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RATIONAL, RECOGNIZABLE, AND APERIODIC SETS IN THE PARTIALLY LOSSY QUEUE MONOID

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Partially lossy queue monoids (or plq monoids) model the behavior of queues that can forget arbitrary parts of their content. While many decision problems on recognizable subsets in the plq monoid are decidable, most of them are undecidable if the sets are rational. In particular, in this monoid the classes of rational and recognizable subsets do not coincide. By restricting multiplication and iteration in the construction of rational sets and by allowing complementation we obtain precisely the class of recognizable sets. From these special rational expressions we can obtain an MSO logic describing the recognizable subsets. Moreover, we provide similar results for the class of aperiodic subsets in the plq monoid.

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1. Introduction

The study of different models of automata along with their expressiveness and algorithmic properties is one of the most important areas in automata theory. Many of these models differ in the mechanism to store their data, e.g., there are finite memories, pushdowns, (blind) counters, and infinite Turing tapes. Another very important mechanism is the so-called fifo queue (or channel), where data can be written to one end and read from the other end of its contents. If we equip these queues with a finite automaton we obtain a Turing complete computation model [3], which results in the undecidability of all non-trivial decision problems on these devices. A surprising result was the decidability of some decision problems like reachability, fair termination, or control-state-maintainability if the fifo queue is allowed to forget any part of its content at any time [1,5,8,18].

To obtain some algebraic results on the behavior of these storage mechanisms we can model them as monoid of transformations. So, a single blind counter induces $(\mathbb{Z}, +, 0)$ and a stack induces a polycyclic monoid [12]. A comprehensive survey about the connection between storage mechanisms and monoids can be found in [29]. Some basic results on the transformation monoid of reliable queues can be found in [11]. Furthermore, in [14] we considered the transformation monoid of lossy queues. When studying the similarities and differences between those two monoids in [16] we found it convenient to join both, the reliable and lossy queues, respectively, into one model: the so-called *partially lossy queues* (or *plq*s for short). Those are given by their underlying alphabet A as well as a subset $U \subseteq A$ of letters that are unforgettable while the letters contained in $A \setminus U$ can be forgotten at any time. We denote the corresponding transformation monoid by $\mathcal{Q}(A, U)$ and call it the *partially lossy queue monoid* or, for short, *plq monoid*. Hence, with the help of plqs we can argue about reliable and lossy queues at the same time, which results in the unification of some proofs considering these two models.

Another main topic in the theory of automata and formal languages is the study of regular languages. This revealed strong relations to logic, combinatorics, and algebra. For example, we can generalize the notion of regularity from free monoids to arbitrary monoids. This generalization results in two notions: the rational subsets, which are a generalization of languages that are described by regular expressions, and recognizable subsets, which are a generalization of sets accepted by finite automata (see, e.g., [2,24]). Kleene's Theorem [13] states that both notions are equivalent in the free monoid.

In Section 4 we consider some algorithmic properties of rational subsets of the plq monoid. Such properties encountered increased attention in recent years, e.g., [17] provides a survey on the membership problem for rational sets. Since the rational sets in the polycyclic monoid (recall that this is the transformation monoid of a pushdown) are exactly the homomorphic images of some very simple regular languages by [26], many decision problems like membership, intersection, universality, inclusion, and recognizability are decidable in the polycyclic monoid. In this paper we will see that the membership problem of the plq monoid is NL-complete, but the other problems are undecidable, which we can prove by reduction from their counterparts in the direct product of $(\mathbb{N}, +, 0)$ and $\{a, b\}^*$ (cf. [9,21]).

If the given subsets are recognizable, all of the considered decision problems in plq monoids are decidable by known constructions from automata theory. Hence, the rational subsets are not effectively recognizable. Especially, we will see that the class of rational subsets in the plq monoid is not closed under intersection implying that the classes of rational and recognizable subsets do not coincide. In contrast, in polycyclic monoids the class of rational sets is closed under Boolean operations. However, the classes of rational and recognizable subsets do not coincide in these monoids since there are only two recognizable sets (the empty set and the monoid itself). But since there are even more recognizable sets in the plq monoid and since each recognizable subset is rational as well due to McKnight's Theorem [19], it is a natural question to ask in which cases a rational subset is recognizable.

For trace monoids, Ochmański could prove in [22] that it suffices to restrict the usage of the Kleene star in an appropriate way to characterize the recognizable subsets in the trace monoid. In Section 5 of this paper we will use an approach similar to Ochmański's to characterize the recognizable sets in terms of special rational sets in the plq monoid. Concretely, we will define some special restrictions on the usage of Kleene star and the concatenation to reach this target.

Another famous characterization of the regular languages is the definability in the monadic second-order logic MSO which was proven by Büchi in [4]. This result gave us an even brighter understanding than rational expressions of the formalization of the behavior of finite automata. Similar results about trace monoids can be found in [7, Chapter 10]. Hence, this motivates to find another MSO logic describing exactly the recognizable subsets in the plq monoid. In this paper we will give such a description.

The last result in this paper regards the connection between the aperiodic subsets, star-free subsets, and first-order logic. Recall that a set is aperiodic if it is accepted by a counter-free finite automaton and a set is star-free if it can be generated from finite sets by application of Boolean operations and concatenation, only. Schützenberger's Theorem [27] states that both classes coincide in the free monoid. This result gives a procedure to decide whether a given regular language is star-free. Additionally, in [10] it was proven that these classes also coincide in trace monoids. In contrast to these two cases this equality does not hold in the plq monoid. But we can characterize the aperiodic subsets in $\mathcal{Q}(A, U)$ with the help of the same restrictions to star-freeness of subsets as in our result regarding the rational subsets. Finally, we prove similar to the results from [7,20] that the aperiodic subsets in the plq monoid can be described by first-order formulas.

Note that this is the full version of the conference contribution [15].

2. Preliminaries

At first, we need some basic definitions. So, let A be an alphabet. A word $v \in A^*$ is a *prefix* of $w \in A^*$ iff $w \in vA^*$. Similarly, v is a *suffix* of w iff $w \in A^*v$ and v is an *infix* of w iff $w \in A^*vA^*$. If $w = a_1 \dots a_\ell$ with $a_1, \dots, a_\ell \in A$ we denote the *infix of w from positions i to j*, where $1 \le i \le \ell$ and $0 \le j \le \ell$ by $w[i, j] = a_i \dots a_j$. In particular, we have $w[i, j] = \varepsilon$ if $i > j$. Furthermore, v is a *subword* of w (denoted by $v \preceq w$) iff there are $\ell \in \mathbb{N}$ and $a_1, \dots, a_\ell \in A$ such that $v = a_1 \dots a_\ell$ and $w \in A^*a_1A^*a_2 \dots A^*a_\ell A^*$. Note that $\preceq$ is a partial ordering on A^*. Let $S \subseteq A$. Then we define the *projection* $\pi_S \colon A^* \to S^*$ on S by

$$\pi_S(\varepsilon) = \varepsilon \quad \text{and} \quad \pi_S(aw) = \begin{cases} a\pi_S(w) & \text{if } a \in S \\ \pi_S(w) & \text{otherwise} \end{cases}$$

for each $a \in A$ and $w \in A^*$.

2.1. *Rationality, recognizability, and aperiodicity*

Let $\mathbb{M}$ be a monoid. A subset $L \subseteq \mathbb{M}$ is called *rational* in $\mathbb{M}$ if it can be constructed from the finite subsets of $\mathbb{M}$ using union, concatenation, and Kleene iteration. The subset $L \subseteq \mathbb{M}$ is *recognizable* in $\mathbb{M}$ if there are a finite monoid $\mathbb{F}$ and a homomorphism $\phi\colon \mathbb{M} \to \mathbb{F}$ such that $L = \phi^{-1}(\phi(L))$, i.e., if L is accepted by a finite automaton on $\mathbb{M}$. It is well-known that the image of a rational set under a homomorphism is rational again and that the homomorphic preimage of a recognizable set also is recognizable. Furthermore, the class of recognizable subsets of $\mathbb{M}$ is closed under Boolean operations. Moreover, in a finitely generated monoid each recognizable set is rational by [19]. For example, this applies to the partially lossy queue monoid $\mathcal{Q}(\mathcal{L})$, which we introduce in the succeeding section. The converse direction is not true in general, e.g., in Theorem 4.5 we prove the existence of a rational subset of the plq monoid which is not recognizable. However, in free monoids generated by some alphabet Γ a subset $L \subseteq \Gamma^*$ is rational if, and only if, it is recognizable by Kleene's Theorem [13]. In this situation, we call L *regular.*

A recognizable set $L \subseteq \mathbb{M}$ is called *aperiodic* in $\mathbb{M}$ if there is $n \in \mathbb{N}$ such that for each $u, v, w \in \mathbb{M}$ we have $uv^n w \in L$ iff $uv^{n+1}w \in L$. By [20] in free monoids $L \subseteq \Gamma^*$ is aperiodic iff it is accepted by a counter-free finite automaton on Γ. It is an easy exercise to prove that the class of aperiodic subsets is closed under Boolean operations and homomorphic preimages. By Schützenberger's Theorem [27] a language $L \subseteq \Gamma^*$ is aperiodic iff it is star-free. Note that a set $L \subseteq \mathbb{M}$ is *star-free* in $\mathbb{M}$ if it can be constructed from finite subsets of $\mathbb{M}$ using union, concatenation, and complementation.

2.2. *Logic and languages*

In this subsection we recall the logics on words and their correspondence to languages known from [28].

Let Γ be an alphabet. By FO we denote the set of first-order formulas built up from the atomic formulas of the form

$$x = y, \quad x < y, \quad \Lambda_a(x) \text{ for } a \in \Gamma$$

where x and y are variables. To simplify notation we write $\Lambda_S(x)$ instead of $\bigvee_{a\in S} \Lambda_a(x)$ for any $S \subseteq \Gamma$. Moreover, we write $x \leq y$ instead of $x < y \vee x = y$.

Now let $w = a_1 \dots a_n \in \Gamma^*$. The *word model* for w is the relational structure

$$\underline{w} = (\mathrm{dom}(w), <^w, (\Lambda_a^w)_{a\in\Gamma})$$

where $\mathrm{dom}(w) = \{1, \dots, n\}$ is the set of letter positions of w, $<^w$ is the natural (strict) order on $\mathrm{dom}(w)$, and $\Lambda_a^w = \{i \in \mathrm{dom}(w) \mid a_i = a\}$ is the set of positions in w labeled with the letter a. Then we write $(\underline{w}, p_1, \dots, p_n) \models \phi(x_1, \dots, x_n)$ for $p_1, \dots, p_n \in \mathrm{dom}(w)$ and a formula $\phi \in \mathsf{FO}$ (i.e., ϕ is satisfied in $\underline{w}$) if ϕ evaluates to true on interpretation of $=, <, \Lambda_a$ as equality, $<^w$, and Λ_a^w, respectively, and on interpretation of the free variables $x_1, \dots, x_n$ in ϕ as $p_1, \dots, p_n$. Then the language defined by the sentence ϕ is $L(\phi) = \{w \in \Gamma^* \mid \underline{w} \models \phi\}$. We say that a language $L \subseteq \Gamma^*$ is FO-*definable* if there is $\phi \in \mathsf{FO}$ with $L = L(\phi)$.

