsymbol{w}(i)$ and $\lambda(i)$ is not a convex optimisation problem, the first question that arises is whether the proposed algorithms will converge to their global minima. The proposed algorithms have been widely tested and we have not observed problems with local minima. This is corroborated by the recent results reported in [26] that shows that alternating optimisation techniques similar to that proposed here converge to the global optimum provided that typical assumptions used for adaptive algorithms such as step size values, forgetting

IET Signal Process., 2014, Vol. 8, Iss. 5, pp. 447–457
doi: 10.1049/iet-spr.2013.0166

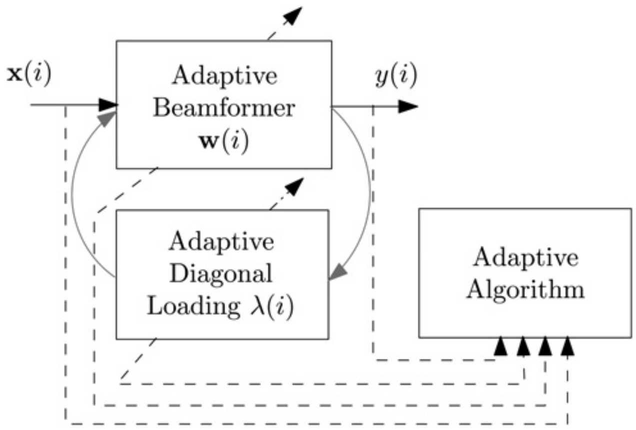

Fig. 1 *Proposed adaptive scheme with alternating updates of beamforming weights and diagonal loading*

factors and the statistical independence of the noise and the source data processed hold.

7.1.1 RCMV design: The proposed low-complexity beamforming algorithms are related to the WC approach [5]. To obtain a design which can be solved with a low complexity, the robust constraint reported in [5] is modified. According to [9], it is sufficient to use the real part of the constraint. In addition, it is assumed that the use of $\tilde{\epsilon}||\boldsymbol{w}||_2^2$ instead of $\epsilon\|\boldsymbol{w}\|_2$ from the conventional constraint has a comparable impact. Finally, the proposed design criterion for the minimum variance case is

$$\min_{\boldsymbol{w}} \boldsymbol{w}^{\mathrm{H}}\boldsymbol{R}_{xx}\boldsymbol{w}, \quad \text{s.t. } \mathrm{Re}\{\boldsymbol{w}^{\mathrm{H}}\boldsymbol{a}\} - \delta \geq \tilde{\epsilon}\|\boldsymbol{w}\|_2^2 \tag{38}$$

Using the method of Lagrange multipliers gives

$$\begin{aligned} L_{\mathrm{CMV}}(\boldsymbol{w},\ \lambda) &= \boldsymbol{w}^{\mathrm{H}}\boldsymbol{R}_{xx}\boldsymbol{w} \\ &+ \lambda\left[\tilde{\epsilon}\boldsymbol{w}^{\mathrm{H}}\boldsymbol{w} - \mathrm{Re}\{\boldsymbol{w}^{\mathrm{H}}\mathrm{ola}\} + \delta\right] \end{aligned} \tag{39}$$

where λ is the Lagrange multiplier. Computing the gradient of (39) with respect to $\boldsymbol{w}^*$, and equating it to zero leads to

$$\boldsymbol{w} = \left(\boldsymbol{R}_{xx} + \tilde{\epsilon}\lambda\boldsymbol{I}\right)^{-1}\lambda\boldsymbol{a}/2 \tag{40}$$

Since there is no known method in the literature that can obtain the Lagrange multiplier in a closed form, here it is proposed a strategy to adjust both the beamformer $\boldsymbol{w}$ and the Lagrange multiplier in an alternating fashion. In this joint optimisation, the Lagrange multiplier is interpreted as a penalty factor and the condition $\lambda > 0$ holds all the time. The adjustment increases the penalty factor when the constraint is not fulfiled and decreases it otherwise. To this end, we devise the following algorithm

$$\begin{aligned} \lambda(i) &= \lambda(i-1) + \mu_\lambda \\ &\times \left(\tilde{\epsilon}\,\|\boldsymbol{w}(i)\|_2^2 - \mathrm{Re}\{\boldsymbol{w}(i)^{\mathrm{H}}\boldsymbol{a}\} + \delta\right) \end{aligned} \tag{41}$$

where μ_λ is the step size. In addition, it is reasonable to define boundaries for the updated term.

To obtain an operation range for the parameter $\tilde{\epsilon}$, the weight vector can be expressed as $\boldsymbol{w} = (\boldsymbol{a}/M) + \boldsymbol{b}$. Rearranging the constraint function leads to the inequality

$$\tilde{\epsilon} \leq \frac{c-\delta}{(1/M)c^2 + \boldsymbol{b}^{\mathrm{H}}\boldsymbol{b}} \leq M\frac{c-\delta}{c^2} \leq \frac{M}{2} \tag{42}$$

which clearly indicates that there is no solution for $\tilde{\epsilon} > (M/2)$. From our experiments, we know that the parameter has to be chosen significantly smaller.

