Boolean combinations of ω-rational trace languages: emptiness, rationality, regularity

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Abstract

This paper studies decision problems for Boolean combinations of ω-rational trace languages. The complexities of these decision problems (emptiness, regularity, rationality) are classified depending on properties of the independence alphabets and of the Boolean combinations allowed. For any of the problems, we obtain a trichotomy ranging from decidable over a low level of the arithmetical hierarchy to a low level of the analytical hierarchy.

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1 Introduction

Finite and infinite Mazurkiewicz traces [32, 10] (cf. [9]) are a formal model for the description of the (nonterminating) behavior of concurrent systems that synchronize via shared actions. They are closely related to communicating finite state machines [5, 16, 17, 30], have been studied in the contexts of model checking (e.g., [12, 3]) and the distributed control synthesis problem [15, 18, 19] and allow to model multi-buffer systems [25] and multi-pushdown systems [26, 27, 29].

The idea behind traces is that sequences of basic actions, i.e., words, form interleavings of the executions of several, concurrently running, processes. Words that are interleavings of one and the same concurrent execution are considered equivalent. Formally, the concurrency in a system is given by an independence alphabet that describes which pairs of actions are executed by disjoint sets of sequential processes and can therefore commute in interleaving executions. As a consequence, questions like "Do two concurrent systems allow the same concurrent executions?" translate into "Does every interleaving execution of one system have an equivalent interleaving execution of the other system?" or "Are the closures of the sets of interleaving executions under the equivalence of words equal?" In other words, one is interested in the closure of a language under this equivalence.

For regular sets of finite words, questions of this type have been studied in detail. Here, e.g., non-inclusion asks for the existence of some finite object (a word) with some decidable property (to belong to the closure of the first regular set, but not to the closure of the second). Hence inclusion belongs to the first universal level of the arithmetical hierarchy Π^0_1. Differently, regularity asks for the existence of a finite object (a finite automaton) such that all finite objects (words) satisfy some decidable property (to belong to the closure if, and only if, they are accepted by the automaton). Hence, this problem belongs to the second existential level of the arithmetical hierarchy Σ^0_2. These easy observations do by no means exclude that inclusion or regularity can be expressed by simpler means, e.g., are decidable or belong to Π^0_1.

The computer science literature provides a clear distinction between the decidable and undecidable instances of these problems. This border is described using two properties of independence alphabets, namely "transitivity" and "diagonality" (which is strictly more general). The known answers are as follows: inclusion and regularity are decidable for transitive independence alphabets, and otherwise complete for Π^0_1 and Σ^0_2, respectively [2, 44, 28]; disjointness is decidable even for diagonal independence alphabets and Π^0_1-complete otherwise [1].

An attempt to extend these results to infinite traces was started in [28]. That paper shows in particular that, for closures of regular languages of infinite traces, regularity for transitive independence alphabets and disjointness for diagonal independence alphabets are in Π^0_1; it was left open as to whether these problems are decidable as they are in the finite case [44, 1]. The results of this paper imply affirmative answers to both these questions.

In detail, we study decision problems for Boolean combinations of closures of regular ω-languages, namely the *emptiness*, the *regularity*, and the *rationality* problem that asks whether a given Boolean combination is the closure of some regular language (this latter problem has been considered for finite words in [2], but does not appear in [28]). For infinite words, all these problems belong to the second universal level of the analytical hierarchy Π^1_2. Depending on graph theoretic properties of the independence alphabet (i.e., the "architecture" of the distributed system) and properties of the Boolean combination, we classify all these problems into decidable, Π^0_1-complete, Σ^0_2-complete, and Π^1_1-hard (cf. Table 1). The only graph properties appearing in this classification are "transitive", "diagonal", and "non-diagonal"; and the only properties of the Boolean combinations are "trivial" (i.e., $\emptyset$ or "all words"), "finite union", "positive", and "non-positive". Looking at the results, one observes two striking similarities between the cases of finite and infinite words:

1. A problem for infinite words is decidable if, and only if, its version for finite words is decidable.
2. If one considers only diagonal independence alphabets, then there is no difference in complexity between a problem for infinite words and for finite words.

One also notes that the regularity problem and the rationality problem behave much the same: the levels of undecidability are almost always the same. Nevertheless, these two problems are quite different: the regularity problem is of maximal difficulty even for the trace closure of a rational language (where the rationality problem has a trivial answer).

On the other hand, there are also two striking differences:

1. Any of the problems on finite words yields a dichotomy while, for infinite words, we always have a trichotomy.
2. If one allows some non-diagonal independence alphabet, then the problem for finite words remains arithmetical while it is Π^1_1-hard for infinite words.

2 Preliminaries

The the set $\mathbb{N}$ of non-negative integers starts with 0 and, for $n \in \mathbb{N}$, we set $[n] = \{1, \dots, n\}$. The powerset of a set X is denoted $\mathcal{P}(X)$.

Words and automata

Let Γ be an alphabet. The set of finite words over Γ is denoted Γ^*, Γ^ω denotes the set of ω-words over Γ, and $\Gamma^\infty = \Gamma^* \cup \Gamma^\omega$. For $x \in \Gamma^\infty$ and $0 \le m \le |x|$, $x[1 \dots m]$ is the prefix of w of length m. For two words or ω-words $x, y \in \Gamma^\infty$, the *shuffle* $x \sqcup\!\sqcup\, y \subseteq \Gamma^\infty$ is the set of all words or ω-words $x_0y_0\, x_1y_1\, x_2y_2 \cdots$ where $x_i, y_i \in \Gamma^*$ are finite words with $x = x_0x_1x_2 \cdots$ and $y = y_0y_1y_2 \cdots$. For $\alpha \in \Gamma^\infty$ and $A \subseteq \Gamma$, the *projection* $\mathrm{proj}_A(\alpha) \in A^\infty$

problem	decidable	otherwise
$\mathrm{EmptP}_*(\mathbb{A},\Phi)$ (Theorem 3.2)	$\mathbb{A} \subseteq \mathrm{TransA}$ or $\mathbb{A} \subseteq \mathrm{DiagA}$ and $\Phi \subseteq \mathrm{monBF}$ or $\Phi \subseteq \mathrm{MAX}$	Π^0_1-complete
$\mathrm{EmptP}_\omega(\mathbb{A},\Phi)$	$\mathbb{A} \subseteq \mathrm{TransA}$ (Thm. 4.1) or $\mathbb{A} \subseteq \mathrm{DiagA}$ and $\Phi \subseteq \mathrm{monBF}$ (Thm. 5.2) or $\Phi \subseteq \mathrm{MAX}$ (Prop. 2.2)	Π^0_1-complete (Thm. 3.2 and Cor. 5.3) (if $\mathbb{A} \not\subseteq \mathrm{TransA}$, $\mathbb{A} \subseteq \mathrm{DiagA}$ and $\Phi \not\subseteq \mathrm{monBF}$) or Π^1_1-complete (Thm. 6.1) (if $\mathbb{A} \not\subseteq \mathrm{DiagA}$, $\Phi \not\subseteq \mathrm{MAX}$, $\Phi \subseteq \mathrm{monBF}$) or (Π^1_1, Π^1_2) (Thm. 6.1) (if $\mathbb{A} \not\subseteq \mathrm{DiagA}$ and $\Phi \not\subseteq \mathrm{monBF}$)
$\mathrm{RatP}_*(\mathbb{A},\Phi)$ (Theorem 3.4)	$\mathbb{A} \subseteq \mathrm{TransA}$ or $\Phi \subseteq \mathrm{MAX}$	Σ^0_2-complete
$\mathrm{RatP}_\omega(\mathbb{A},\Phi)$	$\mathbb{A} \subseteq \mathrm{TransA}$ (Thm. 4.1) or $\Phi \subseteq \mathrm{MAX}$ (Prop. 2.2)	Σ^0_2-complete (Thm. 5.4 and 3.4(2)) (if $\mathbb{A} \not\subseteq \mathrm{TransA}$, $\mathbb{A} \subseteq \mathrm{DiagA}$, $\Phi \not\subseteq \mathrm{MAX}$) or (Π^1_1, Π^1_2) (Thm. 6.1(2)) (if $\mathbb{A} \not\subseteq \mathrm{DiagA}$ and $\Phi \not\subseteq \mathrm{MAX}$)
$\mathrm{RegP}_*(\mathbb{A},\Phi)$ (Theorem 3.4)	$\mathbb{A} \subseteq \mathrm{TransA}$ or $\Phi \subseteq \mathrm{CONST}$	Σ^0_2-complete
$\mathrm{RegP}_\omega(\mathbb{A},\Phi)$	$\mathbb{A} \subseteq \mathrm{TransA}$ (Thm. 4.3) or $\Phi \subseteq \mathrm{CONST}$ (Prop. 2.2)	Σ^0_2-complete (Thm 5.4 and 3.4(2')) (if $\mathbb{A} \not\subseteq \mathrm{TransA}$, $\mathbb{A} \subseteq \mathrm{DiagA}$, $\Phi \not\subseteq \mathrm{CONST}$) or (Π^1_1, Π^1_2) (Thm. 6.1) (if $\mathbb{A} \not\subseteq \mathrm{TransA}$ and $\Phi \not\subseteq \mathrm{CONST}$)

Table 1 Symmary of results.
Here, $\mathbb{A} \subseteq \mathrm{IndA}$ and $\Phi \subseteq \mathrm{BF}$ are decidable sets. In the second column, one finds the conditions on $\mathbb{A}$ and Φ that make the problem decidable. The last column describes the complexity in case $\mathbb{A}$ and Φ do not satisfy the condition in the second column. For ω-words, the last column contains several complexities together with the conditions on $\mathbb{A}$ and Φ that characterize this complexity ("(Π^1_1, Π^1_2)" means "Π^1_1-hard and in Π^1_2").

is obtained by deleting all letters from α that do not belong to A. We write $|\alpha|_A$ for the length of the projection $\mathrm{proj}_A(\alpha)$; $\mathrm{proj}_a(\alpha)$, $|\alpha|_a$, and $|\alpha|$ abbreviate $\mathrm{proj}_{\{a\}}(\alpha)$, $|\alpha|_{\{a\}}$, and $|\alpha|_\Gamma$, respectively.

Subsets of Γ^* are called *finitary languages* or *$*$-languages* while subsets of Γ^ω are *ω-languages*.

Let $K \subseteq \Gamma^*$ and $L \subseteq \Gamma^\infty$. Then the *concatenation* of K and L is $K \cdot L = \{u\,x \mid u \in K \text{ and } x \in L\} \subseteq \Gamma^\infty$. The *Kleene iteration* K^* is the set of words $u_1\,u_2 \cdots u_m$ for some $m \geq 0$ with $u_1, u_2, \ldots, u_m \in K$. The *$\omega$-iteration* K^ω of K is the set of ω-words $u_1\,u_2 \cdots$ with $u_i \in K \setminus \{\varepsilon\}$ for all $i \geq 1$.

Let $K, L \subseteq \Gamma^\infty$. The *shuffle* of K and L is the union of all shuffles $x \sqcup\!\sqcup\, y$ with $x \in K$ and $y \in L$, we will also apply this operation to more than two languages in the obvious manner.

A *nondeterministic finite automaton* or *nfa* is a tuple $\mathcal{A} = (Q, \Gamma, S, T, F)$ where Q is a finite set of "states", Γ an alphabet, $S, F \subseteq Q$ sets of "initial" and "final states", respectively, and $T \subseteq Q \times \Gamma \times Q$ the transition relation. For a state $p \in Q$, we write $\mathcal{A}^{p\to}$ for the nfa $(Q, \Gamma, \{p\}, T, F)$ that only differs from $\mathcal{A}$ in its initial state. For $p, q \in Q$ and $u \in \Gamma^*$, we write $p \xrightarrow{u} q$ if there is some u-labeled path from p to q. If there is such a path that visits some accepting state, we write $p \overset{u}{\Longrightarrow} q$. A word $u \in \Gamma^*$ is *accepted* by $\mathcal{A}$ if there are $p \in S$ and $q \in F$ with $p \xrightarrow{u} q$; $L^*(\mathcal{A})$ denotes the $*$-language of words accepted by $\mathcal{A}$. A $*$-language $L \subseteq \Gamma^*$ is *regular* if there exist an nfa $\mathcal{A}$ with $L = L^*(\mathcal{A})$.

An ω-word α is *accepted* by $\mathcal{A}$ if it splits into finite words $\alpha = u_0\,u_1 \cdots$ such that $p \xrightarrow{u_0} q$ and $q \xrightarrow{u_i} q$ for all $i \geq 1$ for some $p \in S$ and $q \in F$. The set of ω-words accepted by $\mathcal{A}$ is $L^\omega(\mathcal{A})$; an ω-language L is *regular* if there is some nfa $\mathcal{A}$ with $L = L^\omega(\mathcal{A})$.

Finally, we recall some clasical notions on binary relations on Γ^∞ from [22], cf. [4] for the case of relations on finite words.

Let $R \subseteq \Gamma^* \times \Gamma^*$ and $S \subseteq \Gamma^\infty \times \Gamma^\infty$ be two binary relations. Then $R \cdot S = \{(uv, u'v') \mid (u, u') \in R \text{ and } (v, v') \in S\}$ is the *concatenation* of R and S. The *power* of R is defined inductively by $R^0 = \{(\varepsilon, \varepsilon)\}$ and $R^{n+1} = R \cdot R^n$. The *finite iteration* of R is $R^* = \bigcup_{n \geq 0} R^n$; its *$\omega$-iteration* is the set

$$R^\omega = \{(u_1 u_2 u_3 \cdots, v_1 v_2 v_3 \cdots) \in \Gamma^\infty \times \Gamma^\infty \mid (u_i, v_i) \in R \text{ for all } i \geq 1\}\,.$$

The relation $S \subseteq \Gamma^\infty \times \Gamma^\infty$ is *rational* if it can be constructed from finite relations on Γ^* using union, concatenation (with the first factor a relation on Γ^*), and finite and ω-iteration (applied to relations on Γ^*).

Rational relations describe the behavior of transducers: a *transducer* is a tuple $\mathcal{A} = (Q, \Gamma, S, T, F)$ where Q, Γ, S and F are as in nfas and $T \subseteq Q \times (\Gamma \cup \{\varepsilon\})^2 \times Q$ the transition relation. For $p, q \in Q$ and $u, v \in \Gamma^*$, we write $p \xrightarrow{(u,v)} q$ if there is some (u, v)-labeled path from p to q. A pair of possibly infinite words $(\alpha, \beta) \in \Gamma^\infty \times \Gamma^\infty$ is *accepted* by $\mathcal{A}$ if α and β split into $\alpha = u_0\,u_1 \cdots$ and $\beta = v_0\,v_1 \cdots$ such that there are states $p \in S$ and $q \in F$ with $p \xrightarrow{(u_0,v_0)} q$ and $q \xrightarrow{(u_i,v_i)} q$ for all $i \geq 1$ (if $u_i = v_i = \varepsilon$ for all $i > j$, then α and β both are finite and there is a sequence of transitions from q back to q that all are labeled $(\varepsilon, \varepsilon)$). The set of pairs accepted by $\mathcal{A}$ is $R^\infty(\mathcal{A})$. Then a relation on Γ^∞ is rational iff it is accepted by some transducer.

From an nfa $\mathcal{A}$ and a transducer $\mathcal{B}$, one can construct an nfa $\mathcal{A}'$ with $v \in L^*(\mathcal{A}')$ iff there exists $u \in L^*(\mathcal{A})$ with $(u, v) \in R^\infty(\mathcal{B})$. Similarly, from a Büchi-automaton $\mathcal{A}$ and a transducer $\mathcal{B}$, one can construct a Büchi-automaton $\mathcal{A}'$ with $\beta \in L^\omega(\mathcal{A}')$ iff there exists $\alpha \in L^\omega(\mathcal{A})$ with $(\alpha, \beta) \in R^\infty(\mathcal{B})$.

▶ **Convention.** Many objects considered in this paper appear in two forms: word vs. ω-word, *-language vs. ω-language. At places, we blur this distinction and just talk about words and languages assuming that the context explains what version is meant.

Traces

An *independence alphabet* is a pair (Γ, I) of an alphabet Γ and an irreflexive and symmetric *independence relation* $I \subseteq \Gamma \times \Gamma$; its complement is the *dependence relation* $D = \Gamma^2 \setminus I$. The set of all independence alphabets is denoted IndA.

Two words $x, y \in \Gamma^\infty$ are *equivalent* [10] (denoted $x \sim y$) if, for all (not necessarily distinct) letters $a, b \in \Gamma$ with $(a, b) \in D$, we have $\text{proj}_{\{a,b\}}(x) = \text{proj}_{\{a,b\}}(y)$.[1]

For $L \subseteq \Gamma^\infty$, the *closure* is the set $[L] = \{y \in \Gamma^\infty \mid \exists x \in L \colon x \sim y\}$ of all (ω-)words y that are equivalent to some element of L. The set $L \subseteq \Gamma^\infty$ is *closed* if $L = [L]$. If $L = \{x\}$ is a singleton, the closure $[L]$ of L equals the equivalence class $[x]$ of x wrt. $\sim$.

In this paper, we will regularly be concerned with closures of regular sets and with closed regular sets. If L is closed and regular, then it is the closure $[L]$ of the regular set L. Conversely, suppose $(a, b) \in I$. The closure of the regular set $L = \{ab\}^*$ is the set $[L]$ of words $u \in \{a, b\}^*$ with $|u|_a = |u|_b$ and therefore not regular. Thus, regular closed sets are closures of regular sets, but there are closures of regular sets that are not regular.

▶ Remark. Recognizable and rational sets of (infinite) traces [32, 11] correspond to closed regular and closures of regular languages, respectively. We could therefore phrase our results in terms of these sets of traces; the author prefers the current point of view as it only concerns sets of words and avoids the consideration of traces as equivalence classes of words.

Decision problems

This paper deals with decision problems related to Boolean combinations of closures of regular languages, e.g., we will investigate the decidability status of the question whether the language $[L^*(\mathcal{A}_1)] \cap [L^*(\mathcal{A}_2)]$ is empty (is regular, resp.) for two nfas $\mathcal{A}_1$ and $\mathcal{A}_2$. To present all these problems uniformly, we next introduce some notions.

Let $\varphi \colon \{0, 1\}^n \to \{0, 1\}$ be a Boolean function and $L_1, \dots, L_n \subseteq \Gamma^\odot$ $\odot$-languages over Γ. The $\odot$-language $\langle L_1, \dots, L_n \rangle^\varphi$ is the set of words $x \in \Gamma^\odot$ such that $\varphi(\chi_{L_1}(x), \chi_{L_2}(x), \dots, \chi_{L_n}(x)) = 1$ where χ_{L_i} is the characteristic function of L_i.

▶ **Example.** Let Id be the unary identity function, $\wedge$ the binary minimum-function, and $\neg$ the negation. Then, for any languages L_1 and L_2, we have $\langle L_1 \rangle^{\text{Id}} = L_1$, $\langle L_1, L_2 \rangle^\wedge = L_1 \cap L_2$, and $\langle L_1 \rangle^\neg = \Gamma^\odot \setminus L_1$.

We now come to the decision problems studied in this paper.

▶ **Definition 2.1.** *Let $\odot \in \{*, \omega\}$, $\mathbb{A} \subseteq$ IndA be some set of independence alphabets, Φ some set of Boolean functions, and* $\text{GenP} \in \{\text{EmptP}, \text{RatP}, \text{RegP}\}$. *Then $\text{GenP}_\odot(\mathbb{A}, \Phi)$ is the set of tuples $((\Gamma, I), \varphi, \mathcal{A}_1, \dots, \mathcal{A}_n)$ with $(\Gamma, I) \in \mathbb{A}$ an independence alphabet from $\mathbb{A}$, $\varphi \in \Phi$ an n-ary Boolean function from Φ, and $\mathcal{A}_1, \dots, \mathcal{A}_n$ nfas over Γ with*

$$H = \left\langle [L^\odot(\mathcal{A}_1)], [L^\odot(\mathcal{A}_2)], \dots, [L^\odot(\mathcal{A}_n)] \right\rangle^\varphi$$

such that

[1] Two other, equivalent, definitions of $\sim$ have been formulated by Gastin [10].

- $\mathrm{GenP} = \mathrm{EmptP}$ *and* $H = \emptyset$ *(the* emptiness problem*),*
- $\mathrm{GenP} = \mathrm{RatP}$ *and* $H = [L]$ *for some regular language* $L \subseteq \Gamma^{\odot}$ *(the* rationality problem*),*
- *or* $\mathrm{GenP} = \mathrm{RegP}$ *and* H *is regular (the* regularity problem*)*

▶ **Example.** The problems $\mathrm{EmptP}_*((\Gamma, I), \wedge)$ (the "disjointness problem"), $\mathrm{RatP}_*((\Gamma, I), \neg)$, and $\mathrm{RegP}_*((\Gamma, I), \mathrm{Id})$ are undecidable in general [1, 44].

The above problems are parametrized by a set $\mathbb{A}$ of independence alphabets and a set of Boolean functions Φ. We next describe particular sets that are central in our classification.

A Boolean function $\varphi\colon \{0,1\}^n \to \{0,1\}$ is

- *constant* if $\varphi(\bar{b}) = \varphi(\bar{c})$ holds for all $\bar{b}, \bar{c} \in \{0,1\}^n$.
- a *max-function* if it is constant or there exists a non-empty set $A \subseteq [n]$ with $\varphi(b_1, \ldots, b_n) = \max\{b_i \mid i \in A\}$ for all $b_1, \ldots, b_n \in \{0,1\}$.
- *monotone* if $\varphi(b_1, \ldots, b_n) \le \varphi(c_1, \ldots, c_n)$ for any arguments from $\{0,1\}$ with $b_i \le c_i$ for all $i \in [n]$.

The set of all constant (max-, monotone, resp.) functions is denoted CONST (MAX, monBF, resp.); BF denotes the set of all Boolean functions.

It turns out that the regularity problem for constant functions and the emptiness problem and the rationality problem for max-functions are simple (if not trivial).

proof: p. 16

▶ **Proposition 2.2.** *Let* $\odot \in \{*, \omega\}$*. Then* $\mathrm{EmptP}_\odot(\mathrm{IndA}, \mathrm{MAX})$*,* $\mathrm{RatP}_\odot(\mathrm{IndA}, \mathrm{MAX})$*, and* $\mathrm{RegP}_\odot(\mathrm{IndA}, \mathrm{CONST})$ *are decidable.*

After considering constant and max-functions, we next turn attention to Boolean functions outside these classes. Claim (1) in the following lemma ensures that the function Id induces the full difficulty of the regularity problem $\mathrm{RegP}_\odot(\mathbb{A}, \Phi)$ if $\Phi \not\subseteq \mathrm{CONST}$. Similarly, $\wedge \in \mathrm{monBF} \setminus \mathrm{MAX}$ ($\neg \notin \mathrm{monBF}$, resp.) gives rise to the full difficulty of $\mathrm{GenP}_\omega(\mathbb{A}, \Phi)$ if $\Phi \subseteq \mathrm{monBF}$ and $\Phi \not\subseteq \mathrm{MAX}$ ($\Phi \not\subseteq \mathrm{monBF}$).

proof: p. 16

▶ **Lemma 2.3.** *Let* $\odot \in \{*, \omega\}$, $\mathrm{GenP} \in \{\mathrm{EmptP}, \mathrm{RatP}, \mathrm{RegP}\}$*,* $\mathbb{A} \subseteq \mathrm{IndA}$*, and* $\varphi \in \mathrm{BF}$*.*

(1) If $\varphi \notin \mathrm{CONST}$*, then* $\mathrm{RegP}_\odot(\mathbb{A}, \mathrm{Id}) \le \mathrm{RegP}_\odot(\mathbb{A}, \varphi)$*.*
(2) If $\varphi \in \mathrm{monBF} \setminus \mathrm{MAX}$*, then* $\mathrm{GenP}_\odot(\mathbb{A}, \wedge) \le \mathrm{GenP}_\odot(\mathbb{A}, \varphi)$*.*
(3) If $\varphi \notin \mathrm{monBF}$*, then* $\mathrm{GenP}_\odot(\mathbb{A}, \neg) \le \mathrm{GenP}_\odot(\mathbb{A}, \varphi)$*.*

Besides the above classes of Boolean functions, there are also classes of independence alphabets central in this paper. Let (Γ, I) be an independence alphabet. It is

- *complete* if $(a, b) \in I$ for all $a, b \in \Gamma$ distinct.
- *transitive* if P_3 (the path on three vertices) is not an induced subgraph.
- *diagonal* if neither C_4 (the cycle on four vertices) nor P_4 is an induced subgraph.

The term "diagonal" was coined in [46], [1] uses the term "transitive forest" instead.

The set of all complete (transitive, diagonal, resp.) independence alphabets is denoted ComplA (TransA, DiagA, respectively).

Arithmetical and analytical hierarchy

This paper describes the complexity of the above decision problems. But as most of them are undecidable, this requires a yardstick for undecidability. The said hierarchies are such a yardstick. We refer the reader to [37] for definitions and useful properties.

3 Finitary languages

In this section, we study the problems $\mathrm{GenP}_*(\mathbb{A},\Phi)$, i.e., consider finite words. Many of the results follow easily from the literature, an exception is the rationality problem for non-transitive independence alphabets.

We first state upper bounds on some versions of the emptiness problem $\mathrm{EmptP}_*(\mathbb{A},\Phi)$, thanks to the work by Aalbersberg, Welzl, and Hoogeboom, the proofs of claim (2) and (3) are rather straightforward. Claim (1) follows since containment in the closure of a regular language is decidable.

proof: p. 17

▶ **Lemma 3.1.** *(1) The emptiness problem* $\mathrm{EmptP}_*(\mathrm{IndA},\mathrm{BF})$ *is in* Π^0_1.
(2) The emptiness problem $\mathrm{EmptP}_*(\mathrm{TransA},\mathrm{BF})$ *is decidable [2].*
(3) The emptiness problem $\mathrm{EmptP}_*(\mathrm{DiagA},\mathrm{monBF})$ *is decidable [1].*

Now the following dichotomy result follows easily from [1, 31, 44, 35].

proof: p. 18

▶ **Theorem 3.2.** *Let* $\mathbb{A}\subseteq\mathrm{IndA}$ *and* $\Phi\subseteq\mathrm{BF}$ *be decidable sets.*

(1) The emptiness problem $\mathrm{EmptP}_*(\mathbb{A},\Phi)$ *is decidable if* $\mathbb{A}\subseteq\mathrm{TransA}$, *or* $\mathbb{A}\subseteq\mathrm{DiagA}$ *and* $\Phi\subseteq\mathrm{monBF}$, *or* $\Phi\subseteq\mathrm{MAX}$.
(2) Otherwise, the emptiness problem $\mathrm{EmptP}_*(\mathbb{A},\Phi)$ *is* Π^0_1*-hard.*

We next turn to the rationality problem, i.e., the question whether a given Boolean combinations of closures of regular languages is the closure of some regular language.

We start by presenting two upper bounds. Again, citing work by Aalbersberg and Welzl, the first claim follows easily, and also the second follows easily from the definition of the rationality problem.

proof: p. 18

▶ **Lemma 3.3.** *(1) The rationality problem* $\mathrm{RatP}_*(\mathrm{TransA},\mathrm{BF})$ *is decidable [2].*
(2) The rationality problem $\mathrm{RatP}_*(\mathrm{IndA},\mathrm{BF})$ *is in* Σ^0_2.

For a lower bound, one first proves the Σ^0_2-hardness of $\mathrm{RatP}_*((\Gamma,I),\varphi)$ for any non-transitive independence alphabet (Γ,I) and any of the two Boolean functions $\wedge$ and $\neg$. This is achieved in two steps (where $\varphi\in\{\wedge,\neg\}$):

The first step step proves that $\mathrm{RatP}_*(\mathrm{StarA},\varphi)$ is Σ^0_2-hard where StarA is the set of all independence alphabets (Σ,I) that form a star. Prop. B.3

To show this, let $\alpha,\beta\colon A^+\to B^+$ be homomorphisms, $L\subseteq A^+$ regular, $c\notin A\cup B$, and (Σ,I) the star with center c and leaves $A\cup B$. Furthermore, let $K=\{u\in L\mid \alpha(u)=\beta(u)\}$. For $\gamma\in\{\alpha,\beta\}$, let W_γ denote the union of all languages $u\,\gamma(u)\sqcup c^{|\gamma(u)|}$ for $u\in L$. One first shows that K is finite iff $W_\alpha\cap W_\beta$ is the closure of some regular language iff Lemma B.2 $[L\cdot(Bc)^*]\setminus(W_\alpha\cap W_\beta)$ is the closure of some regular language. Following [31, 35], one can construct nfas $\mathcal{A}$ and $\mathcal{B}$ with $\Sigma^*\setminus[L^*(\mathcal{A})]=W_\alpha\cap W_\beta$ and $L^*(\mathcal{B})=L\cdot(Bc)^*$. It follows that K is finite iff $((\Sigma,I),\neg,\mathcal{A})\in\mathrm{RatP}_*(\mathrm{StarA},\neg)$ iff $((\Sigma,I),\wedge,\mathcal{A},\mathcal{B})\in\mathrm{RatP}_*(\mathrm{StarA},\wedge)$. Since, by [28], the finiteness of K is Σ^0_2-hard, the claim follows. Prop. B.1

In the second step, one proves $\mathrm{RatP}_*(\mathrm{StarA},\varphi)\le\mathrm{RatP}_*((\Gamma,I),\varphi)$ for any non-transitive independence alphabet (Γ,I). Prop. B.6

So let $a,b,c\in\Gamma$ be three letters with $(a,c),(b,c)\in I$ and $(a,b)\in D$. And let (Σ,I) be a star with center c and leaves $a_1,\dots,a_n$. Then define the homomorphism $f\colon\Sigma^*\to\Gamma^*$ by $f(c)=c$ and $f(a_i)=ba^i$ for all $i\in[n]$. Then, for any $u,v\in\Sigma^*$, one obtains $u\sim v\iff f(u)\sim f(v)$. From this, it follows that $[K]=[L_1]\cap[L_2]\iff[f(K)]=[f(L_1)]\cap[f(L_2)]$ as well as $[K]=\Sigma^*\setminus[L_1]\iff[f(K)]=[f(\Sigma^*)]\setminus[f(L_1)]$ for any languages $K,L_1,L_2\subseteq\Sigma^*$. Lemma B.4 For the reduction, one also needs that $[L_1]\cap[L_2]$ is the closure of some regular language K

iff this holds for $[f(L_1)] \cap [f(L_2)]$ (and similarly for $\Sigma^* \setminus [L_1]$ and $\Gamma^* \setminus [f(L_1)]$). While the implication "$\Rightarrow$" is straightforward considering $f(K)$, the converse is more involved since, e.g., there can be a regular language $K' \subseteq \Gamma^*$ with $[K'] = [f(L_1)] \cap [f(L_2)]$ and $K' \cap f(\Sigma^*) = \emptyset$. Considering the pre-image of K' under f yields the empty set whose closure need not be the intersection $[L_1] \cap [L_2]$. To prove the implication "$\Leftarrow$", one first constructs a partial rational function $r \colon \Gamma^* \to \Gamma^*$ on Γ that "computes" for every word $U \in [f(\Sigma^*)]$ a "witness" $V \in f(\Sigma^*)$ with $U \sim V$, i.e., satisfies $U \sim r(U) \in f(\Sigma^*)$ whenever $r(U)$ is defined and $r(U)$ is defined for any $U \in [f(\Sigma^*)]$. Then the closure of the regular language $f^{-1}(r(K'))$ equals $[L_1] \cap [L_2]$ and $\Sigma^* \setminus [L_1]$, respectively. As a result, one obtains that also the rationality problems $\mathrm{RatP}_*((\Gamma, I), \wedge)$ and $\mathrm{RatP}_*((\Gamma, I), \neg)$ are Σ^0_2-hard.

Proof of Prop. B.6

Lemma B.5

In summary, we get the following dichotomy that covers arbitrary sets of independence alphabets and Boolean functions. The lower bound holds due to Lemma 2.3(2,3) and the above arguments.

For the regularity problem, the lower bounds can be shown using the arguments for [28, Thm. 3.1] (that build on [44] and the above version of Post's correspondence problem); the upper bounds follow similarly to those of the rationality problem as well as from [2, 44].

proof: p. 24

▶ **Theorem 3.4.** *Let $\mathbb{A} \subseteq \mathrm{IndA}$ and $\Phi \subseteq \mathrm{BF}$ be decidable sets.*

(1) If $\mathbb{A} \subseteq \mathrm{TransA}$ or $\Phi \subseteq \mathrm{MAX}$, then the rationality problem $\mathrm{RatP}_*(\mathbb{A}, \Phi)$ is decidable.
(2) Otherwise, the rationality problem $\mathrm{RatP}_*(\mathbb{A}, \Phi)$ is Σ^0_2-complete.

(1') If $\mathbb{A} \subseteq \mathrm{TransA}$ or $\Phi \subseteq \mathrm{CONST}$, then the regularity problem $\mathrm{RegP}_*(\mathbb{A}, \Phi)$ is decidable.
(2') Otherwise, the regularity problem $\mathrm{RegP}_*(\mathbb{A}, \Phi)$ is Σ^0_2-complete.

4 Transitive independence alphabets

From now on, we only consider the ω-versions of the decision problems.

The "universality problem" $\mathrm{EmptP}_\omega(\mathrm{TransA}, \neg)$ and the "disjointness problem" $\mathrm{EmptP}_\omega(\mathrm{TransA}, \wedge)$ both are decidable [28, Thm. 4.5] and the regularity problem $\mathrm{RegP}_\omega(\mathrm{TransA}, \mathrm{Id})$ is in Σ^0_1 [28, Thm. 4.6]. In this section, we improve on this latter result by demonstrating its decidability. This proof is based on our first result, namely that any Boolean combination of closures of regular languages is the closure of a regular language (assuming the independence alphabet to be transitive). From these two result, we finally infer that all problems $\mathrm{GenP}_\omega(\mathrm{TransA}, \mathrm{BF})$ are decidable.

But before we can do so, we first recall and extend the main tool of the two decidability proofs, namely types (called "characteristicts" in [28]) and type automata. To this aim, let $(\Gamma, I) \in \mathrm{TransA}$ be the disjoint union of the complete independence alphabets (Γ_k, I_k) for $k \in [m]$, let φ be an n-ary Boolean function, and let $\mathcal{A}_i = (Q_i, \Gamma, S_i, T_i, F_i)$ for $i \in [n]$ be Büchi-automata over Γ. Words from $\bigcup_{k \in [m]} \Gamma_k^\infty \setminus \{\varepsilon\}$ are called *mono-alphabetic* since they use letters from only one of the subalphabets Γ_k.

For a finite mono-alphabetic word $u \in \Gamma_k^+$ with $k \in [m]$, define the *type* $\mathrm{tp}_+(u) = \big((R_i, R_i^F)_{i \in [n]}, k\big)$ setting, for $i \in [n]$, $R_i = \{(p,q) \in Q_i^2 \mid \exists u' \sim u \colon p \xrightarrow{u'}_{\mathcal{A}_i} q\}$ and $R_i^F = \{(p,q) \in Q_i^2 \mid \exists u' \sim u \colon p \xRightarrow{u'}_{\mathcal{A}_i} q\}$ [28]. Note that, differently from classical transition profiles (cf. [6]), we collect all pairs of states that are connected by a path whose label is some word from $[u]$. Here, we also define the type of an infinite mono-alphabetic word $\alpha \in \Gamma_k^\omega$: it is $\mathrm{tp}_\omega(\alpha) = \big((A_i)_{i \in [n]}, k\big)$ with $A_i = \{p \in Q_i \mid \exists \alpha' \sim \alpha \colon \alpha' \in L^\omega(\mathcal{A}_i^{p\to})\}$ for all $i \in [n]$. Then A_i is the set of states of the Büchi-automaton $\mathcal{A}_i$ that allow to accept some word from $[\alpha]$.

Note that there are only finitely many types; hence the set of all types of mono-alphabetic Lemma C.10 words is an alphabet that can be computed. From the set of all types and the tuple of Büchi-automata, one can construct *type automata* $\mathcal{A}_i^{\text{mono}}$ and $\mathcal{A}_i^{\text{multi}}$ over the set of all types with the following central properties:

- Suppose $u_1, u_2, \dots \in \Gamma^+$ are mono-alphabetic words. Then the infinite type sequence $\mathrm{tp}_+(u_1)\,\mathrm{tp}_+(u_2)\,\cdots$ is accepted by the Büchi-automaton $\mathcal{A}_i^{\text{multi}}$ iff $u_1 u_2 \cdots \in [L^\omega(\mathcal{A}_i)]$ and none of the factors $u_i u_{i+1}$ is mono-alphabetic (cf. [28, Lemma 4.3]). Lemma C.6
- Suppose $u_1, \dots, u_\ell \in \Gamma^+$ and $\beta \in \Gamma^\infty$ are mono-alphabetic words. Then the finite type sequence $\mathrm{tp}_+(u_1)\,\mathrm{tp}_+(u_2)\,\cdots\,\mathrm{tp}_+(u_\ell)\,\mathrm{tp}_\omega(\beta)$ is accepted by the nfa $\mathcal{A}_i^{\text{mono}}$ iff $u_1 u_2 \cdots u_\ell \beta \in [L^\omega(\mathcal{A}_i)]$ and none of the factors $u_i u_{i+1}$ nor $u_\ell\,\beta$ is mono-alphabetic. Lemma C.7

Using classical constructions from automata theory, one can construct an nfa $\mathcal{B}^{\text{mono}}$ and a Büchi automaton $\mathcal{B}^{\text{multi}}$ over the set of all types accepting $\left\langle L^*(\mathcal{A}_1^{\text{mono}}), \dots, L^*(\mathcal{A}_1^{\text{mono}})\right\rangle^\varphi$ and $\left\langle L^\omega(\mathcal{A}_1^{\text{multi}}), \dots, L^\omega(\mathcal{A}_1^{\text{multi}})\right\rangle^\varphi$, respectively.

Consider some transition in the Büchi-automaton $\mathcal{B}^{\text{multi}}$. Its label is the type t_+ of some finite mono-alphabetic word over Γ_k. Since the subalphabet (Γ_k, I_k) is complete, one can construct an nfa $\mathcal{C}_{t_+}$ such that the *closure* $[L^+(\mathcal{C}_{t_+})]$ of its language is the set of all finite words of type t_+ [2]. In the nfa $\mathcal{B}^{\text{mono}}$, some of the transitions are labeled by the type t_ω of some infinite mono-alphabetic word over Γ_k. Again using the completeness of (Γ_k, I_k), one can also construct a Büchi-automaton $\mathcal{C}_{t_\omega}$ such that the *closure* $[L^\omega(\mathcal{C}_{t_\omega})]$ of its language is the set of all ω-words of type t_ω. Lemma C.9

Now, in the type automata $\mathcal{B}^{\text{mono}}$ and $\mathcal{B}^{\text{multi}}$, replace every t_+-labeled transition with the nfa $\mathcal{C}_{t_+}$. Furthermore, in the nfa $\mathcal{B}^{\text{mono}}$, replace every t_ω-labeled transition from p to some accepting state with the Büchi-automaton $\mathcal{B}_{t_\omega}$. Let $\mathcal{B}$ denote the disjoint union of the resulting two Büchi-automata. One then shows that the closure of the language of $\mathcal{B}$ is the given Boolean combination of the closures of the languages of $\mathcal{A}_i$. In summary, this proves

proof: p. 35

▶ **Theorem 4.1.** *From a transitive independence alphabet (Γ, I), an n-ary Boolean function φ, and Büchi-automata $\mathcal{A}_i$ over Γ for $i \in [n]$, one can construct a Büchi-automaton $\mathcal{B}$ with*

$$[L^\omega(\mathcal{B})] = \left\langle [L^\omega(\mathcal{A}_1)], \dots, [L^\omega(\mathcal{A}_n)]\right\rangle^\varphi .$$

Hence, the problems $\mathrm{RatP}_\omega(\mathrm{TransA}, \mathrm{BF})$ *and* $\mathrm{EmptP}_\omega(\mathrm{TransA}, \mathrm{BF})$ *are decidable.*

Next, we consider the regularity problem $\mathrm{RegP}_\omega(\mathrm{TransA}, \mathrm{Id})$. So let, as before, (Γ, I) be the disjoint union of the complete independence alphabets (Γ_k, I_k) for $k \in [m]$ and let $\mathcal{A}$ be some Büchi-automaton over Γ. We will use types and type automata as defined above, this time for a single automaton, i.e., $n = 1$ and $\mathcal{A}_1 = \mathcal{A}$.

Two mono-alphabetic words u and v are *type-equivalent* (denoted $u \equiv_{\mathcal{A}} v$) if they have the same type. Then the relation $\equiv_{\mathcal{A}}$ is an equivalence relation on $\bigcup_{k\in[m]} \Gamma_k^\infty$ that only relates words from Γ_k^+ or from Γ_k^ω for some $k \in [m]$; such equivalence relations shall be called *multi-equivalences*. A language $L \subseteq \Gamma^\omega$ is *saturated* by the multi-equivalence $\equiv$ iff replacing maximal mono-alphabetic factors in words from L by equivalent ones results in a word from L. More precisely, iff the following hold.

- Let $u_1 u_2 \cdots \in L$ such that all factors u_i, but none of the factors $u_i u_{i+1}$ is mono-alphabetic. Furthermore, let $u_i \equiv v_i$ for all $i \geq 1$. Then $v_1 v_2 \cdots \in L$.
- Let $u_1 u_2 \cdots u_\ell \beta \in L$ such that all factors u_i and β, but none of the factors $u_i u_{i+1}$ and $u_\ell \beta$ is mono-alphabetic. Furthermore, let $u_i \equiv v_i$ and $\beta \equiv \gamma$. Then $v_1 v_2 \cdots v_\ell \gamma \in L$.

Recall that, in the definitions of the type of $x \in \Gamma_k^\infty$, we considered all words equivalent with x. Therefore, the type-equivalence $\equiv_{\mathcal{A}}$ saturates the closure $[L^\omega(\mathcal{A})]$ of the language of $\mathcal{A}$. Furthermore, the equivalence classes of $\equiv_{\mathcal{A}}$ are closures of regular languages. But, in general, these closures need not be regular. The central insight on the regularity of $[L^\omega(\mathcal{A})]$ is the following. Lemma C.9

proof: p. 37

▶ **Proposition 4.2.** *Let $\mathcal{A}$ be a Büchi-automaton. Then $[L^\omega(\mathcal{A})]$ is regular if, and only if, $[L^\omega(\mathcal{A})]$ is saturated by some multi-equivalence $\equiv\, \supseteq\, \equiv_{\mathcal{A}}$ such that all equivalence classes of $\equiv$ are regular.*

Let $\equiv\, \subseteq\, \equiv_{\mathcal{A}}$ be a multi-equivalence and C some equivalence class of $\equiv$. Since $\equiv$ is a multi-equivalence, C is a language over some complete independence alphabet (Γ_k, I_k). Since $\equiv$ extends $\equiv_{\mathcal{A}}$, C is a union of equivalence classes of $\equiv_{\mathcal{A}}$. Note that $\equiv_{\mathcal{A}}$ has only finitely many equivalence classes and recall that any such equivalence class is the closure of some regular language. Hence C is the closure of the union of finitely many regular languages. If C is a set of finite words, then its regularity can be decided by [2], and also in case C Lemma C.2 is a set of infinite words, this is possible. It remains to decide whether $\equiv$ saturates the closure of $L^\omega(\mathcal{A})$. To this aim, we define the *block closure* $\mathrm{blkcl}(L)$ of a language $L \subseteq \Gamma^\omega$ as the minimal language containing L which is saturated by $\equiv$. Since the equivalence classes of $\equiv_{\mathcal{A}}$ are closed languages, the same holds for the equivalence classes of $\equiv$. Hence $\mathrm{blkcl}([L]) = \mathrm{blkcl}(L) = [\mathrm{blkcl}(L)]$. Since the equivalence classes of $\equiv$ are regular, one can Lemma C.13 construct a Büchi-automaton $\mathcal{B}$ that accepts $\mathrm{blkcl}(L^\omega(\mathcal{A}))$. Hence $[L^\omega(\mathcal{A})]$ is saturated by $\equiv$, i.e., $\mathrm{blkcl}([L^\omega(\mathcal{A})]) \subseteq [L^\omega(\mathcal{A})]$ if, and only if, $[L^\omega(\mathcal{B})] \subseteq [L^\omega(\mathcal{A})]$ – which is decidable using Theorem 4.1.