By MSO (the *monadic second-order logic*) we denote the second-order extension of FO where the second-order variables are unary. Again, we say that $L \subseteq \Gamma^*$ is MSO-*definable* if there is $\phi \in \mathsf{MSO}$ with $L = L(\phi)$.

By [4] a language is regular iff it is MSO-definable. Moreover, a language is star-free and, hence, aperiodic iff it is FO-definable by [20].

3. Partially Lossy Queues

The partially lossy queue monoid (or plq monoid) models the behavior of a fifo-queue whose entries come from an alphabet A. The unreliability of the queue stems from the fact that it can forget certain letters that we collect in the set $A \setminus U$. In other words, letters from $U \subseteq A$ are *unforgettable* and those from $A \setminus U$ are *forgettable*. We call the tuple $\mathcal{L} = (A, U)$ a *lossiness alphabet* if A is an alphabet (of possible queue entries) and $U \subseteq A$ (is the set of unforgettable letters).

So, let $\mathcal{L} = (A, U)$ be a lossiness alphabet. The states of the queue are the words from A^*. Furthermore, we have some basic controllable actions on these queues: writing of a symbol $a \in A$ (denoted by a) and reading of $a \in A$ (denoted by $\overline{a}$). In this connection, we assume that the set $\overline{A}$ of all these reading operations $\overline{a}$ is a disjoint copy of A. So, $\Sigma := A \cup \overline{A}$ is the set of all controllable operations on the partially lossy queue. For a word $u = a_1 a_2 \dots a_n \in A^*$ we write $\overline{u}$ for the word $\overline{a_1}\,\overline{a_2} \dots \overline{a_n}$.

Formally, the action $a \in A$ appends the letter a to the state of the queue. The action $\overline{a} \in \overline{A}$ tries to cancel the letter a from the beginning of the current state of the queue. If this state does not start with a then the queue ends up in an error state. The lossiness of the queue is modeled by allowing it to forget arbitrary letters from $A \setminus U$ of its content at any moment.

Before defining the plq monoid we want to identify sequences of operations that have the same effect on any queue. In [16, Theorems 3.5 and 3.15] we proved that $u, v \in \Sigma^*$ act equally if, and only if, they can be transformed into each other by applying the equations from the following definition, only.

Definition 3.1. *Let* $\mathcal{L} = (A, U)$ *be a lossiness alphabet. The binary relation* $\equiv_{\mathcal{L}} \subseteq (\Sigma^*)^2$ *is defined as the least congruence on* Σ^* *satisfying the following equations for* $a, b \in A$ *and* $w \in A^*$*:*

(a) $b\overline{a} \equiv_{\mathcal{L}} \overline{a}b$ *if* $a \neq b$
(b) $a\overline{ab} \equiv_{\mathcal{L}} \overline{a}a\overline{b}$
(c) $bwa\overline{a} \equiv_{\mathcal{L}} bw\overline{a}a$ *if* $b \in U \cup \{a\}$

For $w \in \Sigma^*$*, the equivalence class of* w *is* $[w]_{\equiv_{\mathcal{L}}} := \{v \in \Sigma^* \mid v \equiv_{\mathcal{L}} w\}$*. Then the* partially lossy queue monoid *or* plq monoid *induced by* $\mathcal{L}$ *is the quotient* $\mathcal{Q}(\mathcal{L}) := \Sigma^*/_{\equiv_{\mathcal{L}}} = \{[w]_{\equiv_{\mathcal{L}}} \mid w \in \Sigma^*\}$*. The natural epimorphism on this congruence is* $\eta_{\mathcal{L}} \colon \Sigma^* \to \mathcal{Q}(\mathcal{L}) \colon w \mapsto [w]_{\equiv_{\mathcal{L}}}$*.*

Let $c \in A^*$ be some queue content and $u, v \in \Sigma^*$ be some queue transformation sequences with $u \equiv_{\mathcal{L}} v$. Then a queue with content c reaches some new content $d \in A^*$ after application of the transformations u if, and only if, the queue reaches d after application of v. Hence, the plq monoid $\mathcal{Q}(\mathcal{L})$ is the monoid of transformation sequences on a partially lossy queue with underlying dependence alphabet $\mathcal{L}$.

Remark 3.2. Suppose $A = \{a\}$. Then a partially lossy queue on this alphabet (independently of U) acts like a partially blind counter, the corresponding transformation monoid is the bicyclic semigroup. On the first sight, the equality of these two transformation monoids seems to be counterintuitive. But it comes along with the following simple observation: Let $\mathfrak{A}$ be a partially blind one-counter automaton (i.e., $\mathfrak{A}$ is a PDA with a unary pushdown alphabet). Then $\mathfrak{A}$ can be understood as a finite automaton with a queue over $(\{a\}, \{a\})$. Let $\mathfrak{B}$ be an extension of $\mathfrak{A}$ by some ε-transitions that are decreasing the counter. Alternatively, we can understand $\mathfrak{B}$ as the automaton $\mathfrak{A}$ where a is a forgettable symbol. Then both, $\mathfrak{A}$ and $\mathfrak{B}$, accept the same language.

To handle the equivalence classes of $\equiv_{\mathcal{L}}$ we want to define a normal form on this congruence. We do this by ordering the equations from Definition 3.1 from left to right, which results in a semi-Thue system called $\mathfrak{R}_{\mathcal{L}}$.

Since the rules of $\mathfrak{R}_{\mathcal{L}}$ are length-preserving and move read actions to the left, it is terminating. Moreover, it is locally confluent by [16, Lemma 3.13] and hence confluent. Therefore, for any word $u \in \Sigma^*$ there is a unique, irreducible word $\mathrm{nf}_{\mathcal{L}}(u)$ with $u \to^* \mathrm{nf}_{\mathcal{L}}(u)$, the so-called *normal form* of u.

Example 3.3. Let $\mathcal{L} = (A, U)$ be a lossiness alphabet with $|A| \geq 2$, $a, b \in A$ with $a \neq b$, and $q = aabb\overline{ab}$. If $a \notin U$ then we have

$$aabb\overline{ab} \xrightarrow{(a)} aab\overline{a}b\overline{b} \xrightarrow{(a)} aa\overline{a}bb\overline{b} \xrightarrow{(c)} a\overline{a}abb\overline{b} \xrightarrow{(c)} a\overline{a}ab\overline{b}b$$

and therefore $\mathrm{nf}_{\mathcal{L}}(aabb\overline{ab}) = a\overline{a}ab\overline{b}b$. Otherwise, i.e., if $a \in U$, we can extend this derivation as follows:

$$a\overline{a}ab\overline{b}b \xrightarrow{(c)} a\overline{a}a\overline{b}bb \xrightarrow{(a)} a\overline{a}\overline{b}abb \xrightarrow{(b)} \overline{a}a\overline{b}abb \xrightarrow{(a)} \overline{a}\overline{b}aabb$$

and, hence, we obtain $\mathrm{nf}_{\mathcal{L}}(aabb\overline{ab}) = \overline{ab}aabb$.

From the definition of $\mathfrak{R}_{\mathcal{L}}$ we obtain that a word is in normal form if, and only if, it starts with some read operations followed by a special shuffle of write and read operations where each read action $\overline{a}$ appears directly right from a. At this juncture, the infixes $a\overline{a}$ in these words are separated by words from $(A \setminus (U \cup \{a\}))^*$, only. Formally, such shuffle of $u \in A^*$ and $\overline{v} \in \overline{A}^*$ is defined by

$$\langle\!\langle u, \overline{v} \rangle\!\rangle = w_1 a_1 \overline{a_1} w_2 a_2 \overline{a_2} \dots w_\ell a_\ell \overline{a_\ell} w_{\ell+1} ,$$

where $v = a_1 \dots a_\ell$, $a_1, \dots, a_\ell \in A$, $u = w_1 a_1 \dots w_\ell a_\ell w_{\ell+1}$, $w_i \in (A \setminus (U \cup \{a_i\}))^*$ for each $1 \leq i \leq \ell$, and $w_{\ell+1} \in A^*$. It is easy to see, that for any $u, v \in A^*$ the word $\langle\!\langle u, \overline{v} \rangle\!\rangle$ is defined if, and only if, v is a subword of some prefix u' of u and contains at least all unforgettable letters from U in u'. Formally, this special shuffled word is defined if there is a prefix u' of u with $\pi_U(u') \preceq v \preceq u$ holds. In this case, we call v an *$\mathcal{L}$-prefix* of u which we denote by $v \sqsubseteq_{\mathcal{L}} u$. For example, we have $aa \sqsubseteq_{(A,\{a\})} abaab$ and $aa \not\sqsubseteq_{(A,\{b\})} abaab$. Note that $\sqsubseteq_{\mathcal{L}}$ is a partial ordering. Concretely, $u \sqsubseteq_{(A,\emptyset)} v$ is the subword ordering on A and $u \sqsubseteq_{(A,A)} v$ is the prefix relation on A.

Then the set of all normal forms is

$$\begin{aligned}\mathrm{NF}_{\mathcal{L}} &:= \{\overline{u}\langle\!\langle v, \overline{w}\rangle\!\rangle \mid u, v, w \in A^*, v \sqsubseteq_{\mathcal{L}} w\} \\ &= \overline{A}^* \left(\bigcup_{a\in A} (A \smallsetminus (U \cup \{a\}))^* a\overline{a}\right)^* A^* .\end{aligned}$$

From this equation we can infer that $\mathrm{nf}_{\mathcal{L}}(w) = \overline{w_1}\langle\!\langle w_2, \overline{w_3}\rangle\!\rangle$ is characterized by three components: The first component is the projection to the write actions $\mathrm{wrt}(w) := w_2 = \pi_A(w)$ (note that the transitions of $\mathfrak{R}_{\mathcal{L}}$ preserve the relative ordering of the write operations). Similarly, the second is the projection to the read actions $\mathrm{rd}(w) := w_1w_3$ (note that we suppress the overlines in this projection). Finally, the third component is the *overlap* $\mathrm{rd}_2(w) := w_3$ of w. Note that the characterization of $\mathrm{NF}_{\mathcal{L}}$ from above implies that $\mathrm{rd}_2(w) \sqsubseteq_{\mathcal{L}} \mathrm{wrt}(w)$ holds. Additionally, we can define $\mathrm{rd}_1(w) := w_1$, i.e., $\mathrm{rd}_1(w)$ is the complementary prefix of $\mathrm{rd}(w)$ wrt. $\mathrm{rd}_2(w)$.