7.1.2 RCCM design: In case of constant modulus signals, it has been shown that the constant modulus design performs better than the minimum variance design [14, 15]. Similarly, the robust approach can be combined with the CCM criterion. The corresponding optimisation problem for the iteratively solved constant modulus objective function can be cast as

$$\min_{\boldsymbol{w}} \boldsymbol{w}^{\mathrm{H}}\boldsymbol{R}_a\boldsymbol{w} - 2\gamma\mathrm{Re}\{\boldsymbol{w}^{\mathrm{H}}\boldsymbol{d}\} \tag{43}$$

$$\text{s.t.}\mathrm{Re}\{\boldsymbol{w}^{\mathrm{H}}\boldsymbol{a}\} - \delta \geq \tilde{\epsilon}||\boldsymbol{w}||_2^2 \tag{44}$$

Using the method of Lagrange multipliers gives

$$\begin{aligned} \mathcal{L}_{\mathrm{CCM}}(\boldsymbol{w},\ \lambda) &= \boldsymbol{w}^{\mathrm{H}}\boldsymbol{R}_a\boldsymbol{w} - 2\gamma\mathrm{Re}\{\boldsymbol{w}^{\mathrm{H}}\boldsymbol{d}\} \\ &+ \lambda\left[\tilde{\epsilon}\boldsymbol{w}^{\mathrm{H}}\boldsymbol{w} - \mathrm{Re}\{\boldsymbol{w}^{\mathrm{H}}\boldsymbol{a}\} + \delta\right] \end{aligned} \tag{45}$$

Computing the gradient of (45) with respect to $\boldsymbol{w}^*$, and equating it to zero leads to

$$\boldsymbol{w} = \left[\boldsymbol{R}_a + \tilde{\epsilon}\lambda\boldsymbol{I}\right]^{-1}[\gamma\boldsymbol{d} + \lambda\boldsymbol{a}/2] \tag{46}$$

where $\boldsymbol{I}$ is an M-dimensional identity matrix. The adjustment of the Lagrange multiplier λ can be performed in the same way as in the minimum variance case.

7.2 Adaptive algorithms

To take advantage of the proposed joint optimisation approach of an 'online' MCG method, with one iteration per snapshot is used to solve the resulting problem. Its derivation is based on [27] and it can be interpreted as an extension of the idea in [17].

7.2.1 RCMV-MCG: In the proposed algorithm, an exponentially decayed data window is used to estimate the matrix $\boldsymbol{R}_{xx}$ as described by

$$\hat{\boldsymbol{R}}_{xx}(i) = \mu\hat{\boldsymbol{R}}_{xx}(i-1) + \boldsymbol{x}(i)\boldsymbol{x}^{\mathrm{H}}(i) \tag{47}$$

where μ is the forgetting factor. According to [1]

$$\boldsymbol{R}_{xx} \simeq [1-\mu]\hat{\boldsymbol{R}}_{xx}(i) \tag{48}$$

can be assumed for large i. Replacing $\boldsymbol{R}_{xx}$ in (40), introducing $\hat{\lambda}(i) = (\lambda(i)/1-\mu)$, leads to $\boldsymbol{w}(i) = \left[\hat{\boldsymbol{R}}_{xx}(i) + \tilde{\epsilon}\hat{\lambda}(i)\boldsymbol{I}\right]^{-1}\hat{\lambda}(i)\boldsymbol{a}/2$. Let us introduce the conjugate gradient (CG) weight vector $\boldsymbol{v}(i)$ as follows $\boldsymbol{w}(i) = l\boldsymbol{v}(i)\left(\hat{\lambda}(i)/2\right)$. The conjugate gradient algorithm solves the problem by iteratively updating the CG weight vector

$$\boldsymbol{v}(i) = \boldsymbol{v}(i-1) + \alpha(i)\boldsymbol{p}(i) \tag{49}$$

where $\boldsymbol{p}(i)$ is the direction vector and $\alpha(i)$ is the adaptive step size. One way to realise the conjugate gradient method performing one iteration per snapshot is the application of the degenerated scheme [27]. Under this condition, the adaptive step size $\alpha(i)$ has to fulfil the convergence bound given by

$$0 \le E\{\boldsymbol{p}^{\mathrm{H}}(i)\boldsymbol{g}(i)\} \le 0.5\ E\{\boldsymbol{p}^{\mathrm{H}}(i)\boldsymbol{g}(i-1)\} \tag{50}$$

where $E\{Im\{\boldsymbol{p}^{\mathrm{H}}(i)\boldsymbol{g}(i-1)\}\} \simeq 0$ and $E\{Im\{\boldsymbol{p}^{\mathrm{H}}(i)\boldsymbol{g}(i)\}\} \simeq 0$ can be neglected. The negative gradient vector and its recursive expression are considered in a similar fashion to [17, 27] as described by

$$\begin{aligned} \boldsymbol{g}(i) &= \boldsymbol{a} - [\hat{\boldsymbol{R}}_{xx}(i) + \tilde{\epsilon}\hat{\lambda}(i)\boldsymbol{I}]\boldsymbol{v}(i) \\ &= \boldsymbol{a}[1-\mu] + \mu\boldsymbol{g}(i-1) \\ &\quad - [\boldsymbol{x}\boldsymbol{x}^{\mathrm{H}} + \tilde{\epsilon}(\hat{\lambda}(i) - \mu\hat{\lambda}(i-1))\boldsymbol{I}]\boldsymbol{v}(i-1) \\ &\quad - \alpha(i)[\hat{\boldsymbol{R}}_{xx}(i) + \tilde{\epsilon}\hat{\lambda}(i)\boldsymbol{I}]\boldsymbol{p}(i) \end{aligned} \tag{51}$$