Hence, for any multi-equivalence $\equiv$ extending the type equivalence $\equiv_{\mathcal{A}}$, one can decide whether it witnesses the regularity of $[L^\omega(\mathcal{A})]$ in the sense of Prop. 4.2. Since there are only finitely many such equivalence relations and since this finite set can be computed, we obtain that the regularity of $[L^\omega(\mathcal{A})]$ is decidable. Now Theorem 4.1 implies the decidability of $\mathrm{RegP}_\omega(\mathrm{TransA}, \mathrm{BF})$: to handle an arbitrary function φ, first compute a single automaton $\mathcal{B}$ as in Theorem 4.1 and then check the regularity of the closure $[L^\omega(\mathcal{B})]$ of its language.

Hence we obtain the following extension of Theorem 3.4(2') and therefore of Sakarovich's result [44] from finite words to ω-words.

proof: p. 37

▶ **Theorem 4.3.** *The regularity problem* $\mathrm{RegP}_\omega(\mathrm{TransA}, \mathrm{BF})$ *is decidable.*

5 Diagonal independence alphabets

In this section, we consider the ω-versions of the emptiness, the rationality, and the regularity problem for diagonal independence alphabets. For the emptiness problem, we will distinguish the case of monotone and non-monotone Boolean functions since the problem becomes decidable in the former case and undecidable in the latter. For the other two problems, the monotonicity of the Boolean function does not play any role.

Emptiness for monotone Boolean functions

The aim of this section is to show that the emptiness problem $\mathrm{EmptP}_\omega(\mathrm{DiagA}, \mathrm{monBF})$ for diagonal independence alphabets and monotone Boolean functions is decidable. Since $\mathrm{EmptP}_*(\mathrm{DiagA}, \mathrm{BF})$ is undecidable (cf. Theorem 3.2(2)), the crucial point in the decidability proof of $\mathrm{EmptP}_\omega(\mathrm{DiagA}, \mathrm{monBF})$ is the monotonicity of the Boolean functions considered.

A first simple observation is that it suffices to consider connected diagonal independence Lemma D.1 alphabets; we denote the set of all such independence alphabets with ConDiagA. These independence alphabets can be considered as *order trees*, i.e., a structure $(\Gamma, \preceq)$ where $\preceq$ is a partial order on the finite set Γ with largest element such that, for all $a, b, c \in \Gamma$, $c \preceq a, b$ implies $a \preceq b$ or $b \preceq a$. An independence alphabet (Γ, I) is connected and diagonal if, and only if, there exists an order tree $(\Gamma, \preceq)$ with $(a, b) \in I \iff a \prec b$ or $b \prec a$, i.e., $I = \succ \cup \prec$ [46, 47].

In the following, we will always assume that $(\Gamma, \preceq)$ is an order tree with $I = \succ \cup \prec$.

For $a \in \Gamma$, let $\downarrow a = \{b \in \Gamma \mid b \preceq a\}$ and $\Downarrow a = \{b \in \Gamma \mid b \prec a\}$ be the sets of elements of Γ (properly) below a; $\updownarrow a \subseteq \Gamma$ is the set of letters comparable with a, i.e., a together with all letters $b \in \Gamma$ with $(a, b) \in I$.

We next define relations that are central to our arguments, these definitions are adaptations of similar notions from [1] to Büchi-automata. Let $\overline{\mathcal{A}} = (\mathcal{A}_i)_{i \in [n]}$ be a tuple of nfas $\mathcal{A}_i = (Q_i, \Gamma, I_i, T_i, F_i)$, $\overline{p} = (p_i)_{i \in [n]}$ and $\overline{q} = (q_i)_{i \in [n]}$ tuples of states from $\prod_{i \in [n]} Q_i$, and $a \in \Gamma$. Then $R(\overline{p}, \overline{q}, a)$ denotes the set of tuples of words that lead from p_i to q_i visiting some accepting state and use only letters independent from or equal to a:

$$R(\overline{p}, \overline{q}, a) = \left\{(u_1, \ldots, u_n) \;\middle|\; u_i \in \updownarrow a^*\,,\ p_i \xRightarrow{u_i}_{\mathcal{A}_i} q_i \text{ for all } i \in [n]\right\}$$

Collecting all tuples of ω-words over $\updownarrow a$ accepted by $\mathcal{A}_i$ from p_i, we get the analogous relation

$$R^\omega(\overline{p}, a) = \left\{(\alpha_1, \ldots, \alpha_n) \;\middle|\; \alpha_i \in \updownarrow a^\omega \cap L^\omega(\mathcal{A}_i^{p_i \to}) \text{ for all } i \in [n]\right\}$$

We will be interested in the restrictions of these relations to tuples of words whose projections to $\Downarrow a$ and $\downarrow a$ are mutually equivalent:

$$R_{\Downarrow}(\overline{p}, \overline{q}, a) = \left\{(u_i)_{i \in [n]} \in R(\overline{p}, \overline{q}, a) \;\middle|\; \mathrm{proj}_{\Downarrow a}(u_1) \sim \mathrm{proj}_{\Downarrow a}(u_i) \text{ for all } i \in [n]\right\}$$
$$R^\omega_{\downarrow}(\overline{p}, a) = \left\{(\alpha_i)_{i \in [n]} \in R(\overline{p}, a) \;\middle|\; \mathrm{proj}_{\downarrow a}(\alpha_1) \sim \mathrm{proj}_{\downarrow a}(\alpha_i) \text{ for all } i \in [n]\right\}$$
$$R^\omega_{\Downarrow}(\overline{p}, a) = \left\{(\alpha_i)_{i \in [n]} \in R(\overline{p}, a) \;\middle|\; \mathrm{proj}_{\Downarrow a}(\alpha_1) \sim \mathrm{proj}_{\Downarrow a}(\alpha_i) \text{ for all } i \in [n]\right\}$$

Suppose a is the root of $(\Gamma, \preceq)$ such that $\mathrm{proj}_{\downarrow a}(\alpha) = \alpha$ for all $\alpha \in \Gamma^\omega$. Furthermore, suppose $\varphi(x_1, \ldots, x_n) = \min\{x_1, \ldots, x_n\}$, and let p_i be the (only) initial state of $\mathcal{A}_i$ for all $i \in [n]$. Then $\left\langle [L^\omega(\mathcal{A}_1)], \ldots, [L^\omega(\mathcal{A}_n)] \right\rangle^\varphi$ equals $R^\omega_{\downarrow}(\overline{p}, a)$, so we have to decide the emptiness of this relation. If it was rational, this would easily be possible. Unfortunately, this is not the case in general. For finite words, Aalbersberg and Hoogeboom approximate the relation $R^\omega_{\downarrow}(\overline{p}, a)$ by its Parikh image. In order to also handle ω-words, we generalize this by the notion of a "language version" that is introduced next.

For an alphabet Σ (e.g., $\Sigma = \Gamma \times [n]$), the Parikh mapping $\Psi \colon \Sigma^\infty \to (\mathbb{N} \cup \{\infty\}^\Sigma$ is defined by $\Psi(\alpha) = (|\alpha|_a)_{a \in \Sigma}$ for all $\alpha \in \Sigma^\infty$. We extend the Parikh mapping to tuples of words: the mapping $\Psi_n \colon (\Gamma^\infty)^n \to (\mathbb{N} \cup \{\infty\})^{\Gamma \times [n]}$ is defined by $\Psi_n(x_1, \ldots, x_n)_{(a,i)} = |x_i|_a$, i.e., $\Psi_n(x_1, \ldots, x_n)(a, i)$ is the number of occurrences of the letter a in the word number i. The language $L \subseteq (\Gamma \times [n])^\infty$ is a *language version* of the relation $R \subseteq (\Gamma^\infty)^n$ if $\Psi_n(R) = \Psi(L)$.

The following crucial result will allow to decide the problem $\mathrm{EmptP}_\omega(\mathrm{DiagA}, \mathrm{monBF})$.

proof: p. 40

▶ **Lemma 5.1.** *One can construct a Büchi-automaton $\mathcal{B}$ that accepts a language version of the relation $R = \left\{(\alpha_i)_{i \in [n]} \mid \alpha_i \in L^\omega(\mathcal{A}_i) \text{ and } \alpha_1 \sim \alpha_i \text{ for all } i \in [n]\right\}$.*

Proof idea. By induction on the size of $\downarrow a$, one shows $R^\omega_{\downarrow}(\overline{p}, a) = R^\omega_{\Downarrow}(\overline{p}, a) \cap \mathrm{Eq}^\omega_a$ and that the relation $R^\omega_{\Downarrow}(\overline{p}, a)$ can be constructed from the relations $R_{\Downarrow}(\overline{q}, \overline{r}, a)$ and $R^\omega_{\downarrow}(\overline{q}, b)$ for $b \prec a$ using union, concatenation, and ω-iteration. Lemma D.3

From [1], one gets regular language versions of the relations $R_{\Downarrow}(\overline{p}, \overline{q}, a)$. Note that the union of language versions of two relations is a language version of the union (similarly Lemma D.2

for concatenation and iteration). While this does not hold for all intersections of relations, we can nevertheless construct a language version for the intersection with the very special relation Eq_a^ω and its language version of all ω-words over $\Gamma \times [n]$ whose projection to $\{a\} \times [n]$ belongs to $\big((a,1)(a,2)\dots(a,n)\big)^\infty$. To handle this intersection, the idea is to consider the complete independence relation on $\Gamma \times [n]$ and then to refer to Theorem 4.1. ◀

Since the emptiness of the regular language version of R is decidable, one can infer the main decidability result for diagonal independence alphabets; it is an extension of Lemma 3.1(3) (which follows easily from [1]) to Büchi-automata.

proof: p. 41

▶ **Theorem 5.2.** *The emptiness problem* $\mathrm{EmptP}_\omega(\mathrm{DiagA}, \mathrm{monBF})$ *is decidable.*

Emptiness for general Boolean functions

Next, we study non-monotone Boolean functions φ. The Π_1^0-hardness of $\mathrm{EmptP}_*(\mathrm{DiagA}, \varphi)$ (Theorem 3.2(2)) transfers to $\mathrm{EmptP}_\omega(\mathrm{DiagA}, \varphi)$. To show containment in Π_1^0, one generalizes the proof of [28, Cor. 5.5] that yields the result for the Boolean function $\varphi(x_1, x_2) = x_1 \wedge \neg x_2$.

Thm. D.9 The crucial insight is that, with $H = \big\langle [L^\omega(\mathcal{A}_1)], \dots, [L^\omega(\mathcal{A}_n)] \big\rangle^\varphi$ and the independence alphabet diagonal, $H = \emptyset$ or H contains some ultimately periodic ω-word uv^ω (this is not

Ex. D.4 necessarily the case if the independence alphabet is not diagonal).

Since it is decidable whether an ultimately periodic word is in $[L^\omega(\mathcal{A}_i)]$ [28, Obs. 5.1], we can just search for a witness for non-emptiness. It follows that the non-emptiness problem is semi-decidable.

proof: p. 45

▶ **Corollary 5.3.** *The emptiness problem* $\mathrm{EmptP}_\omega(\mathrm{DiagA}, \mathrm{BF})$ *is in* Π_1^0.

Rationality and regularity problem

By Theorem 3.4, the problems $\mathrm{RatP}_*((\Gamma, I), \neg)$ and $\mathrm{RegP}_*((\Gamma, I), \neg)$ are Σ_2^0-hard for any non-transitive independence alphabet (Γ, I), these lower bounds transfer to the ω-versions of these problems. We can also prove matching upper bounds for the set of all diagonal independence alphabets and all Boolean functions. These proofs rely on the emptiness problem and are almost identical to the corresponding proof for finite words, cf. Lemma 3.3(2).

proof: p. 45

▶ **Theorem 5.4.** *The rationality problem* $\mathrm{RatP}_\omega(\mathrm{DiagA}, \mathrm{BF})$ *and the regularity problem* $\mathrm{RegP}_\omega(\mathrm{DiagA}, \mathrm{BF})$ *are in* Σ_2^0.

6 General independence alphabets

The problems $\mathrm{EmptP}_\omega((\Gamma, I), \varphi)$ for $\varphi \in \{\wedge, \neg\}$ and $\mathrm{RegP}_\omega((\Gamma, I), \mathrm{Id})$ are Π_1^1-hard for any non-diagonal independence alphabet (Γ, I) [28, Thm. 6.4]. In view of Lemma 2.3, it follows that $\mathrm{EmptP}_\omega(\mathbb{A}, \Phi)$ (and $\mathrm{RegP}_\omega(\mathbb{A}, \Phi)$) is Π_1^1-hard if $\mathbb{A} \not\subseteq \mathrm{DiagA}$ and $\Phi \not\subseteq \mathrm{MAX}$ ($\Phi \not\subseteq \mathrm{CONST}$, resp.)

The central point in the hardness proof from [28] is the construction of regular languages H, K, and L such that the regularity of $[H \cap K] \cap [L]$ is Π_1^1-hard (cf. paragraph following

Lemma E.16 [28, Prop. 6.2]). For the very same languages H, K, and L, one can similarly prove that the existence of a regular language M with $[M] = [H \cap K] \cap [L]$ is equivalent to the regularity of $[H \cap K] \cap [L]$ and therefore Π_1^1-hard.

This leads to the lower bounds in the following result, the upper bounds are rather straightforward consequence of the definition of these problems.

Thm. E.1 Props. E.23, E.24, E.26, and Lemma 2.3

▶ **Theorem 6.1.**

(1) *For all* $\mathrm{GenP} \in \{\mathrm{EmptP}, \mathrm{RatP}, \mathrm{RegP}\}$*, the problem* $\mathrm{GenP}_\omega(\mathrm{IndA}, \mathrm{BF})$ *is in* Π^1_2*.*
The problem $\mathrm{EmptP}_\omega(\mathrm{IndA}, \mathrm{monBF})$ *is in* Π^1_1*.*
(2) *Let* $\mathbb{A} \subseteq \mathrm{IndA}$ *with* $\mathbb{A} \not\subseteq \mathrm{DiagA}$ *and* $\Phi \subseteq \mathrm{BF}$ *be decidable sets.*
If $\Phi \not\subseteq \mathrm{MAX}$*, then* $\mathrm{RatP}_\omega(\mathbb{A}, \Phi)$ *and* $\mathrm{EmptP}_\omega(\mathbb{A}, \Phi)$ *are* Π^1_1*-hard.*
If $\Phi \not\subseteq \mathrm{CONST}$*, then* $\mathrm{RegP}_\omega(\mathbb{A}, \Phi)$ *is* Π^1_1*-hard.*

7 Summary and open problems

A discussion of the results of this paper (cf. Table 1) can be found at the end of the introduction. In particular, we solved two of the open problems from [28]: the regularity problem for transitive independence alphabets and the emptiness problem for diagonal independence alphabets and monotone Boolean functions are decidable and not just semi-decidable.

Let (Γ_1, I_1) and (Γ_2, I_2) be two non-diagonal independence alphabets and φ_1, φ_2 be two monotone, but non-max functions. Then the emptiness problems $\mathrm{EmptP}_\omega((\Gamma_1, I_1), \varphi_1)$ and $\mathrm{EmptP}_\omega((\Gamma_2, I_2), \varphi_2)$ both are Π^1_1-complete and therefore inter-reducible. If φ_1 and φ_2 are non-monotone, containment in Π^1_1 is not known. We do not even know whether the two problems are inter-reducible or whether these problems define different (possibly many) degrees of unsolvability.

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A Appendix

A.1 Proofs from Sect. 2

▶ **Proposition 2.2.** *Let* $\odot \in \{*, \omega\}$*. Then* $\mathrm{EmptP}_\odot(\mathrm{IndA}, \mathrm{MAX})$*,* $\mathrm{RatP}_\odot(\mathrm{IndA}, \mathrm{MAX})$*, and* $\mathrm{RegP}_\odot(\mathrm{IndA}, \mathrm{CONST})$ *are decidable.*

Proof. Let $(\Gamma, I) \in \mathrm{IndA}$ be an independence alphabet, φ an n-ary Boolean function, and $\mathcal{A}_1, \dots, \mathcal{A}_n$ nfas over Γ. Let H denote the language $\langle [L^\odot(\mathcal{A}_1)], \dots, [L^\odot(\mathcal{A}_n)] \rangle^\varphi$.

If φ is constant, then H is one of $\Gamma^\odot$ (if $\varphi(0, \dots, 0) = 1$) or $\emptyset$ (otherwise). In any case, H is regular implying that $\mathrm{RegP}_\odot(\mathbb{A}, \Phi)$ is (trivially) decidable for $\Phi \subseteq \mathrm{CONST}$.

Next, let $\varphi \in \mathrm{MAX}$. If φ is even constant, then, as we saw above, H is regular and closed and therefore the closure of some regular language; hence $((\Gamma, I), \varphi, \mathcal{A}_1, \dots, \mathcal{A}_n) \in \mathrm{RatP}_\odot(\mathrm{IndA}, \mathrm{MAX})$. Now suppose $\varphi \in \mathrm{MAX} \setminus \mathrm{CONST}$ is not constant. Then there is some non-empty set $A \subseteq [n]$ with $\varphi(b_1, \dots, b_n) = \max\{b_i \mid i \in A\}$ for all $b_i \in \{0, 1\}$. Hence H is the union of the languages $[L^\odot(\mathcal{A}_i)]$ for $i \in A$ and therefore the closure of $L = \bigcup_{i \in A} L^\odot(\mathcal{A}_i)$. Thus, also in this case, $((\Gamma, I), \varphi, \mathcal{A}_1, \dots, \mathcal{A}_n) \in \mathrm{RatP}_\odot(\mathrm{IndA}, \mathrm{MAX})$. Hence, $\mathrm{RatP}_\odot(\mathrm{IndA}, \mathrm{MAX})$ is trivially decidable.

Finally, we consider the emptiness problem. Above we saw that, for any $\varphi \in \mathrm{MAX}$, the language H is the closure of some (effectively computable) language L (namely $\emptyset$, $\Gamma^\odot$, or $\bigcup_{i \in A} L^\odot(\mathcal{A}_i)$). Since $H = \emptyset \iff L = \emptyset$, the decidability of the emptiness problem $\mathrm{EmptP}_\odot(\mathrm{IndA}, \mathrm{MAX})$ follows. ◀

▶ **Lemma 2.3.** *Let* $\odot \in \{*, \omega\}$*,* $\mathrm{GenP} \in \{\mathrm{EmptP}, \mathrm{RatP}, \mathrm{RegP}\}$*,* $\mathbb{A} \subseteq \mathrm{IndA}$*, and* $\varphi \in \mathrm{BF}$*.*

(1) If $\varphi \notin \mathrm{CONST}$*, then* $\mathrm{RegP}_\odot(\mathbb{A}, \mathrm{Id}) \leq \mathrm{RegP}_\odot(\mathbb{A}, \varphi)$*.*
(2) If $\varphi \in \mathrm{monBF} \setminus \mathrm{MAX}$*, then* $\mathrm{GenP}_\odot(\mathbb{A}, \wedge) \leq \mathrm{GenP}_\odot(\mathbb{A}, \varphi)$*.*
(3) If $\varphi \notin \mathrm{monBF}$*, then* $\mathrm{GenP}_\odot(\mathbb{A}, \neg) \leq \mathrm{GenP}_\odot(\mathbb{A}, \varphi)$*.*

Proof. First, suppose $\varphi \notin \mathrm{CONST}$. For notational simplicity, suppose there is $\overline{b} = (b_i)_{i \in [n-1]} \in \{0, 1\}^{n-1}$ with $\varphi(\overline{b}, 0) \neq \varphi(\overline{b}, 1)$. Let $(\Gamma, I) \in \mathbb{A}$ be some independence alphabet from $\mathbb{A}$ and $\mathcal{A}$ some nfa over Γ. Then we choose, for $i \in [n-1]$, an nfa $\mathcal{A}_i$ with

$$L^\odot(\mathcal{A}_i) = \begin{cases} \Gamma^\odot & \text{if } b_i = 1 \\ \emptyset & \text{if } b_i = 0\,. \end{cases} \tag{1}$$

As a consequence,

$$\langle [L^\odot(\mathcal{A}_1)], \dots, [L^\odot(\mathcal{A}_{n-1})], [L^\odot(\mathcal{A})] \rangle^\varphi = \begin{cases} [L^\odot(\mathcal{A})] & \text{if } \varphi(\overline{b}, 1) = 1 \\ \Gamma^\odot \setminus [L^\odot(\mathcal{A})] & \text{if } \varphi(\overline{b}, 1) = 0\,. \end{cases}$$

Hence $((\Gamma, I), \mathrm{Id}, \mathcal{A}) \in \mathrm{RegP}_\odot(\mathbb{A}, \mathrm{Id})$ if, and only if, $((\Gamma, I), \varphi, \mathcal{A}_1, \dots, \mathcal{A}_{n-1}, \mathcal{A}) \in \mathrm{RegP}_\odot(\mathbb{A}, \varphi)$. In other words, the mapping $((\Gamma, I), \mathrm{Id}, \mathcal{A}) \mapsto ((\Gamma, I), \varphi, \mathcal{A}_1, \dots, \mathcal{A}_{n-1}, \mathcal{A})$ reduces $\mathrm{RegP}_\odot(\mathbb{A}, \mathrm{Id})$ to $\mathrm{RegP}_\odot(\mathbb{A}, \varphi)$.

To show the second claim, let $\varphi \in \mathrm{monBF} \setminus \mathrm{MAX}$ be monotone, but not a max-function. Let $\overline{b_1}, \dots, \overline{b_m} \in \{0, 1\}^n$ be the minimal tuples (wrt. the componentwise order) satisfying $\varphi(\overline{b}) = 1$. Since φ is monotone, we get

$$\varphi(\overline{c}) = 1 \iff \exists j \in [m] \colon \overline{b_j} \leq \overline{c}\,.$$

Now suppose all the tuples $\overline{b_j}$ contain at most one entry 1, say entry number i_j. Then

$$\varphi(\bar{c}) = \max(c_{i_j} \mid j \in [m]).$$

But this contradicts our assumption that φ is not a max-function. Consequently, one of the tuples $\overline{b_j}$ contains $k \geq 2$ entries equal to 1. For notational simplicity, we omit the index j (i.e., set $\bar{b} = \overline{b_j}$) and suppose $\bar{b} = (\underbrace{1,\dots,1}_{k\geq 2 \text{ entries}},0,\dots,0)$, i.e., $b_i = 1 \iff i \leq k$ for all $i \in [n]$.

Now let $(\Gamma, I) \in \mathbb{A}$ be some independence alphabet from $\mathbb{A}$ and $\mathcal{A}_1$ and $\mathcal{A}_2$ be nfas over Γ and set $K = L^{\odot}(\mathcal{A}_1)$ and $L = L^{\odot}(\mathcal{A}_2)$. For all i with $2 < i \leq n$ let $\mathcal{A}_i$ be an nfa satisfying (1). Then, for any word $w \in \Gamma^{\odot}$, one gets $w \in \big\langle [L^{\odot}(\mathcal{A}_1)], \dots, [L^{\odot}(\mathcal{A}_n)] \big\rangle^{\varphi}$ iff $\varphi\big(\chi_1(w), \chi_2(w), b_3, \dots, b_n\big) = 1$ where χ_i is the characteristic function of the language $[L^{\odot}(\mathcal{A}_i)]$. Our choice of the tuple $\bar{b}$ and the nfas $\mathcal{A}_3, \dots, \mathcal{A}_n$ implies that this is equivalent to $\chi_1(w) = \chi_2(w) = 1$, i.e., to $w \in [L^{\odot}(\mathcal{A}_1)] \cap [L^{\odot}(\mathcal{A}_2)]$. Hence we showed $\big\langle [L^{\odot}(\mathcal{A}_1)], \dots, [L^{\odot}(\mathcal{A}_n)] \big\rangle^{\varphi} = \big\langle [L^{\odot}(\mathcal{A}_1)], [L^{\odot}(\mathcal{A}_2)] \big\rangle^{\wedge}$.

It follows that the mapping $((\Gamma, I), \wedge, \mathcal{A}_1, \mathcal{A}_2) \mapsto ((\Gamma, I), \varphi, \mathcal{A}_1, \dots, \mathcal{A}_n)$ is a reduction from $\mathrm{GenP}_{\odot}(\mathbb{A}, \wedge)$ to $\mathrm{GenP}_{\odot}(\mathbb{A}, \varphi)$.

Finally suppose $\varphi \in \mathrm{BF} \setminus \mathrm{monBF}$ not to be monotone, i.e., there are $b_i \leq c_i$ in $\{0,1\}$ for all $i \in [n]$ such that $\varphi(b_1, \dots, b_n) > \varphi(c_1, \dots, c_n)$. There are even $b_i \leq c_i$ in $\{0,1\}$ for all $i \in [n]$ with $\varphi(b_1, \dots, b_n) = 1$, $f(c_1, \dots, c_n) = 0$, and $b_i = c_i$ for all but one $i \in [n]$. For notational simplicity, we assume $b_1 \neq c_1$, i.e., $b_1 = 0$ and $c_1 = 1$ such that $\varphi(0, b_2, \dots, b_n) = 1$ and $\varphi(1, b_2, \dots, b_n) = 0$. For all i with $2 \leq i \leq n$, define an nfa $\mathcal{A}_i$ satisfying (1). Now let $(\Gamma, I) \in \mathbb{A}$ be some independence alphabet from $\mathbb{A}$ and $\mathcal{A}_1$ an nfa over Γ. Then, for any word $w \in \Gamma^{\odot}$, one gets $w \in \big\langle [L^{\odot}(\mathcal{A}_1)], \dots, [L^{\odot}(\mathcal{A}_n)] \big\rangle^{\varphi}$ iff $\varphi\big(\chi(w), b_2, \dots, b_n\big) = 1$ where χ is the characteristic function of the language $[L^{\odot}(\mathcal{A}_1)]$. Our choice of the tuple $\mathcal{A}_2, \dots, \mathcal{A}_n$ ensures that this is the case iff $\chi(w) = 0$, i.e., $w \notin [L^{\odot}(\mathcal{A}_1)]$. Hence we get $\big\langle [L^{\odot}(\mathcal{A}_1)], \dots, [L^{\odot}(\mathcal{A}_n)] \big\rangle^{\varphi} = \Gamma^{\odot} \setminus [L^{\odot}(\mathcal{A}_1)] = \big\langle [L^{\odot}(\mathcal{A}_1)] \big\rangle^{\neg}$. It follows that the mapping $((\Gamma, I), \neg, \mathcal{A}_1) \mapsto ((\Gamma, I), \varphi, \mathcal{A}_1, \dots, \mathcal{A}_n)$ is a reduction of $\mathrm{GenP}_{\odot}(\mathbb{A}, \neg)$ to $\mathrm{GenP}_{\odot}(\mathbb{A}, \varphi)$. ◀

B Proofs from Section 3

▶ **Lemma 3.1.** *(1) The emptiness problem* $\mathrm{EmptP}_*(\mathrm{IndA}, \mathrm{BF})$ *is in* Π^0_1.
(2) The emptiness problem $\mathrm{EmptP}_*(\mathrm{TransA}, \mathrm{BF})$ *is decidable [2].*
(3) The emptiness problem $\mathrm{EmptP}_*(\mathrm{DiagA}, \mathrm{monBF})$ *is decidable [1].*

Proof. Let (Γ, I) be some independence alphabet, φ an n-ary Boolean function, and $\mathcal{A}_i$ for $i \in [n]$ nfas over Γ. For a word $u \in \Gamma^*$, we have $u \in [L^*(\mathcal{A}_i)]$ iff $[u] \cap L^*(\mathcal{A}_i) \neq \emptyset$. Since one can compute the finite set $[u]$ from the word u, it is decidable whether $u \in [L^*(\mathcal{A}_i)]$. Hence $[L^*(\mathcal{A}_i)]$ is a decidable language. Then

$$H := \big\langle [L^*(\mathcal{A}_1)], \dots, [L^*(\mathcal{A}_n)] \big\rangle^{\varphi}$$

is a Boolean combination of decidable languages and therefore decidable.

Now claim (1) follows since $H = \emptyset$ if, and only if, $\forall u \in \Gamma^* \colon u \notin H$, which proves that the emptiness problem $\mathrm{EmptP}_*(\mathrm{IndA}, \mathrm{BF})$ belongs to Π^0_1.

Now let (Γ, I) be transitive. Aalbersberg and Welzl found an algorithm [2] that, from (Γ, I), φ, and the nfas $\mathcal{A}_i$, constructs an nfa $\mathcal{B}$ such that $[L^*(\mathcal{B})] = H$. Hence $H = \emptyset$ iff $L^*(\mathcal{B}) = \emptyset$ which is decidable. Thus, $\mathrm{EmptP}_*(\mathrm{TransA}, \mathrm{BF})$ is decidable.

Finally, let $(\Gamma, I) \in \mathrm{DiagA}$ be diagonal and $\varphi \in \mathrm{monBF}$ monotone. Since φ is monotone, one can construct a finite set $\mathcal{Y} \subseteq \mathcal{P}([n])$ of sets $A \subseteq [n]$ such that

$$H = \bigcup_{A \in \mathcal{Y}} \bigcap_{i \in A} [L^*(\mathcal{A}_i)] \,.$$

It therefore suffices to decide the emptiness of $\bigcap_{i \in A}[L^*(\mathcal{A}_i)]$ for all $A \in \mathcal{Y}$. By [1], this can be done for sets A of size 2, but their proof generalizes to arbitrary finite sets A. Thus, also $\mathrm{EmptP}_*(\mathrm{DiagA}, \mathrm{monBF})$ is decidable. ◀

▶ **Theorem 3.2.** *Let $\mathbb{A} \subseteq \mathrm{IndA}$ and $\Phi \subseteq \mathrm{BF}$ be decidable sets.*

(1) The emptiness problem $\mathrm{EmptP}_*(\mathbb{A}, \Phi)$ is decidable if

(i) $\mathbb{A} \subseteq \mathrm{TransA}$,

(ii) or $\mathbb{A} \subseteq \mathrm{DiagA}$ and $\Phi \subseteq \mathrm{monBF}$,

(iii) or $\Phi \subseteq \mathrm{MAX}$.

(2) Otherwise, the emptiness problem $\mathrm{EmptP}_*(\mathbb{A}, \Phi)$ is Π^0_1-hard.

Proof. The first claim follows from Lemma 3.1(2,3) and Proposition 2.2.

Now suppose none of the properties (i-iii) holds. Since $\mathrm{TransA} \subseteq \mathrm{DiagA}$ and $\mathrm{MAX} \subseteq \mathrm{monBF}$, there are $(\Gamma, I) \in \mathbb{A}$ and $\varphi \in \Phi$ with

(a) $(\Gamma, I) \notin \mathrm{DiagA}$ and $\varphi \notin \mathrm{MAX}$

(b) or $(\Gamma, I) \notin \mathrm{TransA}$ and $\varphi \notin \mathrm{monBF}$.

If (a) holds, the non-halting problem for two-counter automata [33] reduces to $\mathrm{EmptP}_*((\Gamma, I), \wedge)$ [1] which, by Lemma 2.3(2), reduces to $\mathrm{EmptP}_*((\Gamma, I), \varphi)$.

Now suppose (b) holds. Let $f, g \colon A^* \to B^*$ be homomorphisms for two alphabets A and B. To prove Theorem 2 in [35], Muscholl and Petersen produce regular languages L_g and L_h such that $[L_f \cup L_g] = \Gamma^*$ iff there is no non-empty word $u \in A^+$ with $f(u) = f(g)$ (the work by Muscholl and Petersen is based on [31, 44]). Then $L = L_f \cup L_g$ is regular with $\Gamma^* \setminus [L] = \emptyset$ iff $[L] = \Gamma^*$ iff ε is the only word $u \in \Gamma^*$ with $f(u) = g(u)$. This gives a reduction of the complement of Post's correspondence problem [41] to $\mathrm{EmptP}_*((\Gamma, I), \neg)$.

Since the non-halting problem for two-counter automata as well as the complement of Post's correspondence problem is Π^0_1-hard, the second claim follows from Lemma 3.1(1). ◀

▶ **Lemma 3.3.** *(1) The rationality problem $\mathrm{RatP}_*(\mathrm{TransA}, \mathrm{BF})$ is decidable [2].*

(2) The rationality problem $\mathrm{RatP}_*(\mathrm{IndA}, \mathrm{BF})$ is in Σ^0_2.

Proof. Let φ be an n-ary Boolean function and $\mathcal{A}_1, \ldots, \mathcal{A}_n$ nfas over Γ.

First, let $(\Gamma, I) \in \mathrm{TransA}$ be some transitive independence alphabet. Then the language $H := \big\langle [L^*(\mathcal{A}_1)], \ldots, [L^*(\mathcal{A}_n)] \big\rangle^\varphi$ is a Boolean combination of the languages $[L^*(\mathcal{A}_i)]$. Aalbersberg and Welzl [2] showed that there exists a regular language L whose closure equals H. Hence $\mathrm{RatP}_*(\mathrm{TransA}, \mathrm{BF})$ is (trivially) decidable.

Now let (Γ, I) be an arbitrary independence alphabet. In the following, let $X \bigtriangleup Y$ denote the symmetric difference $X \setminus Y \cup Y \setminus X$ of the two sets X and Y; we will use that $X \triangle Y = \emptyset \iff X = Y$. We have $((\Gamma, I), \varphi, \mathcal{A}_1, \ldots, \mathcal{A}_n) \in \mathrm{RatP}_*(\mathrm{IndA}, \mathrm{BF})$ iff

$$\exists \text{ nfa } \mathcal{B} \colon \big[L^*(\mathcal{B})\big] \bigtriangleup \big\langle [L^*(\mathcal{A}_1)], \ldots, [L^*(\mathcal{A}_n)] \big\rangle^\varphi = \emptyset \,.$$

But this is the case iff

$$\exists \text{ nfa } \mathcal{B}\colon ((\Gamma, I), \varphi', \mathcal{A}_1, \dots, \mathcal{A}_n, \mathcal{B}) \in \mathrm{EmptP}_*(\mathrm{IndA}, \mathrm{BF})$$

where φ' is the $(n+1)$-ary Boolean function with $\varphi'(b_1, \dots, b_{n+1}) = 1$ iff $\varphi(b_1, \dots, b_n) \neq b_{n+1}$. Since $\mathrm{EmptP}_*(\mathrm{IndA}, \mathrm{BF}) \in \Pi^0_1$ by Lemma 3.1(1), this statement proves containment in Σ^0_2. ◀

▶ **Proposition B.1** ([28]). *The set* $\mathrm{ePCP}_{\mathrm{fin}}$ *is hard for* Σ^0_2.

▶ **Lemma B.2.** *Let* (A, B, α, β, L) *be an instance,*

$$W_\alpha = \bigcup_{u \in L} u\alpha(u) \sqcup c^{|\alpha(u)|} \text{ and } W_\beta = \bigcup_{u \in L} u\beta(u) \sqcup c^{|\beta(u)|},$$

$\Sigma = A \cup B \cup \{c\}$ *and* I *the symmetric closure of* $(A \cup B) \times \{c\}$. *Then the following are equivalent.*

(i) $\{u \in L \mid \alpha(u) = \beta(u)\}$ *is finite, i.e.,* $(A, B, \alpha, \beta, L) \in \mathrm{ePCP}_{\mathrm{fin}}$
(ii) There exists a regular language K *with* $[K] = W_\alpha \cap W_\beta$.
(iii) There exists a regular language K' *with* $[K'] = [L \cdot (Bc)^*] \setminus (W_\alpha \cap W_\beta)$.
(iv) $\{u\alpha(u) \mid u \in L, \alpha(u) = \beta(u)\}$ *is regular.*

Proof. We first show that (i) implies (ii) and (iii). So suppose the language from (i) to be finite. Then $W_\alpha \cap W_\beta$ is the union of all languages $u\alpha(u) \sqcup c^{|\alpha(u)|}$ with $u \in L$ and $\alpha(u) = \beta(u)$. Since any of these languages is finite and since the language from (i) is finite, the intersection is finite.

Hence $K := W_\alpha \cap W_\beta$ is regular. Since W_α and W_β are closed, we get $[K] = [W_\alpha \cap W_\beta] = W_\alpha \cap W_\beta$, i.e., (ii) holds.

Now set $K' = L \cdot (Bc)^* \setminus (W_\alpha \cap W_\beta)$. Since $W_\alpha \cap W_\beta$ is regular, the language K' is regular. Since $W_\alpha \cap W_\beta$ is closed, we get $[K'] = [L \cdot (Bc)^*] \setminus (W_\alpha \cap W_\beta)$, i.e., (iii) holds.

Next, we show that (ii) implies (iv). So suppose K is regular with $[K] = W_\alpha \cap W_\beta$. Since the letters from $A \cup B = X$ are mutually dependent, we obtain

$$\mathrm{proj}_X(W_\alpha \cap W_\beta) = \mathrm{proj}_X([K]) = \mathrm{proj}_X(K).$$

Since K is regular, this implies the regularity of $\mathrm{proj}_X(W_\alpha \cap W_\beta) = \{u\alpha(u) \mid u \in L, \alpha(u) = \beta(u)\}$, i.e., (iv).

The next paragraph demonstrates that also (iii) implies (iv). So let K' be regular with $[K'] = [L \cdot (Bc)^*] \setminus (W_\alpha \cap W_\beta)$. With $X = A \cup B$, we obtain

$$\begin{aligned}\mathrm{proj}_X(K') &= \mathrm{proj}_X([K']) \\ &= \mathrm{proj}_X([L \cdot (Bc)^*] \setminus (W_\alpha \cap W_\beta)) \\ &= \{uv \mid u \in L, v \in B^*, v \neq \alpha(u) \text{ or } v \neq \beta(u)\}.\end{aligned}$$

Since K' is regular, so is this language and therefore its complement wrt. LB^*, i.e., the language $LB^* \setminus \mathrm{proj}_X(K') = \{uv \mid u \in L, v \in B^*, \alpha(u) = \beta(u) = v\}$. Since this is the language from (iv), we proved that (iv) follows from (iii).

To get the equivalence of all the statements, it remains to be shown that (iv) implies (i). So suppose $H = \{u\alpha(u) \mid u \in L, \alpha(u) = \beta(u)\}$ is regular. Towards a contradiction, suppose the language from (i) is infinite. Since the mapping $u \mapsto u\alpha(u)$ maps the language from (i) injectively to H, it follows that also H is infinite. The regularity of $H \subseteq LB^*$ then implies the existence of infinitely many words $u_1, u_2, \dots \in L$ and a word $v \in B^*$ with $u_i v \in H$ for all $i \geq 1$. But this contradicts $|u| \leq |\alpha(u)|$ for all $u \in A^*$. Hence, indeed, (i) follows from (iv). ◀

▶ **Proposition B.3.** *The rationality problems* $\mathrm{RatP}_*(\mathrm{StarA}, \neg)$ *and* $\mathrm{RatP}_*(\mathrm{StarA}, \wedge)$ *are* Σ_2^0*-hard.*

Proof. Both results are shown by a reduction from $\mathrm{ePCP}_{\mathrm{fin}}$.
Preparation. Let $\mathrm{Ins} = (A, B, \alpha, \beta, L)$ be an instance. Set $X = A \cup B$, $\Sigma = X \uplus \{c\}$, let I be the symmetric closure of $X \times \{c\}$, and let W_α and W_β be the two languages defined in Lemma B.2. Note that these two languages are closed, i.e., $[W_\alpha] = W_\alpha$ and $[W_\beta] = W_\beta$.