Example 3.4. Recall Example 3.3. There, in case of $a \notin U$ we have for $w = aabb\overline{ab}$: $\mathrm{wrt}(w) = aabb$, $\mathrm{rd}(w) = ab$, $\mathrm{rd}_1(w) = \varepsilon$, and $\mathrm{rd}_2(w) = ab$. Otherwise, if $a \in U$ we have $\mathrm{rd}_1(w) = ab$ and $\mathrm{rd}_2(w) = \varepsilon$.

While $\mathrm{rd}_1(w)$ is defined using the semi-Thue system $\mathfrak{R}_{\mathcal{L}}$, it also has a natural meaning: $\mathrm{rd}_1(w)$ is the shortest queue such that there is a run of the plq on execution of w that does not end up in the error state.

By [16, Theorem 3.15] the following holds about $\mathfrak{R}_{\mathcal{L}}$ and $\mathrm{nf}_{\mathcal{L}}(w)$:

Proposition 3.5. *Let $\mathcal{L} = (A, U)$ be a lossiness alphabet with $|A| \geq 2$ and let $v, w \in \Sigma^*$. Then we have*

$$\begin{aligned}v \equiv_{\mathcal{L}} w &\iff \mathrm{nf}_{\mathcal{L}}(v) = \mathrm{nf}_{\mathcal{L}}(w) \\ &\iff (\mathrm{wrt}(v), \mathrm{rd}(v), \mathrm{rd}_2(v)) = (\mathrm{wrt}(w), \mathrm{rd}(w), \mathrm{rd}_2(w)) .\end{aligned}$$

□

With this main property in mind we can also apply wrt, rd, rd_1, and rd_2 to equivalence classes of $\equiv_{\mathcal{L}}$ (i.e., elements from $\mathcal{Q}(\mathcal{L})$) instead of words from Σ^*.

Another question is the description of the word $u\overline{v}$ for any $u, v \in A^*$. We have $\mathrm{wrt}(u\overline{v}) = u$ and $\mathrm{rd}(u\overline{v}) = v$. It remains to describe the overlap $\mathrm{rd}_2(u\overline{v})$:

Lemma 3.6 ([16, Lemma 3.19]). *Let $\mathcal{L} = (A, U)$ be a lossiness alphabet with $|A| \geq 2$ and $u, v \in A^*$. Then $\mathrm{rd}_2(u\overline{v})$ is the longest suffix v' of v that satisfies $v' \sqsubseteq_{\mathcal{L}} u$.* □

Since $\equiv_{\mathcal{L}}$ is a congruence we can infer $w \equiv_{\mathcal{L}} \overline{\mathrm{rd}_1(w)}\,\mathrm{wrt}(w)\,\overline{\mathrm{rd}_2(w)}$ for each $w \in \Sigma^*$ from Lemma 3.6.

4. Algorithmic Properties of Rational Subsets

This section studies decision problems concerning the rational subsets of $\mathcal{Q}(\mathcal{L})$. We will see that the classes of rational and recognizable subsets do not coincide. Especially, we prove that we cannot decide whether a given rational subset of the plq monoid is recognizable. Additionally, we prove that emptiness of intersection and the unique decipherability in $\mathcal{Q}(\mathcal{L})$ are undecidable. Though, we will see first, that the uniform membership problem is NL-complete.

So, let $w \in \Sigma^*$. We will see in the next lemma that the number of left-divisors of $[w]_{\equiv_{\mathcal{L}}}$ in $\mathcal{Q}(\mathcal{L})$ is at most $|w|^3$. Hence, we can obtain a DFA with only $\mathcal{O}(|w|^3)$ many states that accepts $[w]_{\equiv_{\mathcal{L}}}$. Moreover, we prove an even stronger result by using only logarithmic space for the construction of this DFA. Note that the proof of this lemma is very similar to the proof of [11, Lemma 8.1], which states this result for reliable queues.

Lemma 4.1. *Let $\mathcal{L} = (A, U)$ be a lossiness alphabet. From $w \in \Sigma^*$, one can construct a DFA accepting $[w]_{\equiv_{\mathcal{L}}}$ using logarithmic additional space, only.*

Proof. Let $w = a_1 \dots a_n$ and let $0 \leq i, j, k \leq n$ be natural numbers. For the triple $p = (i, j, k)$ we define the words $p_1, p_2, p_3 \in A^*$ as follows:

- $p_1 = \mathrm{rd}(w[1, i])$ and $p_3 = \mathrm{rd}(w[i+1, j])$ as well as
- $p_2 = \mathrm{wrt}(w[1, k])$.

Then p is a state of the DFA if, and only if,

(i) $p_3 \sqsubseteq_{\mathcal{L}} p_2$,
(ii) $i = 0$ or $a_i \in \overline{A}$ and similarly $j = 0$ or $a_j \in \overline{A}$, and
(iii) $k = 0$ or $a_k \in A$.

Hence, we there is some bijection from the states p of the DFA to words $w_p := \overline{p_1}\langle\langle p_2, \overline{p_3}\rangle\rangle$ in normal form. In this connection property (i) ensures that $\langle\langle p_2, \overline{p_3}\rangle\rangle$ is well-defined, whereas properties (ii) and (iii) ensure that i, j, k are unique for each left-divisor of $[w]_{\equiv_{\mathcal{L}}}$.

The initial state of the DFA is $\iota = (0, 0, 0)$, i.e., $w_\iota = \varepsilon$. The state $p = (i, j, k)$ is accepting iff $w_p \equiv_{\mathcal{L}} w$.

Now we want to define the transitions of the automaton such that, after reading of $v \in \Sigma^*$, the automaton reaches a state p with $w_p = \mathrm{nf}_{\mathcal{L}}(v)$, provided that such state exists. Furthermore, we want to make sure that such a state exists whenever $[v]_{\equiv_{\mathcal{L}}}$ is a left-divisor of $[w]_{\equiv_{\mathcal{L}}}$.

So, let $p = (i, j, k)$ be a state and $a \in A$. To define the state reached from p after writing a, let $k' > k$ be the minimal write-position in w after k. In other words, we have $k' > k$, $a_{k'} \in A$, and $w[k+1, k'-1] \in \overline{A}^*$. If there is no such k' or if $a_{k'} \neq a$, then the DFA ends up in an error state. Otherwise, it moves to $q = (i, j, k')$, which is a state of the automaton, again. Then we have

$$\begin{aligned} w_p \cdot a &= \overline{\mathrm{rd}(w[1,i])}\left\langle\!\left\langle \mathrm{wrt}(w[1,k]), \overline{\mathrm{rd}(w[i+1,j])} \right\rangle\!\right\rangle \cdot a \\ &= \overline{\mathrm{rd}(w[1,i])}\left\langle\!\left\langle \mathrm{wrt}(w[1,k])a, \overline{\mathrm{rd}(w[i+1,j])} \right\rangle\!\right\rangle \\ &= \overline{\mathrm{rd}(w[1,i])}\left\langle\!\left\langle \mathrm{wrt}(w[1,k']), \overline{\mathrm{rd}(w[i+1,j])} \right\rangle\!\right\rangle \\ &= w_q \,. \end{aligned}$$

Next, we define which state is reached from p after reading $\overline{a}$. Let $j' > j$ be the minimal read-position in w after j. In other words, we have $j' > j$, $a_{j'} \in \overline{A}$ and, $w[j+1, j'-1] \in A^*$. If there is no such j' or if $a_{j'} \neq \overline{a}$, then the DFA ends up in an error state. So, assume, that such j' exists and $a_{j'} = \overline{a}$. Additionally let $i' \geq i$ with $i' = 0$ or $a_{i'} \in \overline{A}$ be minimal such that $\mathrm{rd}(w[i'+1, j']) \sqsubseteq_{\mathcal{L}} \mathrm{wrt}(w[1,k])$ holds. Then $q = (i', j', k)$ is a state of the DFA and the DFA moves from p to q when reading $\overline{a}$. Then we get

$$\begin{aligned} w_p\,\overline{a} = {} & \overline{\mathrm{rd}(w[1,i])}\left\langle\!\left\langle \mathrm{wrt}(w[1,k]), \overline{\mathrm{rd}(w[i+1,j])} \right\rangle\!\right\rangle \cdot \overline{a} \\ \equiv_{\mathcal{L}} {} & \overline{\mathrm{rd}(w[1,i])}\mathrm{wrt}(w[1,k])\overline{\mathrm{rd}(w[i+1,j])} \cdot \overline{a} && \text{(by Lemma 3.6)} \\ = {} & \overline{\mathrm{rd}(w[1,i])}\mathrm{wrt}(w[1,k])\overline{\mathrm{rd}(w[i+1,j'])} \\ \equiv_{\mathcal{L}} {} & \overline{\mathrm{rd}(w[1,i'])}\mathrm{wrt}(w[1,k])\overline{\mathrm{rd}(w[i'+1,j'])} && \text{(by Lemma 3.6)} \\ \equiv_{\mathcal{L}} {} & \overline{\mathrm{rd}(w[1,i'])}\left\langle\!\left\langle \mathrm{wrt}(w[1,k]), \overline{\mathrm{rd}(w[i'+1,j'])} \right\rangle\!\right\rangle && \text{(by Lemma 3.6)} \\ = {} & w_q \,. \end{aligned}$$

This finishes the construction of the DFA.

Now let $v \in \Sigma^*$. If there is a v-labeled path from $\iota = (0,0,0)$ to some state q, we obtain $v \equiv_{\mathcal{L}} w_q$ by induction on $|v|$ from the above calculations. In particular, any word v accepted by the DFA satisfies $v \equiv_{\mathcal{L}} w$, i.e., $v \in [w]_{\equiv_{\mathcal{L}}}$.

Now let $[v]_{\equiv_{\mathcal{L}}}$ be a left-divisor of $[w]_{\equiv_{\mathcal{L}}}$. Then $\mathrm{wrt}(v)$ and $\mathrm{rd}(v)$ are prefixes of $\mathrm{wrt}(w)$ and $\mathrm{rd}(w)$, respectively, since π and $\overline{\pi}$ are homomorphisms. Then by induction on $|v|$ we obtain a v-labeled path from ι to some state p with $v \equiv_{\mathcal{L}} w_p$. In particular, if $v \in [w]_{\equiv_{\mathcal{L}}}$, then $w_p \equiv_{\mathcal{L}} v \equiv_{\mathcal{L}} w$, i.e., p is accepting. Thus, the DFA accepts $[w]_{\equiv_{\mathcal{L}}}$.

By the construction of the DFA, it is clear that a Turing machine with w on its input tape can, using logarithmic space on its work tape, write the list of all transitions on its one-way output tape. □

Theorem 4.2. *The following* rational subset membership problem *for plq monoids is* NL*-complete:*

> ***Input:*** *A lossiness alphabet $\mathcal{L} = (A, U)$, a word $w \in \Sigma^*$, and an NFA $\mathfrak{A}$ over Σ*
> ***Question:*** *Is there a word $v \in L(\mathfrak{A})$ with $w \equiv_{\mathcal{L}} v$?*

Proof. Let $w \in \Sigma^*$ and let $\mathfrak{A}$ be an NFA over Σ. Let $\mathfrak{B}$ be the DFA from Lemma 4.1 that can be constructed using only logarithmic additional space.