Pre-multiplying with $\boldsymbol{p}^{\mathrm{H}}(i)$, taking expectations on both sides and considering $\boldsymbol{p}(i)$ uncorrelated with $\boldsymbol{a}$, $\boldsymbol{x}(i)$ and $\boldsymbol{v}(i-1)$ lead to

$$\begin{aligned} E\{\boldsymbol{p}^{\mathrm{H}}(i)\boldsymbol{g}(i)\} &\simeq \mu\mathrm{E}\{\boldsymbol{p}^{\mathrm{H}}(i)\boldsymbol{g}(i-1)\} \\ &\quad - E\{\alpha(i)\}E\{\boldsymbol{p}^{\mathrm{H}}(i)[\hat{\boldsymbol{R}}_{xx}(i) + \tilde{\epsilon}\hat{\lambda}(i)\boldsymbol{I}]\boldsymbol{p}(i)\} \end{aligned} \tag{52}$$

Here, it is assumed that the algorithm has converged, which implies $\boldsymbol{a}[1-\mu] - [E\{\boldsymbol{x}\boldsymbol{x}^{\mathrm{H}}\} + \tilde{\epsilon}\hat{\lambda}(i)[1-\mu]\boldsymbol{I}]\boldsymbol{v}(i-1) = 0$, where (48) is taken into account and $\hat{\lambda}(i) \simeq \hat{\lambda}(i-1)$. Introducing $\boldsymbol{p}_R = [\hat{\boldsymbol{R}}_{xx}(i) + \lambda(i)\tilde{\epsilon}\boldsymbol{I}]\boldsymbol{p}(i)$, rearranging (52) and inserting it into (50) determines the step size within its boundaries as follows

$$\alpha(i) = [\boldsymbol{p}^{\mathrm{H}}(i)\boldsymbol{p}_R]^{-1}(\mu - \eta)\boldsymbol{p}^{\mathrm{H}}(i)\boldsymbol{g}(i-1) \tag{53}$$

where $0 \le \eta \le 0.5$. The direction vector is a linear combination of the previous direction vector and the negative gradient given by

$$\boldsymbol{p}(i+1) = \boldsymbol{p}(i) + \beta(i)\boldsymbol{g}(i) \tag{54}$$

where $\beta(i)$ is computed to avoid the reset procedure by employing the Polak–Ribiere approach [18]

$$\beta = [\boldsymbol{g}^{\mathrm{H}}(i-1)\boldsymbol{g}(i-1)]^{-1}[\boldsymbol{g}(i) - \boldsymbol{g}(i-1)]^{\mathrm{H}}\boldsymbol{g}(i) \tag{55}$$

The proposed algorithm, which is termed RCMV-MCG, is described in Table 2.

Note that for the alternating algorithm to adjust the Lagrange multiplier, we divide the update-term by 2, if the Lagrange multiplier is outside a predefined range, as it is described in Table 1. The application of the proposed algorithm corresponds to a computational effort which is quadratic with the number of sensor elements M.

7.2.2 Robust-CCM-MCG: The adaptive algorithm in the case of the CCM criterion is developed analogously to the minimum variance case. The estimates of $\boldsymbol{R}_a$ and $\boldsymbol{d}$ are based on an exponentially decayed data window and are given by

$$\hat{\boldsymbol{R}}_a(i) = \mu\hat{\boldsymbol{R}}_a(i-1) + |y(i)|^2\boldsymbol{x}(i)\boldsymbol{x}^{\mathrm{H}}(i) \tag{56}$$

$$\hat{\boldsymbol{d}}(i) = \mu\hat{\boldsymbol{d}}(i-1) + \boldsymbol{x}(i)y^*(i) \tag{57}$$

Table 2 Proposed RCMV-MCG algorithm

$v(0) = 0;\ p(1) = g(0) = a;\ \hat{\boldsymbol{R}}(0) = \delta\boldsymbol{I};\ \hat{\lambda}(0) = \hat{\lambda}(1) = \hat{\lambda}_0$
'for each time instant' $i = 1, \ldots, N$
$\hat{\boldsymbol{R}}_{xx}(i) = \mu\hat{\boldsymbol{R}}_{xx}(i-1) + x(i)x^{\mathrm{H}}(i)$
$p_R = [\hat{\boldsymbol{R}}_{xx}(i) + \hat{\lambda}(i)\tilde{\epsilon}\boldsymbol{I}]\boldsymbol{p}(i);\quad \nu = [\hat{\lambda}(i) - \mu\hat{\lambda}(i-1)]\tilde{\epsilon}$
$\alpha(i) = [\boldsymbol{p}^{\mathrm{H}}(i)\boldsymbol{p}_R]^{-1}(\mu - \eta)\boldsymbol{p}^{\mathrm{H}}(i)g(i-1);\quad (0 \le \eta \le 0.5)$
$v(i) = v(i-1) + \alpha(i)\boldsymbol{p}(i)$
$g(i) = [1-\mu]a + \mu g(i-1) - \alpha(i)\boldsymbol{p}_R$
$-(x(i)x^{\mathrm{H}}(i) + \nu\boldsymbol{I})v(i-1)$
$\beta(i) = [g^{\mathrm{H}}(i-1)g(i-1)]^{-1}[g(i) - g(i-1)]^{\mathrm{H}}g(i)$
$\boldsymbol{p}(i+1) = g(i) + \beta(i)\boldsymbol{p}(i)$
$w(i) = \lambda(i)v(i)/2$
$\delta_\lambda = \mu_\lambda[\tilde{\epsilon}\Vert w(i)\Vert_2^2 - \mathrm{Re}\{w^{\mathrm{H}}(i)a\} + \delta]$
while $\delta_\lambda \le -\lambda(i)$ or $\delta_\lambda \ge \delta_{\lambda\max}$
$\delta_\lambda \Rightarrow \delta_\lambda/2$
end
$\hat{\lambda}(i+1) = \hat{\lambda}(i) + \delta_\lambda$