Let $W' = \bigcup_{u \in A^*} u\,\alpha(u) \sqcup\!\sqcup\, c^{|\alpha(u)|} \subseteq \Sigma^*$. By [31] (see also [35]), one can construct an nfa accepting a language V with $[V] = \Sigma^* \setminus W'$. Note that $W_\alpha = W' \cap (L\,B^* \sqcup\!\sqcup\, c^*)$ implying $\Sigma^* \setminus W_\alpha = \Sigma^* \setminus W' \cup \Sigma^* \setminus (L\,B^* \sqcup\!\sqcup\, c^*)$. Since the language $\Sigma^* \setminus (L\,B^* \sqcup\!\sqcup\, c^*)$ is regular and closed, one can construct an nfa accepting the language $V_\alpha = V \cup \Sigma^* \setminus (L\,B^* \sqcup\!\sqcup\, c^*)$. Then one has

$$
\begin{aligned}
[V_\alpha] &= [V] \cup [\Sigma^* \setminus (LB^* \sqcup\!\sqcup\, c^*)] \\
&= \Sigma^* \setminus W' \cup \Sigma^* \setminus (LB^* \sqcup\!\sqcup\, c^*) && \text{since } \Sigma^* \text{ and } LB^* \sqcup\!\sqcup\, c^* \text{ are closed} \\
&= \Sigma^* \setminus (W' \cap (LB^* \sqcup\!\sqcup\, c^*)) \\
&= \Sigma^* \setminus [W_\alpha] \\
&= \Sigma^* \setminus W_\alpha
\end{aligned}
$$

where the last equality holds since W_α is closed. Similarly, we get a regular language V_β with $[V_\beta] = \Sigma^* \setminus W_\beta$.
Negation. Hence one can construct an nfa $\mathcal{A}$ with $L^*(\mathcal{A}) = V_\alpha \cup V_\beta$. Then

$$
\Sigma^* \setminus [L^*(\mathcal{A})] = \Sigma^* \setminus [V_\alpha \cup V_\beta] = \Sigma^* \setminus \big([V_\alpha] \cup [V_\beta]\big) = \Sigma^* \setminus [V_\alpha] \cap \Sigma^* \setminus [V_\beta] = W_\alpha \cap W_\beta\,.
$$

By the equivalence of (i) and (ii) in Lemma B.2, there exists a regular language K with $[K] = W_\alpha \cap W_\beta$ (i.e., $[K] = \Gamma^* \setminus [L^*(\mathcal{A})]$) if, and only if, $\mathrm{Ins} \in \mathrm{ePCP}_{\mathrm{fin}}$. Hence the mapping $\mathrm{Ins} \mapsto ((\Sigma, I), \neg, \mathcal{A})$ is a reduction of $\mathrm{ePCP}_{\mathrm{fin}}$ to the rationality problem $\mathrm{RatP}_*(\mathrm{StarA}, \neg)$ implying that this problem is hard for Σ_2^0 by Prop. B.1.
Conjunction. Furthermore, one can construct an nfa $\mathcal{B}$ with $L^*(\mathcal{B}) = L \cdot (Bc)^*$. Then we have

$$
\begin{aligned}
[L^*(\mathcal{A})] \cap [L^*(\mathcal{B})] &= [V_\alpha \cup V_\beta] \cap [L \cdot (Bc)^*] \\
&= \Sigma^* \setminus (W_\alpha \cap W_\beta) \cap [L \cdot (Bc)^*] \\
&= [L \cdot (Bc)^*] \setminus (W_\alpha \cap W_\beta)\,.
\end{aligned}
$$

Now the equivalence of (i) and (iii) in Lemma B.2 implies that there exists a regular language K' with $[K'] = [L^*(\mathcal{A})] \cap [L^*(\mathcal{B})]$ if, and only if, $\mathrm{Ins} \in \mathrm{ePCP}_{\mathrm{fin}}$. Hence the mapping $\mathrm{Ins} \mapsto ((\Sigma, I), \wedge, \mathcal{A}, \mathcal{B})$ is a reduction from $\mathrm{ePCP}_{\mathrm{fin}}$ to the rationality problem $\mathrm{RatP}_*(\mathrm{StarA}, \wedge)$ which is therefore Σ_2^0-hard by Prop. B.1. ◀

▶ **Lemma B.4.** *Let* (Σ, I) *and* (Γ, I) *be independence alphabets,* $\odot \in \{*, \omega\}$*, and* $f\colon \Sigma^\odot \to \Gamma^\odot$ *a mapping satisfying*

$$
u \sim u' \iff f(u) \sim f(u') \tag{2}
$$

for all $u, u' \in \Sigma^\odot$*. Let furthermore* φ *be an* n*-ary Boolean function, and* $K, L_1, \ldots, L_n \subseteq \Sigma^\odot$*. Then*

$$
[K] = \big\langle [L_1], \ldots, [L_n] \big\rangle^\varphi \iff [f(K)] = [f(\Sigma^\odot)] \cap \big\langle [f(L_1)], \ldots, [f(L_n)] \big\rangle^\varphi\,.
$$

Proof. For any $u \in \Sigma^\odot$ and $i \in [n]$, we have $u \in [L_i]$ iff there is some $v \in L_i$ with $u \sim v$. By (2), this is equivalent to the existence of some $v \in L_i$ with $f(u) \sim f(v)$ and therefore to $f(u) \in [f(L_i)]$. Since $\langle [L_1], \dots, [L_n] \rangle^\varphi$ and $\langle [f(L_1)], \dots, [f(L_n)] \rangle^\varphi$ are "analogue" Boolean combinations, we get

$$u \in \langle [L_1], \dots, [L_n] \rangle^\varphi \iff f(u) \in \langle [f(L_1)], \dots, [f(L_n)] \rangle^\varphi \tag{3}$$

for all $u \in \Sigma^\odot$. Consequently,

$$[K] = \langle [L_1], \dots, [L_n] \rangle^\varphi$$

iff

$$\forall u \in \Sigma^\odot \colon \big(u \in [K] \iff u \in \langle [L_1], \dots, [L_n] \rangle^\varphi \big)$$

which is, by Equivalence (3), the case iff

$$\forall u \in \Sigma^\odot \colon \big(f(u) \in [f(K)] \iff f(u) \in \langle [f(L_1)], \dots, [f(L_n)] \rangle^\varphi \big)$$

iff

$$\forall U \in f(\Sigma^\odot) \colon \big(U \in [f(K)] \iff U \in \langle [f(L_1)], \dots, [f(L_n)] \rangle^\varphi \big) .$$

Since both these languages are closed, this is equivalent to

$$\forall U \in [f(\Sigma^\odot)] \colon \big(U \in [f(K)] \iff U \in \langle [f(L_1)], \dots, [f(L_n)] \rangle^\varphi \big) .$$

Since $[f(K)] \subseteq [f(\Sigma^\odot)]$, this is equivalent to

$$[f(K)] = [f(\Sigma^\odot)] \cap \langle [f(L_1)], \dots, [f(L_n)] \rangle^\varphi . \qquad \blacktriangleleft$$

▶ **Lemma B.5.** *Let (Σ, I) and (Γ, I) be independence alphabets, $\odot \in \{*, \omega\}$, and $f \colon \Sigma \to \Gamma^+$ a mapping whose letterwise extension to a mapping $\Sigma^\odot \to \Gamma^\odot$ (that we also denote f) satisfies Equivalence (2) from Lemma B.4. Furthermore, let $R \subseteq \Gamma^\odot \times \Gamma^\odot$ be a rational relation such that*

(a) $\binom{U}{V} \in R$ implies $U \sim V \in f(\Sigma^\odot)$ and
(b) $U \in [f(\Sigma^\odot)]$ ensures the existence of $V \in \Gamma^\odot$ with $\binom{U}{V} \in R$.

Finally, let φ be an n-ary Boolean function and $L_1, \dots, L_n \subseteq \Sigma^\odot$. Then the following are equivalent.

(i) There exists a regular language $K \subseteq \Sigma^\odot$ with $[K] = \langle [L_1], \dots, [L_n] \rangle^\varphi$.
(ii) There exists a regular language $K' \subseteq \Gamma^\odot$ with $[K'] = [f(\Sigma^\odot)] \cap \langle [f(L_1)], \dots, [f(L_n)] \rangle^\varphi$.

Proof. We start with the implication (i)⇒(ii). So suppose $K \subseteq \Sigma^\odot$ is regular with $[K] = \langle [L_1], \dots, [L_n] \rangle^\varphi$.

The language $K' := f(K)$ is regular since it is obtained from the regular language K by applying f letterwise. Then Lemma B.4 implies $[K'] = [f(K)] = [f(\Sigma^\odot)] \cap \langle [f(L_1)], \dots, [f(L_n)] \rangle^\varphi$. Thus, indeed, the implication (i)⇒(ii) holds.

We next verify the converse implication. So let $K' \subseteq \Gamma^\odot$ be regular with $[K'] = [f(K)] = [f(\Sigma^\odot)] \cap \langle [f(L_1)], \dots, [f(L_n)] \rangle^\varphi$. Note that K' need not be a subset of $f(\Sigma^\odot)$ such that setting $K = f^{-1}(K')$ does not make much sense. Instead, we replace K' by some regular language $K'' \subseteq f(\Sigma^\odot)$ that we obtain using the rational relation $R \subseteq \Gamma^\odot \times \Gamma^\odot$. So let

$K'' = \{V \in \Gamma^* \mid \exists U \in K' \colon \binom{U}{V} \in R\}$. Then K'' is regular (as the image of the regular language K' under the rational relation R) and a subset of $f(\Sigma^\odot)$. We verify $[K''] = [K']$. For the inclusion "$\subseteq$", suppose $W \in [K'']$. Then there exists $V \in K''$ with $W \sim V$. Hence there is $U \in K'$ with $\binom{U}{V} \in R$ and therefore $V \sim U$. We therefore get $W \sim V \sim U \in K'$, i.e., $W \in [K']$. Thus, we verified the inclusion "$\subseteq$". To show the converse inclusion, let $W \in [K']$ and therefore $W \in [f(\Sigma^\odot)]$. Hence there is a word V with $\binom{W}{V} \in R$ implying $V \in K''$. Hence we have $W \sim V \in K''$ and therefore $W \in [K'']$. Thus, indeed, $[K''] = [K']$.

Now let $K = \{u \in \Sigma^\odot \mid f(u) \in K''\}$. As the pre-image of the regular language K'' under the letterwise application of $f \colon \Sigma \to \Gamma^+$, the language K is regular. By its very definition and $K'' \subseteq f(\Sigma^\odot)$, we get $f(K) = K''$. Hence it follows that

$$[f(K)] = [K''] = [K'] = [f(\Sigma^\odot)] \cap \big\langle [f(L_1)], \ldots, [f(L_n)] \big\rangle^\varphi .$$

Now Lemma B.4 implies $[K] = \big\langle [L_1], \ldots, [L_n] \big\rangle^\varphi$. Hence, also the implication (ii)⇒(i) holds, i.e., (i) and (ii) are equivalent. ◀

▶ **Proposition B.6.** *Let (Γ, I) be some non-transitive independence alphabet. Then the rationality problems $\mathrm{RatP}_*((\Gamma, I), \wedge)$ and $\mathrm{RatP}_*((\Gamma, I), \neg)$ for (Γ, I) both are Σ^0_2-hard.*

Proof. Since (Γ, I) is non-transitive, there are mutually distinct letters $a, b, c \in \Gamma$ with $(a, c), (b, c) \in I$ and $(a, b) \in D$.

We reduce from the rationality problems $\mathrm{RatP}_*(\mathrm{StarA}, \wedge)$ and $\mathrm{RatP}_*(\mathrm{StarA}, \neg)$ in Prop. B.3. So let $(\Sigma, I) \in \mathrm{StarA}$ be some independence alphabet with $\Sigma = X \cup \{c\}$ and I the symmetric closure of $X \times \{c\}$ and let $\mathcal{A}$, $\mathcal{A}_1$, and $\mathcal{A}_2$ be nfas over Σ. These reductions use Lemma B.5. Therefore, we first provide a mapping f and a rational relation R satisfying Equivalence (2) from Lemma B.4 and conditions (a) and (b) from Lemma B.5.

Preparation. Enumerate the elements of X as $X = \{a_1, \ldots, a_n\}$ and consider the injective homomorphism $f \colon \Sigma^* \to \Gamma^*$ given by $a_i \mapsto ab^i$ and $c \mapsto c$. Since $f(xy) \sim f(yx)$ for all $x, y \in \Sigma$ with $(x, y) \in I$, we have $u \sim v \implies f(u) \sim f(v)$ for all $u, v \in \Sigma^*$. For the converse implication, note that $f(u) \sim f(v)$ implies $f(u), f(v) \in \{a, b, c\}^*$,

$$f(\mathrm{proj}_X(u)) = \mathrm{proj}_{\{a,b\}}(f(u)) = \mathrm{proj}_{\{a,b\}}(f(v)) = f(\mathrm{proj}_X(u)) ,$$

and

$$|u|_c = |f(u)|_c = |f(v)|_c = |v|_c .$$

Hence we have $u \sim v$, i.e., the equivalence (2) from Lemma B.4 holds.

Next, we construct a rational relation R satisfying conditions (a) and (b) of Lemma B.5. For $i \in [n]$, let

$$S_i = \begin{pmatrix} a \\ ab^i \end{pmatrix} \left(\begin{pmatrix} c \\ c \end{pmatrix}^* \begin{pmatrix} b \\ \varepsilon \end{pmatrix} \right)^i \begin{pmatrix} c \\ c \end{pmatrix}^*$$

and set

$$R = \begin{pmatrix} c \\ c \end{pmatrix}^* \Big(\bigcup_{i \in [n]} S_i \Big)^* .$$

Suppose $\binom{U}{V} \in S_i$. Then $U = ac^{m_0}bc^{m_1}b \cdots c^{m_{i-1}}bc^{m_i}$ and $V = ab^ic^{m_0+m_1+\cdots+m_i}$ for some $m_0, \ldots, m_i \in \mathbb{N}$. Hence $U \sim V = f(a_ic^{m_0+m_1+\cdots+m_i}) \in f(\Sigma^*)$. It follows that $\binom{U}{V} \in R$

implies $U \sim V \in f(\Sigma^*)$. Next let $U \in [f(\Sigma^*)] \subseteq \{a,b,c\}^*$. Let $m_0 \in \mathbb{N}$ be maximal such that $U = c^{m_0}U'$. Then also $U' \in [f(\Sigma^*)]$. We have to present some word $W \in f(\Sigma^*)$ with $(U,W) \in R$. There is $v \in \Sigma^*$ with $U' \sim f(v)$. Since U' does not start with c and c is independent from all other letters occuring in U', we can assume that v does not start with c either. Then there are $k \in \mathbb{N}$, $m_1,\dots,m_k \in \mathbb{N}$ and $i_1,i_2,\dots,i_k \in [n]$ such that $U' \sim f(a_{i_1}c^{m_1}a_{i_2}c^{m_2}\cdots a_{i_k}c^{m_k}) = ab^{i_1}c^{m_1}ab^{i_2}c^{m_2}\cdots ab^{i_k}c^{m_k}$.

Since $U' \sim f(v) \in \{a,b,c\}^*$ determines the projections of U' to the alphabets $\{a,b\}$ and $\{c\}$, respectively, we can split the word U' as follows: there are $n_1,\dots,n_k \in \mathbb{N}$ and words $u_1,u_2,\dots,u_k \in \{b,c\}^*$ with $\sum_{j\in[k]} m_j = \sum_{j\in[k]} n_j$ and $u_j \in b^{i_j} \sqcup\!\sqcup\, c^{n_j}$ such that $U' = au_1au_2\cdots au_k$.

Then set $W' = ab^{i_1}c^{n_1}\dots ab^{i_k}c^{n_k}$. From $\binom{au_j}{ab^{i_j}c^{n_j}} \in S_{i_j}$ for all $j \in [k]$, we obtain $U' \sim W' \in f(\Sigma^*)$ and $(U',W') \in R$. Finally, observe that also $(U, c^{m_0}W') = (c^{m_0}U', c^{m_0}W') \in R$.

Finally note that, given the independence alphabet (Σ, I), the mapping f and a transducer for the rational relation R can be computed.

Conjunction. After these preparations, we now compute from $((\Sigma,I),\wedge,\mathcal{A}_1,\mathcal{A}_2)$ a tuple $((\Gamma,I),\wedge,\mathcal{B}_1,\mathcal{B}_2)$ such that the former belongs to $\mathrm{RatP}_*(\mathrm{StarA},\wedge)$ iff the latter is an element of $\mathrm{RatP}_*((\Gamma,I),\wedge)$. We can compute nfas $\mathcal{B}_1$ and $\mathcal{B}_2$ over Γ with $L^*(\mathcal{B}_i) = f(L^*(\mathcal{A}_i))$ for all $i \in [2]$. By Lemma B.5, $((\Sigma,I),\wedge,\mathcal{A}_1,\mathcal{A}_2) \in \mathrm{RatP}_*(\mathrm{StarA},\wedge)$ if, and only if, there exists a regular language $K' \subseteq \Gamma^*$ with $[K'] = [f(\Sigma^*)] \cap [f(L^*(\mathcal{A}_1))] \cap [f(L^*(\mathcal{A}_2))]$. Note that this latter language equals $[L^*(\mathcal{B}_1)] \cap [L^*(\mathcal{B}_2)]$ since $[f(L^*(\mathcal{A}_1))] \subseteq [f(\Sigma^*)]$. Hence we showed the equivalence of $((\Sigma,I),\wedge,\mathcal{A}_1,\mathcal{A}_2) \in \mathrm{RatP}_*(\mathrm{StarA},\wedge)$ and $((\Gamma,I),\wedge,\mathcal{B}_1,\mathcal{B}_2) \in \mathrm{RatP}_*((\Gamma,I),\wedge)$, i.e., that the mapping $((\Sigma,I),\wedge,\mathcal{A}_1,\mathcal{A}_2) \mapsto ((\Gamma,I),\wedge,\mathcal{B}_1,\mathcal{B}_2)$ is a reduction of the rationality problem $\mathrm{RatP}_*(\mathrm{StarA},\wedge)$ to the rationality problem $\mathrm{RatP}_*((\Gamma,I),\wedge)$ which is therefore Σ^0_2-hard.

Negation. It remains to also we compute, from $((\Sigma,I),\neg,\mathcal{A})$, a tuple $((\Gamma,I),\neg,\mathcal{B})$ such that the former belongs to $\mathrm{RatP}_*(\mathrm{StarA},\neg)$ iff the latter is an element of $\mathrm{RatP}_*((\Gamma,I),\neg)$.

We can compute an nfa $\mathcal{B}$ over Γ with $L^*(\mathcal{B}) = f(L^*(\mathcal{A}))$. By Lemma B.5, $((\Sigma,I),\neg,\mathcal{A}) \in \mathrm{RatP}_*(\mathrm{StarA},\neg)$ if, and only if, there exists a regular language $K' \subseteq \Gamma^*$ with $[K'] = [f(\Sigma^*)] \cap \Gamma^* \setminus [f(L^*(\mathcal{A}))]$, i.e., with $[K'] = [f(\Sigma^*)] \setminus [L^*(\mathcal{B})]$.

We show that this is equivalent to the existence of a regular language K'' with $[K''] = \Gamma^* \setminus [L^*(\mathcal{B})]$ (i.e., $((\Gamma,I),\neg,\mathcal{B}) \in \mathrm{RatP}_*((\Gamma,I),\neg)$). But first, note that the closure of $f(\Sigma^*)$ is regular since $u \in [f(\Sigma^*)]$ iff $u \in \{a,b,c\}^*$ and the projection $\mathrm{proj}_{\{a,b\}}(u)$ belongs to the regular langage $\{ab^i \mid i \in [n]\}^*$, i.e., $[f(\Sigma^*)] = \{ab^i \mid i \in [n]\}^* \sqcup\!\sqcup\, c^*$.

Now, suppose $[f(\Sigma^*)] \setminus [L^*(\mathcal{B})]$ is the closure of the regular language K'. Then $K'' = K' \cup \Gamma^* \setminus [f(\Sigma^*)]$ is regular and satisfies

$$\begin{aligned}
[K''] &= [K'] \cup \left[\Gamma^* \setminus [f(\Sigma^*)]\right] \\
&= [f(\Sigma^*)] \setminus [L^*(\mathcal{B})] \cup \Gamma^* \setminus [f(\Sigma^*)] && \text{since } \Gamma^* \setminus [f(\Sigma^*)] \text{ is closed} \\
&= \Gamma^* \setminus [L^*(\mathcal{B})] && \text{since } [L^*(\mathcal{B})] = [f(L^*(\mathcal{A}))] \subseteq [f(\Sigma^*)]\,.
\end{aligned}$$

Conversely, suppose $\Gamma^* \setminus [L^*(\mathcal{B})]$ is the closure of the regular language K''. Then $K' := K'' \cap [f(\Sigma^*)]$ is regular and satisfies

$$\begin{aligned}
[K'] &= [K''] \cap [f(\Sigma^*)] && \text{since } [f(\Sigma^*)] \text{ is closed} \\
&= \Gamma^* \setminus [L^*(\mathcal{B})] \cap [f(\Sigma^*)] \\
&= [f(\Sigma^*)] \setminus [L^*(\mathcal{B})]\,.
\end{aligned}$$

Thus, indeed, $[f(\Sigma^*)] \setminus [L^*(\mathcal{B})]$ is the closure of some regular language K' if, and only if, $\Gamma^* \setminus [L^*(\mathcal{B})]$ is the closure of some regular language K''. Thus, the mapping $((\Sigma, I), \neg, \mathcal{A}) \mapsto ((\Gamma, I), \neg, \mathcal{B})$ is a reduction of the rationality problem $\mathrm{RatP}_*(\mathrm{StarA}, \neg)$ to the rationality problem $\mathrm{RatP}_*((\Gamma, I), \neg)$ which is therefore Σ^0_2-hard. ◀

▶ **Theorem 3.4.** *Let $\mathbb{A} \subseteq \mathrm{IndA}$ and $\Phi \subseteq \mathrm{BF}$ be decidable sets.*

(1) If $\mathbb{A} \subseteq \mathrm{TransA}$ or $\Phi \subseteq \mathrm{MAX}$, then the rationality problem $\mathrm{RatP}_*(\mathbb{A}, \Phi)$ is decidable.
(2) Otherwise, the rationality problem $\mathrm{RatP}_*(\mathbb{A}, \Phi)$ is Σ^0_2-complete.

(1') If $\mathbb{A} \subseteq \mathrm{TransA}$ or $\Phi \subseteq \mathrm{CONST}$, then the regularity problem $\mathrm{RegP}_*(\mathbb{A}, \Phi)$ is decidable.
(2') Otherwise, the regularity problem $\mathrm{RegP}_*(\mathbb{A}, \Phi)$ is Σ^0_2-complete.

Proof. The first statement follows from Lemma 3.3(1) and Prop. 2.2.

If $\mathbb{A} \not\subseteq \mathrm{TransA}$ and $\Phi \not\subseteq \mathrm{MAX}$, then there are $(\Gamma, I) \in \mathbb{A}$ and $\varphi \in \Phi$ such that $(\Gamma, I) \notin \mathrm{TransA}$ and $\varphi \notin \mathrm{MAX}$. Since (Γ, I) is not transitive, the rationality problems $\mathrm{RatP}_*((\Gamma, I), \wedge)$ and $\mathrm{RatP}_*((\Gamma, I), \neg)$ both are Σ^0_2-hard by Proposition B.6. Since φ is not a max-function, Lemma 2.3(2,3) allows to reduce one of them to the rationality problem $\mathrm{RatP}_*((\Gamma, I), \varphi)$ which is therefore Σ^0_2-hard as well. Now $(\Gamma, I) \in \mathbb{A}$ and $\varphi \in \Phi$ implies that also $\mathrm{RatP}_*(\mathbb{A}, \Phi)$ is Σ^0_2-hard. Finally, the Σ^0_2-completeness follows from Lemma 3.3(2).

The rest of this proof deals with the regularity problem $\mathrm{RegP}_*(\mathbb{A}, \Phi)$.

First, suppose $\mathbb{A} \subseteq \mathrm{TransA}$. Let $(\Gamma, I) \in \mathbb{A}$, $\varphi \in \Phi$, and $\mathcal{A}_1, \ldots, \mathcal{A}_n$ nfas over Γ. Since (Γ, I) is transitive, [2] allows to construct an nfa $\mathcal{B}$ over Γ with

$$[L^*(\mathcal{B})] = \big\langle [L^*(\mathcal{A}_1)], \ldots, [L^*(\mathcal{A}_n)] \big\rangle^{\varphi} .$$

Again using the transitivity of (Γ, I), it is decidable whether this language is regular by [44], hence $\mathrm{RegP}_*(\mathbb{A}, \Phi)$ is decidable. If $\mathbb{A} \subseteq \mathrm{CONST}$, the decidability forllows from Prop. 2.2.

To also prove the second claim, let $(\Gamma, I) \in \mathbb{A}$ be non-transitive and $\varphi \in \Phi$ non-constant. Then, by [28], the regularity problem $\mathrm{RegP}_*((\Gamma, I), \mathrm{Id})$ is Σ^0_2-hard. By Lemma 2.3(1), it can be reduced to $\mathrm{RegP}_*((\Gamma, I), \varphi)$ and therefore to $\mathrm{RegP}_*(\mathbb{A}, \Phi)$. Hence, also this problem is Σ^0_2-hard.

It remains to verify containment in Σ^0_2. So let $(\Gamma, I) \in \mathbb{A}$ and $\varphi \in \Phi$. In the following, let $X \bigtriangleup Y$ denote the symmetric difference $X \setminus Y \cup Y \setminus X$ of the two sets X and Y; we will use that $X \triangle Y = \emptyset \iff X = Y$. We have $((\Gamma, I), \varphi, \mathcal{A}_1, \ldots, \mathcal{A}_n) \in \mathrm{RegP}_*(\mathbb{A}, \Phi)$ iff

$$\exists \text{ nfa } \mathcal{B} \colon L^*(\mathcal{B}) \bigtriangleup \big\langle [L^*(\mathcal{A}_1)], \ldots, [L^*(\mathcal{A}_n)] \big\rangle^{\varphi} = \emptyset .$$

But this is the case iff

$$\exists \text{ nfa } \mathcal{B} \colon ((\Gamma, I), \varphi', \mathcal{A}_1, \ldots, \mathcal{A}_n, \mathcal{B}) \in \mathrm{EmptP}_*(\mathrm{IndA}, \mathrm{BF}) \wedge L^*(\mathcal{B}) = [L^*(\mathcal{B})]$$

where φ' is the $(n+1)$-ary Boolean function with $\varphi'(b_1, \ldots, b_{n+1}) = 1$ iff $\varphi(b_1, \ldots, b_n) \neq b_{n+1}$. Since $\mathrm{EmptP}_*(\mathrm{IndA}, \mathrm{BF}) \in \Pi^0_1$ by Lemma 3.1(1) and since the equality $L^*(\mathcal{B}) = [L^*(\mathcal{B})]$ is decidable, this statement proves containment in Σ^0_2. ◀

▶ **Lemma B.7.** *Given a Büchi-automaton $\mathcal{A}$ over Γ, it is decidable whether there is an equivalent deterministic Büchi-automaton $\mathcal{B}$.*

Since, from $\mathcal{A}$, one can also construct a Büchi-automaton $\mathcal{A}'$ accepting the complement of $L^\omega(\mathcal{A})$, it is decidable whether $\mathcal{A}$ is co-deterministic.

Proof. For $\alpha \in \Gamma^\omega$, let $\mathrm{pref}(\alpha) \subseteq \Gamma^*$ denote the set of finite prefixes of α. For a language $Y \subseteq \Gamma^*$, let $\overrightarrow{Y}$ denote the set of all words $\alpha \in \Gamma^\omega$ with infinitely many prefixes in Y. We use that a $L^\omega(\mathcal{A})$ is deterministic regular if, and only if, there exists a regular language Y with $\overrightarrow{Y} = L^\omega(\mathcal{A})$ [40, Cor. I.6.3].

Now let $\mathcal{A}$ be a Büchi-automaton. By Büchi's theorem [7], one can construct an MSO formula $\varphi(X)$ with a single free set variable X such that, for all $\alpha \in \Gamma^\omega$,

$$\Gamma^* \models \varphi(\mathrm{pref}(\alpha))\,. \iff \alpha \in L^\omega(\mathcal{A})$$

Consider the closed formula

$$\psi = \exists Y : \forall B \text{ branch}\colon \varphi(B) \leftrightarrow \forall b \in B \exists y \in Y : b \leq y\,.$$

Here, $B \subseteq \Gamma^*$ is a *branch* if it satisfies $\forall a, b \exists c\colon a \leq b \in B \to a \in B \wedge b < c \in B$, i.e., B is a language of the form $\mathrm{pref}(\alpha)$ for some $\alpha \in \Gamma^\omega$. Hence the formula ψ expresses that there is a set $Y \subseteq \Gamma^*$ with $\overrightarrow{Y} = L^\omega(\mathcal{A})$.

We claim that $L^\omega(\mathcal{A})$ is deterministic regular iff this formula ψ holds in Γ^*. First, suppose $L^\omega(\mathcal{A})$ is deterministic regular. Then there exists a regular language $Y \subseteq \Gamma^*$ with $L^\omega(\mathcal{A}) = \overrightarrow{Y}$, hence the formula ψ holds. Conversely, suppose the formula ψ holds, i.e., there is a (not necessarily regular) language Y with $\overrightarrow{Y} = L^\omega(\mathcal{A})$. Since Y is a set satisfying the part of ψ following the outermost existential quantifier, by [42, Thm. 23], there is also a regular language Y with $\overrightarrow{Y} = L^\omega(\mathcal{A})$. The regularity of Y implies that $L^\omega(\mathcal{A})$ is deterministic regular. ◀

C Proofs from Section 4

C.1 Complete independence alphabets

The following lemma was shown by Aalbersberg and Welzl as an existential statement (see first part of the proof of [2, Lemma 2.3]). We include the proof here to stress that it is effective.

▶ **Lemma C.1.** *From a complete independence alphabet* (Γ, I)*, an* n*-ary Boolean function* φ*, and nfas* $\mathcal{A}_i$ *for* $i \in [n]$*, one can construct an nfa* $\mathcal{B}$ *with* $[L^*(\mathcal{B})] = \big\langle [L^*(\mathcal{A}_1)], \ldots, [L^*(\mathcal{A}_n)] \big\rangle^\varphi$.

Proof. Note that the language $\big\langle [L^*(\mathcal{A}_1)], \ldots, [L^*(\mathcal{A}_n)] \big\rangle^\varphi$ can be build from the languages $[L^*(\mathcal{A}_i)]$ using union and complementation. Since $[L_1] \cup [L_2] = [L_1 \cup L_2]$ and since, from two nfas, one can construct an nfa for the union of the accepted languages, it remains to handle complementation.

Thus, let (Γ, I) be a complete independence alphabet and $\mathcal{A}$ an nfa over Γ. We have to construct an nfa $\mathcal{B}$ with $[L^*(\mathcal{B})] = \Gamma^* \setminus [L^*(\mathcal{A})]$.

We use the Parikh mapping $\Psi\colon \Gamma^* \to \mathbb{N}^\Gamma$ with $\Psi(w) = (|w|_a)_{a \in \Gamma}$ for all $w \in \Gamma^*$. Since (Γ, I) is complete, we have $u \sim v \iff \Psi(u) = \Psi(v)$ and therefore

$$\Gamma^* \setminus [L^*(\mathcal{A})] = \{v \in \Gamma^* \mid \Psi(v) \notin \Psi(L^*(\mathcal{A}))\}\,.$$

By Parikh's theorem [38], the set $\Psi(L^*(\mathcal{A}))$ is effectively semi-linear and the same holds of its complement [20]. In other words, one can compute $m, n_i \geq 1$ and $u_i, v_{ij} \in \Gamma^*$ for all $i \in [m]$ and $j \in [n_i]$ such that

$$\mathbb{N}^\Gamma \setminus \Psi(L^*(\mathcal{A})) = \bigcup_{i \in [m]} \Big(\Psi(u_i) + \sum_{j \in [n_i]} \mathbb{N} \cdot \Psi(v_{ij})\Big)$$

which implies

$$\Gamma^* \setminus [L^*(\mathcal{A})] = \bigcup_{i\in[m]} \left[u_i\, v_{i1}^*\, v_{i2}^* \cdots v_{in_i}^*\right].$$

Since one can construct an nfa $\mathcal{B}$ with $L^*(\mathcal{B}) = \bigcup_{i\in[m]} u_i\, v_{i1}^*\, v_{i2}^* \cdots v_{in_i}^*$, the lemma is demonstrated. ◀

We next want to demonstrate the analogous statement for Büchi-automata.

▶ **Lemma C.2.** *From a complete independence alphabet (Γ, I), an n-ary Boolean function φ, and Büchi-automata $\mathcal{A}_i$ for $i \in [n]$, one can construct a Büchi-automaton $\mathcal{B}$ with*

$$[L^\omega(\mathcal{B})] = \left\langle [L^\omega(\mathcal{A}_1)], \ldots, [L^\omega(\mathcal{A}_n)]\right\rangle^\varphi.$$

Proof. Note that the language $\left\langle [L^\omega(\mathcal{A}_1)], \ldots, [L^\omega(\mathcal{A}_n)]\right\rangle^\varphi$ can be build from the languages $[L^\omega(\mathcal{A}_i)]$ using union and complementation. Since $[L_1] \cup [L_2] = [L_1 \cup L_2]$ and since, from two Büchi-automata, one can construct a Büchi-automaton for the union of the accepted languages, it remains to handle complementation.

Thus, let (Γ, I) be a complete independence alphabet and $\mathcal{A}$ a Büchi-automaton over Γ. We have to construct a Büchi-automaton $\mathcal{B}$ with $[L^\omega(\mathcal{B})] = \Gamma^\omega \setminus [L^\omega(\mathcal{A})]$.

For any non-empty set $A \subseteq \Gamma$, let $\mathrm{Inf}_A = \Gamma^* \sqcup \bigsqcup_{a\in A} a^\omega$ denote the set of words that contain infinitely many occurrences of each letter from A (and only those). Then Inf_A is closed and Γ^ω is the disjoint union of the languages Inf_A. Consequently,

$$\begin{aligned}
\Gamma^\omega \setminus [L^\omega(\mathcal{A})] &= \left(\bigcup_{\emptyset\neq A\subseteq\Gamma} \mathrm{Inf}_A\right) \setminus [L^\omega(\mathcal{A})] \\
&= \bigcup_{\emptyset\neq A\subseteq\Gamma} \left(\mathrm{Inf}_A \setminus [L^\omega(\mathcal{A})]\right).
\end{aligned}$$

Let $A \subseteq \Gamma$ be non-empty. We first construct a Büchi-automaton $\mathcal{A}_A$ such that

$$[L^\omega(\mathcal{A}_A)] = \mathrm{Inf}_A \setminus [L^\omega(\mathcal{A})].$$

Fix some word w_A with $|w_A|_a > 0 \iff a \in A$ for all $a \in \Gamma$. Since any two letters are independent, any word from $\bigsqcup_{a\in A} a^\omega$ is equivalent to $(w_A)^\omega$. Thus, for any $\alpha \in \mathrm{Inf}_A$, we have $\alpha \sim \mathrm{proj}_{\Gamma\setminus A}(\alpha)\,(w_A)^\omega$.

Since $L^\omega(\mathcal{A}) \cap \mathrm{Inf}_A$ is effectively regular, one can construct an nfa $\mathcal{B}_A$ with $L^*(\mathcal{B}_A) = \mathrm{proj}_{\Gamma\setminus A}(L^\omega(\mathcal{A}) \cap \mathrm{Inf}_A)$.

We claim the following equivalence for any $\alpha \in \mathrm{Inf}_A$:

$$\alpha \in [L^\omega(\mathcal{A})] \iff \mathrm{proj}_{\Gamma\setminus A}(\alpha) \in [L^*(\mathcal{B}_A)] \tag{4}$$

So let $\alpha \in \mathrm{Inf}_A$ be arbitrary. Then, as we saw above, $\alpha \sim \mathrm{proj}_{\Gamma\setminus A}(\alpha)\,(w_A)^\omega$.

First, assume $\alpha \in [L^\omega(\mathcal{A})]$. Then there exists $\beta \in L^\omega(\mathcal{A})$ with $\alpha \sim \beta$. Then $\beta \in \mathrm{Inf}_A$ since the set Inf_A is closed and contains α. Hence $\beta \sim \mathrm{proj}_{\Gamma\setminus A}(\beta)\,(w_A)^\omega$ implying

$$\mathrm{proj}_{\Gamma\setminus A}(\alpha)\,(w_A)^\omega \sim \alpha \sim \beta \sim \mathrm{proj}_{\Gamma\setminus A}(\alpha)\,(w_A)^\omega.$$

Since the letters occurring in $\mathrm{proj}_{\Gamma\setminus A}(\alpha)$ and $\mathrm{proj}_{\Gamma\setminus A}(\beta)$ are distinct from those appearing in $(w_A)^\omega$, we obtain $\mathrm{proj}_{\Gamma\setminus A}(\alpha) \sim \mathrm{proj}_{\Gamma\setminus A}(\beta)$. Finally, $\beta \in L^\omega(\mathcal{A}) \cap \mathrm{Inf}_A$ implies $\mathrm{proj}_{\Gamma\setminus A}(\beta) \in$

$\mathrm{proj}_{\Gamma\setminus A}(L^\omega(\mathcal{A})\cap \mathrm{Inf}_A) = L^*(\mathcal{B}_A)$ and therefore $\mathrm{proj}_{\Gamma\setminus A}(\alpha)\in[L^*(\mathcal{B}_A)]$. This proves the implication "$\Rightarrow$" in the equivalence (4).

For the converse implication, suppose $\mathrm{proj}_{\Gamma\setminus A}(\alpha)\in[L^*(\mathcal{B}_A)]$. Then there exists $u\in L^*(\mathcal{B}_A)$ with $\alpha\sim u\,(w_A)^\omega$. Since $u\in L^*(\mathcal{B}_A)=\mathrm{proj}_{\Gamma\setminus A}(L^\omega(\mathcal{A})\cap\mathrm{Inf}_A)$, there exists $\beta\in L^\omega(\mathcal{A})\cap\mathrm{Inf}_A$ with $u=\mathrm{proj}_{\Gamma\setminus A}(\beta)$. It follows that

$$\alpha\sim u\,w_A^\omega=\mathrm{proj}_{\Gamma\setminus A}(\beta)\,w_A^\omega\sim\beta\in L^\omega(\mathcal{A})$$

where the last equivalence holds since $\beta\in\mathrm{Inf}_A$. Hence $\alpha\in[L^\omega(\mathcal{A})]$, i.e., we also showed the converse implication. Thus, the equivalence (4) holds.

Using Lemma C.1, one can construct (from $\mathcal{B}_A$) an nfa $\mathcal{C}_A$ with

$$[L^*(\mathcal{C}_A)]=\Gamma^*\setminus[L^*(\mathcal{B}_A)]\,.$$

Then we have the following for any ω-word $\alpha\in\mathrm{Inf}_A$:

$$\begin{aligned}\alpha\notin[L^\omega(\mathcal{A})]&\iff \mathrm{proj}_{\Gamma\setminus A}(\alpha)\notin[L^*(\mathcal{B}_A)] && \text{by (4)}\\ &\iff \mathrm{proj}_{\Gamma\setminus A}(\alpha)\in[L^*(\mathcal{C}_A)]\\ &\iff \alpha\in[L^*(\mathcal{C}_A)\,(w_A)^\omega]\end{aligned}$$

Finally, one can construct a Büchi-automaton $\mathcal{A}_A$ accepting $L^*(\mathcal{C}_A)\,(w_A)^\omega$ such that (for $\alpha\in\mathrm{Inf}_A$) $\alpha\notin[L^\omega(\mathcal{A})]\iff\alpha\in[L^\omega(\mathcal{A}_A)]$.

Since the sets Inf_A (for $A\subseteq\Gamma$ nonemtpy) are closed and form a partition of Γ^ω, we therefore get the following (where all unions extend over all nonempty subsets A of Γ):

$$\begin{aligned}\Gamma^\omega\setminus[L^\omega(\mathcal{A})]&=\bigcup(\mathrm{Inf}_A\setminus[L^\omega(\mathcal{A})]) && \text{(see above)}\\ &=\bigcup[L^\omega(\mathcal{A}_A)]\\ &=\Big[\bigcup L^\omega(\mathcal{A}_A)\Big]\,.\end{aligned}$$

With $\mathcal{B}$ the disjoint union of the Büchi-automata $\mathcal{A}_A$, we obtain $[L^\omega(\mathcal{B})]=\Gamma^\omega\setminus[L^\omega(\mathcal{A})]$ as required. ◀

The above two lemmas yield the decidability of all problems (considered in this paper) for complete independence alphabets and ω-languages.

▶ **Corollary C.3.** *For all* $\mathrm{GenP}\in\{\mathrm{EmptP},\mathrm{RatP},\mathrm{RegP}\}$*, the problem* $\mathrm{GenP}_\omega(\mathrm{ComplA},\mathrm{BF})$ *is decidable.*

Proof. Let $(\Gamma,I)\in\mathbb{A}$ be a complete independence alphabet, $\varphi\in\Phi$ an n-ary Boolean function, and $\mathcal{A}_i$ for $i\in[n]$ a Büchi-automaton over Γ. Then, by Lemma C.2, we can construct a Büchi-automaton $\mathcal{B}$ with $[L^\omega(\mathcal{B})]=\big\langle[L^\omega(\mathcal{A}_1)],\dots,[L^\omega(\mathcal{A}_n)]\big\rangle^\varphi$. Hence $((\Gamma,I),\varphi,\mathcal{A}_1,\dots,\mathcal{A}_n)\in\mathrm{RatP}_\omega(\mathbb{A},\Phi)$, i.e., this rationality problem is (trivially) decidable. Since the emptiness of the regular language $L^\omega(\mathcal{B})$ is decidable, also the emptiness problem $\mathrm{EmptP}_\omega(\mathbb{A},\Phi)$ is decidable.

To handle the regularity problem, let $L=L^\omega(\mathcal{B})$ and, for $A\subseteq\Gamma$, let $\mathrm{Inf}_A=\Gamma^*\sqcup\bigsqcup_{a\in A}a^\omega$ denote the set of all words over A with infinitely many occurrences of all letters from A. Then the language $[L]$ is regular iff all the languages $[L]\cap\mathrm{Inf}_A$ for $A\subseteq\Gamma$ are regular. So let $A\subseteq\Gamma$. Since Inf_A is closed, we get

$$\begin{aligned}[L]\cap\mathrm{Inf}_A&=[L\cap\mathrm{Inf}_A]\\ &=[\mathrm{proj}_{\Gamma\setminus A}(L\cap\mathrm{Inf}_A)]\sqcup\mathrm{Inf}_A\,.\end{aligned}$$

This language is regular iff $[\mathrm{proj}_{\Gamma\setminus A}(L\cap\mathrm{Inf}_A)]$ is regular. Note that all words from $\mathrm{proj}_{\Gamma\setminus A}(L\cap\mathrm{Inf}_A)$ are finite. Since, from the Büchi-automaton $\mathcal{B}$ and the set $A\subseteq\Gamma$, we can construct an nfa for this language, the decidability follows from [21]. ◀

C.2 Types and type automata

We next introduce notions and prove some results that will be useful later on as well (cf. proof of Theorem D.8), therefore, they are more general than needed at this point. So let (Γ_k, I_k) for $k \in [m]$ be mutually disjoint independence alphabets and let (Γ, I) be their union, i.e., $\Gamma = \bigcup_{k\in[m]} \Gamma_k$ and $I = \bigcup_{k\in[m]} I_k$ (note that we do not require (Γ_k, I_k) to be complete such that (Γ, I) is not necessarily transitive). We call a non-empty finite or infinite word over Γ *mono-alphabetic* if there is $k \in [m]$ such that all letters appearing in the word belong to Γ_k, i.e., the set of mono-alphabetic words equals $\bigcup_{k\in[m]}(\Gamma_k^+ \cup \Gamma_k^\omega)$.

Now let, for $i \in [n]$, $\mathcal{A}_i = (Q_i, \Gamma, S_i, T_i, F_i)$ be a Büchi-automaton over Γ. Our aim is to abstract the behavior of $\alpha \in \Gamma^\omega$ in the automata $\mathcal{A}_i$ for all $i \in [n]$. The word α splits uniquely into maximal mono-alphabetic factors. For any such factor, we will define its "type" and then consider the sequence of these "types" as abstraction of α. There are two possibilities for the splitting of α: it can result in infinitely many finite factors or in finitely many finite ones followed by a single infinite. Since these two cases have to be considered separately, we define the corresponding sets of words α as follows.

Let $\mathrm{MonoInf} := \bigcup_{k\in[m]} \Gamma^* \, \Gamma_k^\omega$ denote the set of words *mono-alphabetic at infinity* and

$$\mathrm{MultiInf} := \Gamma^* \setminus \mathrm{MonoInf} = \bigcup_{1\le k<\ell\le m} \bigsqcup\!\!\bigsqcup(\Gamma^\omega, \Gamma_k^\omega, \Gamma_\ell^\omega)\,.$$

the set of words *multi-alphabetic at infinity.* Note that $\alpha \in \mathrm{MonoInf}$ iff it splits into finitely many maximal mono-alphabetic factors and $\alpha \in \mathrm{MultiInf}$ otherwise.

For a finite mono-alphabetic word $u \in \Gamma_k^+$ with $k \in [m]$, define the *type* $\mathrm{tp}_+(u) = \big((R_i, R_i^F)_{i\in[n]}, k\big)$ as follows (with $i \in [n]$):

- $R_i = \{(p,q) \in Q_i^2 \mid \exists u' \sim u\colon p \xrightarrow{u'}_{\mathcal{A}_i} q\}$
- $R_i^F = \{(p,q) \in Q_i^2 \mid \exists u' \sim u\colon p \overset{u'}{\Longrightarrow}_{\mathcal{A}_i} q\}$

(Recall that $p \overset{u'}{\Longrightarrow}_{\mathcal{A}_i} q$ denotes that there is a u'-labeled path from p to q that visits some accepting state.)

The type of a finite mono-alphabetic word generalizes what is often called the transition profile (cf. [6]) of a word. If $I = \emptyset$ and $i \in [n]$, then R_i is the set of pairs connected by the word u and R_i^F the set of pairs connected by some u-labeled path that visits some accepting state. Thus, in this case, the type is the tuple of transition profiles of u in the automata $\mathcal{A}_i$. If $I \neq \emptyset$, we do not only collect those pairs that are connected by some u-labeled path, but also add those that are connected by some path whose label is equivalent to u. In other words, the type of u is the tuple of "unions" of transition profiles of all words $u' \in [u]$ in the automata $\mathcal{A}_i$.

For an infinite mono-alphabetic word $\alpha \in \Gamma_k^\omega$, define the type $\mathrm{tp}_\omega(\alpha) = \big((A_i)_{i\in[n]}, k\big)$ by

$$A_i = \{p \in Q_i \mid \exists \alpha' \sim \alpha\colon \alpha' \in L^\omega(\mathcal{A}_i^{p\rightarrow})\}$$

where $i \in [n]$. Then A_i is the set of states of the Büchi-automaton $\mathcal{A}_i$ that allow to accept some word from $[\alpha]$.