Then there exists $v \in L(\mathfrak{A})$ with $w \equiv_{\mathcal{L}} v$ if, and only if, $L(\mathfrak{A}) \cap [w]_{\equiv_{\mathcal{L}}} \neq \emptyset$ if, and only if, $L(\mathfrak{A}) \cap L(\mathfrak{B}) \neq \emptyset$. Using an on-the-fly construction of $\mathfrak{B}$, this can be decided non-deterministically in logarithmic space. Hence, the problem is in NL.

Since the free monoid A^* embeds into $\mathcal{Q}(\mathcal{L})$ and since the rational subset membership problem for A^* is NL-hard, we also get NL-hardness for $\mathcal{Q}(\mathcal{L})$. □

Now we will prove some negative algorithmic results on rational subsets of the plq monoid. In [11, Section 8] these undecidabilities for reliable queues could be inferred from an embedding of $\{a,b\}^* \times \{c,d\}^*$ into $\mathcal{Q}(A,A)$ for any at least binary alphabet A. Unfortunately, this does not work in arbitrary plq monoids since this direct product does not embed into $\mathcal{Q}(\{a,b\},\emptyset)$ by [16, Theorem 6.14]. Though, we can prove all the undecidability results considered in [11] for any plq monoid.

Some of these results base on an embedding of the monoid $\{a\}^* \times \{c,d\}^*$ into $\mathcal{Q}(\mathcal{L})$. Unfortunately, this does not help for the following two problems since their counterparts in $\{a\}^* \times \{c,d\}^*$ are decidable. Hence, we have to prove them directly.

The first considered decision problem is the unique decipherability problem in $\mathcal{Q}(\mathcal{L})$, i.e., the question whether a given finite set T freely generates T^*. In this context, for $S,T \subseteq \mathcal{Q}(\mathcal{L})$, the set S is *freely generated by* T if $S = T^*$ and for each $m,n \in \mathbb{N}$ and $s_1,\ldots,s_m,t_1,\ldots,t_n \in T$ with $s_1 \ldots s_m = t_1 \ldots t_n$ we have $m = n$ and $s_i = t_i$ for each $1 \leq i \leq m = n$. We prove that the unique decipherability problem in $\mathcal{Q}(\mathcal{L})$ is undecidable. To this end, we will use the undecidability of this problem in $\{a,b\}^* \times \{c,d\}^*$ by encoding the elements of the given set and adding another item.

Theorem 4.3. *Let $\mathcal{L} = (A,U)$ be a lossiness alphabet. The* unique decipherability problem *in $\mathcal{Q}(\mathcal{L})$ is undecidable:*

Input: *A finite set $T \subseteq \mathcal{Q}(\mathcal{L})$*
Question: *Is T^* freely generated by T?*

Proof. We prove this undecidability by reduction of this question for the monoid $\{a,b\}^* \times \{c,d\}^*$, which is undecidable by [6, Theorem 3.1]. So, let $a,b \in A$ be distinct letters and

$$S := \{(x_1,y_1),\ldots,(x_k,y_k)\} \subseteq \{a,b\}^* \times \{c,d\}^* .$$

Define the embeddings $f\colon \{a,b\}^* \to A^*$ and $g\colon \{c,d\}^* \to A^*$ by $f(a) = g(c) = aa$ and $f(b) = g(d) = ab$. Set $t_0 := [\overline{bbb}bb]_{\equiv_{\mathcal{L}}}$, $t_i := [f(x_i)\overline{g(y_i)}]_{\equiv_{\mathcal{L}}}$ for any $1 \leq i \leq k$, and

$$T := \{t_i \mid 0 \leq i \leq k\} \subseteq \mathcal{Q}(\mathcal{L}) .$$

We show now that S^* is freely generated by S if, and only if, T^* is freely generated by T.

First we assume that S^* is not freely generated by S. Then there are indices $(i_1, \dots, i_m) \neq (j_1, \dots, j_n)$ where $m, n > 0$ such that

$$(x_{i_1} \dots x_{i_m}, y_{i_1} \dots y_{i_m}) = (x_{j_1} \dots x_{j_n}, y_{j_1} \dots y_{j_n}) .$$

Let $\ell = |f(x_{i_1} \dots x_{i_m})|$. Set $p = t_{i_1} \dots t_{i_m} t_0^\ell$ and $q = t_{j_1} \dots t_{j_n} t_0^\ell$. Then we have $\mathrm{wrt}(p) = \mathrm{wrt}(q)$ and $\mathrm{rd}(p) = \mathrm{rd}(q)$. Furthermore, by $\mathrm{rd}_2(p) \sqsubseteq_{\mathcal{L}} \mathrm{wrt}(p)$ we see that $\mathrm{rd}_2(p)$ is at most $b^{3\ell}$ since $\mathrm{wrt}(p)$ contains at most 3ℓ b's. Similarly, $\mathrm{rd}_2(q)$ is at most $b^{3\ell}$. As p and q have the same suffix t_0^ℓ and $\mathrm{wrt}(t_{i_1} \dots t_{i_m}) = \mathrm{wrt}(t_{j_1} \dots t_{j_n})$ holds we can infer that $\mathrm{rd}_2(p) = \mathrm{rd}_2(q)$. By Proposition 3.5 $p = q$ holds and therefore T^* is not freely generated by T.

Now let T^* be not freely generated by T. If $[\varepsilon]_{\equiv_{\mathcal{L}}} \in T$ and hence $(\varepsilon, \varepsilon) \in S$ holds S^* is trivially not freely generated by S. So, we assume that $[\varepsilon]_{\equiv_{\mathcal{L}}} \notin T$ and hence $(\varepsilon, \varepsilon) \notin S$. Then there are indices $(i_1, \dots, i_m) \neq (j_1, \dots, j_n)$ where $m, n > 0$ such that $t_{i_1} \dots t_{i_m} = t_{j_1} \dots t_{j_n}$. Let $(i'_1, \dots, i'_{m'}), (j'_1, \dots j'_{n'})$ be the above sequences after deletion of all 0's. Since t_0 is the only element in T adding bb into the projections of write and read actions, t_0 does not commute with t_i for any $1 \leq i \leq k$. Hence, we still have $(i'_1, \dots, i'_{m'}) \neq (j'_1, \dots j'_{n'})$, $\mathrm{wrt}(t_{i'_1} \dots t_{i'_{m'}}) = \mathrm{wrt}(t_{j'_1} \dots t_{j'_{n'}})$, and $\mathrm{rd}(t_{i'_1} \dots t_{i'_{m'}}) = \mathrm{rd}(t_{j'_1} \dots t_{j'_{n'}})$. Then we have:

$$f(x_{i'_1} \dots x_{i'_{m'}}) = \mathrm{wrt}(t_{i'_1} \dots t_{i'_{m'}}) = \mathrm{wrt}(t_{j'_1} \dots t_{j'_{n'}}) = f(x_{j'_1} \dots x_{j'_{n'}}) ,$$
$$g(y_{i'_1} \dots y_{i'_{m'}}) = \mathrm{rd}(t_{i'_1} \dots t_{i'_{m'}}) = \mathrm{rd}(t_{j'_1} \dots t_{j'_{n'}}) = g(y_{j'_1} \dots y_{j'_{n'}}) .$$

By injectivity of f and g we infer that

$$(x_{i'_1} \dots x_{i'_{m'}}, y_{i'_1} \dots y_{i'_{m'}}) = (x_{j'_1} \dots x_{j'_{n'}}, y_{j'_1} \dots y_{j'_{n'}}) ,$$

i.e., S^* is not freely generated by S. □

The next problem to consider is the emptiness of intersections of rational subsets in the plq monoid. Given two recognizable sets, this problem is decidable since the class of recognizable subsets is closed under intersection. However, we will prove that this decidability does not hold for arbitrary rational subsets. As a corollary we can infer that the class of rational subsets is not effectively closed under intersection. Afterwards we will prove the existence of two rational subsets whose intersection is not rational. In other words, the classes of rational and recognizable subsets do not coincide. Nevertheless, each recognizable set in $\mathcal{Q}(\mathcal{L})$ is rational due to [19] since the plq monoid is finitely generated.

Theorem 4.4. *Let $\mathcal{L} = (A, U)$ be a lossiness alphabet. Then the* intersection problem *for rational sets in $\mathcal{Q}(\mathcal{L})$ is undecidable:*

Input: *Two rational sets $S, T \subseteq \mathcal{Q}(\mathcal{L})$*
Question: *Does $S \cap T = \emptyset$ hold?*

Proof. We prove this by reduction of Post's Correspondence Problem (PCP), which is undecidable by [25]. So, let $a, b \in A$ be distinct letters and $I =$

$((x_1, y_1), \ldots, (x_k, y_k))$ be an instance of the PCP with $x_i, y_i \in A^*$. For $1 \leq i \leq k$ we define the transformation sequences $s_i := [a^i b\overline{x_i}]_{\equiv_\mathcal{L}}$ and $[a^i b\overline{y_i}]_{\equiv_\mathcal{L}}$. Then we can define rational sets as follows:

$$S_I := \{s_i \mid 1 \leq i \leq k\}^+ [\overline{a}\overline{b}^*]_{\equiv_\mathcal{L}} \quad \text{and} \quad Y_I := \{t_i \mid 1 \leq i \leq k\}^+ [\overline{a}\overline{b}^*]_{\equiv_\mathcal{L}} .$$

Now we have to show that $S_I \cap T_I \neq \emptyset$ iff I has a solution. So, let $t \in S_I \cap T_I$. Then by definition of S_I and T_I there are $\ell \in \mathbb{N}$ and indices $i_1, \ldots, i_m$ and $j_1, \ldots, j_n$ with $m, n > 0$ such that $t = s_{i_1} \ldots s_{i_m} [\overline{a}\overline{b}^\ell]_{\equiv_\mathcal{L}} = t_{j_1} \ldots t_{j_n} [\overline{a}\overline{b}^\ell]_{\equiv_\mathcal{L}}$ holds. Then, we have $\mathrm{rd}(t) = x_{i_1} \ldots x_{i_m} ab^\ell = y_{j_1} \ldots y_{j_n} ab^\ell$ implying $x_{i_1} \ldots x_{i_m} = y_{j_1} \ldots y_{j_n}$. By $\mathrm{wrt}(t) = a^{i_1} b \ldots a^{i_m} b = a^{j_1} b \ldots a^{j_n} b$ we can infer $(i_1, \ldots, i_m) = (j_1, \ldots, j_n)$ which is a solution of I.