Following the steps of the derivation of the MCG algorithm and taking into account that

$$\boldsymbol{R}_a \simeq [1-\mu]\hat{\boldsymbol{R}}_a(i) \tag{58}$$

$$\boldsymbol{d} \simeq [1-\mu]\hat{\boldsymbol{d}}(i) \tag{59}$$

lead to the adaptive algorithm. Note that, in contrast to the CMV case, here the beamforming weight vector is the same as the conjugate gradient weight vector, which means $\boldsymbol{w} = [\hat{\boldsymbol{R}}_a + \tilde{\epsilon}\hat{\lambda}\boldsymbol{I}]^{-1}[\gamma\hat{\boldsymbol{d}} + \hat{\lambda}\boldsymbol{a}/2]$. The negative gradient vector and its recursive expression are defined as

$$\begin{aligned} \boldsymbol{g}(i) &= [\gamma\hat{\boldsymbol{d}} + \hat{\lambda}\boldsymbol{a}/2] - [\hat{\boldsymbol{R}}_a + \tilde{\epsilon}\hat{\lambda}\boldsymbol{I}]\boldsymbol{w}(i) \\ &= \mu\boldsymbol{g}(i-1) - \alpha(i)\boldsymbol{p}_R - (|y(i)|^2\boldsymbol{x}(i)\boldsymbol{x}^{\mathrm{H}}(i))\boldsymbol{w}(i-1) \\ &\quad + \gamma\boldsymbol{x}(i)y^*(i) + \nu[\boldsymbol{a}/(2\tilde{\epsilon}) - \boldsymbol{w}(i-1)] \end{aligned} \tag{60}$$

where $\nu = [\hat{\lambda}(i) - \mu\hat{\lambda}(i-1)]\tilde{\epsilon}$. The proposed algorithm, which is termed RCCM-MCG, is described in Table 3.

8 Simulations

In this section, we present a number of simulation examples that illustrate the performance of the proposed robust beamforming algorithms and compare them with existing robust techniques that are representative of the prior work in this area. A uniform linear sensor array is used with $M = 10$ sensors. Specifically, we consider comparisons of the proposed algorithms with the loaded-SMI [3], the optimal SINR [2] (page 54) and the WC-CMV in [5]. We examine scenarios in which the SINR is measured against the parameter ϵ that arises from the WC optimisation, the number of snapshots and the SNR. We also consider a specific situation in which the array steering vector is corrupted by local coherent scattering, and scenarios in which there are changes in the environment and the tracking performance of the beamformers is evaluated. These experiments are important to assess the performance of the proposed algorithms and to illustrate how they perform against existing methods.

Table 3 Proposed RCCM-MCG algorithm

$p(1) = g(0) = a$; $\hat{\boldsymbol{R}}_a(0) = \delta \boldsymbol{I}$; $\hat{d}(0) = 0$;
$\hat{\lambda}(0) = \hat{\lambda}(1) = \hat{\lambda}_0$; $w = a/M$
'for each time instant' $i = 1, \ldots, N$
$\hat{\boldsymbol{R}}_a(i) = \mu \hat{\boldsymbol{R}}_a(i-1) + |y(i)|^2 x(i) x^{\mathrm{H}}(i)$
$\boldsymbol{p}_R = [\hat{\boldsymbol{R}}_a(i) + \hat{\lambda}(i)\tilde{\epsilon}\boldsymbol{I}]\boldsymbol{p}(i)$; $\nu = [\hat{\lambda}(i) - \mu\hat{\lambda}(i-1)]\tilde{\epsilon}$
$\alpha(i) = \left[\boldsymbol{p}^{\mathrm{H}}(i)\boldsymbol{p}_R\right]^{-1}(\mu - \eta)\boldsymbol{p}^{\mathrm{H}}(i)\boldsymbol{g}(i-1)$; $(0 \le \eta \le 0.5)$
$w(i) = w(i-1) + \alpha(i)p(i)$
$g(i) = \mu g(i-1) - \alpha(i)p_R - (|y(i)|^2 x(i)x^{\mathrm{H}}(i))w(i-1)$
$+ \gamma x(i)y^*(i) + \nu[a/(2\tilde{\epsilon}) - w(i-1)]$
$\beta(i) = [g^{\mathrm{H}}(i-1)g(i-1)]^{-1}[g(i) - g(i-1)]^{\mathrm{H}} g(i)$
$\boldsymbol{p}(i+1) = \boldsymbol{g}(i) + \beta(i)\boldsymbol{p}(i)$
$\delta_{\hat{\lambda}} = \mu_{\hat{\lambda}}\left[\tilde{\epsilon}\,\|w(i)\|_2^2 - \mathrm{Re}\left\{w^{\mathrm{H}}(i)a\right\} + \delta\right]$
while $\delta_\lambda \le -\hat{\lambda}(i)$ or $\delta_\lambda \ge \delta_{\lambda\max}$
$\delta_\lambda \Rightarrow \delta_\lambda/2$
end
$\hat{\lambda}(i+1) = \hat{\lambda}(i) + \delta_\lambda$