Let $\mathrm{TP}_+ = \bigcup_{k\in[n]}\{\mathrm{tp}_+(u) \mid u \in \Gamma_k^+\}$ denote the set of all types of mono-alphabetic finite words and $\mathrm{TP}_\omega = \bigcup_{k\in[n]}\{\mathrm{tp}_\omega(\alpha) \mid \alpha \in \Gamma_k^\omega\}$ the set of all types of mono-alphabetic infinite words. Let TP be the union of the sets TP_+ and TP_ω. As (possibly proper) subset of

$$\prod_{i\in[n]} \big(\mathcal{P}(Q_i \times Q_i) \times \mathcal{P}(Q_i \times Q_i)\big) \times [n] \cup \prod_{i\in[n]} \mathcal{P}(Q_i) \times [n]\,,$$

all these sets are finite. It is not difficult to see that $\mathrm{tp}_+(u)$ can be computed from a finite mono-alphabetic word u, the Büchi-automata $\mathcal{A}_i$, and the tuple of independence alphabets (Γ_k, I_k) since all words equivalent with u are of length $|u|$. Since the finite words can be enumerated, it follows that the set TP_+ of all types of finite mono-alphabetic words can be enumerated from the tuples $(\mathcal{A}_i)_{i\in[n]}$ and $(\Gamma_k, I_k)_{k\in[m]}$. But, in general, it cannot be computed from these tuples as this would require to determine the existence of a finite word u for a given tuple $\big((R_i, R_i^F)_{i\in[n]}, k\big)$ of the above direct product; this, in particular, requires to decide the existence of a word $u \in \Gamma_k^+$ with $p \xrightarrow{u}_{\mathcal{A}_i} q$, but not $p' \xrightarrow{v}_{\mathcal{A}_i} q'$ for any $v \sim u$ for $(p,q) \in R_i$ and $(p',q') \notin R_i$, a question that is undecidable in general [1] (it is decidable for transitive independence alphabets, see Lemma C.10 below).

Note that the above consideration does not even yield the semi-decidability of the set TP_ω since we cannot enumerate all infinite mono-alphabetic words $\alpha \in \Gamma_k^\omega$.

Let $\alpha \in \Gamma^\omega$ be some infinite word. Depending on whether $\alpha \in \mathrm{MultiInf}$ or not, we now define the *type sequence* of α as follows:

- Suppose $\alpha = u_1 u_2 u_3 \cdots$ with $u_j \in \Gamma_{k_j}^+$ and $k_j \neq k_{j+1}$ for all $j \geq 1$. Then the type sequence of α is the infinite word
$$\mathrm{tpsqu}(\alpha) = \mathrm{tp}_+(u_1)\,\mathrm{tp}_+(u_2)\,\mathrm{tp}_+(u_3)\cdots \in \mathrm{TP}_+^\omega\ .$$
- Suppose $\alpha = u_1 u_2 \cdots u_\ell\, \beta$ with $u_j \in \Gamma_{k_j}^+$, $\beta \in \Gamma_{k_{\ell+1}}^\omega$ and $k_j \neq k_{j+1}$ for all $i \in [\ell]$. Then the type sequence of α is the finite word
$$\mathrm{tpsqu}(\alpha) = \mathrm{tp}_+(u_1)\,\mathrm{tp}_+(u_2)\,\cdots \mathrm{tp}_+(u_\ell)\,\mathrm{tp}_\omega(\beta) \in \mathrm{TP}_+^*\ \mathrm{TP}_\omega\ .$$

▶ **Lemma C.4.** *Let* $\alpha, \beta \in \Gamma^\omega$ *with* $\alpha \sim \beta$. *Then* $\mathrm{tpsqu}(\alpha) = \mathrm{tpsqu}(\beta)$.

Proof. Suppose $\alpha = u_1 u_2 \cdots$ with $u_j \in \Gamma_{k_j}^+$ and $k_j \neq k_{j+1}$ for all $j \geq 1$. Since letters from distinct alphabets Γ_k are dependent, $\alpha \sim \beta$ implies $\beta = v_1 v_2 \cdots$ for some words $v_j \in \Gamma_{k_j}^+$ with $u_j \sim v_j$ for all $j \geq 1$. But then $\mathrm{tp}_+(u_j) = \mathrm{tp}_+(v_j)$ and therefore $\mathrm{tpsqu}(\alpha) = \mathrm{tpsqu}(\beta)$.

If $\alpha \in \mathrm{MonoInf}$ is mono-alphabetic at infinitely, one can argue similarly. ◀

Let $\mathrm{TPSQU}^{\mathrm{mono}}$ denote the set of words from $\mathrm{TP}_+^*\ \mathrm{TP}_\omega$ that do not contain a factor of the form $\big((R_i, R_i^F)_{i\in[n]}, k\big)\,\big((S_i, S_i^F)_{i\in[n]}, k\big)$ or $\big((R_i, R_i^F)_{i\in[n]}, k\big)\,\big((A_i)_{i\in[n]}, k\big)$ for any $k \in [m]$. Similarly, let $\mathrm{TPSQU}^{\mathrm{multi}}$ denote the set of ω-words from TP_+^ω that do not contain such a factor. Clearly, given the sets of types TP_+ and TP_ω, one can construct an nfa accepting $\mathrm{TPSQU}^{\mathrm{mono}}$ and a Büchi-automaton accepting $\mathrm{TPSQU}^{\mathrm{multi}}$.

▶ **Lemma C.5.** *We have* $\mathrm{TPSQU}^{\mathrm{multi}} = \{\mathrm{tpsqu}(\alpha) \mid \alpha \in \mathrm{MultiInf}\}$ *and* $\mathrm{TPSQU}^{\mathrm{mono}} = \{\mathrm{tpsqu}(\alpha) \mid \alpha \in \mathrm{MonoInf}\}$.

Proof. First, let $\alpha \in \mathrm{MultiInf}$. Then α splits uniquely into mono-alphabetic finite words $u_j \in \Gamma_{k_j}^+$ with $k_j \neq k_{j+1}$ for all $j \geq 1$. Since $\mathrm{tpsqu}^{\mathrm{multi}}(\alpha)$ is the sequence of types $\mathrm{tp}_+(u_j)$ and since $k_j \neq k_{j+1}$ for all $j \geq 1$, we obtain $\mathrm{tpsqu}(\alpha) \in \mathrm{TPSQU}^{\mathrm{multi}}$.

Conversely, let $W = t_1\, t_2 \cdots \in \mathrm{TPSQU}^{\mathrm{multi}}$ with $t_j \in \mathrm{TP}_+$ for all $j \geq 1$. The definition of the set TP_+ implies that there are finite mono-alphabetic words $u_j \in \Gamma_{k_j}^+$ with $t_j = \mathrm{tp}_+(u_j)$ for all $j \geq 1$. Note that the index k_j is determined by the type t_j. Since $W \in \mathrm{TPSQU}^{\mathrm{multi}}$, we obtain $k_j \neq k_{j+1}$ for all $j \geq 1$. With $\alpha = u_1\, u_2\, u_3 \cdots$, we therefore get $\alpha \in \mathrm{MultiInf}$ and $\mathrm{tpsqu}^{\mathrm{multi}}(\alpha) = W$.

Thus, we proved the claim regarding $\mathrm{TPSQU}^{\mathrm{multi}}$, the other claim is shown similarly. ◀

We will next prove that the language $[L^\omega(\mathcal{A}_i)]$ is the inverse image (wrt. the mapping tpsqu) of some regular set. More precisely, this will be shown for the languages $[L^\omega(\mathcal{A}_i)] \cap \mathrm{MultiInf}$ and $[L^\omega(\mathcal{A}_i)] \cap \mathrm{MonoInf}$ separately since the type sequences of the former set are infinite while those of the latter are finite.

We first handle the case of words multi-alphabetic at infinity, i.e., the former language that is a subset of $\mathrm{TPSQU}^{\mathrm{multi}} \subseteq \mathrm{TP}_+^\omega$. From the Büchi-automaton $\mathcal{A}_i$, we define a new Büchi-automaton $\mathcal{A}_i^{\mathrm{multi}} = (Q_i^{\mathrm{multi}}, \mathrm{TP}_+, S_i^{\mathrm{multi}}, T_i^{\mathrm{multi}}, F_i^{\mathrm{multi}})$ over the alphabet TP_+ as follows:

- The set of states is defined by $Q_i^{\mathrm{multi}} = Q_i \times [n] \times \{+,-\}$.
- The set of initial states equals $S_i^{\mathrm{multi}} = S_i \times [n] \times \{+,-\}$.
- The set of accepting states equals $F_i^{\mathrm{multi}} = Q_i \times [n] \times \{+\}$.
- There is a transition $(p,j,a) \xrightarrow{((R_i,R_i^F)_{i\in[n]},k)} (q,\ell,b)$ in T_i^{multi} with $((R_i,R_i^F)_{i\in[n]},k) \in \mathrm{TP}_+$ iff
  - $(p,q) \in R_i$,
  - $j \neq k = \ell$,
  - $b = -$ or $(p,q) \in R_i^F$.

The alphabet of this automaton $\mathcal{A}_i^{\mathrm{multi}}$ is the set TP_+ of types of finite words. Then there is a $\mathrm{tp}_+(u)$-labeled transition from (p,j,a) to (q,ℓ,b) if, for some word v equivalent with u, the automaton $\mathcal{A}_i$ has a v-labeled path from p to q; the component ℓ of (q,ℓ,b) recalls the sub-alphabet Γ_ℓ that the words u and v use; the final component b recalls whether the v-labeled path visits some accepting state or not. Using the second component of states, the automaton $\mathcal{A}_i^{\mathrm{multi}}$ ensures that, in any accepted word, consecutive types t and t' refer to distinct sub-alphabets Γ_k and Γ_ℓ, i.e., $L^\omega(\mathcal{A}_i^{\mathrm{multi}}) \subseteq \mathrm{TPSQU}^{\mathrm{multi}}$.

▶ **Lemma C.6.** *For all $i \in [n]$, we have*

$$[L^\omega(\mathcal{A}_i)] \cap \mathrm{MultiInf} = \{\alpha \in \mathrm{MultiInf} \colon \mathrm{tpsqu}(\alpha) \in L^\omega(\mathcal{A}_i^{\mathrm{multi}})\}\,.$$

Proof. Let $i \in [n]$. Since both sites of the equation are subsets of MultiInf, we consider some $\alpha \in \mathrm{MultiInf}$. Then there are $u_j \in \Gamma_{k_j}^+$ with $k_j \neq k_{j+1}$ for all $j \geq 1$ such that $\alpha = u_1 u_2 \cdots$. For $j \geq 1$, let $\mathrm{tp}_+(u_j) = ((R_{ij}, R_{ij}^F)_{i\in[n]}, k_j)$.

We have $\alpha \in [L^\omega(\mathcal{A}_i)]$ iff

$$\text{there is } \beta \in L^\omega(\mathcal{A}_i) \text{ with } \alpha \sim \beta. \tag{5}$$

Note that any such β is multi-alphabetic at infinity since $\alpha \in \mathrm{MultiInf}$. Therefore, (5) is equivalent to

$$\text{there are words } v_1, v_2, \ldots \text{ with } v_1 v_2 \cdots \in L^\omega(\mathcal{A}_i) \text{ and } u_j \sim v_j \text{ for all } j \geq 1.$$

By the definition of the language of the Büchi-automaton $\mathcal{A}_i$, this is equivalent to the existence of words $v_1, v_2, \ldots$ and states $q_0, q_1, \dots \in Q_i$ of $\mathcal{A}_i$ such that

- $q_0 \in S_i$ is initial in $\mathcal{A}_i$,
- there is a v_j-labeled path from q_{j-1} to q_j in $\mathcal{A}_i$ for all $j \geq 1$,
- such a path visits some accepting state for infinitely many $j \geq 1$, and
- $u_j \sim v_j$ for all $j \geq 1$.

Since $\mathrm{tp}_+(u_j) = ((R_{ij}, R_{ij}^F)_{i\in[n]}, k_j)$ for all $j \geq 1$, this is equivalent to the existence of states $q_0, q_1, \dots \in Q_i$ such that

- $q_0 \in S_i$ is initial in $\mathcal{A}_i$,
- $(q_{j-1}, q_j) \in R_{ij}$ for all $j \geq 1$, and
- $(q_{j-1}, q_j) \in R_{ij}^F$ for infinitely many $j \geq 1$.

But this is equivalent to saying that the Büchi-automaton $\mathcal{A}_i^{\text{multi}}$ accepts $\text{tp}_+(u_1)\,\text{tp}_+(u_2)\cdots$ which equals $\text{tpsqu}(\alpha)$. ◀

We next want to prove that also the language $[L^\omega(\mathcal{A}_i)] \cap \text{MonoInf}$ is the inverse image (wrt. the mapping tpsqu) of some regular set. Since type sequences of words from MonoInf are finite words over TP, we construct the following nfa $\mathcal{A}_i^{\text{mono}} = (Q_i^{\text{mono}}, \text{TP}, S_i^{\text{mono}}, T_i^{\text{mono}}, F_i^{\text{mono}})$ for $i \in [n]$:

- The set of states is defined by $Q_i^{\text{mono}} = (Q_i \times [n]) \cup \{\top\}$.
- The set of initial states equals $S_i^{\text{mono}} = (S_i \times [n])$.
- $\top$ is the only accepting state.
- There is a transition $(p, j) \xrightarrow{((R_i, R_i^F)_{i\in[n]}, k)} (q, \ell)$ in T_i^{mono} with $((R_i, R_i^F)_{i\in[n]}, k) \in \text{TP}_+$ iff $(p, q) \in R_i$ and $j \neq k = \ell$.
- There is a transition $(p, j) \xrightarrow{((A_i)_{i\in[n]}, k)} \top$ in T_i^{mono} with $((A_i)_{i\in[n]}, k) \in \text{TP}_\omega$ iff $p \in A_i$ and $j \neq k$.
- There are no further transitions in T_i^{mono}.

▶ **Lemma C.7.** *For all $i \in [n]$, we have*

$$[L^\omega(\mathcal{A}_i)] \cap \text{MonoInf} = \{\alpha \in \text{MonoInf} : \text{tpsqu}(\alpha) \in L^\omega(\mathcal{A}_i^{\text{mono}})\}\,.$$

Proof. Let $i \in [n]$. Since both sites of the equation are subsets of MonoInf, we consider some $\alpha \in \text{MonoInf}$. Then there are $\ell \geq 0$, $u_j \in \Gamma_{k_j}^+$ and $\alpha' \in \Gamma_{k_{\ell+1}}^\omega$ with $k_j \neq k_{j+1}$ for all $j \in [\ell]$ such that $\alpha = u_1 u_2 \cdots u_\ell \alpha'$. For $j \in [\ell]$, let $\text{tp}_+(u_j) = ((R_{ij}, R_{ij}^F)_{i\in[n]}, k_j)$ and let $\text{tp}_\omega(\alpha') = ((A_i)_{i\in[n]}, k_{\ell+1})$.

We have $\alpha \in [L^\omega(\mathcal{A}_i)]$ iff

$$\text{there is } \beta \in L^\omega(\mathcal{A}_i) \text{ with } \alpha \sim \beta. \tag{6}$$

Note that any such β is mono-alphabetic at infinity since $\alpha \in \text{MonoInf}$. Therefore, (6) is equivalent to

$$\text{there are words } v_1, v_2, \ldots v_\ell, \beta' \text{ with } v_1 v_2 \cdots v_\ell \beta' \in L^\omega(\mathcal{A}_i),\ u_j \sim v_j \text{ for all } j \in [\ell], \text{ and } \alpha' \sim \beta'\,.$$

By the definition of the language of the Büchi-automaton $\mathcal{A}_i$, this is equivalent to the existence of words $v_1, v_2, \ldots, v_\ell, \beta$ and states $q_0, q_1, \ldots, q_\ell \in Q_i$ of $\mathcal{A}_i$ such that

- $q_0 \in S_i$ is initial in $\mathcal{A}_i$,
- there is a v_j-labeled path from q_{j-1} to q_j in $\mathcal{A}_i$ for all $j \in [\ell]$,
- $\beta' \in L^\omega(\mathcal{A}_i^{q_{\ell+1}\to})$, and
- $u_j \sim v_j$ for all $j \in [\ell]$ and $\alpha' \sim \beta'$.

Since $\text{tp}_+(u_j) = ((R_{ij}, R_{ij}^F)_{i\in[n]}, k_j)$ for all $j \in [\ell]$ and $\text{tp}_\omega(\alpha') = ((A_i)_{i\in[n]}, q_\ell)$, this is equivalent to the existence of states $q_0, q_1, \ldots, q_\ell \in Q_i$ such that

- $q_0 \in S_i$ is initial in $\mathcal{A}_i$,
- $(q_{j-1}, q_j) \in R_{ij}$ for all $j \in [\ell]$, and
- $q_\ell \in A_i$.

But this is equivalent to saying that the nfa $\mathcal{A}_i^{\mathrm{mono}}$ accepts $\mathrm{tp}_+(u_1)\mathrm{tp}_+(u_2)\cdots\mathrm{tp}_+(u_\ell)\mathrm{tp}_\omega(\alpha')$ which equals $\mathrm{tpsqu}(\alpha)$. ◀

Now we can summarize our results on types and type sequences.

▶ **Theorem C.8.** *From*

- *mutually disjoint independence alphabets* (Γ_k, I_k) *for* $k \in [m]$,
- *Büchi-automata* $\mathcal{A}_i$ *over* $\Gamma = \bigcup_{k\in[m]} \Gamma_k$ *for* $i \in [n]$,
- *an* n*-ary Boolean function* φ*, and*
- *the sets of types* TP_+ *and* TP_ω,

one can compute a Büchi-automaton $\mathcal{A}^{\mathrm{multi}}$ *and an nfa* $\mathcal{A}^{\mathrm{mono}}$ *such that (for all* $\alpha \in \Gamma^\omega$*)*

$$\alpha \in \big\langle [L^\omega(\mathcal{A}_1)], \ldots, [L^\omega(\mathcal{A}_n)] \big\rangle^{\varphi}$$

if, and only if,

$$\mathrm{tpsqu}(\alpha) \in L^\omega(\mathcal{A}^{\mathrm{multi}}) \cup L^*(\mathcal{A}^{\mathrm{mono}})\,.$$

Proof. From the given data, one can compute the Büchi-automata $\mathcal{A}_i^{\mathrm{multi}}$ and the nfas $\mathcal{A}_i^{\mathrm{mono}}$. Since $\big\langle L^\omega(\mathcal{A}_1^{\mathrm{multi}}), \ldots, L^\omega(\mathcal{A}_n^{\mathrm{multi}}) \big\rangle^{\varphi}$ is a Boolean combination of the languages of the Büchi-automata $\mathcal{A}_i^{\mathrm{multi}}$, one can construct a Büchi-automaton $\mathcal{A}^{\mathrm{multi}}$ with

$$L^\omega(\mathcal{A}^{\mathrm{multi}}) = \big\langle L^\omega(\mathcal{A}_1^{\mathrm{multi}}), \ldots, L^\omega(\mathcal{A}_n^{\mathrm{multi}}) \big\rangle^{\varphi}\,.$$

Now let $\alpha \in \mathrm{MultiInf}$. By Lemma C.6, we have $\alpha \in \big\langle [L^\omega(\mathcal{A}_1)], \ldots, [L^\omega(\mathcal{A}_n)] \big\rangle^{\varphi}$ iff $\mathrm{tpsqu}(\alpha) \in \big\langle [L^\omega(\mathcal{A}_1^{\mathrm{multi}})], \ldots, [L^\omega(\mathcal{A}_n^{\mathrm{multi}})] \big\rangle^{\varphi}$ iff $\mathrm{tpsqu}(\alpha) \in L^\omega(\mathcal{A}^{\mathrm{multi}})$.

Similarly, one can construct an nfa $\mathcal{A}^{\mathrm{mono}}$ with

$$L^*(\mathcal{A}^{\mathrm{mono}}) = \big\langle L^*(\mathcal{A}_1^{\mathrm{mono}}), \ldots, L^*(\mathcal{A}_n^{\mathrm{mono}}) \big\rangle^{\varphi}$$

and obtain, for any $\alpha \in \mathrm{MonoInf}$, $\alpha \in \big\langle [L^\omega(\mathcal{A}_1)], \ldots, [L^\omega(\mathcal{A}_n)] \big\rangle^{\varphi}$ iff $\mathrm{tpsqu}(\alpha) \in L^\omega(\mathcal{A}^{\mathrm{mono}})$ using Lemma C.7. ◀

C.3 Rationality problem

▶ **Lemma C.9.** *Given complete independence alphabets* (Γ_k, I_k) *for* $k \in [m]$*, Büchi-automata* $\mathcal{A}_i = (Q_i, \Gamma, S_i, T_i, F_i)$ *for* $i \in [n]$ *over* $\Gamma = \bigcup_{k\in[m]} \Gamma_k$*,* $I = \bigcup_{k\in[m]} I_k$*, sets* $R_i, R_i^F \subseteq Q_i^2$*,* $A_i \subseteq Q_i$*, and* $k \in [m]$*, one can compute*

- *an nfa* $\mathcal{T}_1$ *over* Γ_k *with* $[L(\mathcal{T}_1)] = \Big\{ u \in \Gamma_k^+ \mid \mathrm{tp}_+(u) = \big((R_i, R_i^F)_{i\in[n]}, k\big) \Big\}$ *and*
- *a Büchi-automaton* $\mathcal{T}_2$ *over* Γ_k *with* $[L^\omega(\mathcal{T}_2)] = \Big\{ \alpha \in \Gamma_k^\omega \mid \mathrm{tp}_\omega(\alpha) = \big((A_i)_{i\in[n]}, k\big) \Big\}$.

Proof. We first construct the nfa $\mathcal{T}_1$. For $i \in [n]$ and $p, q \in Q_i$, let $K_i^{pq} \subseteq \Gamma_k^+$ denote the set of words $u \in \Gamma_k^+$ with $p \xrightarrow{u}_{\mathcal{A}_i} q$ and K_i^{Fpq} the set of words $u' \in \Gamma_k^+$ with $p \overset{u'}{\Longrightarrow}_{\mathcal{A}_i}$. Furthermore, consider the language

$$\begin{aligned} K_i = {} & \bigcap_{(p,q)\in R_i} [K_i^{pq}] \cap \bigcap_{(p,q)\in Q_i^2\setminus R_i} \Gamma_k^+ \setminus [K_i^{pq}] \\ & \cap \bigcap_{(p,q)\in R_i^F} [K_i^{Fpq}] \cap \bigcap_{(p,q)\in Q_i^2\setminus R_i^F} \Gamma_k^+ \setminus [K_i^{Fpq}]\,. \end{aligned}$$

Then $K = \bigcap_{i\in[n]} K_i$ is the set of words $\alpha \in \Gamma_k^+$ whose type equals $((R_i, R_i^F)_{i\in[n]}, k)$.

This is a Boolean combination of closures of regular sets in Γ_k^+. Since (Γ_k, I_k) is complete, one can construct the nfa $\mathcal{T}_1$ as required using Lemma C.1.

The construction of the Büchi-automaton $\mathcal{T}_2$ follows similar lines. For $i \in [n]$ and $p \in Q_i$, let $L_i^p = L^\omega(\mathcal{A}_i^{p\to}) \cap \Gamma_k^\omega$ denote the set of words $\beta \in \Gamma_k^\omega$ accepted by the Büchi-automaton $\mathcal{A}_i$ when started in state p. Furthermore, consider the language

$$L_i = \bigcap_{p\in A_i} [L_i^p] \cap \bigcap_{p\in Q_i\setminus A_i} \Gamma_k^\omega \setminus [L_i^p] .$$

Then $L = \bigcap_{i\in[n]} L_i$ is the set of words $\alpha \in \Gamma_k^\omega$ whose type equals $\big((A_i)_{i\in[n]}, k\big)$. The language L is a Boolean combination of closures of regular sets. Note that all the languages L_i^p are subsets of Γ_k^ω and that (Γ_k, I_k) is a complete independence alphabet. Hence, by Lemma C.2, we can construct a Büchi-automaton $\mathcal{T}_2$ over Γ_k with $L = [L^\omega(\mathcal{T}_2)]$. ◀

▶ **Lemma C.10.** *Given complete independence alphabets (Γ_k, I_k) for $k \in [m]$, Büchi-automata $\mathcal{A}_i = (Q_i, \Gamma, S_i, T_i, F_i)$ for $i \in [n]$ over $\Gamma = \bigcup_{k\in[m]} \Gamma_k$, and $I = \bigcup_{k\in[m]} I_k$, one can compute the sets of types TP_+ and TP_ω.*

Proof. Let $t_1 = ((R_i, R_i^F)_{i\in[n]}, k)$ and $t_2 = ((A_i)_{i\in[n]}, k)$ with $R_i, R_i^F \subseteq Q_i^2$ and $A_i \subseteq Q_i$ for all $i \in [n]$ and $k \in [m]$. Then, by Lemma C.9, one can construct an nfa $\mathcal{T}_{t_1}$ and a Büchi-automaton $\mathcal{T}_{t_2}$ such that $[L^*(\mathcal{T}_{t_1})]$ is the set of finite words of type t_1 and $[L^\omega(\mathcal{T}_{t_2})]$ is the set of infinite words of type t_2. Since the emptiness of these two languages is decidable, we can decide whether $t_1 \in \mathrm{TP}_+$ and $t_2 \in \mathrm{TP}_\omega$, respectively. Since there are only finitely many tuples t_1 and t_2, the result follows. ◀

In view of this lemma and Theorem C.8, we see that one can compute the automata $\mathcal{A}^{\mathrm{multi}}$ and $\mathcal{A}^{\mathrm{mono}}$ that accept the type sequences of words from $\big\langle [L^\omega(\mathcal{A}_1)], \ldots, [L^\omega(\mathcal{A}_n)] \big\rangle^\varphi$. In order to build an automaton for this set itself, we will apply a rational relation to the languages of these two automata.

▶ **Lemma C.11.** *From*

- *mutually disjoint complete independence alphabets (Γ_k, I_k) for $k \in [m]$ and*
- *Büchi-automata $\mathcal{A}_i$ over $\Gamma = \bigcup_{k\in[m]} \Gamma_k$ for all $i \in [n]$,*

one can compute an ω-transducer for a relation R such that the following hold:

- *For any $\alpha \in \Gamma^\omega$, there exists $\beta \sim \alpha$ with $(\mathrm{tpsqu}(\alpha), \beta) \in R$.*
- *For any $(W, \alpha) \in R$ with $W \in \mathrm{TPSQU}$, we have $W = \mathrm{tpsqu}(\alpha)$.*

Proof. The relation R will be the union of relations R_1^ω and $R_1^* H$ where $R_1 \subseteq \mathrm{TP}_+ \times \Gamma^+$ and $H \subseteq \mathrm{TP}_\omega \times \Gamma^\omega$.

By Lemma C.10, one can compute the sets of types TP_+ and TP_ω. From $t \in \mathrm{TP}_+$, one can compute an nfa $\mathcal{T}_1^t$ such that the closure of its languages equals the set of finite words of type t (Lemma C.9). Hence

$$R_1 = \bigcup_{t\in\mathrm{TP}_+} (\{t\} \times L^*(\mathcal{T}_1^t))$$

is rational such that the following hold:

- For any finite mono-alphabetic word $u \in \Gamma_k^+$, there exists $v \sim u$ with $(\mathrm{tp}_+(u), v) \in R_1$.
- For any $(t, u) \in R_1$, we have $t = \mathrm{tp}_+(u)$.

Now consider the rational relation R_1^ω and let $\alpha \in \mathrm{MultiInf}$ be arbitrary. Then we have the following.

- There is $\beta \sim \alpha$ with $(\mathrm{tpsqu}(\alpha), \beta) \in R_1^\omega$.
  To see this, consider the splitting of α into maximal mono-alphabetic factors, i.e., let $u_j \in \Gamma_{k_j}^+$ with $k_j \neq k_{j+1}$ for all $j \geq 1$ such that $\alpha = u_1 u_2 \cdots$. For $j \geq 1$, let $t_j = \mathrm{tp}_+(u_j)$. Then Lemma C.9 ensures the existence of words $v_j \sim u_j$ with $v_j \in L^*(\mathcal{T}_1^{t_j})$ and therefore $(t_j, v_j) \in R_1$ as well as $v_1 v_2 \cdots \sim \alpha$. Hence

$$
\begin{aligned}
(\mathrm{tpsqu}(\alpha), v_1 v_2 \cdots) &= \big(\mathrm{tp}_+(u_1), v_1\big)\,\big(\mathrm{tp}_+(u_2), v_2\big)\big(\mathrm{tp}_+(u_3), v_3\big) \cdots \\
&= \big(\mathrm{tp}_+(v_1), v_1\big)\,\big(\mathrm{tp}_+(v_2), v_2\big)\big(\mathrm{tp}_+(v_3), v_3\big) \cdots \qquad \text{since } u_j \sim v_j \\
&\in R_1^\omega\,.
\end{aligned}
$$

- The only word $W \in \mathrm{TPSQU}$ with $(W, \alpha) \in R_1^\omega$ is $W = \mathrm{tpsqu}(\alpha)$.
  To see this, suppose $(W, \alpha) \in R_1^\omega$ with $W \in \mathrm{TPSQU}$. Then $\alpha = u_1 u_2 \cdots$ and $W = t_1 t_2 \cdots$ with $(t_j, u_j) \in R_1$ for all $j \geq 1$. From the definition of R_1, we get $u_j \in L^*(\mathcal{T}_1^{t_j})$ and therefore $t_j = \mathrm{tp}_+(u_j)$ for all $j \geq 1$. In particular, the factors u_j satisfy $u_j \in \Gamma_{k_j}^+$ where the index $k_j \in [m]$ is determined by the type t_j. Since $W \in \mathrm{TPSQU}$, we obtain $k_j \neq k_{j+1}$ for all $j \geq 1$. Hence the type sequence of α is the sequence of types $\mathrm{tp}_+(u_j) = t_j$, i.e., equals W.

Having considered the relation R_1^ω, i.e., the first part of the relation R, we next define the relation H and study $R_1^*\,H$. From $t \in \mathrm{TP}_\omega$, one can compute a Büchi-automaton $\mathcal{T}_2^t$ such that the closure of its language equals the set of infinite words of type t (Lemma C.9). Hence

$$
H = \bigcup_{t \in \mathrm{TP}_\omega} \big(\{t\} \times L^\omega(\mathcal{T}_1^t)\big)
$$

is rational such that the following hold:

- For any infinite mono-alphabetic word $\alpha' \in \Gamma_k^\omega$, there exists $\beta' \sim \alpha'$ with $(\mathrm{tp}_\omega(\alpha'), \beta') \in H$.
- For any $(t, \alpha') \in H$, we have $t = \mathrm{tp}_\omega(\alpha')$.

Now consider the rational relation $R_1^* \cdot H$ and let $\alpha \in \mathrm{MonoInf}$ be arbitrary. Then we have the following.

- There is $\beta \sim \alpha$ with $(\mathrm{tpsqu}(\alpha), \beta) \in R_2$.
  To see this, consider the splitting of α into maximal mono-alphabetic factors, i.e., let $u_j \in \Gamma_{k_j}^+$ and $\alpha' \in \Gamma_{k_{\ell+1}}^\omega$ with $k_j \neq k_{j+1}$ for all $j \in [\ell]$ such that $\alpha = u_1 u_2 \cdots u_\ell\, \alpha'$. For $j \in [\ell]$, let $t_j = \mathrm{tp}_+(u_j)$ and set $t_{\ell+1} = \mathrm{tp}_\omega(\alpha')$. Then Lemma C.9 ensures the existence of words $v_j \sim u_j$ and $\beta' \sim \alpha'$ with $v_j \in L^*(\mathcal{T}_1^{t_j})$ and $\beta' \in L^*(\mathcal{T}_2^{t_{\ell+1}})$ (and therefore $(t_j, v_j) \in R_1$ and $(t_{\ell+1}, \beta') \in H$ as well as $v_1 v_2 \cdots v_\ell \beta' \sim \alpha$). Hence

$$
\begin{aligned}
(\mathrm{tpsqu}(\alpha), v_1 v_2 \cdots v_\ell \beta') &= \big(\mathrm{tp}_+(u_1), v_1\big) \cdots \big(\mathrm{tp}_+(u_\ell), v_\ell\big)\big(\mathrm{tp}_\omega(\alpha'), \beta'\big) \\
&= \big(\mathrm{tp}_+(v_1), v_1\big) \cdots \big(\mathrm{tp}_+(v_\ell), v_\ell\big)\big(\mathrm{tp}_\omega(\beta'), \beta'\big) \\
&\in R_1^*\, H = R_2\,.
\end{aligned}
$$

- The only word $W \in \mathrm{TPSQU}$ with $(W, \alpha) \in R_1^*\, H$ is $W = \mathrm{tpsqu}(\alpha)$.
  To see this, suppose $(W, \alpha) \in R_1^*\, H$ with $W \in \mathrm{TPSQU}$. Then $\alpha = u_1 u_2 \cdots u_\ell \alpha'$ and $W = t_1 t_2 \cdots t_\ell\, t_{\ell+1}$ with $(t_j, u_j) \in R_1$ for all $j \in [\ell]$ and $(t_{\ell+1}, \alpha') \in H$. From the

definitions of R_1 and H, we get $u_j \in L^*(\mathcal{T}_1^{t_j})$ and therefore $t_j = \mathrm{tp}_+(u_j)$ for all $j \in [\ell]$ as well as $\alpha' \in L^\omega(\mathcal{T}_2^{t_{\ell+1}})$. In particular, the factors u_j and α' satisfy $u_j \in \Gamma_{k_j}^+$ and $\alpha' \in \Gamma_{k_{\ell+1}}^\omega$ where the index $k_j \in [m]$ is determined by the type t_j and $t_{\ell+1}$. Since $W \in \mathrm{TPSQU}$, we obtain $k_j \neq k_{j+1}$ for all $j \in [\ell]$. Hence the type sequence of α is the sequence of types $\mathrm{tp}_+(u_j) = t_j$ followed by $\mathrm{tp}_\omega(\alpha') = t_{\ell+1}$, i.e., equals W.

To finish, let $R = R_1^\omega \cup R^* H$. This relation is effectively rational and has the desired properties. ◀

▶ **Theorem 4.1.** *From a transitive independence alphabet* (Γ, I)*, an* n*-ary Boolean function* φ*, and Büchi-automata* $\mathcal{A}_i$ *over* Γ *for* $i \in [n]$*, one can construct a Büchi-automaton* $\mathcal{B}$ *with*

$$[L^\omega(\mathcal{B})] = \left\langle [L^\omega(\mathcal{A}_1)], \dots, [L^\omega(\mathcal{A}_n)] \right\rangle^\varphi .$$

Hence, the problems $\mathrm{RatP}_\omega(\mathrm{TransA}, \mathrm{BF})$ *and* $\mathrm{EmptP}_\omega(\mathrm{TransA}, \mathrm{BF})$ *are decidable.*

Proof. Let, for $k \in [m]$, (Γ_k, I_k) be the connected components of (Γ, I). Since (Γ, I) is transitive, these independence alphabets are complete.

First, compute the sets of types TP_+ and TP_ω using Lemma C.10. Then, using Theorem C.8, construct the automata $\mathcal{A}^{\mathrm{multi}}$ and $\mathcal{A}^{\mathrm{mono}}$. From Lemma C.11, construct a transducer for the relation R. Finally, let L denote the set of words $\alpha \in \Gamma^\omega$ such that $(\mathrm{tpsqu}(\alpha), \alpha) \in R$ and $\mathrm{tpsqu}(\alpha) \in L^\omega(\mathcal{A}^{\mathrm{multi}}) \cup L^*(\mathcal{A}^{\mathrm{mono}})$. One can construct a Büchi-automaton $\mathcal{B}$ for this language L since R is rational.

It remains to be shown that the closure $[L]$ of L equals $\left\langle [L^\omega(\mathcal{A}_1)], \dots, [L^\omega(\mathcal{A}_n)] \right\rangle^\varphi$. So let $\alpha \in \Gamma^\omega$. Suppose $\alpha \in [L]$. Then there exists $\beta \sim \alpha$ with $\beta \in L$. From the definition of L, we get in particular $\mathrm{tpsqu}(\beta) \in L^\omega(\mathcal{A}^{\mathrm{multi}}) \cup L^*(\mathcal{A}^{\mathrm{mono}})$ and therefore, by Theorem C.8, $\beta \in \left\langle [L^\omega(\mathcal{A}_1)], \dots, [L^\omega(\mathcal{A}_n)] \right\rangle^\varphi$. Since $\alpha \sim \beta$ and since the language $\left\langle [L^\omega(\mathcal{A}_1)], \dots, [L^\omega(\mathcal{A}_n)] \right\rangle^\varphi$ is closed, we obtain $\alpha \in \left\langle [L^\omega(\mathcal{A}_1)], \dots, [L^\omega(\mathcal{A}_n)] \right\rangle^\varphi$.

Conversely, suppose $\alpha \in \left\langle [L^\omega(\mathcal{A}_1)], \dots, [L^\omega(\mathcal{A}_n)] \right\rangle^\varphi$. Then, by Theorem C.8, $\mathrm{tpsqu}(\alpha) \in L^\omega(\mathcal{A}^{\mathrm{multi}}) \cup L^*(\mathcal{A}^{\mathrm{mono}})$. Furthermore, by Lemma C.11, there exists $\beta \sim \alpha$ with $(\mathrm{tpsqu}(\alpha), \beta) \in R$. Note that the type sequences of α and β coincide since $\beta \sim \alpha$; hence $(\mathrm{tpsqu}(\beta), \beta) \in R$ and $\mathrm{tpsqu}(\beta) \in L^\omega(\mathcal{A}^{\mathrm{multi}}) \cup L^*(\mathcal{A}^{\mathrm{mono}})$. But this ensures $\beta \in L$ and therefore $\alpha \in [L]$.

It follows, as in the case of complete independence alphabets, that the rationality problem $\mathrm{RatP}_\omega(\mathrm{TransA}, \mathrm{BF})$ is trivially decidable since the answer is always "yes". ◀

C.4 The regularity problem

The multi-equivalence $\equiv$ induces the *block equivalence* $\equiv^{\mathrm{blk}}$ on Γ^ω as follows. The idea is that $\alpha \equiv^{\mathrm{blk}} \beta$ means that α can be transformed into β by replacing maximal mono-alphabetic factors by equivalent ones (wrt. $\equiv$). The exact definition is as follows. Let $\alpha, \beta \in \Gamma^\omega$. If $\alpha \in \mathrm{MultiInf}$ and $\beta \in \mathrm{MonoInf}$ (or vice versa), then $\alpha \not\equiv^{\mathrm{blk}} \beta$.

- Suppose $\alpha, \beta \in \mathrm{MultiInf}$. Then they factorize uniquely into $\alpha = u_1 u_2 \cdots$ and $\beta = v_1 v_2 \cdots$ with $u_j \in \Gamma_{k_j}^+$, $k_j \neq k_{j+1}$, $v_j \in \Gamma_{\ell_j}^+$ and $\ell_j \neq \ell_{j+1}$ for all $j \geq 1$.
  We set $\alpha \equiv^{\mathrm{blk}} \beta$ if $u_j \equiv v_j$ holds for all $j \geq 1$.
- Suppose $\alpha, \beta \in \mathrm{MonoInf}$. Then they factorize uniquely into $\alpha = u_1 u_2 \cdots u_\ell \alpha'$ and $\beta = v_1 v_2 \cdots v_m \beta'$ where the words u_j, α', v_k and β' for all $j \in [\ell]$ and $k \in [m]$ are mono-alphabetic and none of $u_j u_{j+1}$, $u_\ell \alpha'$, $v_k v_{k+1}$ and $v_m \beta'$ is mono-alphabetic for any $j \in [\ell - 1]$ and $k \in [m-1]$.
  We set $\alpha \equiv^{\mathrm{blk}} \beta$ if $\ell = m$, $u_j \equiv v_j$ for all $j \in [\ell]$, and $\alpha' \equiv \beta'$.

A language $L \subseteq \mathrm{MultiInf}$ is *saturated by* $\equiv$ if $\alpha \equiv^{\mathrm{blk}} \beta \in L$ implies $\alpha \in L$, i.e., if L is a union of equivalence classes of $\equiv^{\mathrm{blk}}$.

For two mono-alphabetic words x and y with $x \sim y$, we have $x \equiv_{\mathcal{A}} y$. Hence the multi-equivalence $\equiv_{\mathcal{A}}$ saturates every closed language, in particular the language $[L^\omega(\mathcal{A})]$.

Finally, the *block closure* of L wrt. $\equiv$ is the minimal language $\mathrm{blkcl}_{\equiv}(L)$ that is saturated by $\equiv$ and contains L. This block closure of L exists since the class of languages saturated by $\equiv$ is closed under arbitrary intersections. It is obtained from the words in L by replacing maximal mono-alphabetic factors by any equivalent ones.

The following example demonstrates that there is not necessarily a largest multi-equivalence that saturates a given language $L \subseteq \mathrm{MultiInf}$.

▶ **Example.** Let $\Gamma_1 = \{a\}$ and $\Gamma_2 = \{b\}$ (with $I_1 = I_2 = \emptyset$) and consider the language L of all words

$$\alpha = a^{m_1}\, b^{n_1}\, a^{m_2}\, b^{n_2} \cdots$$

with $m_i, n_i \geq 1$ for all $i \geq 1$ and $\{i \mid n_i = n \text{ or } m_i = n\}$ finite for all $n \geq 1$ (i.e., any "block size" appears only finitely often). Now let $\equiv_1$ and $\equiv_2$ be the multi-equivalences given by

- for any $c \in \{a, b\}$, $c^m \equiv_1 c^n$ iff $|m - n| \geq 1$ and $\min(m, n)$ is even and
- for any $c \in \{a, b\}$, $c^m \equiv_2 c^n$ iff $|m - n| \geq 1$ and $\min(m, n)$ is odd.

Suppose $\alpha \equiv_1^{\mathrm{blk}} \beta \in L$. Then the number of blocks of size m in α is at most the number of blocks of size $m-1$, m, or $m+1$ in β. From $\beta \in L$, we obtain that this number is finite and therefore $\alpha \in L$. Consequently, $\equiv_1$ and similarly $\equiv_2$ both saturate L. Note that the only multi-equivalence containing them both is the total multi-equivalence $\equiv$ with equivalence classes a^+, b^+, $\{a^\omega\}$, and $\{b^\omega\}$. But this multi-equivalence does not saturate L since $L \not\ni (ab)^\omega \equiv^{\mathrm{blk}} a^1 b^2 a^3 b^4 a^5 b^6 \cdots \in L$.

This cannot happen if L is saturated by some multi-equivalence of finite index.

▶ **Lemma C.12.** *Let $L \subseteq \Gamma^\omega$ be saturated by the multi-equivalences $\equiv_1$ and $\equiv_2$ and suppose there are only finitely many equivalence classes wrt. $\equiv_1$. Then L is saturated by some multi-equivalence $\equiv\, \supseteq\, \equiv_1 \cup \equiv_2$.*

Proof. Let $\equiv$ be the least equivalence relation with $\equiv_1 \cup \equiv_2 \subseteq \equiv$; then $\equiv$ is a multi-equivalence.

Consider two mono-alphabetic words x and z with $x \equiv z$. Since the relations $\equiv_1$ and $\equiv_2$ are reflexive and transitive, there are $n \geq 0$ and mono-alphabetic words $y_0, y_1, \ldots, y_{2n}$ such that

- $x = y_0$
- for all $j \in [n]$, we have $y_{2j-2} \equiv_1 y_{2j-1} \equiv_2 y_{2j}$, and
- $y_{2n} = z$.