For the converse implication we assume that I has a solution $(i_1, \ldots, i_n)$ where $n > 0$. Set $s' = s_{i_1} \ldots s_{i_n}$ and $t' = t_{i_1} \ldots t_{i_n}$ and set $s = s'[\overline{ab^n}]_{\equiv_\mathcal{L}} \in X_I$ and $t = t'[\overline{ab^n}]_{\equiv_\mathcal{L}} \in Y_I$. Then we have $\mathrm{wrt}(s) = \mathrm{wrt}(t)$ and $\mathrm{rd}(s) = \mathrm{rd}(t)$. Furthermore, by Proposition 3.5 we have

$$s = \left[\overline{\mathrm{rd}_1(s')}\mathrm{wrt}(s)\overline{\mathrm{rd}_2(s')}\,\overline{ab^n}\right]_{\equiv_\mathcal{L}} \quad \text{and} \quad t = \left[\overline{\mathrm{rd}_1(t')}\mathrm{wrt}(t)\overline{\mathrm{rd}_2(t')}\,\overline{ab^n}\right]_{\equiv_\mathcal{L}} .$$

Due to Lemma 3.6 $\mathrm{rd}_2(s)$ is the longest suffix of $\mathrm{rd}(s')ab^n$ that is a $\mathcal{L}$-prefix of $\mathrm{wrt}(s)$. By definition of s this is b^n or an even shorter suffix of $\mathrm{wrt}(s)$ since it contains exactly n b's. Similarly this holds for t as well. Hence, we have $\mathrm{rd}_2(s) = \mathrm{rd}_2(t)$ and therefore $s = t$ by Proposition 3.5, i.e., $S_I \cap T_I \neq \emptyset$. Hence, we reduced PCP to this emptiness problem which is therefore undecidable. □

To prove that the rational subsets are not closed under intersection and to prove the undecidability of the next problems we use an embedding of $\{a\}^* \times \{b, c\}^*$ into the plq monoid. Assume that $a, b \in A$ holds and a and b are distinct letters. Such an embedding is $\psi\colon \{a\}^* \times \{b, c\}^* \to \mathcal{Q}(\mathcal{L})$ with $\psi(a, \varepsilon) = [a]_{\equiv_\mathcal{L}}$, $\psi(\varepsilon, b) = [\overline{ab}]_{\equiv_\mathcal{L}}$, and $\psi(\varepsilon, c) = [\overline{abb}]_{\equiv_\mathcal{L}}$ by [16, Lemma 6.17].

Theorem 4.5. *Let $\mathcal{L} = (A, U)$ be a lossiness alphabet. The set of rational subsets of $\mathcal{Q}(\mathcal{L})$ is not closed under intersection. In particular, there is a rational subset of $\mathcal{Q}(\mathcal{L})$ which is not recognizable.*

Proof. Consider the following rational relations:

$$R_1 = \{(a^m, b^m c^n) \mid m, n \in \mathbb{N}\} \quad \text{and} \quad R_2 = \{(a^m, b^n c^m) \mid m, n \in \mathbb{N}\} .$$

Then $\psi(R_1)$ and $\psi(R_2)$ are rational in $\mathcal{Q}(\mathcal{L})$. Suppose that $\psi(R_1) \cap \psi(R_2)$ is rational. Then there is a regular language $L \subseteq \Sigma^*$ with $\eta_\mathcal{L}(L) = \psi(R_1) \cap \psi(R_2)$. Since ψ is injective we have

$$\psi(R_1) \cap \psi(R_2) = \psi(R_1 \cap R_2) = \psi(\{(a^n, b^n c^n) \mid n \in \mathbb{N}\}) .$$

Hence, by definition of ψ we have $\mathrm{rd}(L) = \{(ab)^n (abb)^n \mid n \in \mathbb{N}\}$ which would be regular since rd is a homomorphism. But this is a contradiction to the Pumping Lemma. □

By $\mathbb{I} \subseteq \mathcal{Q}(\mathcal{L})$ we denote the image of ψ, i.e., ψ is an isomorphism from $\{a\}^* \times \{b,c\}^*$ onto $\mathbb{I}$. It is easy to see that $\mathbb{I} = \{[a]_{\equiv_\mathcal{L}}, [a\overline{b}]_{\equiv_\mathcal{L}}, [a\overline{bb}]_{\equiv_\mathcal{L}}\}^*$. In the following lemma we prove that recognizability in this submonoid implies recognizability in the whole plq monoid.

Lemma 4.6. *Let $T \subseteq \mathbb{I}$ be recognizable in $\mathbb{I}$. Then T is recognizable in $\mathcal{Q}(\mathcal{L})$.*

Proof. Let $T \subseteq \mathbb{I}$ be recognizable in $\mathbb{I}$. Then $\psi^{-1}(T) \subseteq \{a\}^* \times \{b,c\}^*$ is recognizable in $\{a\}^* \times \{b,c\}^*$. Due to Mezei's Theorem [2, Theorem III.1.5] there are regular languages $V_i \subseteq \{a\}^*$ and $W_i \subseteq \{b,c\}^*$ with

$$\psi^{-1}(T) = \bigcup_{1 \leq i \leq k} V_i \times W_i \,.$$

Now we define homomorphisms $g\colon \{a\}^* \to A^*$ by $g(a) = a = \psi(a,\varepsilon)$ and $h\colon \{b,c\}^* \to A^*$ by $h(b) = a\overline{b} = \psi(\varepsilon, b)$ and $h(c) = a\overline{bb} = \psi(\varepsilon, c)$. Then $g(V_i), h(W_i) \subseteq A^*$ are regular as well. Hence, $\mathrm{wrt}^{-1}(g(V_i))$ and $\mathrm{rd}^{-1}(h(W_i))$ are recognizable in $\mathcal{Q}(\mathcal{L})$ and therefore

$$T = \bigcup_{1 \leq i \leq k} \mathrm{wrt}^{-1}(g(V_i)) \cap \mathrm{rd}^{-1}(h(W_i))$$

also is recognizable. □

To prove the undecidability of the universality and the recognizability problem we use the embedding ψ and the results from [9] which state that their counterparts in $\{a\}^* \times \{b,c\}^*$ are undecidable.

Theorem 4.7. *Let $\mathcal{L} = (A, U)$ be a lossiness alphabet. Then the following statements hold:*

(1) *The* universality problem *for rational sets in $\mathcal{Q}(\mathcal{L})$ is undecidable:*

   Input: *A rational set $T \subseteq \mathcal{Q}(\mathcal{L})$*
   Question: *Does $T = \mathcal{Q}(\mathcal{L})$ hold?*

   Consequently, the inclusion problem *and the* equality problem *for rational sets in $\mathcal{Q}(\mathcal{L})$ are undecidable.*

(2) *The* recognizability problem *for rational sets in $\mathcal{Q}(\mathcal{L})$ is undecidable:*

   Input: *A rational set $T \subseteq \mathcal{Q}(\mathcal{L})$*
   Question: *Is T recognizable in $\mathcal{Q}(\mathcal{L})$?*

Proof.

(1) Let $T \subseteq \{a\}^* \times \{b,c\}^*$ be rational. Then $\psi(T)$ is rational in $\mathcal{Q}(\mathcal{L})$. Due to Lemma 4.6, the set $\mathbb{I}$ is recognizable in $\mathcal{Q}(\mathcal{L})$. Therefore, $\mathcal{Q}(\mathcal{L}) \setminus \mathbb{I}$ is recognizable and, hence, rational in $\mathcal{Q}(\mathcal{L})$ since this is finitely generated. Consequently, $\psi(T) \cup (\mathcal{Q}(\mathcal{L}) \setminus \mathbb{I})$ is rational as well. This set equals $\mathcal{Q}(\mathcal{L})$ iff $\psi(T) = \mathbb{I}$, i.e., $T = \{a\}^* \times \{b,c\}^*$. But this latter question is undecidable by [9, Theorem 2(Q4)].

(2) Let $T \subseteq \{a\}^* \times \{b,c\}^*$ be rational. Then $\psi(T)$ is rational. By Lemma 4.6 $\psi(T)$ is recognizable in $\mathcal{Q}(\mathcal{L})$ iff it is recognizable in $\mathbb{I}$. This is the case iff T is recognizable in $\{a\}^* \times \{b,c\}^*$. But this latter question is undecidable by [9, Theorem 2(Q6)]. □

5. Characterizations of the Recognizable Subsets

In Section 4 we have shown many decision problems on rational subsets of the plq monoid to be undecidable. We know that all of these problems are decidable if the given subsets are recognizable from the known constructions in automata theory. Here, we want to give some characterizations of the recognizable subsets to determine in which cases a rational subset also is recognizable. Concretely, we prove characterizations in the manner of Kleene's and Büchi's Theorem [13,4], i.e., we characterize the recognizable sets as certain rational sets and by logical means.

At first, we state our main theorem. Later in this section we give the concrete definitions of q-rational subsets and $\mathsf{MSO_q}$ and prove the correctness of this theorem.

Theorem 5.1 (Main Theorem). *Let $\mathcal{L} = (A,U)$ be a lossiness alphabet and $T \subseteq \mathcal{Q}(\mathcal{L})$. Then the following are equivalent:*

(A) T is recognizable.
(B) T is q-rational.
(C) T is $\mathsf{MSO_q}$-definable.

5.1. *Some helpful characterizations*

Before we prove Theorem 5.1 we prove two further characterizations which turned out to be convenient for simplification of the proof of Theorem 5.1. We know these characterizations from [11] for the recognizable subsets in the reliable queue monoid $\mathcal{Q}(A,A)$. Concretely, we generalize those equivalences to plq monoids $\mathcal{Q}(\mathcal{L})$ with arbitrary lossiness alphabet $\mathcal{L} = (A,U)$. On the one hand, we prove the correspondence of recognizability in the plq monoid to regularity in the underlying free monoid. On the other hand, we show that each recognizable subset is a Boolean combination of sets $\mathrm{wrt}^{-1}(R)$, $\mathrm{rd}^{-1}(R)$ where $R \subseteq A^*$ is regular and some special sets Ω_ℓ for any $\ell \in \mathbb{N}$:

Definition 5.2. *Let $\mathcal{L} = (A,U)$ be a lossiness alphabet and $t \in \mathcal{Q}(\mathcal{L})$. Then the* overlap's bounded width *of t is*

$$\mathrm{obw}(t) := \inf\left\{ |\mathrm{rd}_2(s)| \;\middle|\; \begin{array}{c} s \in \mathcal{Q}(\mathcal{L}), \mathrm{wrt}(s) = \mathrm{wrt}(t), \mathrm{rd}(s) = \mathrm{rd}(t), \\ |\mathrm{rd}_2(s)| > |\mathrm{rd}_2(t)| \end{array} \right\}.$$

Similarly, for $w \in \Sigma^*$ we may define $\mathrm{obw}(w) := \mathrm{obw}([w]_{\equiv_\mathcal{L}})$. Furthermore, for $\ell \in \mathbb{N}$ set

$$\Omega_\ell := \{t \in \mathcal{Q}(\mathcal{L}) \mid \mathrm{obw}(t) > \ell\}.$$

The overlap's bounded width specifies the minimal length of the overlap of a word with the same projections having a longer overlap. If no such word exists then we set this value to ∞.