8.1 Proposed WC-CCM algorithm

In this part of the simulations, the WC-CCM design algorithm of Table 1 that uses a SOC programme [28] is compared with the loaded-SMI [3], the optimal SINR [2] and the WC optimisation-based CMV algorithm [5]. In the next simulations, it is considered that $|s_1| = 1$, $\delta = 1$, $\gamma = 1$, $\epsilon = 2.1$ and $\mu = 0.995$ unless otherwise specified. In addition to user 1, the desired signal, there are four interferers, the powers (P) relative to user 1 and DoA in degrees of which are detailed in Table 4.

The array steering vector is corrupted by local coherent scattering

$$\boldsymbol{a}_1 = \boldsymbol{a} + \sum_{k=1}^{4} \mathrm{e}^{\mathrm{j}\Phi_k} \boldsymbol{a}_{\mathrm{sc}}(\theta_k) \tag{61}$$

where Φ_k is uniformly distributed between zero and 2π and θ_k is uniformly distributed with a standard deviation of two degrees with the assumed direction as the mean. The mismatch changes for every realisation and is fixed over the snapshots of each simulation trial.

Fig. 2 shows the SINR as a function of the design parameter ϵ for different levels of mismatch, where its level corresponds to the standard deviation of the local scattering. In Fig. 3, no mismatch is considered but different noise levels. Both simulations show performance degradations when ϵ is chosen close to $\sqrt{M}$, especially for high SNR values. The simulations corroborate the analysis and show that the optimal value for ϵ depends on the SNR.

Table 4 Interference scenario

Snapshot	*P*, dB relative to user1/DoA				
	User 1 (desired user)	User 2	User 3	User 4	User 5
1–1000	0/93	13/120	1/140	22/67	10/157
1001–2000	0/93	30/120	25/170	4/104	9/68

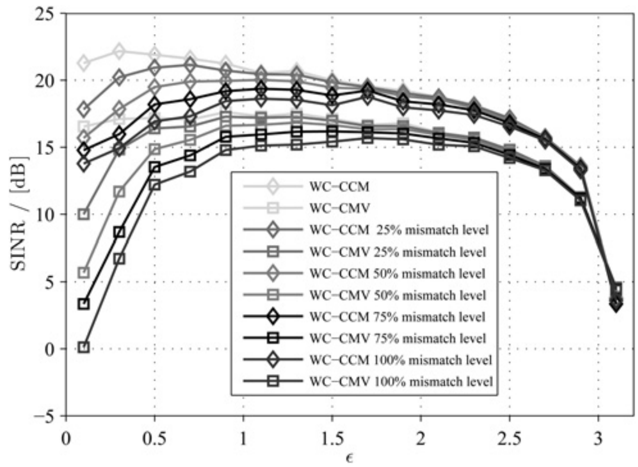

Fig. 2 *SINR against* $\in$*, SNR = 15 dB, M = 10 and i = 200*

Fig. 4 presents the SINR performance over the snapshots in the presence of local coherent scattering. At time index $i = 1000$, the interference scenario changes according to Table 4 and interferers assume different power levels and DoAs. With this change, the beamformers must adapt to the new environment and their tracking performance is assessed by a plot showing the SINR performance against the snapshots. The proposed WC-CCM algorithm shows in Fig. 4 a significantly better SINR performance than the WC-CMV [5] and the loaded-SMI algorithm. In terms of tracking performance, the proposed WC-CCM algorithm of Table 1 is able to effectively adjust to the new environment. Fig. 5 shows the SINR performance against the SNR for $i = 500$ snapshots. The curves show that the proposed WC-CCM algorithm is more robust against mismatch problems than the existing WC-CMV and loaded-SMI algorithms.