Let n be minimal with these properties. Then there cannot exist $0 \leq i < j \leq n$ with $y_{2i-2} \equiv_1 y_{2j-2}$ for otherwise

$$x = y_0, y_1, \ldots, y_{2i-2}, y_{2j-1}, y_{2j}, \ldots, y_{2n} = z$$

would be a shorter sequence witnessing $x \equiv z$. Hence n is at most twice the number N of equivalence classes of $\equiv_1$.

Now let $\alpha \equiv^{\mathrm{blk}} \beta \in L$. By the above arguments, there are words $\alpha_0, \alpha_1, \ldots, \alpha_{2n}$ with $0 \leq n \leq N$ such that

- $\alpha = \alpha_0$,
- for all $j \in [n]$, we have $\alpha_{2j-2} \equiv_1^{\mathrm{blk}} \alpha_{2j-1} \equiv_2^{\mathrm{blk}} \alpha_{2j}$ and
- $\alpha_{2n} = \beta$.

Since L is saturated by both multi-equivalences $\equiv_1$ and $\equiv_2$, we obtain by induction $\alpha_i \in L$ for all $i \in [2n]$ and therefore in particular $\alpha \in L$. Hence, indeed, L is saturated by $\equiv$. ◀

The following lemma shows that the block closure wrt. $\equiv$ of a regular language is regular. The condition for this is that all equivalence classes of $\equiv$ are regular. If these equivalence classes are giben by finite automata, then a Büchi-automaton for the block closure is computable – this is the actual content of the following lemma.

▶ **Lemma C.13.** *From a Büchi-automaton $\mathcal{A}$, a tuple of nfas $(\mathcal{B}_i)_{i\in[M]}$, and a tuple of Büchi-automata $(\mathcal{C}_i)_{i\in[N]}$, one can construct a Büchi-automaton $\mathcal{B}$ with the following property. If there is a multi-equivalence $\equiv$ with set of equivalence classes $\{L^*(\mathcal{B}_i) \mid i \in [M]\} \cup \{L^\omega(\mathcal{C}_i \mid i \in [N]\}$, then $L^\omega(\mathcal{B}) = \mathrm{blkcl}_\equiv(L^\omega(\mathcal{A}))$.*

Proof. Since the equivalence classes of $\equiv$ are regular, the relations $R_1 = \{(u, v) \mid u, v \in \Gamma^+, u \equiv v\} = \bigcup_{i\in[M]} L^*(\mathcal{B}_i) \times L^*(\mathcal{B}_i)$ and $R_2 = \{(\alpha, \beta) \mid \alpha, \beta \in \Gamma^\omega, \alpha \equiv \beta\} = \bigcup_{i\in[N]} L^\omega(\mathcal{C}_i) \times L^\omega(\mathcal{C}_i)$ are rational (more precisely: transducers for these relations can be computed). It follows that also a transducer for the block equivalence $\equiv^{\mathrm{blk}}$ can be constructed. Hence one can compute a Büchi-automaton accepting the block closure $\mathrm{blkcl}_\equiv(L^\omega(\mathcal{A}))$ of $L^\omega(\mathcal{A})$. ◀

▶ **Proposition 4.2.** *Let $\mathcal{A}$ be a Büchi-automaton. Then $[L^\omega(\mathcal{A})]$ is regular if, and only if, $[L^\omega(\mathcal{A})]$ is saturated by some multi-equivalence $\equiv \supseteq \equiv_\mathcal{A}$ such that all equivalence classes of $\equiv$ are regular.*

Proof. First, suppose $[L^\omega(\mathcal{A})]$ is regular. By [13, Thm. 4.6] and the proof of [13, Prop. 3.4], there exists a Büchi-automaton $\mathcal{B}$ such that $L^\omega(\mathcal{B}) = [L^\omega(\mathcal{A})]$ and all equivalence classes of $\equiv_\mathcal{B}$ are regular. By Lemma C.12, there exists a multi-equivalence $\equiv \supseteq \equiv_\mathcal{A} \cup \equiv_\mathcal{B}$ that saturates $[L^\omega(\mathcal{A})]$. Since the equivalence classes of $\equiv$ are (finite) unions of equivalence classes of $\equiv_\mathcal{B}$, they are regular.

Conversely, suppose $\equiv \supseteq \equiv_\mathcal{A}$ saturates $[L^\omega(\mathcal{A})]$ and all equivalence classes of $\equiv$ are regular. Then, by Lemma C.13, the block closure $\mathrm{blkcl}_\equiv(L^\omega(\mathcal{A}))$ of $L^\omega(\mathcal{A})$ is regular. Since $\equiv$ saturates $[L^\omega(\mathcal{A})]$, we get $[L^\omega(\mathcal{A})] \subseteq \mathrm{blkcl}_\equiv([L^\omega(\mathcal{A})]) \subseteq [L^\omega(\mathcal{A})]$. Furthermore, $\sim \subseteq \equiv^{\mathrm{blk}}$ implies $\mathrm{blkcl}_\equiv([L^\omega(\mathcal{A})]) = \mathrm{blkcl}_\equiv(L^\omega(\mathcal{A}))$. Hence $[L^\omega(\mathcal{A})]$ equals the regular language $\mathrm{blkcl}_\equiv(L^\omega(\mathcal{A}))$. ◀

▶ **Theorem 4.3.** *The regularity problem $\mathrm{RegP}_\omega(\mathrm{TransA}, \mathrm{BF})$ is decidable.*

Proof. Let $(\Gamma, I) \in \mathrm{TransA}$, $\varphi \in \mathrm{BF}$ n-ary, and $\mathcal{A}_1, \ldots, \mathcal{A}_n$ Büchi-automata over Γ. By Theorem 4.1, one can construct a Büchi-automaton $\mathcal{A}$ with $[L^\omega(\mathcal{A})] = \big\langle [L^\omega(\mathcal{A}_1)], \ldots, [L^\omega(\mathcal{A}_n)] \big\rangle^\varphi$. To prove the theorem, it remains to decide whether $[L^\omega(\mathcal{A})]$ is regular.

Let (Γ, I) be the disjoint union of the complete independence alphabets (Γ_k, I_k) for $k \in [m]$. By Lemma C.10, one can compute the sets of types TP_+ and TP_ω. By Lemma C.9, one can then compute nfas $\mathcal{B}_t$ for $t \in \mathrm{TP}_+$ and Büchi-automata $\mathcal{C}_t$ for $t \in \mathrm{TP}_\omega$ such that the languages $[L^*(\mathcal{B}_t)]$ and $[L^\omega(\mathcal{C}_t)]$ are the equivalence classes of the multi-equivalence $\equiv_\mathcal{A}$.

For an equivalence relation $\approx$ on the set TP of all types, define the equivalence $\equiv$ on the set of mono-alphabetic words by $x \equiv y$ iff the types of x and y are equivalent wrt. $\approx$. Note that then $\equiv \supseteq \equiv_\mathcal{A}$ and that any equivalence relation extending $\equiv_\mathcal{A}$ arises this way. Thus, the language $[L^\omega(\mathcal{A})]$ is regular if, and only if, there is an equivalence relation $\approx$ on the set of all types such that the induced equivalence relation $\equiv$ is a multi-equivalence, saturates $[L^\omega(\mathcal{A})]$, and has only regular equivalence classes (cf. Prop. 4.2). Since there are only finitely many types, there are only finitely many equivalence relations $\approx$ that we will test one after the other.

So let $\approx$ be an equivalence relation on the set of all types and let $\equiv$ denote the induced equivalence relation on the set of mono-alphabetic words. Then the equivalence classes of $\approx$ are unions of languages of the form $[L^*(\mathcal{B}_t)]$ and $[L^\omega(\mathcal{C}_t)]$. Note that $\equiv$ is a multi-equivalence iff, for each equivalence class C of $\equiv$, there exists $k \in [m]$ with $C \subseteq \Gamma_k^+$ or $C \subseteq \Gamma_k^\omega$. Since C is a union of languages $[L^*(\mathcal{B}_t)]$ and $[L^\omega(\mathcal{C}_t)]$, this is decidable. Assuming this, one has to decide whether the equivalence classes are regular. Since the independence alphabets (Γ_k, I_k) all are complete, this is decidable by Corollary C.3. If, indeed, all equivalence classes are regular, Lemmas C.1 and C.2 allow to compute automata accepting these equivalence classes. Finally, Lemma C.13 allows to compute a Büchi-automaton $\mathcal{B}$ accepting the language $\mathrm{blkcl}_\equiv(L^\omega(\mathcal{A}))$. It remains to decide whether $\equiv$ saturates $[L^\omega(\mathcal{A})]$ which is the case iff $\mathrm{blkcl}_\equiv([L^\omega(\mathcal{A})]) \subseteq [L^\omega(\mathcal{A})]$. Since $x \sim y$ for mono-alphabetic words x and y implies $x \equiv_\mathcal{A} y$ and therefore $x \equiv y$, we get $\sim \subseteq \equiv^{\mathrm{blk}}$. Hence

$$\mathrm{blkcl}_\equiv([L^\omega(\mathcal{A})]) = \mathrm{blkcl}_\equiv(L^\omega(\mathcal{A})) = [\mathrm{blkcl}_\equiv(L^\omega(\mathcal{A}))] = [L^\omega(\mathcal{B})]\,,$$

i.e., $\equiv$ saturates $[L^\omega(\mathcal{A})]$ iff $[L^\omega(\mathcal{B})] \subseteq [L^\omega(\mathcal{A})]$, which is equivalent to the emptiness of $[L^\omega(\mathcal{A})] \setminus [L^\omega(\mathcal{B})]$. But this is decidable by Theorem 4.1. ◀

D Proofs from Sect. 5

▶ **Lemma D.1.** *Let $\Phi \subseteq \mathrm{monBF}$ be a set of monotone Boolean functions. The emptiness problem $\mathrm{EmptP}_\omega(\mathrm{DiagA}, \Phi)$ for arbitrary diagonal independence alphabets can be reduced to $\mathrm{EmptP}_\omega(\mathrm{ConDiagA}, \Phi)$ for connected diagonal independence alphabets.*

Proof. Let $(\Gamma, I) \in \mathrm{DiagA}$, $\varphi \in \Phi$ n-ary, and $\mathcal{A}_1, \dots, \mathcal{A}_n$ Büchi-automata over Γ.

To (Γ, I), add a letter a that is independent from all letters from Γ, i.e., set $\Sigma = \Gamma \cup \{a\}$ and $I_\Sigma = I \cup (\{a\} \times \Gamma) \cup (\Gamma \times \{a\})$. The independence alphabet (Σ, I_Σ) is connected and diagonal.

Then, for any language $L \subseteq \Gamma^\omega$, the closure $[L]$ remains the same no matter whether we construct it wrt. (Γ, I) or (Σ, I_Σ).

Since $\varphi \in \Phi \subseteq \mathrm{monBF}$ is a monotone Boolean function, the language $\langle [L^\omega(\mathcal{A}_1)], \dots, [L^\omega(\mathcal{A}_n)] \rangle^\varphi$ can be constructed from the languages $[L^\omega(\mathcal{A}_i)]$ by union and intersection. Hence also the language $\langle [L^\omega(\mathcal{A}_1)], \dots, [L^\omega(\mathcal{A}_n)] \rangle^\varphi$ remains the same no matter whether we construct it wrt. (Γ, I) or (Σ, I_Σ). Consequently, $\big((\Gamma, I), \varphi, \mathcal{A}_1, \dots, \mathcal{A}_n\big) \in \mathrm{EmptP}_\omega(\mathrm{DiagA}, \Phi)$ if, and only if, $\big((\Sigma, I_\Sigma), \varphi, \mathcal{A}_1, \dots, \mathcal{A}_n\big) \in \mathrm{EmptP}_\omega(\mathrm{ConDiagA}, \Phi)$. ◀

▶ **Lemma D.2.** *From a connected diagonal independence alphabet (Γ, I), a letter $a \in \Gamma$, nfas $\mathcal{A}_i = (Q_i, \Gamma, I_i, T_i, F_i)$ for $i \in [n]$, and tuples of states $\overline{p} = (p_i)_{i\in[n]}$ and $\overline{q} = (q_i)_{i\in[n]}$ from $\prod_{i\in[n]} Q_i$, one can compute an nfa over $\Gamma \times [n]$ that accepts a language version of the relation $R_\Downarrow(\overline{p}, \overline{q}, a)$.*

Proof. We first show that the set $\Psi_n\big(R_\Downarrow(\overline{p}, \overline{q}, a)\big) \subseteq \mathbb{N}^{\Gamma\times[n]}$ is semi-linear. To this aim, we define auxiliary nfas $\mathcal{B}_i = (Q_i \times [2], \Gamma, \emptyset, T_i', \emptyset)$ setting

$$\big((p, b), a, (q, c)\big) \in T_i' \iff \Big((p, a, q) \in T_i \text{ and } (c = 2 \iff b = 2 \lor p \in F_i \lor q \in F_i)\Big)$$

What is important here is that, for any word $u_i \in \Gamma^+$, we have

$$p_i \xRightarrow{u_i}_{\mathcal{A}_i} q_i \text{ iff } (p_i,1) \xrightarrow{u_i}_{\mathcal{B}_i} (q_i,2)\,.$$

Hence, for any tuple $\overline{u} = (u_i)_{i\in[n]}$ of non-empty words over Γ, we have $\overline{u} \in R_{\Downarrow}(\overline{p},\overline{q},a)$ if, and only if,

- $(p_i,1) \xrightarrow{u_i}_{\mathcal{B}_i} (q_i,2)$,
- $\mathrm{proj}_{\Downarrow a}(u_1) \sim \mathrm{proj}_{\Downarrow a}(u_i)$, and
- $u_i \in \updownarrow a^*$

for all $i \in [n]$. Now [1, Claim 2.7] implies that, indeed, $\Psi_n(R_{\Downarrow}(\overline{p},\overline{q},a))$ is effectively semilinear. In other words, one can compute finitely many tuples $(x_0,x_1,\dots,x_t)$ with $x_s \in \mathbb{N}^{\Gamma\times[n]}$ for all $s \in \{0,\dots,t\}$ such that $\Psi_n(R_{\Downarrow}(\overline{p},\overline{q},a))$ is the union of the sets $x_0 + \sum_{s\in[t]} \mathbb{N}\cdot x_t$. For any of the tuples $(x_0,\dots,x_t)$, let $w_0,w_1,\dots,w_t \in (\Gamma\times[n])^*$ be words with Parikh image $x_0,x_1,\dots,x_t$, respectively. Then

$$L = w_0\, w_1^*\, w_2^* \cdots w_t^*$$

is regular with Parikh image $x_0 + \sum_{s\in[t]} \mathbb{N}\cdot x_t$. Hence a finite union of languages of the form L form a language version of the relation $R_{\Downarrow}(\overline{p},\overline{q},a)$. Note that the finite set of the tuples $(x_0,\dots,x_t)$ can be computed from the automata $\mathcal{B}_i$, the tuple of states $\overline{p}$ and $\overline{q}$, and the letter a. Hence one can compute an nfa for this regular language version of $R_{\Downarrow}(\overline{p},\overline{q},a)$. ◀

▶ **Lemma D.3.** *Let $(\Gamma,\preceq)$ be an order tree and $I = \succ \cup \prec$. Let furthermore $\overline{\mathcal{A}} = (\mathcal{A}_i)_{i\in[n]}$ be a tuple of Büchi-automata $\mathcal{A}_i = (Q_i,\Gamma,I_i,T_i,F_i)$.*

For all $\overline{p} \in \prod_{i\in[n]} Q_i$ and $a \in \Gamma$, the following hold:

$$R^\omega_{\downarrow}(\overline{p},a) = R^\omega_{\Downarrow}(\overline{p},a) \cap \mathrm{Eq}^\omega_a$$

$$R^\omega_{\Downarrow}(\overline{p},a) = \bigcup_{\overline{q}\in \prod\limits_{i\in[n]} Q_i} R_{\Downarrow}(\overline{p},\overline{q},a)\cdot\left(R_{\Downarrow}(\overline{q},\overline{q},a)^\omega \cup \bigcup_{b\in\max(\Downarrow a)} R^\omega_{\downarrow}(\overline{q},b)\right)$$

Proof. Let $\overline{\alpha} = (\alpha_i)_{i\in[n]}$ be a tuple of ω-words $\alpha_i \in \Gamma^\omega$ for all $i \in [n]$.

Since the letter a is independent from all letters in $\Downarrow a$, we get $\mathrm{proj}_{\downarrow a}(\alpha_1) \sim \mathrm{proj}_{\downarrow a}(\alpha_i)$ if, and only if, $\mathrm{proj}_{\Downarrow a}(\alpha_1) \sim \mathrm{proj}_{\Downarrow a}(\alpha_i)$ and $\mathrm{proj}_a(\alpha_1) \sim \mathrm{proj}_a(\alpha_i)$ (which is equivalent to $|\alpha_1|_a = |\alpha_i|_a$ and therefore to $\overline{\alpha} \in \mathrm{Eq}^\omega_a$). This proves the first equation.

We now turn to the second equation. Note that the independence alphabet $(\Downarrow a, I)$ is the disjoint union of the independence alphabets $(\downarrow b, I)$ for $b \in \max(\Downarrow a,\preceq)$. Hence the projection of α_1 to $\Downarrow a$ is a word over the disjoint union of these independence alphabets. As in Section C.2, we will distinguish the cases of this projection to be multi-alphabetic at infinity or mono-alphabetic at infinity.

multi-alphabetic projection. So, first, suppose $\mathrm{proj}_{\Downarrow a}(\alpha_1)$ is multi-alphabetic at infinity, i.e., contains infinitely many occurrences of letters from $\downarrow b$ and $\downarrow c$ for two distinct letters $b,c \in \max(\Downarrow a)$. Note the following.

- Since $\mathrm{proj}_{\Downarrow a}(\alpha_1)$ is multi-alphabetic at infinity, we have that $\mathrm{proj}_{\Downarrow a}(\alpha_1) \sim \mathrm{proj}_{\Downarrow a}(\alpha_i)$ for all $i \in [n]$ is equivalent to the existence of words $v_i^j \in \downarrow b_j^+$ with $b_j \in \max(\Downarrow a)$, $\mathrm{proj}_{\Downarrow a}(\alpha_i) = v_i^1 v_i^2 \cdots$, $v_1^j \sim v_i^j$, and $b_j \neq b_{j+1}$ for all $i \in [n]$ and $j \geq 1$. Since the factorisation of the projection $\mathrm{proj}_{\Downarrow a}(\alpha_i)$ corresponds to some factorisation of α_i, the above is equivalent to the existence of words $u_i^j \in \updownarrow b_j^+$ with $b_j \in \max(\Downarrow a)$, $\alpha_i = u_i^1 u_i^2 \cdots$, $\mathrm{proj}_{\Downarrow a}(u_1^j) \sim \mathrm{proj}_{\Downarrow a}(u_i^j)$, and $b_j \neq b_{j+1}$ for all $i \in [n]$ and $j \geq 1$.

- Let $u_i^j \in \Gamma^+$ with $\alpha_i = u_i^1 u_i^2 \cdots$ for all $i \in [n]$ and $j \geq 1$. Then, for all $i \in [n]$, $\alpha_i \in \updownarrow a^\omega$ if, and only if, $u_i^j \in \updownarrow a^+$ for all $j \geq 1$.
- Let, again, $u_i^j \in \Gamma^+$ with $\alpha_i = u_i^1 u_i^2 \cdots$ for all $i \in [n]$ and $j \geq 1$. Then $\alpha_i \in L^\omega(\mathcal{A}_i^{p_i \to})$ for all $i \in [n]$ if, and only if, there are $\overline{q} = (q_i)_{i\in[n]} \in \prod_{i\in[n]} Q_i$ and natural numbers $1 \leq n_1 < n_2 < n_3 \cdots$ such that, for all $i \in [n]$ and $\ell \geq 1$,

$$p_i \xrightarrow{u_i^1\, u_i^2 \cdots u_i^{n_1}}_{\mathcal{A}_i} q_i \text{ and } q_i \xrightarrow{u_i^{n_\ell+1}\, u_i^{n_\ell+2} \cdots u_i^{n_{\ell+1}}}_{\mathcal{A}_i} q_i\,.$$

In summary, we have (in case $\mathrm{proj}_{\Downarrow a}(\alpha_1)$ is multi-alphabetic at infinity)

$$\overline{\alpha} \in R^\omega_{\downarrow}(\overline{p}, a) \iff \overline{\alpha} \in \bigcup_{\overline{q}\in\prod_{i\in[n]} Q_i} R_{\Downarrow}(\overline{p}, \overline{q}, a) \cdot R_{\Downarrow}(\overline{q}, \overline{q}, a)^\omega\,.$$

mono-alphabetic projection. It remains to consider the case that $\mathrm{proj}_{\Downarrow a}(\alpha_1)$ is mono-alphabetic at infinity, i.e., belongs to $(\Downarrow a)^*(\downarrow b)^\omega$ for some $b \in \max(\Downarrow a)$. Note the following.

- Since $\mathrm{proj}_{\Downarrow a}(\alpha_1)$ is mono-alphabetic at infinity, we have that $\mathrm{proj}_{\Downarrow a}(\alpha_1) \sim \mathrm{proj}_{\Downarrow a}(\alpha_i)$ for all $i \in [n]$ is equivalent to the existence of words $v_i^j \in \downarrow b_j^+$ and $\beta_i \in \downarrow b_{\ell+1}^\omega$ with $b_j, b_{\ell+1} \in \max(\Downarrow a)$, $\mathrm{proj}_{\Downarrow a}(\alpha_i) = v_i^1 v_i^2 \cdots v_i^\ell \beta_i$, $v_1^j \sim v_i^j$, $\beta_1 \sim \beta_i$, $b_j \neq b_{j+1}$ for all $i \in [n]$ and $j \in [n]$. Since the factorisation of the projection $\mathrm{proj}_{\Downarrow a}(\alpha_i)$ corresponds to some factorisation of α_i, the above is equivalent to the existence of words $u_i^j \in \updownarrow b_j^+$ and $\gamma_i \in \updownarrow b_{\ell+1}^\omega$ with $b_j, b_{\ell+1} \in \max(\Downarrow a)$, $\alpha_i = u_i^1 u_i^2 \cdots u_i^\ell \gamma_i$, $\mathrm{proj}_{\Downarrow a}(u_1^j) \sim \mathrm{proj}_{\Downarrow a}(u_i^j)$, $\mathrm{proj}_{\Downarrow a}(\gamma_1) \sim \mathrm{proj}_{\Downarrow a}(\gamma_i)$, and $k_j \neq k_{j+1}$ for all $i \in [n]$ and $j \in [\ell]$.
- Let $u_i^j \in \Gamma^+$ and $\gamma_i \in \Gamma^\omega$ with $\alpha_i = u_i^1 u_i^2 \cdots u_i^\ell \gamma_i$ for all $i \in [n]$ and $j \in [\ell]$. Then, for all $i \in [n]$, $\alpha_i \in \updownarrow a^\omega$ if, and only if, $u_i^j \in \updownarrow a^+$ for all $j \in [\ell]$ and $\gamma_i \in \updownarrow a^\omega$.
- Let, again, $u_i^j \in \Gamma^+$ and $\gamma_i \in \Gamma^\omega$ with $\alpha_i = u_i^1 u_i^2 \cdots u_i^\ell \gamma_i$ for all $i \in [n]$ and $j \geq\in [\ell]$. Then $\alpha_i \in L^\omega(\mathcal{A}_i^{p_i\to})$ for all $i \in [n]$ if, and only if, there is $\overline{q} = (q_i)_{i\in[n]} \in \prod_{i\in[n]} Q_i$ such that, for all $i \in [n]$,

$$p_i \xrightarrow{u_i^1\, u_i^2 \cdots u_i^\ell}_{\mathcal{A}_i} q_i \text{ and } \gamma_i \in L^\omega(\mathcal{A}_i^{q_i\to})\,.$$

In summary, we have (in case $\mathrm{proj}_{\Downarrow a}(\alpha_1)$ is mono-alphabetic at infinity)

$$\overline{\alpha} \in R^\omega_{\downarrow}(\overline{p}, a) \iff \overline{\alpha} \in \bigcup_{\substack{\overline{q}\in\prod_{i\in[n]} Q_i \\ b\in\max(\Downarrow a)}} R_{\Downarrow}(\overline{p}, \overline{q}, a) \cdot R^\omega_{\downarrow}(\overline{q}, b)\,.$$

Thus, we proved that the second equation holds. ◀

▶ **Lemma 5.1.** *From a connected diagonal independence alphabet (Γ, I) and Büchi-automata $\mathcal{A}_i$ for all $i \in [n]$, one can construct a Büchi-automaton $\mathcal{B}$ that accepts a regular language version of the relation*

$$R = \left\{(\alpha_i)_{i\in[n]} \mid \alpha_i \in L^\omega(\mathcal{A}_i) \text{ and } \alpha_1 \sim \alpha_i \text{ for all } i \in [n]\right\}.$$

Proof. Let $(\Gamma, \preceq)$ be some order tree with $I = \prec \cup \succ$ and let r be the root of this tree. Then R is the union of the relations $R^\omega_{\downarrow}((\iota_i)_{i\in[n]}, r)$ where ι_i is initial in $\mathcal{A}_i$ (for all $i \in [n]$). Hence, by Lemma D.3, the relation R is constructed from the relations $R_{\Downarrow}(\overline{p}, \overline{q}, a, [n])$ with $\overline{p}, \overline{q} \in \prod_{i\in[n]} Q_i$, $a \in \Gamma$ and $Y \subseteq [n]$ using union $\cup$ of relations on Γ^ω, concatenation $\cdot$ of relations on Γ^* and on Γ^ω, ω-iteration ${}^\omega$ of relations on Γ^*, and intersection with Eq_a^ω for $a \in \Gamma$.

We construct the language version of R along this construction building on Lemma D.2 that allows to construct regular language versions of the relations $R_\Downarrow(\overline{p}, \overline{q}, a)$.

To handle the operations $\cup$, $\cdot$, and ${}^\omega$, note the following. Let L_j be a language version of R_j (for $j \in [2]$). Then $L_1 \cup L_2$ is a language version of $R_1 \cup R_2$, $L_1 \cdot L_2$ is a language version of $R_1 \cdot R_2$, and L_1^ω is a language version of R_1^ω. Hence, provided R_1 and R_2 have regular language versions, so do $R_1 \cup R_2$, $R_1 \cdot R_2$, and R_1^*.

Finally, we have to handle intersection with Eq_a^ω for $a \in \Gamma$. So suppose $K \subseteq \big(\Gamma \times [n]\big)^\omega$ is a regular language version of $R \subseteq (\Gamma^\omega)^n$ and $a \in \Gamma$. On the alphabet $\Gamma \times [n]$, consider the complete independence relation J, i.e., $\big((b,i),(c,j)\big) \in J \iff (b,i) \neq (c,j)$.

Now consider the regular language

$$\begin{aligned} L \quad = \quad & \big((a,1)(a,2)\dots(a,n)\big)^* \cdot \big(\Gamma \setminus \{a\} \times [n]\big)^\omega \\ & \cup \big((a,1)(a,2)\dots(a,n)\big)^\omega \sqcup\!\sqcup \big(\Gamma \setminus \{a\} \times [n]\big)^\infty . \end{aligned}$$

It is a language version of Eq_a^ω. Its closure $[L]_J$ wrt. the independence relation J is the set of words $V \in \big(\Gamma \times [n]\big)^\omega$ with $|V|_{(a,1)} = |V|_{(a,i)}$ for all $i \in [n]$.

Since J is the complete independence relation on $\Gamma \times [n]$, Lemma C.2 allows to construct a regular language $H \subseteq \big(\Gamma \times [n]\big)^\omega$ with $[H] = [K] \cap [L]$. Now let $V \in \big(\Gamma \times [n]\big)^\omega$ be any word. Then $V \in [H]$ iff $V \in [K]$ and $V \in [L]$. But this is the case if, and only if, the following hold:

(i) there is some $W \in K$ such that V and W have the same Parikh image (since J is complete)

(ii) $|V|_{(a,1)} = |V|_{(a,i)}$ for all $i \in [n]$

Since K is a language version of R, (i) holds iff there is some tuple $\overline{u} \in R$ with $|V|_{(b,i)} = |u_i|_b$ for all $(b,i) \in \Gamma \times [n]$. But this is equivalent to saying that the Parikh image of the word V belongs to $\Psi_n(R)$. Thus, we showed $V \in [H]$ iff the Parikh image of V belongs to $\Psi_n(R)$ and (ii) holds. Hence, $[H]$ is a language version of $R \cap \mathrm{Eq}_a$. Since H and $[H]$ have the same Parikh image, we showed that H is a regular language version of $R \cap \mathrm{Eq}_a$. ◀

▶ **Theorem 5.2.** *The emptiness problem* $\mathrm{EmptP}_\omega(\mathrm{DiagA}, \mathrm{monBF})$ *is decidable.*

Proof. First, let (Γ, I) be a connected diagonal independence alphabet, φ a monotone Boolean function, and $\mathcal{B}_1, \dots, \mathcal{B}_m$ Büchi-automata over Γ. Since φ is monotone, there is a set $\mathcal{Y} \subseteq \mathcal{P}([m])$ such that

$$\big\langle [L^\omega(\mathcal{B}_1)], \dots, [L^\omega(\mathcal{B}_m)] \big\rangle^\varphi = \bigcup_{A \in \mathcal{Y}} \bigcap_{i \in A} [L^\omega(\mathcal{B}_i)] \,.$$

Even more, the set $\mathcal{Y}$ can be computed from the function φ.

Now let $A \in \mathcal{Y}$ and suppose $\overline{\mathcal{A}} = (\mathcal{A}_1, \dots, \mathcal{A}_n)$ is the tuple of automata $(\mathcal{B}_i)_{i \in A}$. By Lemma 5.1, one can construct a regular language version L of the relation $R = \{(\alpha_i)_{i \in [n]} \mid \alpha_i \in L^\omega(\mathcal{A}_i)$ and $\alpha_1 \sim \alpha_i$ for all $i \in [n]\}$. Then $\bigcap_{i \in A}[L^\omega(\mathcal{B}_i)] = \bigcap_{i \in [n]}[L^\omega(\mathcal{A}_i)]$ is empty iff L is empty; and the emptiness of L is decidable.

Verifying all sets $A \in \mathcal{Y}$ in this way, we can decide whether $((\Gamma, I), \varphi, \mathcal{B}_1, \dots, \mathcal{B}_m)$ belongs to $\mathrm{EmptP}_\omega(\mathrm{ConDiagA}, \mathrm{monBF})$, i.e., this problem for connected diagonal independence alphabets is decidable. Hence, by Lemma D.1, also the more general problem $\mathrm{EmptP}_\omega(\mathrm{DiagA}, \mathrm{monBF})$ for arbitrary diagonal independence alphabets is decidable. ◀

▶ **Example D.4.** Let $\Gamma = \{a, b, c, d\}$ and I the symmetric closure of $\{a, b\} \times \{c, d\}$. Then (Γ, I) is not diagonal since it contains C_4. Consider the following regular languages:

$$L_1 = \big(a(bc)^*d\big)^\omega \text{ and } L_2 = a\big(a(bc)^*bd\big)^\omega$$

Then an ω-word α belongs to the intersection $[L_1] \cap [L_2]$ iff

- $\mathrm{proj}_{\{a,b\}}(\alpha) = ab^{m_0}ab^{m_1}\cdots$ and $\mathrm{proj}_{\{c,d\}}(\alpha) = c^{m_0}dc^{m_1}d\cdots$ for some $m_i \geq 1$ and
- $\mathrm{proj}_{\{c,d\}}(\alpha) = c^{n_0}dc^{n_1}d\cdots$ and $\mathrm{proj}_{\{a,b\}}(\alpha) = aab^{n_0+1}ab^{n_1+1}\cdots$ for some $n_i \geq 0$

But this holds iff $m_0 = 0$, $m_i = n_i$, and $m_{i+1} = n_i + 1$ for all $i \geq 0$. This is equivalent to saying $m_i = n_i = i$. Hence none of the words from $[L_1] \cap [L_2]$ is ultimately periodic.

Since, as explained above, we are interested in the ultimately periodic words in $[L]$, we start with the following simple decidability result that holds for all independence alphabets.

▶ **Observation D.5.** *Given an independence alphabet (Γ, I), a Büchi-automaton $\mathcal{A}$ over Γ, and two non-empty words $u, v \in \Gamma^+$, it is decidable whether $uv^\omega \in [L^\omega(\mathcal{A})]$. Furthermore, if this is the case, then there are nonempty words $x, y \in \Gamma^+$ with $uv^\omega \sim xy^\omega \in L^\omega(\mathcal{A})$.*

Proof. We have $uv^\omega \in [L^\omega(\mathcal{A})]$ if, and only if, $[uv^\omega] \cap L^\omega(\mathcal{A}) \neq \emptyset$.

We show that $[uv^\omega] \subseteq \Gamma^\omega$ is effectively regular. An ω-word $\alpha \in \Gamma^\omega$ belongs to $[uv^\omega]$ if, and only if, $\alpha \sim uv^\omega$. But this is the case iff $\mathrm{proj}_{\{a,b\}}(\alpha) = \mathrm{proj}_{\{a,b\}}(u)\,\mathrm{proj}_{\{a,b\}}(v)^\omega$ holds for all letters $a, b \in \Gamma$ with $(a, b) \notin I$. Hence $[uv^\omega]$ is the finite intersection of all languages

$$\big(\mathrm{proj}_{\{a,b\}}(u)\,\mathrm{proj}_{\{a,b\}}(v)^\omega \sqcup (\Gamma \setminus \{a, b\})^\infty\big) \cap \Gamma^\omega$$

for $(a, b) \notin I$. Thus, indeed, $[uv^\omega]$ is effectively regular.

Hence the non-emptiness of the intersection $[uv^\omega] \cap L^\omega(\mathcal{A})$ of the two regular ω-languages $[uv^\omega]$ and $L^\omega(\mathcal{A})$ is decidable. Since this intersection is regular and any non-empty regular ω-language contains some ultimately periodic word, we also find x and y as required. ◀

It therefore suffices to show that, indeed, any non-empty set of the form $\big\langle [L^\omega(\mathcal{A}_1)], \ldots, [L^\omega(\mathcal{A}_n)]\big\rangle^\varphi$ contains some ultimately periodic word.

▶ **Definition D.6.** *Let (Γ, I) be some independence alphabet. It* guarantees witnesses for non-emptiness *if, for any n-ary Boolean function φ and any Büchi-automata $\mathcal{A}_i$ for $i \in [n]$ with $\big\langle [L^\omega(\mathcal{A}_1)], \ldots, [L^\omega(\mathcal{A}_n)]\big\rangle^\varphi \neq \emptyset$, there exist $u, v \in \Gamma^+$ with $uv^\omega \in \big\langle [L^\omega(\mathcal{A}_1)], \ldots, [L^\omega(\mathcal{A}_n)]\big\rangle^\varphi$.*

Wolk showed that an independence alphabet is diagonal if, and only if, it can be constructed from singleton alphabets by disjoint union and addition of single letters that are independent from all existing ones [46, 47]. Singleton alphabets clearly guarantee witnesses for non-emptiness since $\{a\}^\omega = \{a^\omega\}$. We therefore have to show that the disjoint union as well as the addition of new letters preserves this property. These two proofs can be found in Proposition D.8 and D.7, respectively.

▶ **Proposition D.7.** *Let (Γ, I) be an independence alphabet and $a \in \Gamma$ with $\{a\} \times (\Gamma \setminus \{a\}) \subseteq I$. Set $\Delta = \Gamma \setminus \{a\}$. If $(\Delta, I \cap \Delta^2)$ guarantees witnesses for non-emptiness, then also (Γ, I) guarantees witnesses for non-emptiness.*

Proof. Let φ be any n-ary Boolean function, let $\mathcal{A}_i$ be a Büchi-automaton over Γ for $i \in [n]$, and let $\alpha \in \Gamma^\omega$ with $\alpha \in \big\langle [L^\omega(\mathcal{A}_1)], \ldots, [L^\omega(\mathcal{A}_n)]\big\rangle^\varphi$. We distinguish two cases depending on the number of occurrences of letters from Δ.

Suppose $|\alpha|_\Delta < \omega$. Then $\alpha = ua^\omega$ for some word $u \in \Gamma^+$. Setting $v = a$, we therefore get $uv^\omega = \alpha \in \big\langle [L^\omega(\mathcal{A}_1)], \dots, [L^\omega(\mathcal{A}_n)] \big\rangle^\varphi$.

Next suppose $|\alpha|_\Delta = \omega$ and let $\varkappa = |\alpha|_a \in \mathbb{N} \cup \{\infty\}$ denote the number of occurrences of the letter a in α. In a first step, we build Büchi-automata $\mathcal{A}_i^1$ with $L^\omega(\mathcal{A}_i^1) = L^\omega(\mathcal{A}_i) \cap (\Delta^\omega \sqcup a^\varkappa)$, i.e., they accept those ω-words accepted by $\mathcal{A}_i$ that have $\varkappa$ many occurrences of a. In particular, $\alpha \in [L^\omega(\mathcal{A}_i)] \iff \alpha \in [L^\omega(\mathcal{A}_i^1)]$ holds for all $i \in [n]$.

From the Büchi-automaton $\mathcal{A}_i^1 = (Q, \Gamma, S, T, F)$ over Γ, we construct a new Büchi-automaton $\mathcal{A}_i^2$ over the smaller alphabet Δ with $L^\omega(\mathcal{A}_i^2) = \mathrm{proj}_\Delta(L^\omega(\mathcal{A}_i^1))$.

▷ Claim. For all $i \in [n]$ and $\beta \in \Gamma^\omega$ with $|\beta|_a = \varkappa$, we have $\beta \in [L^\omega(\mathcal{A}_i)] \iff \mathrm{proj}_\Delta(\beta) \in [L^\omega(\mathcal{A}_i^2)]$.

Proof of claim. We have $\beta \in [L^\omega(\mathcal{A}_i)]$ iff there exists $\gamma \in L^\omega(\mathcal{A}_i)$ with $\beta \sim \gamma$. Since $\varkappa$ is the number of occurrences of a in β and a is independent from all letters in Δ, this is equivalent to

$$\text{there exists } \gamma \in L^\omega(\mathcal{A}_i) \text{ with } |\gamma|_a = \varkappa \text{ and } \mathrm{proj}_{\{b,c\}}(\beta) = \mathrm{proj}_{\{b,c\}}(\gamma) \text{ for all } b, c \in \Delta \text{ with } (b,c) \notin I\,.$$

Since $\mathrm{proj}_{\{b,c\}}(\beta) = \mathrm{proj}_{\{b,c\}}(\mathrm{proj}_\Delta(\beta))$ for any $b, c \in \Delta$ (and similarly for γ), this is equivalent to

$$\text{there exists } \gamma \in L^\omega(\mathcal{A}_i) \text{ with } |\gamma|_a = \varkappa \text{ and } \mathrm{proj}_\Delta(\beta) \sim \mathrm{proj}_\Delta(\gamma)\,.$$

Note that a word is accepted by $\mathcal{A}_i^2$ if, and only if, it can be enriched with $\varkappa$ occurrences of a to yield a word accepted by $\mathcal{A}_i$. Hence the last statement is equivalent to

$$\text{there exists } \delta \in L^\omega(\mathcal{A}_i^2) \text{ with } \mathrm{proj}_\Delta(\beta) \sim \delta$$

and therefore to

$$\mathrm{proj}_\Delta(\beta) \in [L^\omega(\mathcal{A}_i^2)]\,.$$

q.e.d.

This claim together with $\alpha \in \big\langle [L^\omega(\mathcal{A}_1)], \dots, [L^\omega(\mathcal{A}_n)] \big\rangle^\varphi$, implies $\mathrm{proj}_\Delta(\alpha) \in \big\langle [L^\omega(\mathcal{A}_1^2)], \dots, [L^\omega(\mathcal{A}_n^2)] \big\rangle^\varphi$; in particular, this last set is not empty. Since $(\Delta, I \cap \Delta^2)$ guarantees witnesses for non-emptiness, there are $\overline{u}, \overline{v} \in \Delta^+$ with

$$\overline{u}\,\overline{v}^\omega \in \big\langle [L^\omega(\mathcal{A}_1^2)], \dots, [L^\omega(\mathcal{A}_n^2)] \big\rangle^\varphi\,.$$

Depending on whether $\varkappa$ is finite or not, we define an ultimately periodic word uv^ω: if $\varkappa \in \mathbb{N}$, then set $u = \overline{u}a^\varkappa$ and $v = \overline{v}$; if $\varkappa = \omega$, then set $u = \overline{u}$ and $v = \overline{v}\,a$. In any case, $\mathrm{proj}_\Delta(uv^\omega) = \overline{u}\,\overline{v}^\omega$ and $|uv^\omega|_a = \varkappa$. Using the claim, again, we get (for all $i \in [n]$)

$$uv^\omega \in [L^\omega(\mathcal{A}_i)] \iff \overline{u}\,\overline{v}^\omega \in [L^\omega(\mathcal{A}_i^2)]\,.$$

Hence $\overline{u}\,\overline{v}^\omega \in \big\langle [L^\omega(\mathcal{A}_1^2)], \dots, [L^\omega(\mathcal{A}_n^2)] \big\rangle^\varphi$ implies $u\,v^\omega \in \big\langle [L^\omega(\mathcal{A}_1)], \dots, [L^\omega(\mathcal{A}_n)] \big\rangle^\varphi$. Thus, indeed, (Γ, I) guarantees witnesses for non-emptiness. ◀

Our next aim is to show that also the disjoint union of two independence alphabets guarantees witnesses for non-emptiness (provided the two independence alphabets do).

▶ **Proposition D.8.** *Let (Γ_1, I_1) and (Γ_2, I_2) be disjoint independence alphabets that both guarantee witnesses for non-emptiness. Set $\Gamma = \Gamma_1 \cup \Gamma_2$ and $I = I_1 \cup I_2$. Then the independence alphabet (Γ, I) guarantees witnesses for non-emptiness.*

Proof. Let φ be any n-ary Boolean function, $\mathcal{A}_i$ for $i \in [n]$ a Büchi-automaton over Γ, and let $\alpha \in \langle [L^\omega(\mathcal{A}_1)], \ldots, [L^\omega(\mathcal{A}_n)] \rangle^\varphi$. Furthermore, let $\mathcal{A}^{\text{multi}}$ and $\mathcal{A}^{\text{mono}}$ be the automata from Theorem C.8 (here, we only need the existence of these automata). Then $\text{tpsqu}(\alpha) \in L^\omega(\mathcal{A}^{\text{multi}}) \cup L^*(\mathcal{A}^{\text{mono}})$.

In the former case, the Büchi-automaton $\mathcal{A}^{\text{multi}}$ accepts some word. Since any non-empty regular ω-language contains some ultimately periodic word, there are types $t_1, \ldots, t_\ell \in \text{TP}_+$ with

$$t_1\, t_2\, \cdots\, t_j\, (t_{j+1}\, \cdots\, t_\ell)^\omega \in L^\omega(\mathcal{A}^{\text{multi}})$$

for some $j \in [\ell]$. The definition of the set of types ensures the existence of finite words $u_i \in \Gamma_{k_i}^+$ with $t_i = \text{tp}_+(u_i)$ for all $i \in [\ell]$. Since the indices k_i originate from the types t_i and since the sequence of types is accepted by $\mathcal{A}^{\text{multi}}$, we get $k_i \neq k_{i+1}$ for all $i \in [\ell - 1]$ and $k_\ell \neq k_{j+1}$. Hence

$$\text{tpsqu}(u_1 u_2 \cdots u_j (u_{j+1} \cdots u_\ell)^\omega) = t_1\, t_2\, \cdots\, t_j\, (t_{j+1}\, \cdots\, t_\ell)^\omega$$

which is accepted by the Büchi-automaton $\mathcal{A}^{\text{multi}}$. It follows from Theorem C.8 that the ultimately periodic word $u_1 u_2 \cdots u_j (u_{j+1} \cdots u_\ell)^\omega$ belongs to $\langle [L^\omega(\mathcal{A}_1)], \ldots, [L^\omega(\mathcal{A}_n)] \rangle^\varphi$.