Example 5.3. Let $A = U = \{a, b\}$ and $t = [\overline{abab}a\overline{a}b\overline{b}abab]_{\equiv_\mathcal{L}}$. Then there are two words with the same projections and longer overlaps:

$$s_1 = [\overline{ab}a\overline{a}b\overline{b}a\overline{a}b\overline{b}ab]_{\equiv_\mathcal{L}} \qquad \text{and} \qquad s_2 = [a\overline{a}b\overline{b}a\overline{a}b\overline{b}a\overline{a}b\overline{b}]_{\equiv_\mathcal{L}} .$$

We have $|\mathrm{rd}_2(s_1)| = 4$ and $|\mathrm{rd}_2(s_2)| = 6$. Therefore, we have $\mathrm{obw}(t) = 4$, $\mathrm{obw}(s_1) = 6$, and $\mathrm{obw}(s_2) = \infty$. Hence, $t \in \Omega_3 \setminus \Omega_4$ holds.

From [11, Observation 9.1] we know that any non-trivial property of the overlap's width $|\mathrm{rd}_2(t)|$ cannot be recognized in $\mathcal{Q}(A, A)$. An appropriate alternative for the generators of the Boolean algebra of recognizable subsets was found in such kind of "approximation" of the overlap's length (note that $\mathrm{obw}(t) > |\mathrm{rd}_2(t)|$ holds). Additionally, the following observations provide some more motivation of this notion:

Observation 5.4. *Every $t \in \mathcal{Q}(\mathcal{L})$ is completely described by* $\mathrm{wrt}(t)$*,* $\mathrm{rd}(t)$*, and* $\mathrm{obw}(t)$*.*

Proof. Fix $u, v \in A^*$. Let $t \in \mathcal{Q}(\mathcal{L})$ with $\mathrm{wrt}(t) = u$ and $\mathrm{rd}(t) = v$. Furthermore, let $m = \mathrm{obw}(t)$. Due to Proposition 3.5 it suffices to provide $\mathrm{rd}_2(t)$ in terms of u, v, and m. To this end, let $w \in A^*$ be the longest suffix of v being an $\mathcal{L}$-prefix of u and satisfies $|w| < m$. We show that $\mathrm{rd}_2(t) = w$ or rather $|\mathrm{rd}_2(t)| = |w|$ since both words are suffixes of v.

We have $|\mathrm{rd}_2(t)| < \mathrm{obw}(t) = m$ by definition of $\mathrm{obw}(t)$. By definition of $\mathrm{rd}_2(t)$, this word is a suffix of v and an $\mathcal{L}$-prefix of u. Due to the choice of w we have $|\mathrm{rd}_2(t)| \le |w|$. Moreover, there is $s \in \mathcal{Q}(\mathcal{L})$ with $\mathrm{wrt}(s) = \mathrm{wrt}(t)$, $\mathrm{rd}(s) = \mathrm{rd}(t)$, and $\mathrm{rd}_2(p) = w$. Hence, we have $|\mathrm{rd}_2(t)| \le |\mathrm{rd}_2(s)| < \mathrm{obw}(t)$. But from definition of $\mathrm{obw}(t)$ we infer that $|\mathrm{rd}_2(t)| = |\mathrm{rd}_2(s)| = |w|$. □

Observation 5.5. *Let $\ell \in \mathbb{N}$ and $w \in \Sigma^*$. Then* $\mathrm{obw}(w) \le \ell$ *if, and only if, there is $u \in A^{\le \ell}$ with* $\mathrm{rd}(w) \in A^*u$ *and $u \sqsubseteq_\mathcal{L}$* $\mathrm{wrt}(w)$ *such that* $|\mathrm{rd}_2(w)| < |u|$*.*

Proof. First, suppose $\mathrm{obw}(w) > \ell$. Let $u \in A^{\le \ell}$ such that $\mathrm{rd}(w) \in A^*u$ and $u \sqsubseteq_\mathcal{L} \mathrm{wrt}(w)$. Furthermore, let $x \in A^*$ with $\mathrm{rd}(w) = xu$. Set $t := [\overline{x}\,\mathrm{wrt}(w)\,\overline{u}]_{\equiv_\mathcal{L}}$. Then we have $\mathrm{wrt}(w) = \mathrm{wrt}(t)$, $\mathrm{rd}(w) = \mathrm{rd}(t)$, and $|\mathrm{rd}_2(t)| = |u| \le k$. Since $|\mathrm{rd}_2(w)| < |\mathrm{rd}_2(t)|$ would imply $\mathrm{obw}(t) \le |\mathrm{rd}_2(t)| \le \ell$, we have $|\mathrm{rd}_2(w)| \ge |\mathrm{rd}_2(t)| = |u|$.

Now assume $\mathrm{obw}(w) \le \ell$. Then there is $t \in \mathcal{Q}(\mathcal{L})$ with $\mathrm{wrt}(w) = \mathrm{wrt}(t)$, $\mathrm{rd}(w) = \mathrm{rd}(t)$, and $|\mathrm{rd}_2(w)| < |\mathrm{rd}_2(t)| \le \ell$. Consider $u = \mathrm{rd}_2(t)$. Then we have $|u| = |\mathrm{rd}_2(t)| \le \ell$, $\mathrm{rd}(w) = \mathrm{rd}(t) \in A^*u$, and $u \sqsubseteq_\mathcal{L} \mathrm{wrt}(t) = \mathrm{wrt}(w)$. □

Now we can state the following equivalences:

Theorem 5.6. *Let $\mathcal{L} = (A, U)$ be a lossiness alphabet and $T \subseteq \mathcal{Q}(\mathcal{L})$. Then the following are equivalent:*

(i) T is recognizable.
(ii) $\eta_{\mathcal{L}}^{-1}(T) \cap \overline{A}^* A^* \overline{A}^*$ is regular.
(iii) T is a Boolean combination of sets of the form $\mathrm{wrt}^{-1}(R)$ or $\mathrm{rd}^{-1}(R)$ for some regular $R \subseteq A^*$ and the sets Ω_ℓ for some $\ell \in \mathbb{N}$.

Remark 5.7. Let $\mathcal{L} = (A, U)$ be a lossiness alphabet. If $A = U$ holds (i.e., we are considering a reliable queue), we can extend Theorem 5.6 by

(iv) $\eta_{\mathcal{L}}^{-1}(T) \cap A^* \overline{A}^* A^*$ is regular.

These equivalences hold also if A is a singleton (i.e., $|A| = 1$).

However, in the other cases (i.e., $|A| \geq 2$ and $A \neq U$) statement (iv) is not equivalent to the other ones: for example, let $a, b \in A$ be distinct with $a \in U$. Then $T = \{[\overline{a}^n a^n b\overline{b}]_{\equiv_{\mathcal{L}}} \mid n \geq 1\}$ is not recognizable, since the language

$$\eta_{\mathcal{L}}^{-1}(T) = \{\overline{a}^n a^n b\overline{b} \mid n \geq 1\}$$

is not regular. However, the set $\eta_{\mathcal{L}}^{-1}(S) \cap A^* \overline{A}^* A^*$ is empty and, hence, regular.

Moreover, this example demonstrates that the theorem cannot be expanded by $\eta_{\mathcal{L}}^{-1}(T) \cap L$ where L is $\overline{A}^* A^*$, $A^* \overline{A}^*$, or the union of both sets.

5.1.1. *The implication "(i)⇒(ii)" in Theorem 5.6*

The first implication in Theorem 5.6 that we want to prove is very simple:

Proposition 5.8. *Let $\mathcal{L} = (A, U)$ be a lossiness alphabet and $T \subseteq \mathcal{Q}(\mathcal{L})$ be recognizable. Then $\eta_{\mathcal{L}}^{-1}(T) \cap \overline{A}^* A^* \overline{A}^*$ is regular.*

Proof. Since T is recognizable, the set $\eta_{\mathcal{L}}^{-1}(T)$ is recognizable and hence regular. Since the class of regular languages is closed under intersection, $\eta_{\mathcal{L}}^{-1}(T) \cap \overline{A}^* A^* \overline{A}^*$ is regular. □

5.1.2. *The implication "(ii)⇒(iii)" in Theorem 5.6*

Fix some lossiness alphabet $\mathcal{L} = (A, U)$. To prove the next implication we partition the subsets $T \subseteq \mathcal{Q}(\mathcal{L})$, that are satisfying regularity of $\eta_{\mathcal{L}}^{-1}(T) \cap \overline{A}^* A^* \overline{A}^*$, and show that these can be generated by the Boolean operations from the sets $\mathrm{wrt}^{-1}(R)$, $\mathrm{rd}^{-1}(R)$ for some regular $R \subseteq A^*$ and from the sets Ω_ℓ for some $\ell \in \mathbb{N}$. To this end, we show that we can extract a sequence of ℓ reading symbols from the right end of any queue transformation sequence from Ω_ℓ.

Lemma 5.9. *Let $\ell \in \mathbb{N}$, $t \in \Omega_\ell$, and $v \in A^\ell$ be a suffix of $\mathrm{rd}(t)$. Then there is $s \in \mathcal{Q}(\mathcal{L})$ such that $t = s\,[\overline{v}]_{\equiv_{\mathcal{L}}}$.*

Proof. Let $t \in \Omega_\ell$. By Lemma 3.6 we have $t = [\overline{\mathrm{rd}_1(t)}\,\mathrm{wrt}(t)\,\overline{\mathrm{rd}_2(t)}]_{\equiv_\mathcal{L}}$. If $|\mathrm{rd}_2(t)| \geq \ell$ we set $s := [\overline{\mathrm{rd}_1(t)}\,\mathrm{wrt}(t)\,\overline{u}]_{\equiv_\mathcal{L}}$ where $\mathrm{rd}_2(t) = uv$ and we have $t = s\,[\overline{v}]_{\equiv_\mathcal{L}}$.

If $|\mathrm{rd}_2(t)| < \ell$ then $\mathrm{rd}_2(t)$ is the longest suffix of $\mathrm{rd}(t)$ that is an $\mathcal{L}$-prefix of $\mathrm{wrt}(t)$ by $t \in \Omega_\ell$. Hence, by Lemma 3.6 we have $t = [\mathrm{wrt}(t)\,\overline{\mathrm{rd}(t)}]_{\equiv_\mathcal{L}}$. Now set $s := [\mathrm{wrt}(t)\,\overline{u}]_{\equiv_\mathcal{L}}$ where $\mathrm{rd}(t) = uv$. Then we have $t = s\,[\overline{v}]_{\equiv_\mathcal{L}}$ again. □

Now, we prove that the aforementioned partitions of $T \subseteq \mathcal{Q}(\mathcal{L})$ can be generated by Boolean operations from the generators $\mathrm{wrt}^{-1}(R)$, $\mathrm{rd}^{-1}(R)$, and Ω_ℓ. Note that by swapping projections this lemma generalizes the proofs of [11, Lemmas 9.9-9.11] which state these results for reliable queue monoid, only.