8.2 Low-complexity robust adaptive beamforming

In this subsection, we assess the SINR performance of the proposed low-complexity robust beamforming algorithms in Tables 2 and 3 that are devised for an online operation. In the simulations, the same parameters of the previous subsection are used and, in addition, the step sizes are $\mu_\lambda(\mathrm{CMV}) = 800$ and $\mu_\lambda(\mathrm{CCM}) = 100$. The limitation on the

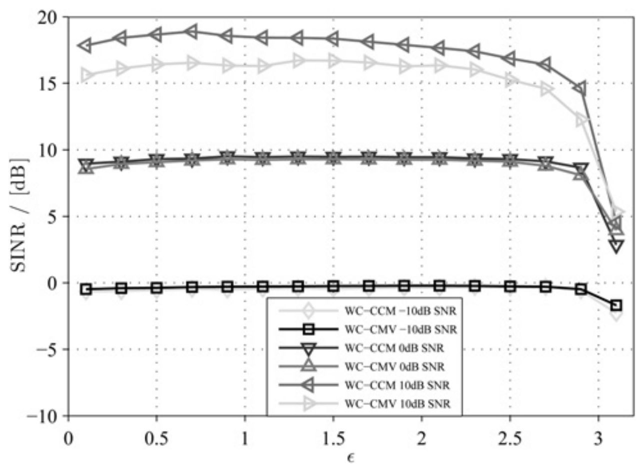

Fig. 3 *SINR against* $\in$*, perfect array steering vector (ASV) and M = 10*

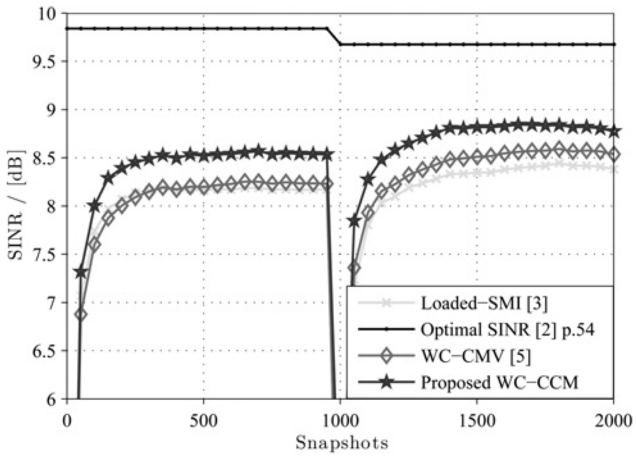

Fig. 4 *SINR against snapshots, SNR = 0 dB and local coherent scattering*

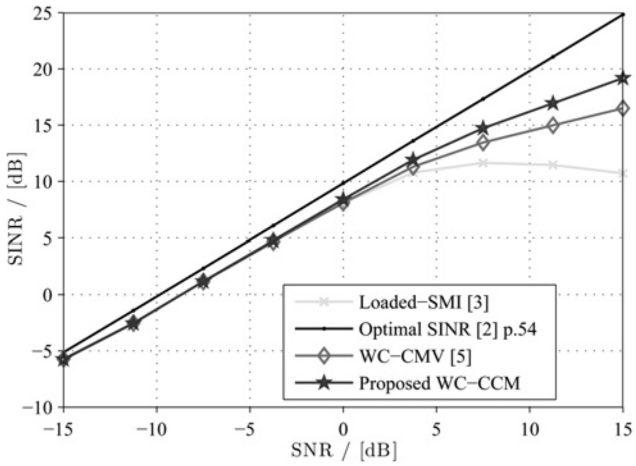

Fig. 5 *SINR against SNR, local coherent scattering, i = 1500 and M = 10*

update is set to $\delta_{\lambda\max} = 200$. For the robust constraints, we employ $\epsilon = \tilde{\epsilon} = 2.1$ and the parameters $|s| = 1$, $\delta = 1$ and $\gamma = 1$. According to the different constraint functions, the equality is a special case for $M = 10$ and cannot be generalised. In addition to the desired user (user 1), there are four interferers whose relative powers (P) with respect to the desired user and DoA in degrees are detailed in Table 5. At time index $i = 1000$, the adaptive beamforming algorithms are confronted with a change of scenario given in Table 4 and the interferers assume different power levels and DoAs. In this situation, the adaptive beamforming algorithms must adapt to the new conditions and their tracking performance is evaluated.

Fig. 6 shows the SINR performance as a function of the number of snapshots in the presence of local coherent scattering. The results of Fig. 6 show that the proposed

Table 5 Interference scenario

Snapshot	*P*, dB relative to user1/DoA				
	User 1 (desired user)	User 2	User 3	User 4	User 5
1–1000	0/93	10/120	5/140	10/150	7/105
1001–2000	0/93	30/120	34/170	6/104	9/68

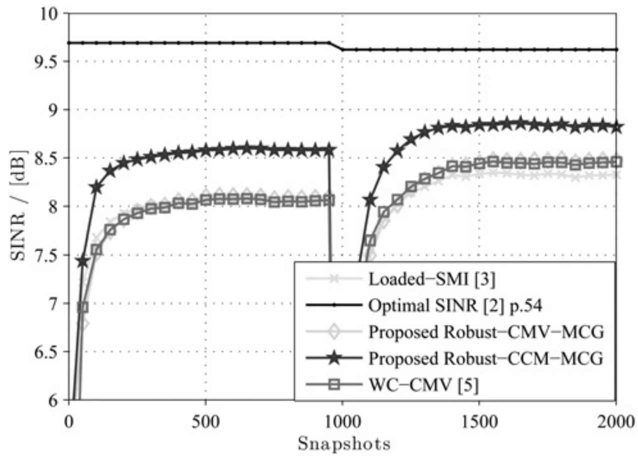

Fig. 6 *SINR against snapshots, local coherent scattering, SNR = 0 dB and $|s_1| = 1$ and $\delta = 1$*