Now suppose $\text{tpsqu}(\alpha)$ is accepted by the nfa $\mathcal{A}^{\text{mono}}$. Then $\text{tpsqu}(\alpha) = t_1 t_2 \cdots t_\ell t$ with $t_i \in \text{TP}_+$ for all $i \in [\ell]$ and $t \in \text{TP}_\omega$. As before, there are words $u_i \in \Gamma_{k_i}^+$ and $\alpha' \in \Gamma_{k_{\ell+1}}^\omega$ with $\text{tp}_+(u_i) = t_i$ for all $i \in [\ell]$ and $\text{tp}_\omega(\alpha') = t$. Consider the language

$$L = \{\gamma \in \Gamma_{k_{\ell+1}}^\omega \mid \text{tp}_\omega(\gamma) = t\}\,.$$

Then $\beta \in L$ witnesses that L is not empty. In the proof of Lemma C.9, we saw that L is a Boolean combination of languages $[K]$ with $K \subseteq \Gamma_{k_{\ell+1}}^\omega$ regular.[2] Since the indendence alphabet $(\Gamma_{k_{\ell+1}}, I_{k_{\ell+1}})$ guarantees witnesses for non-emptiness, there are finite words $v, w \in \Gamma^+$ with $vw^\omega \in L$. Note that

$$\text{tpsqu}(u_1 \cdots u_\ell v w^\omega) = t_1 \cdots t_\ell t$$

and is therefore accepted by the nfa $\mathcal{A}^{\text{mono}}$. Theorem C.8 implies

$$u_1 \cdots u_\ell v w^\omega \in \langle [L^\omega(\mathcal{A}_1)], \ldots, [L^\omega(\mathcal{A}_n)] \rangle^\varphi\,.$$

In any of the two cases, we found an ultimately periodic word that belongs to $\langle [L^\omega(\mathcal{A}_1)], \ldots, [L^\omega(\mathcal{A}_n)] \rangle^\varphi$. Thus, indeed, the independence alphabet (Γ, I) guarantees witnesses for non-emptiness. ◀

We can therefore now show that every diagonal independence alphabet guarantees witnesses for non-emptiness.

▶ **Theorem D.9.** *Every diagonal independence alphabet guarantees witnesses for non-emptiness.*

Proof. Recall [46, 47] that the set of diagonal independence alphabets is the least set containing the singleton alphabets that is closed under finite disjoint unions and addition of single letters that are independent from all other letters. Hence the claim follows immediately from Propositions D.8 and D.7. ◀

[2] Strictly speaking, Lemma C.9 only talks about transitive independence alphabets, but the presentation of L as described there remains valid also in this case.

▶ **Corollary 5.3.** *The emptiness problem* $\mathrm{EmptP}_\omega(\mathrm{DiagA}, \mathrm{BF})$ *is in* Π^0_1.

Even more, there is an algorithm that, on input of a diagonal independence alphabet (Γ, I)*, an* n*-ary Boolean function* φ*, and Büchi-automata* $\mathcal{A}_i$ *for all* $i \in [n]$ *over* Γ *halts iff* $\left\langle [L^\omega(\mathcal{A}_1)], \ldots, [L^\omega(\mathcal{A}_n)] \right\rangle^\varphi \neq \emptyset$ *and, if this is the case, outputs nonempty words* $u, v \in \Gamma^+$ *with* $uv^\omega \in \left\langle [L^\omega(\mathcal{A}_1)], \ldots, [L^\omega(\mathcal{A}_n)] \right\rangle^\varphi$.

Proof. The algorithm guesses a pair of words $u, v \in \Gamma^+$, checks (for all $i \in [n]$) whether $uv^\omega \in [L^\omega(\mathcal{A}_i)]$ (which is possible by Observation D.5) and then evaluates the Boolean function φ. By Theorem D.9, such a pair (u, v) exists if, and only if, $\left\langle [L^\omega(\mathcal{A}_1)], \ldots, [L^\omega(\mathcal{A}_n)] \right\rangle^\varphi \neq \emptyset$. ◀

▶ **Theorem 5.4.** *The rationality problem* $\mathrm{RatP}_\omega(\mathrm{DiagA}, \mathrm{BF})$ *and the regularity problem* $\mathrm{RegP}_\omega(\mathrm{DiagA}, \mathrm{BF})$ *are in* Σ^0_2.

Proof. Let (Γ, I) be a diagonal independence alphabet, φ an n-ary Boolean function, and $\mathcal{A}_i$ for $i \in [n]$ a Büchi-automaton over Γ. In the following, let $X \triangle Y$ denote the symmetric difference $X \setminus Y \cup Y \setminus X$ of the two sets X and Y. We have $((\Gamma, I), \varphi, \mathcal{A}_1, \ldots, \mathcal{A}_n) \in \mathrm{RatP}_\omega(\mathrm{IndA}, \varphi)$ iff

$$\exists \text{ Büchi-automaton } \mathcal{B} \colon [L^\omega(\mathcal{B})] \triangle \left\langle [L^\omega(\mathcal{A}_1)], \ldots, [L^\omega(\mathcal{A}_n)] \right\rangle^\varphi = \emptyset\,.$$

But this is the case iff

$$\exists \text{ Büchi-automaton } \mathcal{B} \colon ((\Gamma, I), \varphi', \mathcal{A}_1, \ldots, \mathcal{A}_n, \mathcal{B}) \in \mathrm{EmptP}_\omega((\Gamma, I), \mathrm{BF})$$

where φ' is the $(n+1)$-ary Boolean function with $\varphi'(b_1, \ldots, b_{n+1}) = 0$ if, and only if, $\varphi(b_1, \ldots, b_n) = b_{n+1}$. From Cor. 5.3, we obtain that this is a statement in Σ^0_2 (note that φ' is not monotone even if φ is).

To handle the regularity problem, note that the language $\left\langle [L^\omega(\mathcal{A}_1)], \ldots, [L^\omega(\mathcal{A}_n)] \right\rangle^\varphi$ is closed. Hence it is regular iff it equals $[L^\omega(\mathcal{B})]$ for some Büchi-automaton $\mathcal{B}$ with $L^\omega(\mathcal{B}) = [L^\omega(\mathcal{B})]$, i.e. such that the language $L^\omega(\mathcal{B})$ is closed. Hence we have $((\Gamma, I), \varphi, \mathcal{A}_1, \ldots, \mathcal{A}_n) \in \mathrm{RatP}_\omega((\mathrm{DiagA}, \mathrm{BF})$ iff

$$\exists \text{ Büchi-automaton with closed language } \mathcal{B} \colon [L^\omega(\mathcal{B})] \triangle \left\langle [L^\omega(\mathcal{A}_1)], \ldots, [L^\omega(\mathcal{A}_n)] \right\rangle^\varphi = \emptyset\,.$$

which is the case iff

$$\exists \text{ Büchi-automaton with closed language } \mathcal{B} \colon ((\Gamma, I), \varphi', \mathcal{A}_1, \ldots, \mathcal{A}_n, \mathcal{B}) \in \mathrm{EmptP}_\omega((\Gamma, I), \mathrm{BF})$$

where φ' is the $(n+1)$-ary Boolean function with $\varphi'(b_1, \ldots, b_{n+1}) = 0$ if, and only if, $\varphi(b_1, \ldots, b_n) = b_{n+1}$. As shown in [39, 34], it is decidable whether a Büchi-automaton accepts a closed language. We therefore obtain from Cor. 5.3, that the above is a statement in Σ^0_2. ◀

E Proofs from Sect. 6

In this section, we consider the ω-versions of the emptiness, the rationality, and the regularity problem for general independence alphabets.

The upper bounds Π^1_1 and Π^1_2 are relatively easy consequences of the definition of these problems. The claimed lower bounds Π^1_1 are more demanding. To demonstrate them, we first consider two-counter machines with Büchi-acceptance whose emptiness problem is Π^1_1-hard. The emptiness problem for these machines is then reduced to the emptiness, regularity, and rationality problem of socalled “standard independence alphabets” and the Boolean functions $\wedge$ and $\neg$ (Id in case of regularity). These “standard independence alphabets” (cf. Def. E.4) consist of the commands of two-counter machines (i.e., incrementation, decrementation, and zero-test of each counter) extended by three letters to encode and relate counter values before and after the execution of a single command. This step from counter machines to finite automata requires an involved coding that is presented in Sections E.2.2 and E.2.3. From these standard independence alphabets, we then reduce to arbitrary non-diagonal independence alphabets without changing the Boolean function. To obtain the lower bound for any not too simple Boolean function, we finally refer to Lemma 2.3.

E.1 Upper bounds

The upper bounds for the emptiness, the regularity, and the rationality problem for arbitrary independence alphabets are analytical and therefore much higher than those for diagonal independence alphabets. Later, they will turn out to be almost optimal since we also prove analytical lower bounds.

▶ **Theorem E.1.** *(1) The emptiness problem* $\mathrm{EmptP}_\omega(\mathrm{IndA}, \mathrm{monBF})$ *is in* Π^1_1.
(2) For all $\mathrm{GenP} \in \{\mathrm{EmptP}, \mathrm{RatP}, \mathrm{RegP}\}$*, the problem* $\mathrm{GenP}_\omega(\mathrm{IndA}, \mathrm{BF})$ *is in* Π^1_2.

Proof. Let (Γ, I) be some independence alphabet, φ an n-ary Boolean function, and $\mathcal{A}_i = (Q_i, \Gamma, S_i, T_i, F_i)$ for $i \in [n]$ Büchi-automata over Γ. We will show that the non-emptiness of $\big\langle [L^\omega(\mathcal{A}_1)], \ldots, [L^\omega(\mathcal{A}_n)] \big\rangle^\varphi$ can be expressed by a Σ^1_1-formula. Formally, we assume Γ and Q_i to be finite sets of natural numbers such that, e.g., an ω-word is a function $\alpha \colon \mathbb{N} \to \mathbb{N}$ with $\alpha(n) \in \Gamma$ for all $n \in \mathbb{N}$. To start with, we express by a Σ^1_1 formula $f_i(\alpha)$ that $\mathcal{A}_i$ accepts some ω-word that is equivalent to α:

$$
\begin{aligned}
&\exists \beta, \rho_i \colon \mathbb{N} \to \mathbb{N} \colon \\
&\qquad \forall n \colon \beta(n) \in \Gamma \wedge \rho_i(n) \in Q_i && (0)\\
&\quad \wedge \rho_i(0) \in S_i && (1)\\
&\quad \wedge \forall n \colon \big(\rho_i(n), \beta(n), \rho_i(n+1)\big) \in T_i && (2)\\
&\quad \wedge \forall k \, \exists n > k \colon \rho_i(n) \in F_i && (3)\\
&\quad \wedge \forall a, b, m \colon a, b \in \Gamma \wedge (a, b) \notin I \to \exists n \colon \mathrm{proj}_{\{a,b\}}(\alpha[1 \ldots m]) = \mathrm{proj}_{\{a,b\}}(\beta[1 \ldots n]) && (4)\\
&\quad \wedge \forall a, b, n \colon a, b \in \Gamma \wedge (a, b) \notin I \to \exists m \colon \mathrm{proj}_{\{a,b\}}(\alpha[1 \ldots m]) = \mathrm{proj}_{\{a,b\}}(\beta[1 \ldots n]) && (5)
\end{aligned}
$$

The mapping β is meant to code an ω-word over Γ while ρ_i shall map positions in this word to states of $\mathcal{A}_i$. Note that line (0) ensures precisely this. Line (2) expresses that ρ_i is a run of $\mathcal{A}_i$ on β. Line (1) ensures that this run starts in an initial state and (3) that it infinitely often visits some accepting state. Hence, (1-3) together with the existentially quantified mapping ρ_i state that the ω-word β is accepted by the Büchi-automaton $\mathcal{A}_i$.

Now let $a, b \in \Gamma$ be dependent letters and consider line (4). It states that the projection of α to $\{a, b\}$ is a prefix of the corresponding projection of β. Since, in (5), the roles of α and β are exchanged, (4) and (5) together ensure that the projections of α and β to $\{a, b\}$ coincide. As this is required to hold for any pair $(a, b) \notin I$ (including the case $a = b$), (4) and (5) express that α and β are equivalent wrt. the independence relation I.

In summary, the formula f_i above expresses the existence of an ω-word β that is accepted by the Büchi-automaton $\mathcal{A}_i$ and equivalent to α, i.e., $\alpha \in [L^\omega(\mathcal{A}_i)]$.

From the function φ, one can compute a finite set $\mathcal{Y}$ of pairs (A,B) with $A,B\subseteq[n]$ such that

$$\big\langle [L^\omega(\mathcal{A}_1)],\ldots,[L^\omega(\mathcal{A}_n)]\big\rangle^\varphi = \bigcup_{(A,B)\in\mathcal{Y}} \left(\bigcap_{i\in A} [L^\omega(\mathcal{A}_i)] \cap \bigcap_{i\in B} \Gamma^\omega \setminus [L^\omega(\mathcal{A}_i)] \right) .$$

Hence the non-emptiness of this set is expressed as

$$\delta = \exists \alpha\colon \mathbb{N}\to\mathbb{N}\colon \left(\forall n\colon \alpha(n)\in\Gamma \wedge \bigvee_{(A,B)\in\mathcal{Y}} \left(\bigwedge_{i\in A} f_i \wedge \bigwedge_{i\in B} \neg f_i \right)\right) .$$

Since the formulas f_i belong to Σ^1_1, this formula δ belongs to Σ^1_2. Furthermore, it can be computed from the independence alphabet (Γ,I), the monotone Boolean function φ, and the Büchi-automata $\mathcal{A}_i$. Thus, we showed that the emptiness problem $\mathrm{EmptP}_\omega(\mathrm{IndA},\mathrm{BF})$ is in Π^1_2.

If the Boolean function φ is even monotone, then $\mathcal{Y}$ can be chosen as a set of pairs $(A,\emptyset)$. Hence, in the formula δ, the Σ^1_1-formules f_i appear only positively implying that, in this case, $\delta\in\Sigma^1_1$. Thus, we showed that the emptiness problem $\mathrm{EmptP}_\omega(\mathrm{IndA},\mathrm{monBF})$ for monotone Boolean functions is in Π^1_1.

The remaining proofs regarding the rationality and the regularity problem are similar to the proof of Theorems 5.4. They use that any statement of the form $\exists$ nfa $\mathcal{B}\,\forall\beta\colon\mathbb{N}\to\mathbb{N}\cdots$ can be reformulated into an equivalent statement of the from $\forall\beta'\colon\mathbb{N}\to\mathbb{N}\cdots$ (cf. [43, Thm. 16-III]). ◀

E.2 Intermezzo

To show lower bounds, we will reduce from the emptiness problem for deterministic two-counter Büchi-automata. The first section of this intermezzo introduces them and proves the Π^1_1-hardness of their emptiness problem (cf. Cor. E.3). The other two sections prepare the reduction to our problems $\mathrm{GenP}_\omega((\Sigma,I),\varphi)$ for socalled standard independence alphabets (Σ,I) and certain simple Boolean functions φ. To this aim, we prove that the set Val of valid counter operations is the projection of both, the intersection (Section E.2.2) and the complement (Section E.2.3) of closures of certain regular languages.

The lower bounds are even shown under the restriction to deterministic and co-deterministic Büchi-automata. Here, a Büchi-automaton $\mathcal{A}=(Q,\Gamma,S,T,F)$ is *deterministic* if $(p,a,q),(p,a,r)\in T$ implies $q=r$. Note that, for simplicity, we allow several initial states as well as "missing transitions" (i.e., there can be $p\in Q$ and $a\in\Gamma$ such that $(p,a,q)\notin T$ for all $q\in Q$). An ω-language is *deterministic regular* if it is accepted by some deterministic Büchi-automaton. It is *co-deterministic regular* if its complement is deterministic regular. Similarly, a Büchi-automaton is co-deterministic if its language is co-derterministic regular.

E.2.1 Two-counter automata with Büchi-acceptance

To define two-counter Büchi-automata, fix the alphabet $\mathrm{Op}=\{\mathrm{inc}_1,\mathrm{inc}_2,\mathrm{dec}_1,\mathrm{dec}_2,\mathrm{zero}_1,\mathrm{zero}_2\}$. The idea is that, whenever the Büchi-automaton "reads" the "letter" inc_1 or dec_1, it increments or decrements its first counter, the "letter" zero_1 can only be read if the first counter's value equals 0. Since we assume the counters to be initially 0, no ω-word of the form $\mathrm{inc}_1\,\mathrm{zero}_1\,\alpha$ can be executed since, after one step, the first counter holds the value 1 while, in the second step, the counter is required to be 0. Since we assume that counters hold non-negative values, only, also no ω-word starting with $\mathrm{inc}_2\,\mathrm{dec}_2\,\mathrm{dec}_2$ can be executed.

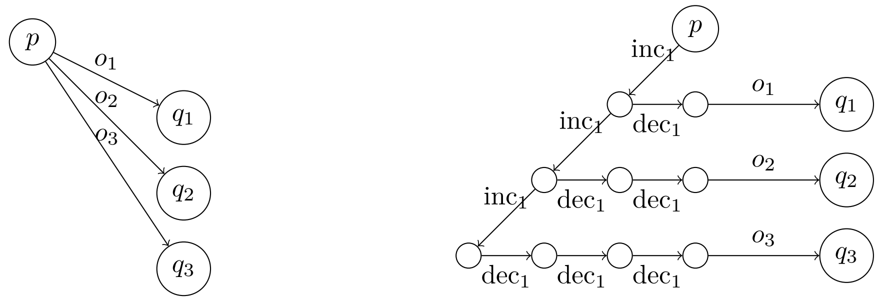

Figure 1 Proof of Corollary E.3.
Left: transitions in $\mathcal{M}$, right: transitions in $\mathcal{A}$

To define the executable or valid words, let $\alpha = a_0 a_1 \cdots \in \mathrm{Op}^\omega$ with $a_i \in \mathrm{Op}$ for all $i \geq 0$. For $n \geq 0$ and $k \in [2]$, let

$$\mathrm{value}_k(\alpha, n) = |a_0 a_1 \cdots a_{n-1}|_{\mathrm{inc}_k} - |a_0 a_1 \cdots a_{n-1}|_{\mathrm{dec}_k}$$

denote the value of the counter k after executing the first n operations of α. The ω-word α is *valid* if, for all $n \geq 0$ and $k \in [2]$, the following hold:

- If $a_n = \mathrm{zero}_k$, then $\mathrm{value}_k(\alpha, n) = 0$.
- If $a_n = \mathrm{dec}_k$, then $\mathrm{value}_k(\alpha, n) > 0$.

Let $\mathrm{Val} \subseteq \mathrm{Op}^\omega$ denote the set of all valid ω-words.

The emptiness problem for nondeterministic Turing machines with Büchi-acceptance is complete for Π^1_1 [23]. In the same way that deterministic two-counter machines simulate deterministic Turing machines [33] (cf., e.g., [24]), we can simulate nondeterministic Turing machines with Büchi-acceptance by Büchi-automata over Op which yields the following result.

▶ **Theorem E.2.** *The set of Büchi-automata $\mathcal{M}$ over* Op *with $L^\omega(\mathcal{M}) \cap \mathrm{Val} = \emptyset$ is complete for Π^1_1.*

Note that the emptiness problem for deterministic Turing machines with Büchi-acceptance is arithmetical since it amounts to the non-existence of some input w such that, for any natural number n, there is a finite (incomplete) computation with input w that visits some accepting state at least n times. Differently, for the above problem on Büchi automata, the hardness already holds for deterministic automata.

▶ **Corollary E.3.** *The set of deterministic Büchi-automata $\mathcal{A}$ over* Op *with $L^\omega(\mathcal{A}) \cap \mathrm{Val} = \emptyset$ is complete for Π^1_1.*

Proof. We reduce from the problem in Theorem E.2. So let $\mathcal{M}$ be a (possibly nondeterministic) Büchi-automaton over Op. This automaton gets transformed into a deterministic Büchi-automaton $\mathcal{A}$ by doing the following for any state p. Let (p, o_i, q_i) for $1 \leq i \leq n$ denote the set of transitions starting in p (see left of Fig. 1 for $n = 3$). We replace these transitions by the gadget on the right of Fig. 1. The result is a deterministic automaton $\mathcal{A}$. Note that there is an obvious bijection between the infinite runs in $\mathcal{M}$ and in $\mathcal{A}$ that carry words from Val. Hence $L^\omega(\mathcal{M}) \cap \mathrm{Val} \neq \emptyset \iff L^\omega(\mathcal{A}) \cap \mathrm{Val} \neq \emptyset$. Thus, the described construction is a reduction from the problem in Theorem E.2 to that of this corollary. ◀

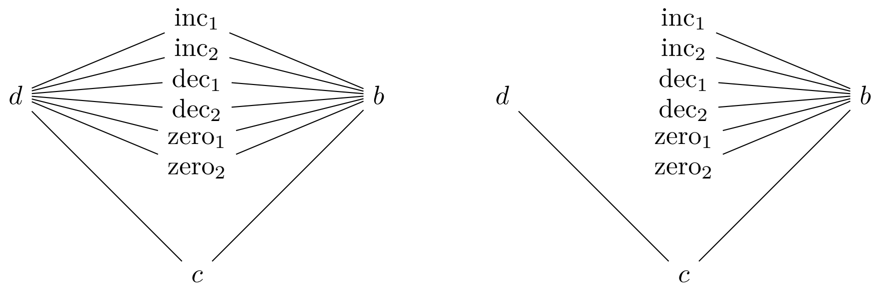

Figure 2 The two standard independence alphabets

E.2.2 Val as intersection of closures of regular languages

In this section, we prove that the set $\mathrm{Val} \subseteq \mathrm{Op}^\omega$ of valid words corresponds to the intersection of the closures $[K]$ and $[L]$ of certain deterministic regular ω-languages K and L (cf. Prop. E.8); this coding of the set Val builds on the analogous coding by Aalbersberg and Hoogeboom [1].

The said languages K and L use the letters from Op and, in addition, the letters b, c, and d. We define two independence relations on this alphabet that both are non-diagonal (the first one contains C_4, the second P_4). The set Val will then be the projection of $[K] \cap [L]$ to the alphabet Op (where the closures are constructed wrt. any of the two non-diagonal independence alphabets).

▶ **Definition E.4.** *The following two independence alphabets* (Σ, I_1) *and* (Σ, I_2) *(that differ in the independence relation, only) are* standard independence alphabets*:* $\Sigma = \mathrm{Op} \cup \{b, c, d\}$ *and, for all* $s \in [2]$*,*

- $\mathrm{Op} \times \{b\} \subseteq I_s$*,* $(b, c), (c, d) \in I_s$*,*
- $\mathrm{Op} \times (\mathrm{Op} \cup \{c\}) \subseteq D_s$*,* $(b, d) \in D_s$*,*
- $\mathrm{Op} \times \{d\} \subseteq I_1$*, and* $\mathrm{Op} \times \{d\} \subseteq D_2$*.*

Note that all letters from Op behave the same: they are all independent from b, dependent on c and on each other, and independent from or dependent on d, see Fig. 2. The standard independence alphabet (Σ, I_s) therefore contains C_4 for $s = 1$ and P_4 for $s = 2$.

▶ **Stipulation E.5.** In the following, Σ will always denote the alphabet $\mathrm{Op} \cup \{b, c, d\}$, i.e., the set of letters of some standard independence alphabet.

We next define the two ω-languages K and L and describe their closures wrt. the two standard independence alphabets. First, consider the regular languages

$$\begin{aligned} K_0 = {} & \mathrm{Op}\, d\, (cb)^+ \subseteq \Sigma^+ \text{ and} \\ L_0 = {} & \mathrm{inc}_1\, (b\, c^5\, c^5)^+\, d \cup \mathrm{inc}_2\, (b\, c^5\, c^5\, c^5)^+\, d \\ & \cup \mathrm{dec}_1\, (b\, b\, c^5)^+\, d \cup \mathrm{dec}_2\, (b\, b\, b\, c^5)^+\, d \\ & \cup \mathrm{zero}_1\, (b\, c^5\, b\, c^5)^*\, b\, c^5\, d \cup \mathrm{zero}_2\, (b\, c^5\, b\, c^5\, b\, c^5)^*\, \{b\, c^5\, b\, c^5, b\, c^5\}\, d \end{aligned}$$

and the regular ω-languages

$$K = b\, K_0^\omega \subseteq \Sigma^\omega \text{ and } L = L_0^\omega\,. \tag{6}$$

Before characterizing the closures of these two ω-languages, some words are in order about the idea behind these definitions.

Note that words from K have the form

$$\alpha = b^{m_0}\, a_0 d(cb)^{m_1}\, a_1 d(cb)^{m_2} \cdots$$

with $a_i \in \mathrm{Op}$ and $m_0 = 1$. Let $\beta = \mathrm{proj}_{\mathrm{Op}}(\alpha) = a_0 a_1 \cdots$ denote the sequence of operations in this word from K. The idea is that the number m_n (for $n \geq 0$) encodes the counter values and the step counter before the execution of a_n. More precisely, we will ensure that

$$m_n = 2^{\mathrm{value}_1(\beta,n)} \cdot 3^{\mathrm{value}_2(\beta,n)} \cdot 5^{n-1}$$

holds in case α belongs to the closure of L as well.

To understand why the language L contains the necessary information, note that $\alpha \in K$ implies

$$\alpha' = a_0 b^{m_0} c^{m_1} d\, a_1 b^{m_1} c^{m_2} d\, \cdots \in [K]$$

since these two words are equivalent. In order for α' to also be in the closure of $L = L_0^\omega$, one expects that $w_n = a_n b^{m_n} c^{m_{n+1}} d$ belongs to the closure of L_0 for all $n \geq 0$. Suppose that $n \geq 0$ and m_n encodes the counter values and step counter as described above before the execution of a_n. If, e.g., $a_n = \mathrm{inc}_1$, then $w_n \in [L_0]$ is equivalent to $2 \cdot 5 \cdot m_n = m_{n+1}$. Hence m_{n+1} encodes the counter values and the step counter after the execution of a_n. Similarly, if $a_n = \mathrm{zero}_2$, then $w_n \in [L_0]$ iff $5 \cdot m_n = m_{n+1} \in 15 \cdot \mathbb{N} + \{10, 5\}$. Hence, when going from m_n to m_{n+1}, the two counter values are left unchanged and the step counter has increased by one. Furthermore, there is $k \geq 0$ such that m_{n+1} is $15k + 10$ or $15k + 5$, i.e., $m_n \in \{3k+2, 3k+1\}$ which is equivalent to saying that m_n is not divisible by 3. But this means that the second counter's value encoded in m_n is 0.

Since we want to show that an ω-word β is valid iff it belongs to the projection of the intersection of $[K]$ and $[L]$, the following two lemmas characterise these two closures.

▶ **Lemma E.6.** *Let (Σ, I_s) be a standard independence alphabet, i.e., $s \in [2]$. An ω-word $\alpha \in \Sigma^\omega$ belongs to the closure $[K]$ of K if, and only if, there are letters $a_0, a_1, \dots \in \mathrm{Op}$ and positive integers $m_0, m_1, m_2, \dots \geq 1$ with $m_0 = 1$ such that*

(i) $\mathrm{proj}_{\{b,d\}}(\alpha) = b^{m_0}\, db^{m_1}\, db^{m_2}\, db^{m_3} \cdots$,
(ii) $\mathrm{proj}_{\mathrm{Op}\cup\{c\}}(\alpha) = a_0 c^{m_1}\, a_1 c^{m_2}\, a_2 c^{m_3} \cdots$,
(iii) and, in case $s = 2$, $\mathrm{proj}_{\mathrm{Op}\cup\{d\}} \in (\mathrm{Op}\, d)^\omega$.

Proof. Let $\alpha \in [K]$. Then there is $\beta \in K$ with $\beta \sim \alpha$. Since $\beta \in K$, there are $m_0, m_1, \dots \geq 1$ with $m_0 = 1$ and $a_0, a_1, \dots \in \mathrm{Op}$ such that $\beta = b^{m_0}\, a_0\, d\, (cb)^{m_1}\, a_1\, d\, (cb)^{m_2} \cdots$. From $\alpha \sim \beta$, the two words α and β have the same projections to $\{b, d\}$, to $\mathrm{Op} \cup \{c\}$ and, in case $s = 2$, to $\mathrm{Op} \cup \{d\}$. Since these projections of β satisfy (i-iii), we showed that any element of $[K]$ has the desired properties.

Conversely, suppose $\alpha \in \Sigma^\omega$, $m_0, m_1, \dots \geq 1$ with $m_0 = 1$ and $a_0, a_1, \dots \in \mathrm{Op}$ such that (i-iii) hold. Consider the ω-word $\beta = b^{m_0}\, a_0\, d\, (cb)^{m_1}\, a_1\, d\, (cb)^{m_2} \cdots$. Then $\beta \in K$ and the relevant projections of α and β coincide, i.e., $\alpha \sim \beta$. Hence $\alpha \in [K]$ which proves the converse implication. ◀

Next, we present a similar characterisation of the closure of the language L. Since the definition of that language is more involved than that of K, also the characterisation is a bit cumbersome.

▶ **Lemma E.7.** *Let (Σ, I_s) be a standard independence alphabet, i.e., $s \in [2]$. An ω-word α over Σ belongs to $[L]$ if, and only if, there are $a_0, a_1, \dots \in \mathrm{Op}$ and positive integers $k_0, k_1, k_2, \dots, \ell_1, \ell_2, \dots \geq 1$ such that*

(i) $\mathrm{proj}_{\{b,d\}}(\alpha) = b^{k_0}\, d\, b^{k_1}\, d\, b^{k_2} \cdots$,
(ii) $\mathrm{proj}_{\mathrm{Op}\cup\{c\}}(\alpha) = a_0 (c^5)^{\ell_1}\, a_1 (c^5)^{\ell_2}\, a_2 (c^5)^{\ell_3} \cdots$,
(iii) *and, in case* $s = 2$, $\mathrm{proj}_{\mathrm{Op}\cup\{d\}} \in (\mathrm{Op}\, d)^\omega$

where, for all $i \geq 0$*, the letter* $a_i \in \mathrm{Op}$ *and the number* $k_i \geq 1$ *determine the number* ℓ_{i+1} *according to the following rules for all* $k \in [2]$*:*

(L1) If $a_i = \mathrm{inc}_1$*, then* $\ell_{i+1} = 2 \cdot k_i$.
(L2) If $a_i = \mathrm{dec}_1$*, then* $2 \cdot \ell_{i+1} = k_i$.
(L3) If $a_i = \mathrm{zero}_1$*, then* $\ell_{i+1} = k_i$ *is odd.*
(L4) If $a_i = \mathrm{inc}_2$*, then* $\ell_{i+1} = 3 \cdot k_i$.
(L5) If $a_i = \mathrm{dec}_2$*, then* $3 \cdot \ell_{i+1} = k_i$.
(L6) If $a_i = \mathrm{zero}_2$*, then* $\ell_{i+1} = k_i \notin 3\mathbb{N}$.

Proof. Let $\alpha \in [L]$. Then there is $\beta \in L$ with $\beta \sim \alpha$. Since $\beta \in L$, there are $a_0, a_1, \cdots \in \mathrm{Op}$ and $w_0, w_1, \cdots \in \{b, c^5\}^+$ such that

$$\beta = a_0\, w_0\, d\, a_1\, w_1\, d\, a_2\, w_2\, d \cdots$$

and $a_i\, w_i\, d \in L_0$ for all $i \geq 0$. For $i \geq 0$, set

$$k_i = |w_i|_b \text{ and } \ell_{i+1} = \frac{1}{5} \cdot |w_i|_c$$

(since $w_i \in \{b, c^5\}^+$, $|w_i|_c$ is a multiple of 5 such that ℓ_{i+1} is a natural number). From $\alpha \sim \beta$, we get that the ω-words α and β agree on their projections to $\{b, d\}$, to $\mathrm{Op} \cup \{c\}$, and, in case $s = 2$, to $\mathrm{Op} \cup \{d\}$. Hence α satisfies (i-iii). Now let $i \geq 0$ be arbitrary.

(L1) If $a_i = \mathrm{inc}_1$, then $w_i \in (b\, c^5\, c^5)^+$ since $a_i\, w_i\, d \in L_0$. Hence $\ell_{i+1} = \frac{1}{5}|w_i|_c = 2 \cdot \frac{1}{10}|w_i|_c = 2 \cdot |w_i|_b = 2 \cdot k_i$.
(L2) If $a_i = \mathrm{dec}_1$, then $w_i \in (b\, b\, c^5)^+$ since $a_i\, w_i\, d \in L_0$. Hence $2 \cdot \ell_{i+1} = \frac{2}{5}|w_i|_c = |w_i|_b = k_i$.
(L3) If $a_i = \mathrm{inc}_2$, then $w_i \in (b\, c^5\, c^5\, c^5)^+$ since $a_i\, w_i\, d \in L_0$. Hence $\ell_{i+1} = \frac{1}{5}|w_i|_c = 3 \cdot |w_i|_b = 3 \cdot k_i$.
(L4) If $a_i = \mathrm{dec}_2$, then $w_i \in (b\, b\, b\, c^5)^+$ since $a_i\, w_i\, d \in L_0$. Hence $3 \cdot \ell_{i+1} = \frac{3}{5}|w_i|_c = |w_i|_b = k_i$.
(L5) If $a_i = \mathrm{zero}_1$, then $w_i \in (b\, b\, c^5\, c^5)^+\, b\, c^5$ since $a_i\, w_i\, d \in L_0$. Hence $\ell_{i+1} = \frac{1}{5}|w_i|_c = |w_i|_b = k_i$ is odd.
(L6) If $a_i = \mathrm{zero}_2$, then $w_i \in (b\, b\, b\, c^5\, c^5\, c^5)^+ \cdot \{b\, b\, c^5\, c^5, b\, c^5\}$ since $a_i\, w_i\, d \in L_0$. Hence $\ell_{i+1} = \frac{1}{5}|w_i|_c = |w_i|_b = k_i$ is not divisible by three.

Thus, we showed that any element of $[L]$ has the desired properties.

Conversely, suppose $\alpha \in \Sigma^\omega$, $k_0, k_1, \cdots, \ell_1, \ell_2 \cdots \geq 1$ and $a_0, a_1, \cdots \in \mathrm{Op}$ such that (i-iii) as well as (L(L1))-(L(L6)) hold.

For $i \geq 0$, we define words $w_i \in \{b, c^5\}^+$ as follows:

$$w_i = \begin{cases} (b\, c^5\, c^5)^{k_i} & \text{if } a_i = \mathrm{inc}_1 \\ (b\, c^5\, c^5\, c^5)^{k_i} & \text{if } a_i = \mathrm{inc}_2 \\ (b\, b\, c^5)^{\ell_{i+1}} & \text{if } a_i = \mathrm{dec}_1 \\ (b\, b\, b\, c^5)^{\ell_{i+1}} & \text{if } a_i = \mathrm{dec}_2 \\ (b\, c^5)^{k_i} & \text{if } a_i = \mathrm{zero}_1 \text{ or } a_i = \mathrm{zero}_2 \end{cases}$$

Consider the ω-word

$$\beta = a_0\, w_0\, d\, a_1\, w_1\, d \cdots$$

and let $i \geq 0$. If $a_i = \mathrm{inc}_1$, then one easily sees that $a_i w_i d$ belongs to L_0. The same holds for $a_i \in \{\mathrm{inc}_2, \mathrm{dec}_1, \mathrm{dec}_2\}$. Now let $a_i = \mathrm{zero}_1$. Then, by (L(L3)), k_i is odd implying

$a_i\, w_i\, d \in L_0$ (in case $a_i = \mathrm{zero}_2$, one can argue similarly). It follows that β belongs to L. By (L(L1))-(L(L6)) and the above definition of w_i, we get $|w_i|_b = k_i$ and $|w_i|_c = 5 \cdot \ell_{i+1}$ for all $i \geq 0$. Hence $\mathrm{proj}_{\{b,d\}}(\alpha) = \mathrm{proj}_{\{b,d\}}(\beta)$, $\mathrm{proj}_{\mathrm{Op}\cup\{c\}}(\alpha) = \mathrm{proj}_{\mathrm{Op}\cup\{c\}}(\beta)$, and (in case $s = 2$), $\mathrm{proj}_{\mathrm{Op}\cup\{d\}}(\alpha) = \mathrm{proj}_{\mathrm{Op}\cup\{d\}}(\beta)$. Hence $\alpha \sim \beta \in L$ which ensures $\alpha \in [L]$ and therefore the converse implication. ◀

We now come to the main result of this section that relates the set Val of valid words and the regular ω-languages K and L.

▶ **Proposition E.8.** *Let (Σ, I_s) be a standard independence alphabet, i.e., $s \in [2]$, and K and L the regular ω-languages defined by* (6).

The set Val *of valid words equals the projection of the intersection $[K] \cap [L]$, i.e.,* $\mathrm{Val} = \mathrm{proj}_{\mathrm{Op}}([K] \cap [L])$.

Proof. By the above Lemmas E.6 and E.7, an ω-word $\alpha \in \Sigma^\omega$ belongs to $[K] \cap [L]$ if, and only if, it satisfies the characterisation of elements of $[L]$ from Lemma E.7 with $k_0 = 1$ and $k_i = 5 \cdot \ell_i$ for all $i \geq 1$. By induction, it then follows that

$$k_i = 2^{\mathrm{value}_1(\mathrm{proj}_{\mathrm{Op}}(\alpha),i)} \cdot 3^{\mathrm{value}_2(\mathrm{proj}_{\mathrm{Op}}(\alpha),i)} \cdot 5^i$$

for all $i \geq 0$.

Recall further that the pair of counter values (c_1, c_2), i.e., of natural numbers, is encoded as a number $2^{c_1} \cdot 3^{c_2} \cdot m$ with $m \notin 2 \cdot \mathbb{N} \cup 3 \cdot \mathbb{N}$. Thus, an ω-word $\beta \in \mathrm{Op}^\omega$ is valid if, and only if, it equals the projection of some ω-word $\alpha \in [K] \cap [L]$ to Op. In other words, $\mathrm{Val} = \mathrm{proj}_{\mathrm{Op}}([K] \cap [L])$. ◀

E.2.3 Val as complement of the closure of a regular language

In this section, we prove that the set $\mathrm{Val} \subseteq \mathrm{Op}^\omega$ of valid words corresponds to the complement of the closure $[K' \cup L']$ of certain co-deterministic regular ω-languages K' and L' (cf. Prop. E.14).

To this aim, we will construct K' and L' such that their closures are the complements of the closures of K and L, respectively. To appreciate these technical results, note that, in general, the complement of the closure of a regular ω-language is closed, but not necessarily the closure of any regular ω-language.

After these introductory words, we start with a result on the shuffle and deterministic Büchi-automata. In particular, we are interested in ω-languages of the form $X = L^\omega(\mathcal{A}) \sqcup \Delta^\infty$ for some deterministic Büchi-automaton $\mathcal{A}$. To ensure that X is accepted by some deterministic Büchi-automaton, we only consider the case that the alphabet of $\mathcal{A}$ is disjoint from Δ. Then $L^\omega(\mathcal{A}) \sqcup \Delta^\omega$ is accepted by some deterministic Büchi-automaton, but no deterministic Büchi-automaton accepts $L^\omega(\mathcal{A}) \sqcup \Delta^*$ (consider the example $L^\omega(\mathcal{A}) = \{a^\omega\}$ and $\Delta = \{b\}$ [45, Example 4.2] or [40, Example I.6.2]). Nevertheless, we have the following result.

▶ **Lemma E.9.** *Let Γ and Δ be disjoint alphabets and $L \subseteq \Gamma^\omega$ accepted by some deterministic Büchi-automaton. Then there is a deterministic Büchi-automaton accepting $L \sqcup \Delta^\infty$.*

Proof. Let $\mathcal{A} = (Q_\mathcal{A}, \Gamma, S_\mathcal{A}, T_\mathcal{A}, F_\mathcal{A})$ be a deterministic Büchi-automaton accepting L. We construct the deterministic Büchi-automaton $\mathcal{B}$ as follows. The idea is to simulate $\mathcal{A}$ on letters from Γ and to recall the alphabet of the last letter read. More precisely, let $\mathcal{B} = (Q_\mathcal{B}, \Gamma \cup \Delta, S_\mathcal{B}, T_\mathcal{B}, F_\mathcal{B})$ where

- $Q_{\mathcal{B}} = Q_{\mathcal{A}} \times \{\Gamma, \Delta\}$,
- $S_{\mathcal{B}} = S_{\mathcal{A}} \times \{\Gamma\}$,
- $\big((p,x), a, (q,y)\big) \in T_{\mathcal{B}}$ iff
  - $a \in \Gamma$, $(p,a,q) \in T_{\mathcal{A}}$, and $y = \Gamma$ or
  - $a \in \Delta$, $p = q$, and $y = \Delta$, and
- $F_{\mathcal{B}} = F_{\mathcal{A}} \times \{\Gamma\}$.

Note that this Büchi-automaton is deterministic. Then an ω-word $\alpha \in (\Gamma \cup \Delta)^\omega$ is accepted by this deterministic Büchi-automaton iff infinitely many prefixes u of α

- end with a letter from Γ and
- $\mathrm{proj}_\Gamma(u)$ leads $\mathcal{A}$ to some accepting state from $F_{\mathcal{A}}$.

But this is the case iff $\alpha \in L \sqcup \Delta^\infty$. ◀

When, in Lemmas E.11 and E.12 below, we will prove that the (not yet defined) languages K' and L' are co-deterministic regular, we will have to prove their complements to be deterministic regular. It turns out that the following, rather special, language is useful there.

▶ **Lemma E.10.** *Let Σ be some alphabet, $d \in \Sigma$, $\Sigma_d = \Sigma \setminus \{d\}$ and $A, B \subseteq \Sigma_d^+$ regular and disjoint. Then*

$$(\Sigma^* d)^\omega \setminus (Ad)^* \, (Bd) \, (Ad \cup Bd)^\omega$$

is deterministic regular.

Proof. A word $\alpha \in \Sigma^\omega$ belongs to this difference iff $\alpha = u_1 d u_2 d \cdots$ with $u_i \in \Sigma_d^*$ for all $i \geq 1$ such that $u_i \notin A \cup B$ for some $i \geq 1$ or $u_i \in A$ for all $i \geq 1$. Since $\Sigma_d^+ \setminus (A \cup B)$ and A are accepted by deterministic finite automata, one obtains a deterministic Büchi-automaton for this language. ◀

We now come to the definition of the first ω-language K'. As starting point, let $K_0' = \mathrm{Op}\, d\, (cb)^+ \, (c^+ \cup b^+) \subseteq \Sigma^+$. Note that this language is somewhat like the "opposite" of $K_0 = \mathrm{Op}\, d(cb)^+$, more precisely, $[K_0'] = [\mathrm{Op}\, dc^+ b^+] \setminus [K_0]$. Now consider the following ω-languages:

- $K_1' = b(db^+)^\omega \sqcup (\mathrm{Op} \cup \{c\})^\infty$
- $K_2' = \{b,d\}^\infty \sqcup (\mathrm{Op}\, c^+)^\omega$
- $K_3' = \begin{cases} \Sigma^\omega & \text{if } s = 1 \\ (\mathrm{Op}\, d)^\omega \sqcup \{b,c\}^\infty & \text{if } s = 2 \end{cases}$
- $K_4' = b\, K_0^* \, K_0' \, (K_0 \cup K_0')^\omega$

Finally, the ω-language K' is defined by

$$K' = \Sigma^\omega \setminus K_1' \cup \Sigma^\omega \setminus K_2' \cup \Sigma^\omega \setminus K_3' \cup K_4' \,. \tag{7}$$

We next prove that the complement $\Sigma^\omega \setminus K'$ can indeed be accepted by a deterministic Büchi-automaton and that the closure $[K']$ of K' is the complement of the closure $[K]$ of K.