Lemma 5.10. *Let $\ell \in \mathbb{N}$ and $T \subseteq \mathcal{Q}(\mathcal{L})$ such that $\eta_\mathcal{L}^{-1}(T) \cap \overline{A}^* A^* \overline{A}^*$ is recognized by a monoid with ℓ elements. Then all of the following sets satisfy property (iii) of Theorem 5.6:*

(i) $T \cap \mathrm{rd}^{-1}\big(A^{<\ell}\big) \cap \Omega_\ell$.
(ii) $T \cap \mathrm{rd}^{-1}\big(A^{\geq\ell}\big) \cap \Omega_\ell$.
(iii) $T \cap \Omega_k \setminus \Omega_{k+1}$ *for any* $k \in \mathbb{N}$.

Proof. Let $L = \eta_\mathcal{L}^{-1}(T) \cap \overline{A}^* A^* \overline{A}^*$ and $\phi\colon \Sigma^* \to \mathbb{F}$ be a homomorphism recognizing L where $\mathbb{F}$ is a finite monoid with $\ell := |\mathbb{F}|$. Furthermore, let $\mu, \overline{\mu}\colon A^* \to \mathbb{F}$ such that $\mu(w) = \phi(w)$ and $\overline{\mu}(w) = \phi(\overline{w})$.

(i) We show the first statement by establishing the following equation

$$T \cap \mathrm{rd}^{-1}\big(A^{<\ell}\big) \cap \Omega_\ell = \bigcup_{\substack{v \in A^{<\ell}, \alpha \in \mathbb{F}:\\ \alpha\overline{\mu}(v) \in \phi(L)}} \mathrm{rd}^{-1}(v) \cap \mathrm{wrt}^{-1}\big(\mu^{-1}(\alpha)\big) \cap \Omega_\ell\,.$$

We denote the left and right hand side of this equation by Y and Z, respectively. Clearly, $Y, Z \subseteq \mathrm{rd}^{-1}\big(A^{<\ell}\big) \cap \Omega_\ell$. Hence, it suffices to show, given $t \in \mathrm{rd}^{-1}\big(A^{<\ell}\big) \cap \Omega_\ell$, that $t \in Y$ iff $t \in Z$ holds.

So, let $v := \mathrm{rd}(t) \in A^{<\ell}$. Then by $\Omega_\ell \subseteq \Omega_{|v|}$ and Lemma 5.9 there is $s \in \mathcal{Q}(\mathcal{L})$ with $t = s\,[\overline{v}]_{\equiv_\mathcal{L}}$. We can infer that $\mathrm{wrt}(s) = \mathrm{wrt}(t)$ and $\mathrm{rd}(s) = \varepsilon$ holds. Hence, we have

$$\begin{aligned} t \in Y &\iff t = [\mathrm{wrt}(t)\,\overline{v}]_{\equiv_\mathcal{L}} \in T \\ &\iff \phi(\mathrm{wrt}(t)\,\overline{v}) = \mu(\mathrm{wrt}(t))\,\overline{\mu}(v) \in \phi(L) \\ &\iff t \in Z\,. \end{aligned}$$

(ii) We show this by establishing the following equation

$$T \cap \mathrm{rd}^{-1}\big(A^{\geq\ell}\big) \cap \Omega_\ell = \bigcup_{\substack{v \in A^{\ell}, \alpha,\beta \in \mathbb{F}:\\ \alpha\beta\overline{\mu}(v) \in \phi(L)}} \mathrm{rd}^{-1}\big(\overline{\mu}^{-1}(\alpha)v\big) \cap \mathrm{wrt}^{-1}\big(\mu^{-1}(\beta)\big) \cap \Omega_\ell\,.$$

We denote the left and right hand side of this equation by Y and Z, respectively. Clearly, $Y, Z \subseteq \mathrm{rd}^{-1}(A^{\geq \ell}) \cap \Omega_\ell$. Hence, it suffices to show, given $t \in \mathrm{rd}^{-1}(A^{\geq \ell}) \cap \Omega_\ell$, that $t \in Y$ iff $t \in Z$ holds.

So, let $v \in A^\ell$ such that $\mathrm{rd}(t) \in A^* v$. Then by Lemma 5.9 there is $s \in \mathcal{Q}(\mathcal{L})$ such that $t = s\,[\overline{v}]_{\equiv_\mathcal{L}}$. By Proposition 3.5 and Lemma 3.6 there are words $x, y, z \in A^*$ such that $s = [\overline{x}y\overline{z}]_{\equiv_\mathcal{L}}$ and, hence, $t = [\overline{x}y\overline{z}\overline{v}]_{\equiv_\mathcal{L}}$. Furthermore, by $|\mathbb{F}| = \ell$ there is $y_0 \in A^{\leq \ell}$ with $\phi(y_0) = \phi(y)$. Due to $|y_0| \leq \ell = |v|$ and Lemma 3.6 we have $\overline{x}y_0\overline{z}\overline{v} \equiv_\mathcal{L} \overline{x}\overline{z}y_0\overline{v}$. Then we have

$$\begin{aligned}
t \in T &\iff \phi(\overline{x}y\overline{z}\overline{v}) \in \phi(L) && (\text{since } t = [\overline{x}y\overline{z}\overline{v}]_{\equiv_\mathcal{L}})\\
&\iff \phi(\overline{x}y_0\overline{z}\overline{v}) \in \phi(L) && (\text{since } \phi(y) = \phi(y_0))\\
&\iff [\overline{x}y_0\overline{z}\overline{v}]_{\equiv_\mathcal{L}} \in T\\
&\iff [\overline{x}\overline{z}y_0\overline{v}]_{\equiv_\mathcal{L}} \in T && (\text{since } \overline{x}y_0\overline{z}\overline{v} \equiv_\mathcal{L} \overline{x}\overline{z}y_0\overline{v})\\
&\iff \phi(\overline{x}\overline{z}y_0\overline{v}) \in \phi(L)\\
&\iff \phi(\overline{x}\overline{z}y\overline{v}) \in \phi(L) && (\text{since } \phi(y) = \phi(y_0))
\end{aligned}$$

Hence, we can infer

$$\begin{aligned}
t \in Y &\iff t \in T\\
&\iff \phi(\overline{x}\overline{z}y\overline{v}) \in \phi(L)\\
&\iff \overline{\mu}(xz)\mu(y)\overline{\mu}(v) \in \phi(L)\\
&\iff t \in Z\,.
\end{aligned}$$

(iii) We show the last statement by establishing the following equation

$$T \cap \Omega_k \setminus \Omega_{k+1} = \bigcup_{\substack{v \in A^k, \alpha, \beta \in \mathbb{F}:\\ \alpha\beta\overline{\mu}(v) \in \phi(L)}} \mathrm{rd}^{-1}\big(\overline{\mu}^{-1}(\alpha)v\big) \cap \mathrm{wrt}^{-1}\big(\mu^{-1}(\beta)\big) \cap \Omega_k \setminus \Omega_{k+1}\,.$$

We denote the left and right hand side of this equation by Y and Z, respectively. Clearly, $Y, Z \subseteq \Omega_k \setminus \Omega_{k+1}$. Hence, it suffices to show, given $t \in \Omega_k \setminus \Omega_{k+1}$, that $t \in Y$ iff $t \in Z$ holds.

By definition of Ω_{k+1} there is $s_0 \in \mathcal{Q}(\mathcal{L})$ with $\mathrm{wrt}(s_0) = \mathrm{wrt}(t)$, $\mathrm{rd}(s_0) = \mathrm{rd}(t)$, and $k + 1 \geq |\mathrm{rd}_2(s_0)| > |\mathrm{rd}_2(t)|$. By $t \in \Omega_k$ we can infer that $|\mathrm{rd}_2(s_0)| = k + 1$. Hence, there is $v \in A^k$ and $a \in A$ with $\mathrm{rd}(t) = \mathrm{rd}(s_0) \in A^* av$ and $av \sqsubseteq_\mathcal{L} \mathrm{wrt}(s_0) = \mathrm{wrt}(t)$. Furthermore, by Lemma 5.9 there is $s \in \mathcal{Q}(\mathcal{L})$ such that $t = s\,[\overline{v}]_{\equiv_\mathcal{L}}$. By Proposition 3.5 and Lemma 3.6 there are words $x, y, z \in A^*$ such that $s = [\overline{x}y\overline{z}]_{\equiv_\mathcal{L}}$ and, hence, $t = [\overline{x}y\overline{z}\overline{v}]_{\equiv_\mathcal{L}}$. By $|\mathrm{rd}_2(t)| < k+1$ we can assume that $z = \varepsilon$. Then we have

$$t \in Y \iff \phi(\overline{x}y\overline{v}) \in \phi(L) \iff \overline{\mu}(x)\mu(y)\overline{\mu}(v) \in \phi(L) \iff t \in Z\,. \quad \square$$

Proposition 5.11. *Let $\mathcal{L} = (A, U)$ be a lossiness alphabet and $T \subseteq \mathcal{Q}(\mathcal{L})$ such that $\eta_\mathcal{L}^{-1}(T) \cap \overline{A}^* A^* \overline{A}^*$ is regular. Then T is a Boolean combination of sets of the form $\mathrm{wrt}^{-1}(R)$ or $\mathrm{rd}^{-1}(R)$ for some regular $R \subseteq A^*$ and the sets Ω_ℓ for $\ell \in \mathbb{N}$.*

Proof. Let $\eta_{\mathcal{L}}^{-1}(T) \cap \overline{A}^* A^* \overline{A}^*$ be recognizable by a monoid with ℓ elements. Since $\mathcal{Q}(\mathcal{L}) = \Omega_0 \supseteq \Omega_1 \supseteq \cdots \supseteq \Omega_\ell$ we have

$$T = \big(T \cap \mathrm{rd}^{-1}\big(A^{<\ell}\big) \cap \Omega_\ell\big) \cup \big(T \cap \mathrm{rd}^{-1}\big(A^{\geq \ell}\big) \cap \Omega_\ell\big) \cup \bigcup_{0 \leq k < \ell} T \cap \Omega_k \smallsetminus \Omega_{k+1}\,.$$

By Lemma 5.10 the right hand side is a finite union of sets satisfying property (iii) of Theorem 5.6. □