RCCM-MCG algorithm has a superior SINR performance to the existing WC-CMV [5] algorithm, the proposed RCMV-MCG algorithm and the loaded-SMI algorithm. The RCMV-MCG algorithm has a comparable performance with the WC-CMV [5] algorithm, but the latter has a significantly higher computational cost. The SINR performance against the SNR is presented in Fig. 7. Although the proposed RCMV-MCG algorithm shows an equivalent performance to the WC-CMV [5], the proposed RCCM-MCG algorithm exploits the constant modulus property and performs better than existing approaches. Fig. 8 shows the SINR performance against the number of snapshots for the same scenario as in Fig. 6 with different values of γ while keeping delta fixed. The results show that for certain values the convexity constraint is satisfied and the algorithm converges to a higher SINR value, whereas for smaller values of gamma the algorithm converges to lower values of SINR, suggesting that a local minimum of the constant modulus cost function might have been reached. Therefore the values of γ should be set appropriately in order to ensure an optimised performance. This adjustment could be performed with either some prior knowledge about the energy of the signal or with the help of a procedure that computes the energy of the signal online.

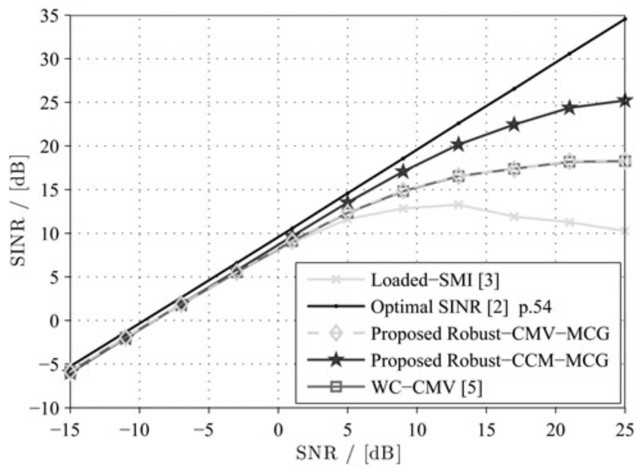

Fig. 7 *SINR against SNR, local coherent scattering, i = 1500 and M = 10*

IET Signal Process., 2014, Vol. 8, Iss. 5, pp. 447–457
doi: 10.1049/iet-spr.2013.0166

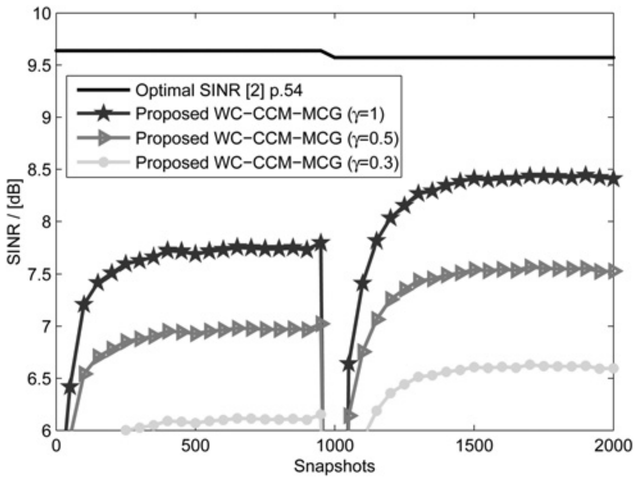

Fig. 8 *SINR against snapshots with different γ, local coherent scattering and SNR = 0 dB*

9 Conclusion

We have proposed a robust beamforming algorithm based on the WC constraint and the CCM design criterion which is called WC constant modulus criterion (WC-CCM). The proposed approach exploits the constant modulus property of the desired signal. The problem can be solved iteratively, where each iteration is effectively solved by a SOC programme. Compared with the conventional WC optimisation-based approach using the minimum variance design, the proposed algorithm shows better results especially in the high SNR regime.

In addition to the WC-CCM algorithm, we have also developed two low-complexity robust adaptive beamforming algorithms, namely, the RCMV-MCG and the RCCM-MCG. The proposed algorithms use a constraint similar to the WC optimisation-based approach. It has been shown that the joint optimisation approach allows the exploitation of highly efficient 'online' algorithms like the MCG method which performs just one iteration per snapshot taking advantage of previous computations. As a result, the complexity is reduced by more than an order of magnitude compared with the WC optimisation-based beamformer which is solved with a SOC programme. Although the proposed RCMV-MCG performs equivalently, the proposed RCCM-MCG algorithm based on the CCM design criterion shows a better performance which takes advantage of the constant modulus property of the signal amplitude of the desired user.