▶ **Lemma E.11.** *Let (Σ, I_s) be a standard independence alphabet, i.e., $s \in [2]$, and K and K' the ω-language defined by (6) and (7), respectively. Then K' is co-deterministic regular and $[K'] = \Sigma^\omega \setminus [K]$.*

Proof. We first show that the complement of K' is deterministic regular. To this aim, note that we have

$$\begin{aligned}\Sigma^\omega \setminus K' &= K_1' \cap K_2' \cap K_3' \cap \Sigma^\omega \setminus K_4'\\ &= K_1' \cap K_2' \cap K_3' \cap \big((\Sigma^* d)^\omega \cap \Sigma^\omega \setminus K_4'\big)\end{aligned}$$

where the second equation holds since $K_1' \subseteq (\Sigma^* d)^\omega$.

First, we verify that the four ω-languages in this intersection all are deterministic regular.

1. There is a deterministic Büchi-automata accepting $b(db^+)^\omega$. Hence, by Lemma E.9, K_1' is deterministic regular. Since also $(\mathrm{Op}\, c^+)^\omega$ and $(\mathrm{Op}\, d)^\omega$ are deterministic regular, Lemma E.9 also proves that K_2' and K_3' are deterministic regular.
2. Next consider the intersection of $(\Sigma^*\, d)^\omega$ and the complement of K_4', i.e., the ω-language $(\Sigma^* d)^\omega \setminus K_4'$. Note that

$$\begin{aligned}K_4' &= b\, K_0^*\, K_0'\, (K_0 \cup K_0')^\omega\\ &= b\, (\mathrm{Op}\, d(cb)^+)^*\, (\mathrm{Op}\, d(cb)^+(c^+ \cup b^+))\, ((\mathrm{Op}\, d(cb)^+) \cup (\mathrm{Op}\, d(cb)^+(c^+ \cup b^+)))^\omega\\ &= b\, \mathrm{Op}\, d\, \big((cb)^+ \mathrm{Op}\, d\big)^*\, \big((cb)^+(c^+ \cup b^+)\mathrm{Op}\, d\big)\big((cb)^+(c^* \cup b^*)\mathrm{Op}\, d\big)^\omega\\ &= b\, \mathrm{Op}\, d(Ad)^*(Bd)(Ad \cup Bd)^\omega\end{aligned}$$

   where $A = (cb)^+\mathrm{Op}$ and $B = (cb)^+(c^+ \cup b^+)\mathrm{Op}$. Since the two languages A and B are regular and disjoint, Lemma E.10 provides a deterministic Büchi-automaton $\mathcal{B}$ with

$$L^\omega(\mathcal{B}) = (\Sigma^* d)^\omega \setminus (Ad)^*(Bd)(Ad \cup Bd)^\omega\,.$$

   Hence the ω-language

$$\Sigma^\omega \setminus b\, \mathrm{Op}\, d\, \Sigma^\omega \cup b\, \mathrm{Op}\, d \cdot L^\omega(\mathcal{B})$$

   is deterministic regular. Finally note that this language equals $(\Sigma^* d)^\omega \setminus K_4'$.

Since the class of deterministic regular ω-languages is closed under finite intersections by Choueka's flag construction [8] (cf. also proof of [45, Lemma 1.2] or [40, Prop. I.6.4]), it follows that $\Sigma^\omega \setminus K'$ is deterministic regular. Hence K' is co-deterministic regular, i.e., we proved the first claim.

It remains to be shown that the closure $[K']$ of K' is the complement $\Sigma^\omega \setminus [K]$ of the closure of K.

To prove the inclusion $[K'] \subseteq \Sigma^\omega \setminus [K]$, it suffices to verify $K' \subseteq \Sigma^\omega \setminus [K]$ since the complement of $[K]$ is closed. But this is equivalent to $[K] \subseteq \Sigma^\omega \setminus K'$. So let $\alpha \in [K]$. Then Lemma E.6 yields $\mathrm{proj}_{\{b,d\}}(\alpha) \in b\, (db^+)^\omega$ and $\mathrm{proj}_{\mathrm{Op} \cup \{c\}}(\alpha) \in (\mathrm{Op}\, c^+)^\omega$. The definition of the ω-languages K_1' and K_2' yields $\alpha \in K_1' \cap K_2'$ or, equivalently, $\alpha \notin \Sigma^\omega \setminus K_1' \cup \Sigma^\omega \setminus K_2'$. If $s = 1$, we definitely get $\alpha \in \Sigma^\omega = K_3'$. If $s = 2$, Lemma E.6 yields $\mathrm{proj}_{\mathrm{Op} \cup \{d\}}(\alpha) \in (\mathrm{Op}\, d)^\omega$ and therefore $\alpha \in K_3'$ also in this case. Thus, in any case, $\alpha \notin \Sigma^\omega \setminus K_3'$. Finally, from $\alpha \in [K]$ we obtain an ω-word $\beta \in K$ with $\alpha \sim \beta$. Note that $K \cap K_4' = \emptyset$ and that elements of $K \cup K_4' = b(K_0 \cup K_0')^\omega$ are mutually non-equivalent. Hence $\alpha \sim \beta \in K$ implies $\alpha \notin K_4'$. In summary, we got that α is neither in the complements of K_1', K_2', or K_3', nor in the ω-language K_4', i.e., $\alpha \notin K'$. Thus, indeed, $[K] \subseteq \Sigma^\omega \setminus K'$ and therefore $[K'] \subseteq \Sigma^\omega \setminus [K]$.

We now have to verify the converse inclusion $[K'] \supseteq \Sigma^\omega \setminus [K]$ and therefore $[K] \cup [K'] \supseteq \Sigma^\omega$. So let $\alpha \in \Sigma^\omega$ be an arbitrary ω-word. Suppose $\alpha \notin K_i'$ for some $i \in [3]$. Then $\alpha \in \Sigma^\omega \setminus K_i \subseteq K' \subseteq [K']$. So, from now on, assume $\alpha \in K_1' \cap K_2' \cap K_3'$. Then there are positive natural numbers $k_0, k_1, \dots, \ell_1, \ell_2 \dots \geq 1$ with $k_0 = 1$ and letters $a_0, a_1, \dots \in \mathrm{Op}$ such that

- $\mathrm{proj}_{\{b,d\}}(\alpha) = b^{k_0}\, d\, b^{k_1}\, d\, b^{k_2} \cdots$,
- $\mathrm{proj}_{\mathrm{Op}\cup\{c\}}(\alpha) = a_0\, c^{\ell_1}\, a_1\, c^{\ell_2} \cdots$, and,
- if $s = 2$, also $\mathrm{proj}_{\mathrm{Op}\cup\{d\}}(\alpha) \in (\mathrm{Op}\, d)^\omega$.

For $i \geq 1$ define

$$m_i = \min(k_i, \ell_i) \text{ and } w_i = \begin{cases} b^{k_i-\ell_i} & \text{if } k_i \geq \ell_i \\ c^{\ell_i-k_i} & \text{otherwise} \end{cases}$$

which ensures $a_{i-1}\, d\, (cb)^{m_i}\, w_i \in K_0$ iff $k_i = \ell_i$. Furthermore, let

$$\beta = b^{k_0}\, a_0\, d\, (cb)^{m_0} w_0\, a_1\, d\, (cb)^{m_1} w_1\, a_2\, d\, (cb)^{m_2} w_2 \cdots$$

Looking at the projections of α and β to the alphabets $\{b, d\}$, $\mathrm{Op}\cup\{c\}$, and (if $s = 2$) $\mathrm{Op}\cup\{d\}$, we obtain $\alpha \sim \beta$. Furthermore, if $w_i = \varepsilon$ for all $i \geq 1$, the ω-word β belongs to K, otherwise, it belongs to K'. Thus, in any case $\alpha \sim \beta \in K \cup K'$ and therefore $[K'] \supseteq \Sigma^\omega \setminus [K]$. ◀

Recall the definition of the language L_0. The following language L'_0 is similar, but has disjoint closure:

$$\begin{aligned} L'_0 = \;& \mathrm{inc}_1\, (b\, c^5\, c^5)^*\, \big(b\, c^{<10} \cup b^+ \cup c^+\big)\, d \\ &\cup \mathrm{inc}_2\, (b\, c^5\, c^5\, c^5)^*\, \big(b\, c^{<15} \cup b^+ \cup c^+\big)\, d \\ &\cup \mathrm{dec}_1\, (b\, b\, c^5)^*\, (bbc^{<5} \cup b\, c^* \cup c^+)\, d \\ &\cup \mathrm{dec}_2\, (b\, b\, b\, c^5)^*\, \big(b\, b\, b\, c^{<5} \cup b\, b\, c^* \cup b\, c^* \cup c^+\big)\, d \\ &\cup \mathrm{zero}_1\, \big\langle (b\, c^5)^*\, (b\, c^{<5} \cup b^+ \cup c^+) \cup c^*\, (b\, c^*\, b\, c^*)^* \big\rangle\, d \\ &\cup \mathrm{zero}_2\, \big\langle (b\, c^5)^*\, (b\, c^{<5} \cup b^+ \cup c^+) \cup c^*\, (b\, c^*\, b\, c^*\, b\, c^*)^* \big\rangle\, d \end{aligned}$$

Let $w \in \{b, c\}^+$. Then $\mathrm{inc}_1\, w\, d \in [L_0]$ iff $10 \cdot |w|_b = |w|_c$, and $\mathrm{inc}_1\, w\, d \in [L'_0]$ iff $10 \cdot |w|_b \neq |w|_c$. Similarly, $\mathrm{zero}_1\, w\, d \in [L_0]$ iff $5 \cdot |w|_b = |w|_c$ and $|w|_b$ is odd. On the other hand, $\mathrm{zero}_1\, w\, d \in [L'_0]$ iff $5 \cdot |w|_b \neq |w|_c$ or $|w|_b$ is even. Since similar observations hold for all letters from Op, the closures $[L_0]$ and $[L'_0]$ are disjoint and $[\mathrm{Op}\, \{b, c\}^+\, d] = [L_0] \cup [L'_0]$.

Now consider the following ω-languages:

- $L'_1 = b(db^+)^\omega \sqcup (\mathrm{Op} \cup \{c\})^\infty$
- $L'_2 = \{b, d\}^\infty \sqcup (\mathrm{Op}\, (c^5)^+)^\omega$
- $L'_3 = \begin{cases} \Sigma^\omega & \text{if } s = 1 \\ (\mathrm{Op}\, d)^\omega \sqcup \{b, c\}^\infty & \text{if } s = 2 \end{cases}$
- $L'_4 = L_0^*\, L'_0\, (L_0 \cup L'_0)^\omega$

Finally, the ω-language L' is defined, analogously to K', by

$$L' = \Sigma^\omega \setminus L'_1 \cup \Sigma^\omega \setminus L'_2 \cup \Sigma^\omega \setminus L'_3 \cup L'_4 \,. \tag{8}$$

Similarly to the case of K and K', the complement $\Sigma^\omega \setminus L'$ is accepted by some deterministic Büchi-automaton and the closure $[L']$ of L' is the complement of the closure of L. The proof of these facts is very similar to the proof of Lemma E.11 and is therefore placed in the appendix.

▶ **Lemma E.12.** *Let (Σ, I_s) be a standard independence alphabet, i.e., $s \in [2]$, and L and L' the ω-language defined by (6) and (8), respectively. Then L' is co-deterministic regular and $[L'] = \Sigma^\omega \setminus [L]$.*

Proof of Lemma E.12. We first show that the complement of L' is deterministic regular. To this aim, note that we have

$$\begin{aligned}\Sigma^\omega \setminus L' &= L'_1 \cap L'_2 \cap L'_3 \cap \Sigma^\omega \setminus L'_4 \\ &= L'_1 \cap L'_2 \cap L'_3 \cap \big((\Sigma^* d)^\omega \cap \Sigma^\omega \setminus L'_4\big)\end{aligned}$$

where the second equation holds since $L'_1 \subseteq (\Sigma^* d)^\omega$.

First, we verify that the four ω-languages in this intersection all are deterministic regular.

1. There is a deterministic Büchi-automata accepting $b(db^+)^\omega$. Hence, by Lemma E.9, L'_1 is deterministic regular. Since also $(\mathrm{Op}\,(c^5)^+)^\omega$ and $(\mathrm{Op}\,d)^\omega$ are deterministic regular, Lemma E.9 also proves that L'_2 and L'_3 are deterministic regular.
2. Next consider the intersection of $(\Sigma^*\, d)^\omega$ and the complement of L'_4, i.e., the ω-language $(\Sigma^* d)^\omega \setminus L'_4$. Define the residual languages

   $$A = \{u \mid ud \in L_0\} \text{ and } B = \{u \mid ud \in L'_0\}\,.$$

   Since any word from L_0 or L'_0 ends with an occurrence of d, we get $L_0 = Ad$ and $L'_0 = Bd$. Consequently, we obtain

   $$\begin{aligned}L'_4 &= L_0^*\, L'_0\, (L_0 \cup L'_0)^\omega \\ &= (Ad)^*\, Bd\, (Ad \cup Bd)^\omega\,.\end{aligned}$$

   Since the two languages A and B are regular, disjoint, and have no occurrences of the letter d, Lemma E.10 provides a deterministic Büchi-automaton $\mathcal{B}$ with

   $$L^\omega(\mathcal{B}) = (\Sigma^* d)^\omega \setminus (Ad)^*(Bd)(Ad \cup Bd)^\omega = (\Sigma^* d)^\omega \setminus L'_4\,.$$

Since the class of deterministic regular ω-languages is closed under finite intersections by Choueka's flag construction [8] (cf. also proof of [45, Lemma 1.2] or [40, Prop. I.6.4]), it follows that $\Sigma^\omega \setminus L'$ is deterministic regular. Hence L' is co-deterministic regular, i.e., we proved the first claim.

It remains to be shown that the closure $[L']$ of L' is the complement $\Sigma^\omega \setminus [L]$ of the closure of L.

To prove the inclusion $[L'] \subseteq \Sigma^\omega \setminus [L]$, it suffices to verify $L' \subseteq \Sigma^\omega \setminus [L]$ since the complement of $[L]$ is closed. Thus, we have to prove that $[L] \subseteq \Sigma^\omega \setminus L'$. So let $\alpha \in [L]$. Then Lemma E.7 yields $\mathrm{proj}_{\{b,d\}}(\alpha) \in b\,(db^+)^\omega$ and $\mathrm{proj}_{\mathrm{Op}\cup\{c\}}(\alpha) \in (\mathrm{Op}\,(c^5)^+)^\omega$. The definition of the ω-languages L'_1 and L'_2 yields $\alpha \in L'_1 \cap L'_2$ or, equivalently, $\alpha \notin \Sigma^\omega \setminus L'_1 \cup \Sigma^\omega \setminus L'_2$. If $s = 1$, we definitely get $\alpha \in \Sigma^\omega = L'_3$. If $s = 2$, Lemma E.7 yields $\mathrm{proj}_{\mathrm{Op}\cup\{d\}}(\alpha) \in (\mathrm{Op}\,d)^\omega$ and therefore $\alpha \in L'_3$ also in this case. Thus, in any case, $\alpha \notin \Sigma^\omega \setminus L'_3$. Finally, from $\alpha \in [L]$ we obtain an ω-word $\beta \in L$ with $\alpha \sim \beta$. Note that $L \cap L'_4 = \emptyset$ and that elements of $L \cup L'_4 = (L_0 \cup L'_0)^\omega$ are mutually non-equivalent. Hence $\alpha \sim \beta \in L$ implies $\alpha \notin L'_4$. In summary, we got that α is neither in the complements of L'_1, L'_2, or L'_3, nor in the ω-language L'_4, i.e., $\alpha \notin L'$. Thus, indeed, $[L'] \subseteq \Sigma^\omega \setminus [L]$.

We now have to verify the converse inclusion $[L'] \supseteq \Sigma^\omega \setminus [L]$ and therefore $[L] \cup [L'] \supseteq \Sigma^\omega$. So let $\alpha \in \Sigma^\omega$ be an arbitrary ω-word. Suppose $\alpha \notin L'_i$ for some $i \in [3]$. Then $\alpha \in \Sigma^\omega \setminus L_i \subseteq L' \subseteq [L']$. So, from now on, assume $\alpha \in L'_1 \cap L'_2 \cap L'_3$. Then there are positive natural numbers $k_0, k_1, \dots, \ell_1, \ell_2 \dots \geq 1$ with $k_0 = 1$ and letters $a_0, a_1, \dots \in \mathrm{Op}$ such that

- $\mathrm{proj}_{\{b,d\}}(\alpha) = b^{k_0}\, d\, b^{k_1}\, d\, b^{k_2} \cdots$,
- $\mathrm{proj}_{\mathrm{Op}\cup\{c\}}(\alpha) = a_0\, (c^5)^{\ell_1}\, a_1\, (c^5)^{\ell_2} \cdots$, and,

- if $s = 2$, also $\mathrm{proj}_{\mathrm{Op}\cup\{d\}}(\alpha) \in (\mathrm{Op}\, d)^\omega$.

Let

$$\beta = a_0\, b^{k_0}\, (c^5)^{\ell_1}\, d\, a_1\, b^{k_1}\, (c^5)^{\ell_2}\, d\, a_2\, b^{k_2}\, (c^5)^{\ell_3}\, d \cdots$$

Looking at the projections of α and β to the alphabets $\{b, d\}$, $\mathrm{Op} \cup \{c\}$, and (if $s = 2$), to $\mathrm{Op} \cup \{d\}$, we obtain $\alpha \sim \beta$. Note that any of the factors $w_i := a_i\, b^{k_i}\, (c^5)^{\ell_{i+1}}\, d$ belongs to $\mathrm{Op}\,\{b, c\}^+\, d$ and therefore to $[L_0] \cup [L_0']$. Suppose $a_i\, b^{k_i}\, (c^5)^{\ell_{i+1}}\, d \in [L_0]$ for all $i \geq 0$. Then $\beta \in [L_0]^\omega \subseteq [L']$. If, alternatively, there is $i \in \mathbb{N}$ such that $a_i\, b^{k_i}\, (c^5)^{\ell_{i+1}}\, d \notin [L_0]$, then $a_i\, b^{k_i}\, (c^5)^{\ell_{i+1}}\, d \in [L_0']$. Thus, in any case $\alpha \sim \beta \in [L] \cup [L']$ and therefore $[L'] \supseteq \Sigma^\omega \setminus [L]$. ◀

▶ **Lemma E.13.** *Given a Büchi-automaton $\mathcal{A}$ over Γ, it is decidable whether there is an equivalent deterministic Büchi-automaton $\mathcal{B}$.*

Since, from $\mathcal{A}$, one can also construct a Büchi-automaton $\mathcal{A}'$ accepting the complement of $L^\omega(\mathcal{A})$, it is decidable whether $\mathcal{A}$ is co-deterministic.

Proof. For $\alpha \in \Gamma^\omega$, let $\mathrm{pref}(\alpha) \subseteq \Gamma^*$ denote the set of finite prefixes of α. For a language $Y \subseteq \Gamma^*$, let $\overrightarrow{Y}$ denote the set of all words $\alpha \in \Gamma^\omega$ with infinitely many prefixes in Y. We use that a $L^\omega(\mathcal{A})$ is deterministic regular if, and only if, there exists a regular language Y with $\overrightarrow{Y} = L^\omega(\mathcal{A})$ [40, Cor. I.6.3].

Now let $\mathcal{A}$ be a Büchi-automaton. By Büchi's theorem [7], one can construct an MSO formula $\varphi(X)$ with a single free set variable X such that, for all $\alpha \in \Gamma^\omega$,

$$\Gamma^* \models \varphi(\mathrm{pref}(\alpha))\,. \iff \alpha \in L^\omega(\mathcal{A})$$

Consider the closed formula

$$\psi = \exists Y\colon \forall B \text{ branch}\colon \varphi(B) \leftrightarrow \forall b \in B \exists y \in Y\colon b \leq y\,.$$

Here, $B \subseteq \Gamma^*$ is a *branch* if it satisfies $\forall a, b \exists c\colon a \leq b \in B \to a \in B \wedge b < c \in B$, i.e., B is a language of the form $\mathrm{pref}(\alpha)$ for some $\alpha \in \Gamma^\omega$. Hence the formula ψ expresses that there is a set $Y \subseteq \Gamma^*$ with $\overrightarrow{Y} = L^\omega(\mathcal{A})$.

We claim that $L^\omega(\mathcal{A})$ is deterministic regular iff this formula ψ holds in Γ^*. First, suppose $L^\omega(\mathcal{A})$ is deterministic regular. Then there exists a regular language $Y \subseteq \Gamma^*$ with $L^\omega(\mathcal{A}) = \overrightarrow{Y}$, hence the formula ψ holds. Conversely, suppose the formula ψ holds, i.e., there is a (not necessarily regular) language Y with $\overrightarrow{Y} = L^\omega(\mathcal{A})$. Since Y is a set satisfying the part of ψ following the outermost existential quantifier, by [42, Thm. 23], there is also a regular language Y with $\overrightarrow{Y} = L^\omega(\mathcal{A})$. The regularity of Y implies that $L^\omega(\mathcal{A})$ is deterministic regular. ◀

We now come to the main result of this section: the set of valid words corresponds to the complement of the closure of some regular language.

▶ **Proposition E.14.** *Let (Σ, I_s) be a standard independence alphabet, i.e., $s \in [2]$, and K' and L' the regular ω-languages defined by (7) and (8), respectively.*

The set Val *of valid words equals the projection of the complement of $[K' \cup L']$, i.e.,* $\mathrm{Val} = \mathrm{proj}_{\mathrm{Op}}(\Sigma^\omega \setminus [K' \cup L'])$.

Proof. We have

$$\begin{aligned}
[K' \cup L'] &= [K'] \cup [L'] \\
&= \Sigma^\omega \setminus [K] \cup \Sigma^\omega \setminus [L] && \text{by Lemmas E.11 and E.12} \\
&= \Sigma^\omega \setminus \big([K] \cap [L]\big)
\end{aligned}$$

and therefore

$$\begin{aligned}\mathrm{proj}_{\mathrm{Op}}(\Sigma^\omega \setminus [K' \cup L']) &= \mathrm{proj}_{\mathrm{Op}}([K] \cap [L]) \\ &= \mathrm{Val} && \text{by Prop. E.8}\,.\end{aligned}$$

◀

E.3 Lower bounds

In this section, we will prove analytical lower bounds for the problems $\mathrm{GenP}_\omega(\mathrm{IndA}, \Phi)$. As it turns out, the lower bounds can even be shown when restricting to deterministic regular and co-deterministic regular languages. We therefore introduce the following restricted versions of our problems:

▶ **Definition E.15.** *Let* $\mathrm{GenP} \in \{\mathrm{EmptP}, \mathrm{RatP}, \mathrm{RegP}\}$*,* $\mathbb{A} \subseteq \mathrm{IndA}$ *a set of independence alphabets and* $\Phi \subseteq \mathrm{BF}$ *a set of Boolean functions.*

- $\mathrm{GenP}_\omega^{det}(\mathbb{A}, \Phi)$ *is the set of all tuples* $((\Gamma, I), \varphi, \mathcal{A}_1, \dots, \mathcal{A}_n)$ *from* $\mathrm{GenP}_\omega(\mathbb{A}, \Phi)$ *where the Büchi-automata* $\mathcal{A}_i$ *are deterministic.*
- $\mathrm{GenP}_\omega^{co-det}(\mathbb{A}, \Phi)$ *is the set of all tuples* $((\Gamma, I), \varphi, \mathcal{A}_1, \dots, \mathcal{A}_n)$ *from* $\mathrm{GenP}_\omega(\mathbb{A}, \Phi)$ *where the languages* $L^\omega(\mathcal{A}_i)$ *are co-deterministic.*[3]

E.3.1 Standard independence alphabets

All lower bounds will be shown by a reduction from the problem in Corollary E.3, i.e., from the emptiness problem for $L(\mathcal{M}) \cap \mathrm{Val}$ where $\mathcal{M}$ is a deterministic Büchi-automaton over Op. The following lemma prepares these reductions by relating the said emptiness to closures of certain regular ω-languages.

▶ **Lemma E.16.** *Let* $\mathcal{M}$ *be some deterministic Büchi-automaton over* Op *and* $H = L^\omega(\mathcal{M}) \sqcup \{b, c, d\}^\infty \subseteq \Sigma^\omega$ *denote the set of* ω*-words whose projection to* Op *is accepted by* $\mathcal{M}$*. Furthermore, let* (Σ, I_s) *be a standard independence alphabet, i.e.,* $s \in [2]$*. Finally, let* K*,* L*,* K'*, and* L' *be the* ω*-languages defined by* (6)*,* (7)*, and* (8)*, respectively.*

Then

(1) $[H \cap K] \cap [L] = \Sigma^\omega \setminus [\Sigma^\omega \setminus H \cup K' \cup L']$ *and*
(2) the following are equivalent:
  (i) $L^\omega(\mathcal{M}) \cap \mathrm{Val} = \emptyset$*.*
  (ii) $[H \cap K] \cap [L] = \emptyset$*.*
  (iii) $[H \cap K] \cap [L]$ *is regular.*
  (iv) There is a regular language $M \subseteq \Sigma^\omega$ *with* $[M] = [H \cap K] \cap [L]$*.*

Proof. Before we start the actual proof, we show

$$[H \cap X] = H \cap [X]\,. \tag{9}$$

for all $X \subseteq \Sigma^\omega$ as this equation will be used several times. To this aim, first note that H is closed, i.e., $H = [H]$. Hence, for any ω-word $\alpha \in \Sigma^\omega$, we have $\alpha \in [H \cap X]$ iff $\alpha \in \big[[H] \cap X\big]$.

[3] The author is convinced that the set of co-deterministic Büchi-automata is decidable, but he could not find a proof of this fact in the literature. To be on the safe side, such a proof can be found in the appendix as Lemma E.13.

But this is the case iff there is $\beta \sim \alpha$ with $\beta \in [H]$ and $\beta \in X$. This, in turn, is equivalent to $\alpha \in [H] \cap [X]$. Since $[H] = H$, we therefore get Equation (9).

Next, we verify (1). To this aim, first note that

$$\begin{aligned}
\Sigma^\omega \setminus [H \cap K] &= \Sigma^\omega \setminus \big(H \cap [K]\big) && \text{by Eq. (9)}\\
&= \Sigma^\omega \setminus H \cup \Sigma^\omega \setminus [K] \\
&= \Sigma^\omega \setminus H \cup [K'] && \text{by Lemma E.11}\\
&= [\Sigma^\omega \setminus H \cup K'] && \text{since } \Sigma^\omega \setminus H \text{ is closed}\,.
\end{aligned}$$

As a consequence,

$$\begin{aligned}
\Sigma^\omega \setminus \big([H \cap K] \cap [L]\big) &= \Sigma^\omega \setminus [H \cap K] \cup \Sigma^\omega \setminus [L] \\
&= [\Sigma^\omega \setminus H \cup K'] \cup [L'] && \text{by above calculations and Lemma E.12}\\
&= [\Sigma^\omega \setminus H \cup K' \cup L']\,.
\end{aligned}$$

Thus, indeed, claim (1) holds.

To show the equivalence of (i) and (ii), we demonstrate that the projection of $[H \cap K] \cap [L] = H \cap [K] \cap [L]$ equals $L^\omega(\mathcal{M}) \cap \mathrm{Val}$. For the first inclusion, let $f \in \mathrm{proj}_{\mathrm{Op}}(H \cap [K] \cap [L])$. Then there is $\alpha \in H \cap [K] \cap [L]$ with $f = \mathrm{proj}_{\mathrm{Op}}(\alpha)$. From $\alpha \in H$, we get $\mathrm{proj}_{\mathrm{Op}}(\alpha) \in L^\omega(\mathcal{M})$. From $\alpha \in [K] \cap [L]$, Proposition E.8 implies $\mathrm{proj}_{\mathrm{Op}}(\alpha) \in \mathrm{Val}$. Now $f = \mathrm{proj}_{\mathrm{Op}}(\alpha)$ implies $f \in L^\omega(\mathcal{M}) \cap \mathrm{Val}$. For the inverse inclusion, suppose $f \in L^\omega(\mathcal{M}) \cap \mathrm{Val}$. Since $f \in \mathrm{Val}$, there exists $\alpha \in [K] \cap [L]$ with $f = \mathrm{proj}_{\mathrm{Op}}(\alpha)$. Since f is accepted by $\mathcal{M}$, this implies $\alpha \in H$. Hence $\alpha \in H \cap [K] \cap [L]$ implying $f \in \mathrm{proj}_{\mathrm{Op}}(H \cap [K] \cap [L])$.

Thus, indeed, $\mathrm{proj}_{\mathrm{Op}}([H \cap K] \cap [L]) = L^\omega(\mathcal{M}) \cap \mathrm{Val}$ implying the equivalence of (i) and (ii).

The implication (ii)$\Rightarrow$(iii) is trivial since $\emptyset$ is regular.

For the implication (iii)$\Rightarrow$(iv), suppose $M := [H \cap K] \cap [L]$ is regular. As the intersection of two closed sets, M closed. Hence $[H \cap K] \cap [L] = M = [M]$ which ensures (iv).

The final implication (iv)$\Rightarrow$(ii) is shown by contradiction. So suppose (iv) holds, but $[H \cap K] \cap [L] \neq \emptyset$. In other words, there is a non-empty regular language $M \subseteq \Gamma^\omega$ with $[H \cap K] \cap [L] = [M]$. Since M is regular, there are nonempty words $u, v \in \Gamma^+$ with $uv^\omega \in M$. But then $uv^\omega \in M \subseteq [M] = [H \cap K] \cap [L] \subseteq [K] \cap [L]$. From Lemma E.6, we obtain $\mathrm{proj}_{\{b,d\}}(uv^\omega) = b^{k_0}\, d\, b^{k_1}\, d\, b^{k_3} \cdots$ and $\mathrm{proj}_{\mathrm{Op} \cup \{c\}}(uv^\omega) = a_0 c^{5 \cdot \ell_1} a_1 c^{5 \cdot \ell_2} \cdots$ with $k_0 = 1$ and $k_i = 5 \cdot \ell_i$ for all $i > 0$. From $uv^\omega \in [L]$ and Lemma E.7, we obtain $\ell_{i+1} \geq \frac{1}{3} k_i = \frac{5}{3} \cdot \ell_i > \ell_i$ for all $i \geq 0$. But this contradicts that $\mathrm{proj}_{\mathrm{Op} \cup \{c\}}(uv^\omega)$ is ultimately periodic. ◀

We can now use the above lemma to show lower bounds for all the problems in consideration over the standard independence alphabets.

▶ **Corollary E.17.** *Let (Σ, I_s) be a standard independence alphabet, i.e., $s \in [2]$. The following problems are Π^1_1-hard:*

(1) The regularity problems $\mathrm{RegP}^{det}_\omega((\Sigma, I_s), \wedge)$ and $\mathrm{RegP}^{co-det}_\omega((\Sigma, I_s), \mathrm{Id})$.
(2) The emptiness problems $\mathrm{EmptP}^{det}_\omega((\Sigma, I_s), \wedge)$ and $\mathrm{EmptP}^{co-det}_\omega((\Sigma, I_s), \neg)$.
(3) The rationality problems $\mathrm{RatP}^{det}_\omega((\Sigma, I_s), \wedge)$ and $\mathrm{RatP}^{co-det}_\omega((\Sigma, I_s), \neg)$.

In particular, all three problems for deterministic Büchi-automata are hard for the conjunction function $\wedge$. For co-deterministic Büchi-automata, the regularity problem is already Π^1_1-hard for the identity function, the other two problems are hard for the complementation function $\neg$ (and easily decidable for the identity function).

Proof. All these hardness results are derived from Corollary E.3. So let $\mathcal{M}$ be a deterministic Büchi-automaton over Op. Let H, K, L, K', and L' as in Lemma E.16.

From $\mathcal{M}$, one can construct a deterministic Büchi-automaton $\mathcal{A}_H$ with $L^\omega(\mathcal{A}_H) = L^\omega(\mathcal{M}) \sqcup \{b,c,d\}^\infty$ (cf. Lemma E.9). One easily sees that there are deterministic Büchi-automata $\mathcal{A}_K$ and $\mathcal{A}_L$ such that $K = L^\omega(\mathcal{A}_K)$ and $L = L^\omega(\mathcal{A}_L)$.

From Lemmas E.11 and E.12, we obtain deterministic Büchi-automata $\mathcal{B}_{K'}$ and $\mathcal{B}_{L'}$ such that $K' = \Sigma^\omega \setminus L^\omega(\mathcal{B}_{K'})$ and $L' = \Sigma^\omega \setminus L^\omega(\mathcal{B}_{L'})$.

From the deterministic automata $\mathcal{A}_H$ and $\mathcal{A}_K$, one can construct a deterministic Büchi-automaton $\mathcal{A}_{HK}$ with $L^\omega(\mathcal{A}_{HK}) = H \cap K$. Lemma E.16(2) translates into the equivalence of the following statements:

(i) $L^\omega(\mathcal{M}) \cap \mathrm{Val} = \emptyset$
(ii) $[L^\omega(\mathcal{A}_{HK})] \cap [L^\omega(\mathcal{A}_L)] = \emptyset$, i.e., $((\Sigma, I_s), \wedge, \mathcal{A}_{HK}, \mathcal{A}_L) \in \mathrm{EmptP}_\omega^{det}((\Sigma, I_s), \wedge)$
(iii) $[L^\omega(\mathcal{A}_{HK})] \cap [L^\omega(\mathcal{A}_L)]$ is regular, i.e., $((\Sigma, I_s), \wedge, \mathcal{A}_{HK}, \mathcal{A}_L) \in \mathrm{RegP}_\omega^{det}((\Sigma, I_s), \wedge)$.
(iv) There exists $M \subseteq \Sigma^\omega$ regular such that $[M] = [L^\omega(\mathcal{A}_{HK})] \cap [L^\omega(\mathcal{A}_L)]$, i.e., $((\Sigma, I_s), \wedge, \mathcal{A}_{HK}, \mathcal{A}_L) \in \mathrm{RatP}_\omega^{det}((\Sigma, I_s), \wedge)$.

Thus, the mapping $\mathcal{M} \mapsto ((\Sigma, I_s), \wedge, \mathcal{A}_{HK}, \mathcal{A}_L)$ is a reduction from the Π_1^1-complete problem in Cor. E.3 to all the problems $\mathrm{GenP}_\omega^{det}((\Sigma, I_s), \wedge)$ for $\mathrm{GenP} \in \{\mathrm{RegP}, \mathrm{EmptP}, \mathrm{RatP}\}$.

To also verify the remaining claims, note the following:

$$\begin{aligned}\Sigma^\omega \setminus (\Sigma^\omega \setminus H \cup K' \cup L') &= H \cap \Sigma^\omega \setminus K' \cap \Sigma^\omega \setminus L' \\ &= L^\omega(\mathcal{A}_H) \cap L^\omega(\mathcal{B}_{K'}) \cap L^\omega(\mathcal{B}_{L'})\,.\end{aligned}$$

Since all these Büchi-automata are deterministic, one can construct a Büchi-automaton $\mathcal{A}$ with

$$L^\omega(\mathcal{A}) = \Sigma^\omega \setminus H \cup K' \cup L'\,.$$

This automaton is co-deterministic since the complement of $L^\omega(\mathcal{A})$ is deterministic regular. From Lemma E.16, we obtain $\Sigma^\omega \setminus [L^\omega(\mathcal{A})] = [H \cap K] \cap [L]$ implying that also the following statements are equivalent to statement (i) above:

(v) $\Sigma^\omega \setminus [L^\omega(\mathcal{A})] = \emptyset$, i.e., $((\Sigma, I_s), \neg, \mathcal{A}) \in \mathrm{EmptP}_\omega^{co-det}((\Sigma, I_s), \neg)$
(vi) $\Sigma^\omega \setminus [L^\omega(\mathcal{A})]$ is regular.
(vii) There exists $M \subseteq \Sigma^\omega$ regular such that $[M] = \Sigma^\omega \setminus [L^\omega(\mathcal{A})]$, i.e., $((\Sigma, I_s), \neg, \mathcal{A}) \in \mathrm{RatP}_\omega^{co-det}((\Sigma, I_s), \neg)$.

Thus, the mapping $\mathcal{M} \mapsto ((\Sigma, I_s), \neg, \mathcal{A})$ is a reduction from the Π_1^1-complete problem in Cor. E.3 to the problems $\mathrm{EmptP}_\omega^{co-det}((\Sigma, I_s), \neg)$ and $\mathrm{RatP}_\omega^{co-det}((\Sigma, I_s), \neg)$.

Finally note that statement (vi) is equivalent to the regularity of $[L^\omega(\mathcal{A})$. Hence the mapping $\mathcal{M} \mapsto ((\Sigma, I_s), \mathrm{Id}, \mathcal{A})$ reduces the Π_1^1-complete problem in Cor. E.3 to the regularity problem $\mathrm{RegP}_\omega^{co-det}((\Sigma, I_s), \mathrm{Id})$. ◀

E.3.2 Non-diagonal independence alphabets

From the hardness for the standard independence alphabets, we now infer the same hardness results for each and every non-diagonal independence alphabet (Γ, I). For this, we first prove two preparatory lemmas that are then used to reduce the problems on the standard independence alphabets to arbitrary non-diagonal independence alphabets.

In order to simplify the wording of the following lemmas, we collect repeating assumptions as follows.

▶ **Stipulation E.18.**
- (Γ, I) is a non-diagonal independence alphabet and $a, b, c, d \in \Gamma$ with $(a, b), (b, c), (c, d) \in I$ and $(a, c), (b, d) \notin I$; it is possible that $(d, a) \in I$, but also that $(d, a) \notin I$.
- $s \in [2]$ with $s = 1$ iff $(a, d) \in I$, i.e., (Σ, I_s) is the standard independence alphabet with $(a, d) \in I$ iff $\mathrm{Op} \times \{d\} \subseteq I_s$.
- $f : \Sigma \to \{cac, caac, \ldots, ca^6c, b, c, d\}$ is a bijection with $f(b) = b$, $f(c) = c$, and $f(d) = d$.

Note that this stipulation also ensures $f(o) \in \{cac, caac, \ldots, ca^6c\}$ for all $o \in \mathrm{Op}$ since f is required to be a bijection. We denote the letterwise application of f to words from Σ^∞ by f as well.

E.3.2.1 Preparation

Since we want to apply Lemma B.4, we first ensure its applicability.

▶ **Lemma E.19.** *Let (Γ, I), a, b, c, d, s, and f satisfy Stipulation E.18. The mapping $f : \Sigma^\omega \to \Gamma^\omega$ is injective and, for $\alpha, \beta \in \Sigma^\omega$, we have $\alpha \sim \beta \iff f(\alpha) \sim f(\beta)$, i.e., f satisfies Equivalence* (2) *from Lemma B.4.*

Proof. Let $x_i, y_i \in \Sigma$ with $f(x_1x_2\cdots) = f(y_1y_2\cdots)$. Then $f(x_1x_2)$ is a prefix of $f(y_1y_2)$ or vice versa. An inspection of all possible cases yields $x_1 = y_1$. Hence $f(x_2x_3\cdots) = f(y_2y_3\cdots)$ and we can proceed by induction getting $x_1x_2\cdots = y_1y_2\cdots$. Hence, indeed, f is injective.

Now let $\alpha, \beta \in \Sigma^\omega$. We have to show that $\alpha \sim \beta$ holds if, and only if, $f(\alpha) \sim f(\beta)$. For this, we will show the following equivalences:

1. $\mathrm{proj}_{\{b,d\}}(\alpha) = \mathrm{proj}_{\{b,d\}}(\beta)$ iff $\mathrm{proj}_{\{b,d\}}(f(\alpha)) = \mathrm{proj}_{\{b,d\}}(f(\beta))$
2. $\mathrm{proj}_{\mathrm{Op}\cup\{c\}}(\alpha) = \mathrm{proj}_{\mathrm{Op}\cup\{c\}}(\beta)$ iff $\mathrm{proj}_{\{a,c\}}(f(\alpha)) = \mathrm{proj}_{\{a,c\}}(f(\beta))$
3. Suppose $(a, d) \in D$ (and therefore $\mathrm{Op} \times \{d\} \subseteq D_s$, i.e., $s = 2$) as well as $\mathrm{proj}_{\mathrm{Op}\cup\{c\}}(\alpha) = \mathrm{proj}_{\mathrm{Op}\cup\{c\}}(\beta)$. Then $\mathrm{proj}_{\{a,d\}}(\alpha) = \mathrm{proj}_{\{a,d\}}(\beta)$ iff $\mathrm{proj}_{\mathrm{Op}\cup\{d\}}(f(\alpha)) = \mathrm{proj}_{\mathrm{Op}\cup\{d\}}(f(\beta))$

Before we do so, we verify that these three properties imply the equivalence of $\alpha \sim \beta$ and $f(\alpha) \sim f(\beta)$.

- Suppose $\alpha \sim \beta$. Then α and β have the same projections to the alphabets $\{b, d\}$, $\mathrm{Op}\cup\{c\}$, and (if $s = 2$) to $\mathrm{Op} \cup \{d\}$. Hence, by the above, $f(\alpha)$ and $f(\beta)$ agree on their projections to the alphabets $\{b, d\}$, $\{a, c\}$, and $\{a, d\}$ (provided $s = 2$). But this ensures $f(\alpha) \sim f(\beta)$.
- Conversely, suppose $f(\alpha) \sim f(\beta)$. Then these two words have the same projections to the alphabets $\{b, d\}$, $\{a, c\}$, and (if $s = 2$) $\{a, d\}$. The above ensures that α and β agree in their projections to $\{b, d\}$, $\mathrm{Op} \cup \{c\}$ and therefore to $\mathrm{Op} \cup \{d\}$ (if $s = 2$). Hence $\alpha \sim \beta$.

All that remains to be shown are the claims 1-3.

ad 1. For all $x \in \Sigma$, we have $\mathrm{proj}_{\{b,d\}}(x) = \mathrm{proj}_{\{b,d\}}(f(x))$. Hence $\mathrm{proj}_{\{b,d\}}(\alpha) = \mathrm{proj}_{\{b,d\}}(f(\alpha))$ and $\mathrm{proj}_{\{b,d\}}(\beta) = \mathrm{proj}_{\{b,d\}}(f(\beta))$ ensures the first equivalence.

ad 2. For all $x \in \Sigma$, we have $\mathrm{proj}_{\{a,c\}}(f(x)) = f(\mathrm{proj}_{\mathrm{Op}\cup\{c\}}(x))$. Hence $\mathrm{proj}_{\{a,c\}}(f(\alpha)) = f(\mathrm{proj}_{\mathrm{Op}\cup\{c\}}(\alpha))$ and $\mathrm{proj}_{\{a,c\}}(f(\beta)) = f(\mathrm{proj}_{\mathrm{Op}\cup\{c\}}(\beta))$. Now the second statement follows from the injectivity of f.

ad 3. Suppose $(a, d) \in D$ and therefore $s = 2$ and $\mathrm{Op} \times \{d\} \subseteq D_s$. Furthermore, suppose $\mathrm{proj}_{\mathrm{Op}\cup\{c\}}(\alpha) = \mathrm{proj}_{\mathrm{Op}\cup\{c\}}(\beta)$. We show the two implications seperately.