10 References

1 Trees, H.V.: 'Optimum array processing' (John Wiley, 2002)
2 Li, J., Stoica, P.: 'Robust adaptive beamforming' (Wiley: Hoboken, NJ, 2006)
3 Cox, H., Zeskind, R.M., Owen, M.H.: 'Robust adaptive beamforming', *IEEE Trans. Acoust. Speech Signal Process.*, 1987, **ASSP-35**, pp. 1365–1376
4 Gershman, A.B.: 'Robust adaptive beamforming in sensor arrays', *Int. J. Electron. Commun.*, 1999, **53**, pp. 305314
5 Vorobyov, S.A., Gershman, A.B., Luo, Z.-Q.: 'Robust adaptive beamforming using worst-case performance optimization: a solution to the signal mismatch problem', *IEEE Trans. Signal Process.*, 2003, **51**, pp. 313–324
6 Stoica, P., Wang, Z., Li, J.: 'Robust capon beamforming', *IEEE Signal Process. Lett.*, 2003, **10**, (6), pp. 172–175
7 Li, J., Stoica, P.: 'On robust capon beamforming and diagonal loading', *IEEE Trans. Signal Process.*, 2003, **51**, (7), pp. 1702–1715
8 Li, J., Stoica, P., Wang, Z.: 'Doubly constrained robust capon beamformer', *IEEE Trans. Signal Process.*, 2004, **52**, (9), pp. 2407–2423
9 Lorenz, R., Boyd, S.: 'Robust minimum variance beamforming', *IEEE Trans. Signal Process.*, 2005, **53**, pp. 1684–1696
10 Chen, C.-Y., Vaidyanathan, P.P.: 'Quadratically constrained beamforming robust against direction-of-arrival mismatch', *IEEE Trans. Signal Process.*, 2007, **55**, (8), pp. 4139–4150
11 Hassanien, A., Vorobyov, S.A.: 'Robust adaptive beamforming using sequential quadratic programming: an iterative solution to the mismatch problem', *IEEE Signal Process. Lett.*, 2008, **15**, pp. 733–736
12 Gershman, A.B., Sidiropoulos, N.D., Shahbazpanahi, S., Bengtsson, M., Ottersten, B.: 'Convex optimization-based beamforming', *IEEE Signal Process. Mag.*, 2010, **27**, (3), pp. 62–75
13 Khabbazibasmenj, A., Vorobyov, S., Hassanien, A.: 'Robust adaptive beamforming based on steering vector estimation with as little as possible prior information', *IEEE Trans. Signal Process.*, 2012, **60**, (6), pp. 2974–2987
14 de Lamare, R.C., Sampaio-Neto, R.: 'Blind adaptive code-constrained constant modulus algorithms for CDMA interference suppression in multipath channels', *IEEE Signal Process. Lett.*, 2005, **9**, (4), pp. 334–336
15 de Lamare, R.C., Haardt, M., Sampaio-Neto, R.: 'Blind adaptive constrained reduced-rank parameter estimation based on constant modulus design for CDMA interference suppression', *IEEE Trans. Signal Process.*, 2008, **56**, (2), pp. 2470–2482
16 Chen, Y., Le-Ngoc, T., Champagne, B., Xu, C.: 'Recursive least squares constant modulus algorithm for blind adaptive array', *IEEE Trans. Signal Process.*, 2004, **52**, (5), pp. 1452–1456
17 Wang, L., de Lamare, R.C.: 'Constrained adaptive filtering algorithms based on conjugate gradient techniques for beamforming', *IET Signal Process.*, 2010, **4**, pp. 686–697
18 Luenberger, D.G.: 'Linear and nonlinear programming' (Addison-Wesley, Reading, MA, 1984, 2nd edn.)
19 Feldman, D.D., Griffiths, L.J.: 'A projection approach to robust adaptive beamforming', *IEEE Trans. Signal Process.*, 1994, **42**, pp. 867–876
20 de Lamare, R.C., Wang, L., Fa, R.: 'Adaptive reduced-rank LCMV beamforming algorithms based on joint iterative optimization of filters: design and analysis', *Signal Process.*, 2010, **90**, (2), pp. 640–652
21 Vorobyov, S.A., Gershman, A.B., Rong, Y.: 'On the relationship between the worst-case optimization-based and probability-constrained approaches to robust adaptive beamforming'. Proc. IEEE ICASSP, 2, Honolulu, HI, Apr. 2007, pp. 977–980
22 Wang, L., de Lamare, R.C.: 'Low-complexity adaptive step size constrained constant modulus sg algorithms for blind adaptive beamforming', *Signal Process.*, 2009, **89**, (12), pp. 2503–2513
23 de Lamare, R.C., Sampaio-Neto, R., Haardt, M.: 'Blind adaptive constrained constant-modulus reduced-rank interference suppression algorithms based on interpolation and switched decimation', *IEEE Trans. Signal Process.*, 2011, **59**, (2), pp. 681–695
24 Meng, Y., You, M., Liu, J.: 'Robust noise suppression least square constant modulus interference cancellation'. Proc. 2009 IEEE 20th Int. Symp. Personal, Indoor and Mobile Radio Communications, September 2009, pp. 526–530
25 Sturm, J.F.: 'Using sedumi 1.02, a matlab toolbox for optimization over symmetric cones', *Optim. Meth. Softw.*, 1999, **11**, (12), pp. 625–653
26 Niesen, U., Shah, D., Wornell, G.W.: 'Adaptive Alternating Minimization Algorithms', 2009, 55, no. 3
27 Chang, P.S., Willson, A.N.: 'Analysis of conjugate gradient algorithms for adaptive filtering', *IEEE Trans. Signal Process.*, 2000, **48**, pp. 409–418
28 Grant, M., Boyd, S.: 'CVX: matlab software for disciplined convex programming, version 1.21'. http://www.cvxr.com/cvx, January 2011