First, suppose $\mathrm{proj}_{\mathrm{Op}\cup\{d\}}(f(\alpha)) = \mathrm{proj}_{\mathrm{Op}\cup\{d\}}(f(\beta))$. For all $x \in \Sigma$, we have $\mathrm{proj}_{\{a,d\}}(f(x)) = \mathrm{proj}_{\{a,d\}}(f(\mathrm{proj}_{\mathrm{Op}\cup\{d\}}(x)))$. Therefore, we obtain

$$\begin{aligned}
\mathrm{proj}_{\{a,d\}}(f(\alpha)) &= \mathrm{proj}_{\{a,d\}}\big(f(\mathrm{proj}_{\mathrm{Op}\cup\{d\}}(\alpha))\big)\\
&= \mathrm{proj}_{\{a,d\}}\big(f(\mathrm{proj}_{\mathrm{Op}\cup\{d\}}(\beta))\big)\\
&= \mathrm{proj}_{\{a,d\}}\big(f(\beta)\big).
\end{aligned}$$

Conversely, suppose $\mathrm{proj}_{\{a,d\}}(f(\alpha)) = \mathrm{proj}_{\{a,d\}}(f(\beta))$. We have to show $\mathrm{proj}_{\mathrm{Op}\cup\{d\}}(\alpha) = \mathrm{proj}_{\mathrm{Op}\cup\{d\}}(\beta)$. Towards a contradiction, suppose this is not the case. Let α_1 and β_1 denote the maximal prefixes of α and β, respectively, with $\mathrm{proj}_{\mathrm{Op}\cup\{d\}}(\alpha_1) = \mathrm{proj}_{\mathrm{Op}\cup\{d\}}(\beta_1)$. If both prefixes are infinite, then $\alpha_1 = \alpha$ and $\beta_1 = \beta$ which contradicts the assumption $\mathrm{proj}_{\mathrm{Op}\cup\{d\}}(\alpha) \neq \mathrm{proj}_{\mathrm{Op}\cup\{d\}}(\beta)$. Hence, at least one of α_1 and β_1 is finite; suppose the former. Since α_1 is finite, there are $\alpha_2 \in \Sigma$ and $\alpha_3 \in \Sigma^\omega$ with $\alpha = \alpha_1\,\alpha_2\,\alpha_3$. The maximality assumption on α_1 ensures $\alpha_2 \in \mathrm{Op}\cup\{d\}$. Suppose β_1 is infinite, i.e., $\beta_1 = \beta$. Then $\mathrm{proj}_{\mathrm{Op}\cup\{d\}}(\beta) = \mathrm{proj}_{\mathrm{Op}\cup\{d\}}(\beta_1) = \mathrm{proj}_{\mathrm{Op}\cup\{d\}}(\alpha_1) \neq \mathrm{proj}_{\mathrm{Op}\cup\{d\}}(\alpha_1\,\alpha_2\,\alpha_3) = \mathrm{proj}_{\mathrm{Op}\cup\{d\}}(\alpha)$, contradicting $\alpha \sim \beta$. Thus, β_1 cannot be infinite either, i.e., there are $\beta_2 \in \Sigma$ and $\beta_3 \in \Sigma^\omega$ with $\beta = \beta_1\,\beta_2\,\beta_3$. By the maximality assumption on β_1, we obtain $\beta_2 \in \mathrm{Op}\cup\{d\}$ as well as $\alpha_2 \neq \beta_2$. Now note that

$$\begin{aligned}
\mathrm{proj}_{\{a,d\}}\big(f(\alpha_1\,\alpha_2)\big) &= \mathrm{proj}_{\{a,d\}}\big(f(\alpha_1)\big)\,\mathrm{proj}_{\{a,d\}}\big(f(\alpha_2)\big) \text{ and}\\
\mathrm{proj}_{\{a,d\}}\big(f(\beta_1\,\beta_2)\big) &= \mathrm{proj}_{\{a,d\}}\big(f(\beta_1)\big)\,\mathrm{proj}_{\{a,d\}}\big(f(\beta_2)\big)\\
&= \mathrm{proj}_{\{a,d\}}\big(f(\mathrm{proj}_{\mathrm{Op}\cup\{d\}}(\beta_1))\big)\,\mathrm{proj}_{\{a,d\}}\big(f(\beta_2)\big)\\
&= \mathrm{proj}_{\{a,d\}}\big(f(\mathrm{proj}_{\mathrm{Op}\cup\{d\}}(\alpha_1))\big)\,\mathrm{proj}_{\{a,d\}}\big(f(\beta_2)\big)\\
&= \mathrm{proj}_{\{a,d\}}\big(f(\alpha_1)\big)\,\mathrm{proj}_{\{a,d\}}\big(f(\beta_2)\big)
\end{aligned}$$

and that both these words are prefixes of $\mathrm{proj}_{\{a,d\}}(f(\alpha)) = \mathrm{proj}_{\{a,d\}}(f(\beta))$. Hence $\mathrm{proj}_{\{a,d\}}(f(\alpha_2))$ is a prefix of $\mathrm{proj}_{\{a,d\}}(f(\beta_2))$ or vice versa. Since $\alpha_2 \neq \beta_2$ are elements of $\mathrm{Op}\cup\{d\}$, we obtain $\alpha_2, \beta_2 \in \mathrm{Op}$. Consequently,

$$\begin{aligned}
\mathrm{proj}_{\mathrm{Op}}\big(\mathrm{proj}_{\mathrm{Op}\cup\{d\}}(\alpha_1)\big)\,\alpha_2\,\mathrm{proj}_{\mathrm{Op}}(\alpha_3) &= \mathrm{proj}_{\mathrm{Op}}(\alpha)\\
&= \mathrm{proj}_{\mathrm{Op}}(\beta) \qquad \text{since } \mathrm{proj}_{\mathrm{Op}\cup\{c\}}(\alpha) = \mathrm{proj}_{\mathrm{Op}\cup\{c\}}(\beta)\\
&= \mathrm{proj}_{\mathrm{Op}}\big(\mathrm{proj}_{\mathrm{Op}\cup\{d\}}(\beta_1)\big)\,\beta_2\,\mathrm{proj}_{\mathrm{Op}}(\beta_3)\\
&= \mathrm{proj}_{\mathrm{Op}}\big(\mathrm{proj}_{\mathrm{Op}\cup\{d\}}(\alpha_1)\big)\,\beta_2\,\mathrm{proj}_{\mathrm{Op}}(\beta_3)\,.
\end{aligned}$$

But this contradicts $\alpha_2 \neq \beta_2$. Thus, $\mathrm{proj}_{\mathrm{Op}\cup\{d\}}(\alpha) \neq \mathrm{proj}_{\mathrm{Op}\cup\{d\}}(\beta)$ is impossible, i.e., we have the desired equality. ◀

We need another preparation for the reduction from the problems on the standard independence alphabets (Σ, I_s) to arbitrary independence alphabets. Namely, that images (under f) of deterministic regular languages are deterministic regular and similarly for co-deterministic regular languages.

▶ **Lemma E.20.** *From (Γ, I), $a, b, c, d \in \Gamma$, and f satisfying Stipulation E.18, and a Büchi-automaton $\mathcal{A}$ over Σ, one can compute a Büchi-automaton $\mathcal{A}^f$ over Γ with $L^\omega(\mathcal{A}^f) = f(L^\omega(\mathcal{A}))$ such that*

(i) $\mathcal{A}^f$ is deterministic if $\mathcal{A}$ is deterministic and
(ii) $\mathcal{A}^f$ is co-deterministic if $\mathcal{A}$ is co-deterministic.

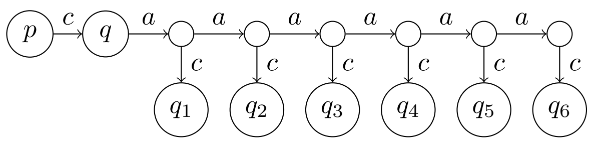

Figure 3 Proof of Lemma E.20

Proof. Let $\mathcal{A} = (Q, \Sigma, I, T, F)$ be a Büchi-automaton. For all $p \in Q$ and $o \in \mathrm{Op}$, we assume, w.l.o.g., the existence of a state $q \in Q$ with $(p, o, q) \in T$. Let $\mathrm{Op} = \{o_1, \ldots, o_6\}$ with $f(o_i) = ca^ic$ for all $i \in [6]$. To construct the Büchi-automaton $\mathcal{A}^f$, let $p \in Q$ and let $q_i \in Q$ with $(p, c, q) \in T$ and $(p, o_i, q_i) \in T$ for all $i \in [6]$. Then replace the transitions (p, o_i, q_i) by the gatget in Fig. 3. The result is a Büchi-automaton $\mathcal{A}^f$ with $L^\omega(\mathcal{A}^f) = f(L^\omega(\mathcal{A}))$. Note that, assumeing $\mathcal{A}$ to be deterministic, also $\mathcal{A}^f$ is deterministic.

Now suppose $\mathcal{A}$ is co-deterministic. We have to show that $\mathcal{A}^f$ is co-deterministic, i.e., that the language $\Gamma^\omega \setminus L^\omega(\mathcal{A}^f)$ is deterministic regular. Since $\mathcal{A}$ is co-deterministic, there is a deterministic Büchi-automaton $\mathcal{B}$ with $L^\omega(\mathcal{B}) = \Sigma^\omega \setminus L^\omega(\mathcal{A})$. Since the mapping f is injective by Lemma E.19, we get

$$L^\omega(\mathcal{B}^f) = f(L^\omega(\mathcal{B})) = f(\Sigma^\omega \setminus L^\omega(\mathcal{A})) = f(\Sigma^\omega) \setminus f(L^\omega(\mathcal{A})) = f(\Sigma^\omega) \setminus L^\omega(\mathcal{A}^f).$$

Note that $f(\Sigma^\omega)$ equals $\{cac, caac, \ldots, ca^6c, b, c, d\}^\omega \subseteq \Gamma^\omega$. Hence there is a deterministic Büchi-automaton $\mathcal{C}$ with $L^\omega(\mathcal{C}) = \Gamma^\omega \setminus f(\Sigma^\omega)$. Since the class of deterministic regular languages is closed under union, the set

$$L^\omega(\mathcal{C}) \cup L^\omega(\mathcal{B}^f) = \Gamma^\omega \setminus f(\Sigma^\omega) \cup f(\Sigma^\omega) \setminus L^\omega(\mathcal{A}^f) = \Gamma^\omega \setminus L^\omega(\mathcal{A}^f)$$

is deterministic regular. Hence, indeed, $\mathcal{A}^f$ is co-deterministic. ◀

E.3.2.2 Regularity problem

Our next aim is to reduce the regularity problem for a standard independence alphabet to the regularity problem of the non-diagonal independence alphabet (Γ, I). As a preparation, Lemma E.22 relates the regularity of $[L_1] \cap [L_2]$ and $[f(L_1)] \cap [f(L_2)]$ for $L_1, L_2 \subseteq \Sigma^\omega$. Its proof uses the following notion and result. For a finite word $w \in \Delta^*$, let $\mathrm{Alph}(w) \subseteq \Delta$ denote the set of letters appearing in w; a language $V \subseteq \Delta^*$ is *mono-alphabetic* if any two words from V agree on the letters appearing. Gastin, Petit, and Zielonka found the following characterisation of closed and regular ω-languages.

▶ **Theorem E.21** ([14]). *Let (Δ, I) be some independence alphabet and $L \subseteq \Delta^\omega$ be an ω-language. Then the following are equivalent.*

(1) $[L]$ is regular.
(2) $[K] = [L]$ for some finite union $K = \bigcup_{i \in [n]} U_i V_i^\omega$ where $U_i, V_i \subseteq \Delta^+$ are regular, closed, and mono-alphabetic.

▶ **Lemma E.22.** *Let (Γ, I), a, b, c, d, s, and f satisfy Stipulation E.18.*
Let $L_1, L_2 \subseteq \Sigma^\omega$. Then $[L_1] \cap [L_2]$ is regular if, and only if, $[f(L_1)] \cap [f(L_2)]$ is regular.

Proof. First, suppose that $[f(L_1)] \cap [f(L_2)]$ is regular and let $\alpha \in \Sigma^\omega$. Then $\alpha \in [L_1] \cap [L_2]$ iff, for all $i \in [2]$, there exist $\beta_i \in L_i$ with $\alpha \sim \beta_i$. By Lemma E.19, this is the case iff there exist $\beta_i \in L_i$ with $f(\alpha) \sim f(\beta_i)$ for all $i \in [2]$. But this is equivalent to $f(\alpha) \in [f(L_1)] \cap [f(L_2)]$.

Thus, $[L_1] \cap [L_2]$ is the preimage of the regular language $[f(L_1)] \cap [f(L_2)]$ under the mapping f and therefore regular as well.

Conversely, suppose $[L_1] \cap [L_2]$ is regular. By Theorem E.21, there exists a union $K = \bigcup_{i\in[n]} U_i V_i^\omega$ with $[K] = [L_1] \cap [L_2]$ where $U_i, V_i \subseteq \Sigma^+$ are regular, closed, and mono-alphabetic. Consider the ω-language $H = \bigcup_{i\in[n]} [f(U_i)]\,[f(V_i)]^\omega$. For $u, v \in \Sigma^+$ with $\mathrm{Alph}(u) = \mathrm{Alph}(v)$, we get $\mathrm{Alph}(f(u)) = \mathrm{Alph}(f(v))$, hence the languages $[f(U_i)]$ and $[f(V_i)]$ are mono-alphabetic and closed. Now let $x, y \in \Sigma$ with $(x, y) \in D$. Then the set $\mathrm{Alph}(f(xy)) \subseteq \Gamma$ of letters appearing in the image of $f(x)$ or $f(y)$ induces a connected subgraph of (Γ, D). Since $U_i = [U_i]$ is regular, also the language $[f(U_i)]$ is regular by [36] (and similarly for V_i). Thus, the ω-language H is regular. Applying Theorem E.21, we obtain that the closure $[H]$ is also regular.

We show $[H] = [f(L_1)] \cap [f(L_2)]$ which will complete the proof. In other words, we have to show $\alpha \in [H] \iff \alpha \in [f(L_1)] \cap [f(L_2)]$ for all $\alpha \in \Gamma^\omega$.

First, suppose $\alpha \in [H]$. By the definition of H, there exist $i \in [n]$, $u \in U_i$ and $v_1, v_2, \dots \in V_i$ with $\alpha \sim f(u)\,f(v_1)\,f(v_2)\cdots = f(uv_1v_2\cdots)$. From $uv_1v_2\cdots \in U_i V_i^\omega$, we get $uv_1v_2\cdots \in K$. Since $[K] = [L_1] \cap [L_2]$, there exist $\beta_i \in L_i$ with $\beta_i \sim uv_1v_2\cdots$ for all $i \in [2]$. Now Lemma E.19 ensures

$$\alpha \sim f(uv_1v_2\cdots) \sim f(\beta_i) \in f(L_i)$$

and therefore $\alpha \in [f(L_1)] \cap [f(L_2)]$.

Conversely, suppose $\alpha \in [f(L_1)] \cap [f(L_2)]$. Then there are $\beta_i \in L_i$ with $\alpha \sim f(\beta_i)$ for all $i \in [2]$. Lemma E.19 ensures $\beta_1 \sim \beta_2 \in L_2$ and therefore $\beta_1 \in [L_1] \cap [L_2] = [K]$. Hence there is $\gamma \in K$ with $\beta_1 \sim \gamma$ and therefore, by Lemma E.19, $\alpha \sim f(\gamma)$. But $\gamma \in K$ implies the existence of $i \in [n]$, $u \in U_i$, and $v_1, v_2 \dots \in V_i$ with $\gamma = u\,v_1\,v_2\cdots$. Hence we have

$$\alpha \sim f(\gamma) = f(u)\,f(v_1)\,f(v_2)\cdots \in [f(U_i)]\,[f(V_i)]^\omega \subseteq H$$

and therefore $\alpha \in [H]$. ◀

Now we can use the above preparations to demonstrate the hardness of the regularity problem.

▶ **Proposition E.23.** *Let (Γ, I) be a non-diagonal independence alphabet. Then $\mathrm{RegP}_\omega^{det}((\Gamma, I), \wedge)$ and $\mathrm{RegP}_\omega^{co-det}((\Gamma, I), \mathrm{Id})$ are Π^1_1-hard.*

Proof. To obtain these lower bounds, we reduce from the corresponding problems for the standard independence alphabets in Corollary E.17. So let a, b, c, d, s, and f satisfy Stipulation E.18.

Let $\mathcal{A}_1$ and $\mathcal{A}_2$ be deterministic and $\mathcal{B}$ a co-deterministic Büchi-automaton over Σ. Construct, using Lemma E.20 (co-)deterministic Büchi-automata $\mathcal{A}_1^f$, $\mathcal{A}_2^f$, and $\mathcal{B}^f$ over Γ with $L^\omega(\mathcal{A}_i^f) = f(L^\omega(\mathcal{A}_i))$ and $L^\omega(\mathcal{B}^f) = f(L^\omega(\mathcal{B}))$. From Lemma E.22, we obtain the following:

- $[L^\omega(\mathcal{A}_1)] \cap [L^\omega(\mathcal{A}_2)]$ is regular iff $[L^\omega(\mathcal{A}_1^f)] \cap [L^\omega(\mathcal{A}_2^f)]$ is regular.
- $[L^\omega(\mathcal{B})]$ is regular iff $[L^\omega(\mathcal{B})] \cap [L^\omega(\mathcal{B})]$ is regular iff $[L^\omega(\mathcal{B}^f)] \cap [L^\omega(\mathcal{B}^f)]$ is regular iff $[L^\omega(\mathcal{B}^f)]$ is regular.

Hence the mappings $((\Sigma, I_s), \wedge, \mathcal{A}_1, \mathcal{A}_2) \mapsto ((\Sigma, I_s), \wedge, \mathcal{A}_1^f, \mathcal{A}^f)$ and $((\Sigma, I_s), \mathrm{Id}, \mathcal{B}) \mapsto ((\Sigma, I_s), \mathrm{Id}, \mathcal{B}^f)$ prove $\mathrm{RegP}_\omega^{det}((\Sigma, I_s), \wedge) \leq \mathrm{RegP}_\omega^{det}((\Gamma, I), \wedge)$ and $\mathrm{RegP}_\omega^{co-det}((\Sigma, I_s), \mathrm{Id}) \leq \mathrm{RegP}_\omega^{co-det}((\Gamma, I), \mathrm{Id})$. Now the claims follow from Cor. E.17. ◀

E.3.2.3 Emptiness problem

We next show that also the emptiness problem for non-diagonal independence alphabets is hard. Note that, differently from the regularity problem, we consider the Boolean function $\neg$ as opposed to Id when studying co-deterministic regular languages.

▶ **Proposition E.24.** *Let* (Γ, I) *be a non-diagonal independence alphabet. Then* $\mathrm{EmptP}_\omega^{det}((\Gamma, I), \wedge)$ *and* $\mathrm{EmptP}_\omega^{co-det}((\Gamma, I), \neg)$ *both are* Π_1^1*-hard.*

Proof. To obtain these lower bounds, we reduce from the corresponding problems in Corollary E.17(1). So let a, b, c, d, s, and f satisfy Stipulation E.18.

We start with $\mathrm{EmptP}_\omega^{det}((\Gamma, I), \wedge)$. So let $\mathcal{A}_1$ and $\mathcal{A}_2$ be deterministic Büchi-automata over the alphabet Σ. For $i \in [2]$, construct the deterministic Büchi-automaton $\mathcal{A}_i^f$ from Lemma E.20(i) with $L^\omega(\mathcal{A}_i^f) = f(L^\omega(\mathcal{A}_i))$. By Lemma E.19, we can apply Lemma B.4 which yields $[L^\omega(\mathcal{A}_1)] \cap [L^\omega(\mathcal{A}_2)] = \emptyset$ iff $[f(\Sigma^\omega)] \cap [L^\omega(\mathcal{A}_1^f)] \cap [L^\omega(\mathcal{A}_2^f)] = \emptyset$. Since $L^\omega(\mathcal{A}_1^f) = f(L^\omega(\mathcal{A}_1)) \subseteq f(\Sigma^\omega)$, this is the case iff $[L^\omega(\mathcal{A}_1^f)] \cap [L^\omega(\mathcal{A}_2^f)] = \emptyset$. Consequently, the mapping $((\Sigma, I_s), \wedge, \mathcal{A}_1, \mathcal{A}_2) \mapsto ((\Gamma, I), \wedge, \mathcal{A}_1^f, \mathcal{A}_2^f)$ is a reduction of $\mathrm{EmptP}_\omega^{det}((\Sigma, I_s), \wedge)$ to $\mathrm{EmptP}_\omega^{det}((\Gamma, I), \wedge)$.

To also prove the hardness of $\mathrm{EmptP}_\omega^{co-det}((\Gamma, I), \neg)$, let $\mathcal{A}$ be a co-deterministic Büchi-automaton over Σ. Construct, using Lemma E.20(ii) a co-deterministic Büchi-automaton $\mathcal{A}^f$ over Γ with $L^\omega(\mathcal{A}^f) = f(L^\omega(\mathcal{A}))$. From Lemmas E.19 and B.4, we obtain $[\Sigma^\omega] \setminus [L^\omega(\mathcal{A})] = \emptyset$ iff $[f(\Sigma^\omega)] \cap \Gamma^\omega \setminus [L^\omega(\mathcal{A}^f)] = \emptyset$.

Since $\Gamma^\omega \setminus f(\Sigma^\omega)$ is regular, one can construct a Büchi-automaton $\mathcal{B}$ with $L^\omega(\mathcal{B}) = L^\omega(\mathcal{A}^f) \cup \Gamma^\omega \setminus f(\Sigma^\omega)$. Even more, the languages $\Gamma^\omega \setminus L^\omega(\mathcal{A}^f)$ and $f(\Sigma^\omega)$ are deterministic regular implying that $\mathcal{B}$ is co-deterministic (since the class of deterministic regular languages is closed under intersection). Now we have

$$
\begin{aligned}
\Gamma^\omega \setminus [L^\omega(\mathcal{A}_f)] \cap [f(\Sigma^\omega)] &= \Gamma^\omega \setminus [L^\omega(\mathcal{A}_f)] \cap \Gamma^\omega \setminus \big(\Gamma^\omega \setminus [f(\Sigma^\omega)]\big) \\
&= \Gamma^\omega \setminus \big([L^\omega(\mathcal{A}_f)] \cup \Gamma^\omega \setminus [f(\Sigma^\omega)]\big) \\
&= \Gamma^\omega \setminus [L^\omega(\mathcal{A}_f) \cup \Gamma^\omega \setminus f(\Sigma^\omega)] \\
&= \Gamma^\omega \setminus [L^\omega(\mathcal{B})]
\end{aligned}
$$

and therefore

$$
\Sigma^\omega \setminus [L^\omega(\mathcal{A})] = \emptyset \iff [f(\Sigma^\omega)] \cap \Gamma^\omega \setminus [L^\omega(\mathcal{A}_f)] = \emptyset \iff \Gamma^\omega \setminus [L^\omega(\mathcal{B})] = \emptyset\,.
$$

Hence, the mapping $((\Sigma, I_s), \neg, \mathcal{A}) \mapsto ((\Gamma, I), \neg, \mathcal{B})$ is a reduction of the Π_1^1-hard emptiness problem $\mathrm{EmptP}_\omega^{co-det}((\Sigma, I_s), \neg)$ to the emptiness problem $\mathrm{EmptP}_\omega^{co-det}((\Gamma, I), \neg)$. ◀

E.3.2.4 Rationality problem

The above propositions provide lower bounds for the regularity and the emptiness problem. We next prepare the similar result for the rationality problem. The proof relies on Lemma B.5 whose applicability is assured by the following preparatory result.

▶ **Lemma E.25.** *Let* (Γ, I), a, b, c, d, s, *and* f *satisfy Stipulation E.18.*

There is a rational relation R *that satisfies conditions (a) and (b) of Lemma B.5.*

Proof. First, suppose $(a, d) \in I$. For $i \in [6]$, we set

$$
S_i = \left\{ \begin{pmatrix} U \\ ca^i c\, v \end{pmatrix} \mid U \sim ca^i c\, v \text{ and } v \in \{b, d\}^* \right\}
$$

and, for all $U, V \in \Gamma^\omega$, $\binom{U}{V} \in R$ iff

- $U = U_1 U_2 \cdots$,
- $V = V_1 V_2 \cdots$, and
- $\binom{U_j}{V_j} \in \bigcup_{i\in[6]} S_i \cup \left\{ \binom{b}{b}, \binom{c}{c}, \binom{d}{d} \right\}$ for all $j \geq 1$

First, we verify that the relations S_i are rational (for all $i \in [6]$). By definition $\binom{U}{V} \in S_i$ iff there exists $v \in \{b,d\}^*$ with $V = ca^i c\, v$,

(i) $\mathrm{proj}_{\{a,c\}}(U) = \mathrm{proj}_{\{a,c\}}(V)$, and
(ii) $\mathrm{proj}_{\{b,d\}}(U) = \mathrm{proj}_{\{b,d\}}(V)$.

Assuming $V = ca^i c\, v$, the first condition is equivalent to $\mathrm{proj}_{\{a,c\}}(U) = ca^i c$. Hence the statement $\binom{U}{V} \in S_i$ is the conjunction of

- a regular condition on V (namely $V \in ca^i c \{b,d\}^*$),
- a regular condition on U (namely $U \in ca^i c \sqcup\!\sqcup \{b,d\}^*$), and
- a rational condition on the pair U and V (namely $\mathrm{proj}_{\{b,d\}}(U) = \mathrm{proj}_{\{b,d\}}(V)$).

Hence the relation S_i is rational.

Before we look at the relation R, the following aspect of the relation S_i is important. There is some $o \in \mathrm{Op}$ with $f(o) = ca^i c$. Since the mapping f fixes the letters b and d, we get

$$S_i = \left\{ \begin{pmatrix} U \\ f(w) \end{pmatrix} \mid U \sim f(w) \text{ and } w \in o\{b,d\}^* \right\}$$

and therefore

$$S := \bigcup_{i\in[6]} S_i = \left\{ \begin{pmatrix} U \\ f(w) \end{pmatrix} \mid U \sim f(w) \text{ and } w \in \mathrm{Op}\,\{b,d\}^* \right\} .$$

Now let $\binom{U}{V} \in R$. Then U and V factorize into $U = U_1 U_2 \cdots$ and $V = V_1 V_2 \cdots$ with

$$\begin{pmatrix} U_j \\ V_j \end{pmatrix} \in S \cup \left\{ \begin{pmatrix} b \\ b \end{pmatrix}, \begin{pmatrix} c \\ c \end{pmatrix}, \begin{pmatrix} d \\ d \end{pmatrix} \right\}$$

for all $j \geq 1$. Since f fixes b, c, and d, this implies $U_j \sim V_j \in f(\Sigma^+)$ for all $j \geq 1$ and therefore $U \sim V \in f(\Sigma^\omega)$. This ensures condition (a) from Lemma B.5.

To also verify the condition (b) from Lemma B.5, let $U \in [f(\Sigma^\omega)]$. Then $\mathrm{proj}_{\{a,c\}}(U) \in \{c, ca^1c, ca^2c, \ldots, ca^6c\}^\infty$. Hence U factorizes into $U = U_1 U_2 \cdots$ with

$$U_j \in \{b,c,d\} \cup \bigcup_{i\in[6]} \left(ca^i c \sqcup\!\sqcup \{b,d\}^* \right)$$

for all $j \geq 1$. Set

$$V_j = \begin{cases} U_j & \text{if } U_j \in \{b,c,d\} \\ ca^i c\, \mathrm{proj}_{\{b,d\}}(U_j) & \text{if } U_j \in ca^i c \sqcup\!\sqcup \{b,d\}^* . \end{cases}$$

such that $\binom{U_j}{V_j} \in \left\{ \binom{b}{b}, \binom{c}{c}, \binom{d}{d} \right\} \cup S$ holds for all $j \geq 1$. With $V = V_1 V_2 \cdots$, we therefore have $\binom{U}{V} \in R$ and $V \in f(\Sigma^\omega)$.

Thus, indeed, the rational relation R satisfies conditions (a) and (b) of Lemma B.5.

It remains to consider the case $(a,d) \notin I$.

For $i \in [6]$, we set

$$S_i = \left\{ \begin{pmatrix} cxyzc \\ x\,\mathrm{proj}_b(y)\,ca^ic\,z \end{pmatrix} \mid x, z \in \{b,d\}^*, y \in a^i \sqcup\!\sqcup\, b^* \right\}$$

and $S = \bigcup_{i\in[6]} S_i$. Then a pair of words U and V belongs to S_i iff they can be factorized into $cxyzc$ and $x\,\mathrm{proj}_b(y)\,ca^ic\,z$ with the properties given in the definition of S_i, i.e., iff

$$\begin{pmatrix} U \\ V \end{pmatrix} \in \begin{pmatrix} c \\ \varepsilon \end{pmatrix} \underbrace{\left\{ \begin{pmatrix} b \\ b \end{pmatrix}, \begin{pmatrix} d \\ d \end{pmatrix} \right\}^*}_{\hat{=}\binom{x}{x}} \underbrace{\left(\begin{pmatrix} b \\ b \end{pmatrix}^* \begin{pmatrix} a \\ \varepsilon \end{pmatrix} \right)^i \begin{pmatrix} b \\ b \end{pmatrix}^*}_{\hat{=}\binom{y}{\mathrm{proj}_b(y)}} \begin{pmatrix} \varepsilon \\ ca^ic \end{pmatrix} \underbrace{\left\{ \begin{pmatrix} b \\ b \end{pmatrix}, \begin{pmatrix} d \\ d \end{pmatrix} \right\}^*}_{\hat{=}\binom{z}{z}} \begin{pmatrix} c \\ \varepsilon \end{pmatrix}.$$

Thus, this rational relation equals S_i implying that S is rational. It can easily be checked that $\binom{U}{V} \in S$ implies that the projections of U and V to $\{a,c\}$, $\{a,d\}$, and $\{b,d\}$ are equal, i.e., that $U \sim V$ holds. Furthermore, then $V \in \{b,d\}^* f(\mathrm{Op})\,\{b,d\}^* \subseteq f(\Sigma^+)$ since f fixes b and d.

Furthermore let the relation R over Γ^ω consist of all pairs $\binom{U}{V}$ with

- $U = U_1U_2\cdots$,
- $V = V_1V_2\cdots$, and
- $\binom{U_j}{V_j} \in S \cup \left\{ \binom{b}{b}, \binom{c}{c}, \binom{d}{d} \right\}$ for all $j \geq 1$

Since $\binom{U_j}{V_j} \in S$ implies $U_j \sim V_j \in f(\Sigma^+)$ and since f fixes b, c, and d, we get $U \sim V \in f(\Sigma^\omega)$ which ensures condition (a) from Lemma B.5.

To also verify condition (b) from Lemma B.5, let $U \in [f(\Sigma^\omega)]$. Then there exists $w \in \Sigma^\omega$ with $U \sim f(w)$. It follows that their projections to $\{a,c,d\}$ are equivalent as well: $\mathrm{proj}_{\{a,c,d\}}(U) \sim \mathrm{proj}_{\{a,c,d\}}(f(w))$ (note that these two projections need not be equal since $(c,d) \in I$). Since $f(\Sigma^\omega)$ is the set of sequences of words ca^ic for $i \in [6]$, b, c, and d, and since $w \in \Sigma^\omega$, the word $\mathrm{proj}_{\{a,c,d\}}(f(w))$ factorizes into words ca^ic with $i \in [6]$, c, and d. Depending on whether the number of factors of the form ca^ic is zero, finite (and non-zero), or infinite, one of the following hold:

- $\mathrm{proj}_{\{a,c,d\}}(f(w)) \in \{c,d\}^\infty$.
- $\mathrm{proj}_{\{a,c,d\}}(f(w)) = x_0a^{i_1}x_1a^{i_2}x_2\cdots a^{i_k}x_k$ with $k \geq 1$, $i_j \in [6]$ for all $j \in [k]$, and

$$x_j \in \begin{cases} \{c,d\}^*cd^* & \text{for } j = 0 \\ d^*c\{c,d\}^*cd^* & \text{for } j \in [k-1] \\ d^*c\{c,d\}^\infty & \text{for } j = k\,. \end{cases}$$

- $\mathrm{proj}_{\{a,c,d\}}(f(w)) = x_0a^{i_1}x_1a^{i_2}\cdots$ with $x_0 \in \{c,d\}^*cd^*$, $x_j \in d^*c\{c,d\}^*cd^*$ and $i_j \in [6]$ for all $j \geq 1$.

Since a is dependent from both, c and d and since $\mathrm{proj}_{\{a,c,d\}}(U) \sim \mathrm{proj}_{\{a,c,d\}}(f(w))$, also the word $\mathrm{proj}_{\{a,c,d\}}(U)$ has one of these three forms. In other words, is factorizes into words of the form $cd^*a^id^*c$ for $i \in [6]$, c, and d. Since U belongs to the shuffle of its projection and b^∞, it factorizes into $U = U_1U_2\cdots$ with $U_j \in \{b,c,d\} \cup \bigcup_{i\in[6]}(cd^*a^id^*c \sqcup\!\sqcup\, b^*)$ for all $j \geq 1$.

We next define V_j such that

$$\begin{pmatrix} U_j \\ V_j \end{pmatrix} \in \left(S \cup \left\{ \begin{pmatrix} b \\ b \end{pmatrix}, \begin{pmatrix} c \\ c \end{pmatrix}, \begin{pmatrix} d \\ d \end{pmatrix} \right\} \right)^+.$$

This is trivial if $U_j \in \{b,c,d\}$: just set $V_j = U_j$. Otherwise, there is $i \in [6]$ with $U_j \in cd^*a^id^*c \sqcup b^*$. Hence $U_j = x_0cxyzcx_1$ with $x_0, x_1 \in b^*$, $x \in \{b,d\}^*$, $y \in a^i \sqcup b^*$, and $z \in \{b,d\}^*$. Then set $V_j = x_0x\,\mathrm{proj}_b(y)\,ca^iczx_1$. Since $\binom{cxyzc}{x\mathrm{proj}_b(y)ca^icz} \in S$ and $x_0, x_1 \in \{b,c,d\}^*$, we found the word V_j as required. Setting $V = V_1V_2\cdots$, we obtain that $\binom{U}{V}$ factorizes into factors from

$$S \cup \left\{ \begin{pmatrix} b \\ b \end{pmatrix}, \begin{pmatrix} c \\ c \end{pmatrix}, \begin{pmatrix} d \\ d \end{pmatrix} \right\}$$

implying that it belongs to R.

Thus, also in case $(a,d) \notin I$ there is a rational relation R satisfying conditions (a) and (b) in Lemma B.5. ◀

Finally, we can also prove the hardness of the rationality problems for any non-diagonal independence alphabet.

▶ **Proposition E.26.** *Let (Γ, I) be a non-diagonal independence alphabet. Then the rationality problems* $\mathrm{RatP}_\omega^{det}((\Gamma,I),\wedge)$ *and* $\mathrm{RatP}_\omega^{co-det}((\Gamma,I),\neg)$ *both are Π_1^1-hard.*

Proof. To obtain these lower bounds, we reduce from the rationality problems in Corollary E.17(3). Both these reductions are based on Lemma B.5.

So let a, b, c, d, and f satisfy Stipulation E.18. Furthermore, set $s = 1$ if $(a,d) \in I$ and $s = 2$ if $(a,d) \notin I$, i.e., (Σ, I_s) is the standard independence alphabet with $(\mathrm{inc}_1, d) \in I_s \iff (a,d) \in I$. Then, by Lemma E.19, Equivalence (2) from Lemma B.4 holds. Lemma E.25 ensures the existence of a rational relation R such that conditions (a) and (b) of Lemma B.5 hold.

Conjunction. After these preparations, we can now reduce $\mathrm{RatP}_\omega^{det}((\Sigma,I_s),\wedge)$ to the rationality problem $\mathrm{RatP}_\omega^{det}((\Gamma,I),\wedge)$. So let $\mathcal{A}_1$ and $\mathcal{A}_2$ be two deterministic Büchi-automata over Σ. Let $\mathcal{A}_1^f$ and $\mathcal{A}_2^f$ denote the deterministic Büchi-automata with $L^\omega(\mathcal{A}_i) = f(L^\omega(\mathcal{A}_i))$ that we can compute by Lemma E.20(i). By Lemma B.5, $((\Sigma,I_s),\wedge,\mathcal{A}_1,\mathcal{A}_2) \in \mathrm{RatP}_\omega^{det}((\Sigma,I_s),\wedge)$ if, and only if, there exists a regular language $K' \subseteq \Gamma^*$ with $[K'] = [f(\Sigma^\omega)] \cap [f(L^\omega(\mathcal{A}_1))] \cap [f(L^\omega(\mathcal{A}_2))]$. Note that this latter language equals $[L^\omega(\mathcal{A}_1^f)] \cap [L^\omega(\mathcal{A}_2^f)]$ since $[f(L^\omega(\mathcal{A}_1))] \subseteq [f(\Sigma^\omega)]$. Hence we showed the equivalence of $((\Sigma,I_s),\wedge,\mathcal{A}_1,\mathcal{A}_2) \in \mathrm{RatP}_\omega^{det}((\Sigma,I_s),\wedge)$ and $((\Gamma,I),\wedge,\mathcal{A}_1^f,\mathcal{A}_2^f) \in \mathrm{RatP}_\omega^{det}((\Gamma,I),\wedge)$, i.e., that the mapping $((\Sigma,I),\wedge,\mathcal{A}_1,\mathcal{A}_2) \mapsto ((\Gamma,I),\wedge,\mathcal{A}_1^f,\mathcal{A}_2^f)$ is a reduction of the rationality problem $\mathrm{RatP}_\omega^{det}((\Sigma,I_s),\wedge)$ to the rationality problem $\mathrm{RatP}_\omega^{det}((\Gamma,I),\wedge)$ which is therefore Π_1^1-hard by Corollary E.17(3).

Negation. It remains to also reduce $\mathrm{RatP}_\omega^{co-det}((\Sigma,I_s),\neg)$ to the rationality problem $\mathrm{RatP}_\omega^{co-det}((\Gamma,I),\neg)$. So let $\mathcal{A}$ be a co-deterministic Büchi-automaton over Σ and $\mathcal{A}^f$ the co-deterministic Büchi-automaton with $L^\omega(\mathcal{A}^f) = f(L^\omega(\mathcal{A}))$. By Lemma B.5, $((\Sigma,I_s),\neg,\mathcal{A}) \in \mathrm{RatP}_\omega^{co-det}((\Sigma,I_s),\neg)$ if, and only if, there exists a regular language $K' \subseteq \Gamma^\omega$ with $[K'] = [f(\Sigma^\omega)] \cap \Gamma^\omega \setminus [f(L^\omega(\mathcal{A}))]$, i.e., with $[K'] = [f(\Sigma^\omega)] \setminus [L^\omega(\mathcal{A}^f)]$.

We show that this is equivalent to the existence of a regular language K'' with $[K''] = \Gamma^\omega \setminus [L^\omega(\mathcal{A}^f)]$ (i.e., $((\Gamma,I),\neg,\mathcal{A}^f) \in \mathrm{RatP}_\omega^{co-det}((\Gamma,I),\neg)$). But first, note that the closure of $f(\Sigma^\omega)$ is regular since $U \in [f(\Sigma^\omega)]$ iff the projection $\mathrm{proj}_{\{a,c\}}(U)$ belongs to $\{c, ca^1c, ca^2c, \ldots, ca^6c\}^\infty$.

Now, suppose $[f(\Sigma^\omega)] \setminus [L^\omega(\mathcal{A}^f)]$ is the closure of the regular language K'. Then $K'' = K' \cup \Gamma^\omega \setminus [f(\Sigma^\omega)]$ is regular and satisfies

$$\begin{aligned}
[K''] &= [K'] \cup \left[\Gamma^\omega \setminus [f(\Sigma^\omega)]\right] \\
&= [f(\Sigma^\omega)] \setminus [L^*(\mathcal{A}^f)] \cup \Gamma^\omega \setminus [f(\Sigma^\omega)] && \text{since } \Gamma^\omega \setminus [f(\Sigma^\omega)] \text{ is closed} \\
&= \Gamma^\omega \setminus [L^\omega(\mathcal{A}^f)] && \text{since } [L^\omega(\mathcal{A}^f)] = [f(L^\omega(\mathcal{A}))] \subseteq [f(\Sigma^\omega)]\,.
\end{aligned}$$

Conversely, suppose $\Gamma^\omega \setminus [L^\omega(\mathcal{A}^f)]$ is the closure of the regular language K''. Then $K' := K'' \cap [f(\Sigma^\omega)]$ is regular and satisfies

$$\begin{aligned}
[K'] &= [K''] \cap [f(\Sigma^\omega)] && \text{since } [f(\Sigma^\omega)] \text{ is closed}\\
&= \Gamma^\omega \setminus [L^\omega(\mathcal{A}^f)] \cap [f(\Sigma^\omega)]\\
&= [f(\Sigma^\omega)] \setminus [L^\omega(\mathcal{A}^f)]\,.
\end{aligned}$$

Thus, indeed, $[f(\Sigma^\omega)] \setminus [L^\omega(\mathcal{A}^f)]$ is the closure of some regular language K' if, and only if, $\Gamma^\omega \setminus [L^\omega(\mathcal{A}^f)]$ is the closure of some regular language K''. Thus, the mapping $((\Sigma, I_s), \neg, \mathcal{A}) \mapsto ((\Gamma, I), \neg, \mathcal{A}^f)$ is a reduction of the rationality problem $\mathrm{RatP}_\omega^{co-det}((\Sigma, I_s), \neg)$ to the rationality problem $\mathrm{RatP}_\omega^{co-det}((\Gamma, I), \neg)$ which is therefore Π^1_1-hard by Corollary E.17(3). ◀

So far, we proved the hardness of $\mathrm{GenP}_\omega((\Gamma, I), \varphi)$ for any non-diagonal independence alphabets and specific functions $\varphi \in \{\mathrm{Id}, \wedge, \neg\}$.

Using the reductions from Lemma 2.3, these hardness results can be extended to (almost) all sets of functions and independence alphabets.

▶ **Corollary E.27.** *Let $\mathbb{A} \subseteq \mathrm{IndA}$ and $\Phi \subseteq \mathrm{BF}$ be decidable sets of independence alphabets and Boolean functions, respectively.*

(1) If $\mathbb{A} \not\subseteq \mathrm{DiagA}$ and $\Phi \not\subseteq \mathrm{CONST}$, then $\mathrm{RegP}_\omega(\mathbb{A}, \Phi)$ is Π^1_1-hard.
(2) If $\mathbb{A} \not\subseteq \mathrm{DiagA}$ and $\Phi \not\subseteq \mathrm{MAX}$, then $\mathrm{RatP}_\omega(\mathbb{A}, \Phi)$ and $\mathrm{EmptP}_\omega(\mathbb{A}, \Phi)$ are Π^1_1-hard.

Proof. Let $(\Gamma, I) \in \mathbb{A}$ be non-diagonal.

(1) Suppose $\varphi \in \Phi \setminus \mathrm{CONST}$. Then $\mathrm{RegP}_\omega((\Gamma, I), \mathrm{Id})$ is Π^1_1-hard by Prop. E.23 and can be reduced to $\mathrm{RegP}_\omega(\mathbb{A}, \Phi)$ by Lemma 2.3(1). Hence, also the latter problem is Π^1_1-hard.
(2) Now let $\varphi \in \Phi \setminus \mathrm{MAX}$ and $\mathrm{GenP} \in \{\mathrm{RatP}, \mathrm{EmptP}\}$.
Then $\mathrm{GenP}_\omega((\Gamma, I), \wedge)$ and $\mathrm{GenP}_\omega((\Gamma, I), \neg)$ both are Π^1_1-hard by Props. E.24 and E.26, respectively.
Lemma 2.3 allows to reduce at least one of these two problems to $\mathrm{GenP}_\omega((\Gamma, I), \varphi)$ and therefore to $\mathrm{GenP}_\omega(\mathbb{A}, \Phi)$. Hence, also this latter problem is Π^1_1-hard.

◀