5.1.3. *The implication "(iii)⇒(i)" in Theorem 5.6*

Again, fix some lossiness alphabet $\mathcal{L} = (A, U)$. We have to prove now that $T \subseteq \mathcal{Q}(\mathcal{L})$ is recognizable if it is a Boolean combination of sets $\mathrm{wrt}^{-1}(R)$, $\mathrm{rd}^{-1}(R)$ for some regular $R \subseteq A^*$, and Ω_ℓ for some $\ell \in \mathbb{N}$. Since the homomorphic preimages of regular languages are recognizable and the set of recognizable subsets of a monoid is closed under Boolean operations, we only have to show that Ω_ℓ is recognizable (and even aperiodic) for any $\ell \in \mathbb{N}$. To this end, we prove that $\eta_{\mathcal{L}}^{-1}(\Omega_\ell)$ is FO-definable and, due to [20], aperiodic. We first define two FO-formulas prefix_ℓ and $\mathsf{overlap}_\ell$ for $\ell \in \mathbb{N}$ which describe the following properties: The word model $\underline{w}$ of a queue transformation sequence $w \in \Sigma^*$ satisfies prefix_ℓ if, and only if, there is a suffix of $\mathrm{rd}(w)$ of length ℓ that is an $\mathcal{L}$-prefix of $\mathrm{wrt}(w)$. The formula $\mathsf{overlap}_\ell$ strengthens this and describes the queue actions with an overlap of at least ℓ letters such that the last ℓ read actions are an $\mathcal{L}$-prefix of the write operations. We do this by assigning these ℓ read actions to variables $x_1, \ldots, x_\ell$ (where x_1 is the position of the last read action and x_ℓ the ℓ^{th} last read action) and the corresponding write actions to variables $y_1, \ldots, y_\ell$. So, let $\ell \in \mathbb{N}$ and $x_1, \ldots, x_\ell, y_1, \ldots, y_\ell$ be variables. Then we define the following formulas:

(i) $\phi_1 := x_\ell < x_{\ell-1} < \cdots < x_1 \wedge y_\ell < y_{\ell-1} < \cdots < y_1$ - This formula guarantees that the x_i's are mutually distinct and in descending order and the same holds for the y_i's.

(ii) $\phi_2 := \bigwedge_{i=1}^{\ell} \bigvee_{a \in A} (\Lambda_{\overline{a}}(x_i) \wedge \Lambda_a(y_i))$ - This formula ensures that x_i reads the same letter from the queue as y_i writes into it.

(iii) $\phi_3 := \forall z \colon \Big((x_\ell \leq z \wedge \Lambda_{\overline{A}}(z)) \to \bigvee_{i=1}^{\ell} x_i = z\Big)$ - Satisfaction of this formula requires the x_i's to be the last ℓ read actions in w.

(iv) $\phi_4 := \bigwedge_{i=1}^{\ell-1} \bigwedge_{a \in A} \forall z \colon ((y_{i+1} < z < y_i \wedge \Lambda_a(y_i)) \to \Lambda_{\Sigma \smallsetminus (U \cup \{a\})}(z))$ and $\phi_5 := \bigwedge_{a \in A} \forall z \colon ((z < y_\ell \wedge \Lambda_a(y_\ell)) \to \Lambda_{\Sigma \smallsetminus (U \cup \{a\})}(z))$ - These formulas assure that the infix w_i of w between y_{i+1} and y_i does not contain the same letter as y_i and no letters from U. Hence, together with the formulas above, these ones enforce the last ℓ read actions to be an $\mathcal{L}$-prefix of the write actions.

(v) $\phi_6 := \bigwedge_{i=1}^{\ell} y_i < x_i$ - This formula guarantees that each x_i appears right from y_i.

By conjuction of the formulas above we obtain the announced formulas:

$$\mathsf{prefix}_\ell := \exists x_1, \dots, x_\ell, y_1, \dots, y_\ell \colon \phi_1 \wedge \phi_2 \wedge \phi_3 \wedge \phi_4 \wedge \phi_5 \,,$$
$$\mathsf{overlap}_\ell := \exists x_1, \dots, x_\ell, y_1, \dots, y_\ell \colon \phi_1 \wedge \phi_2 \wedge \phi_3 \wedge \phi_4 \wedge \phi_5 \wedge \phi_6 \,.$$

Example 5.12. Let $w = abba\overline{aba}$. If $U = \{a\}$ then we have $w \in L(\mathsf{overlap}_1)$ and $w \in L(\mathsf{overlap}_3)$, but $w \notin L(\mathsf{overlap}_2)$ since each assignment of y_2 to b violates ϕ_5 because of the leading a in w. If $U = \{b\}$ then we have $w \in L(\mathsf{overlap}_1)$, but $w \notin L(\mathsf{overlap}_2)$ and $w \notin L(\mathsf{overlap}_3)$ since each assignment of y_1 or y_2, respectively, to a violates ϕ_4 because of the b at the third position in w.

Now let $w' = \overline{aba}abba$. Then $w' \in L(\mathsf{prefix}_\ell)$ if, and only if, $w \in L(\mathsf{prefix}_\ell)$ for any $\ell \in \mathbb{N}$. But $w' \in L(\mathsf{overlap}_0)$ and $w' \notin L(\mathsf{overlap}_\ell)$ for any $\ell \geq 0$.

In the following lemma we describe the words satisfying the formulas prefix_ℓ and $\mathsf{overlap}_\ell$ for any $\ell \in \mathbb{N}$. As announced in the introduction of this subsection in the first case these are the words where the last ℓ read actions are an $\mathcal{L}$-prefix of the write operations. Furthermore, the words satisfying $\mathsf{overlap}_\ell$ also satisfy prefix_ℓ and have an overlap of at least ℓ symbols.

Note that this lemma is no contradiction to the non-recognizability of the sets $L_k = \{t \in \mathcal{Q}(A, A) \mid |\mathrm{rd}_2(t)| > k\}$ which was proven in [11, Observation 9.1] since there are words $w \in \Sigma^*$ with $|\mathrm{rd}_2(w)| \geq \ell$ such that the last ℓ read actions are not an (A, A)-prefix of $\mathrm{wrt}(w)$.

Lemma 5.13. *Let $\ell \in \mathbb{N}$ and $w \in \Sigma^*$. Then the following holds:*

(i) $w \in L(\mathsf{prefix}_\ell)$ if, and only if, there is $u \in A^\ell$ with $u \sqsubseteq_{\mathcal{L}} \mathrm{wrt}(w)$ and $\mathrm{rd}(w) \in A^* u$.
(ii) $w \in L(\mathsf{overlap}_\ell)$ if, and only if, there is $u \in A^\ell$ with $u \sqsubseteq_{\mathcal{L}} \mathrm{wrt}(w)$ and $\mathrm{rd}_2(w) \in A^* u$.

Proof.

(i) First, let $w \in L(\mathsf{prefix}_\ell)$. Then there are letters $a_1, \dots, a_\ell, b_1, \dots, b_\ell \in \Sigma$ contained in w such that their positions $p_1, \dots, p_\ell, q_1, \dots, q_\ell$ in w satisfy ϕ_1-ϕ_5. By ϕ_2 we have $a_\ell \dots a_1 = \overline{b_\ell \dots b_1}$. Due to ϕ_1 and ϕ_3 we have $\mathrm{rd}(w) \in A^* b_\ell \dots b_1$ and $b_\ell \dots b_1 \preceq \mathrm{wrt}(w)$. From $\phi_4 \wedge \phi_5$ we infer that $b_\ell \dots b_1 \sqsubseteq_{\mathcal{L}} \mathrm{wrt}(w)$.

For the converse implication, let $u = b_\ell \dots b_1 \in A^\ell$ such that $u \sqsubseteq_{\mathcal{L}} \mathrm{wrt}(w)$ and $\mathrm{rd}(w) \in A^* u$. Since there is no restriction to the order of x_i and y_i, the set $L(\mathsf{prefix}_\ell)$ contains a word only if it contains all words with the same projections. Hence, we can assume that $\mathrm{rd}_2(w) = u$ and $w = \mathrm{nf}_{\mathcal{L}}(w)$. Let $p_1, \dots, p_\ell$ be the positions of the last ℓ read actions in w (in descending order) and let $\overline{b_1}, \dots, \overline{b_\ell} \in \overline{A}$ be the letters on these positions. Then by definition of $\mathrm{nf}_{\mathcal{L}}(w)$ the positions $p_1 - 1, \dots, p_\ell - 1$ are labeled with $b_1, \dots, b_\ell$. Then it is easy to see, that

$$(\underline{w}, p_1, \dots, p_\ell, p_1 - 1, \dots, p_\ell - 1) \models \phi_1 \wedge \phi_2 \wedge \phi_3 \wedge \phi_4 \wedge \phi_5 \,,$$

i.e., $w \in L(\mathsf{prefix}_\ell)$.

(ii) Let $w \in L(\mathsf{overlap}_\ell)$. Then by (i) there is $u = b_\ell \dots b_1 \in A^\ell$ such that $\mathrm{rd}(w) \in A^* u$ and $u \sqsubseteq_{\mathcal{L}} \mathrm{wrt}(w)$. By satisfaction of ϕ_6 we obtain that each b_i is left from $\overline{b_i}$ and hence $\mathrm{rd}_2(w) \in A^* u$.

Conversely, let $u = b_\ell \dots b_1 \in A^k$ such that $u \sqsubseteq_{\mathcal{L}} \mathrm{wrt}(w)$ and $\mathrm{rd}_2(w) \in A^* u$. Then by (i) we have $w \in L(\mathsf{prefix}_\ell)$. Hence, we only have to check the satisfaction of ϕ_6. We prove this by induction on the minimal length n of a derivation $w \rightarrow^n_{\mathfrak{R}_{\mathcal{L}}} \mathrm{nf}_{\mathcal{L}}(w)$ where $\mathfrak{R}_{\mathcal{L}}$ is the semi-Thue system we obtain by ordering the equations from Definition 3.1 from left to right.

If $n = 0$ then we have $w = \mathrm{nf}_{\mathcal{L}}(w)$. Set $k := |\mathrm{rd}_2(w)| \geq \ell$. Let $p_1, \dots, p_k$ be the positions of the last k read actions (in descending order) in w and let $\overline{b_1}, \dots, \overline{b_k} \in \overline{A}$ be the letters on these positions. Due to $b_k \dots b_1 = \mathrm{rd}_2(w) \sqsubseteq_{\mathcal{L}} \mathrm{wrt}(w)$ there is a factorization $w_k b_k w_{k-1} b_{k-1} \dots w_1 b_1 w_0 = \mathrm{wrt}(w)$ with $w_i \in (A \setminus (U \cup \{b_i\}))^*$ for any $1 \leq i \leq k$ and $w_0 \in A^*$. Then, by $w = \mathrm{nf}_{\mathcal{L}}(w)$ the positions $q_i := p_i - 1$ are labeled with b_i and, hence, we have $\mathrm{wrt}(w[q_{i+1}{+}1, q_i]) = w_i b_i$ for each $1 \leq i \leq k$ (where $q_{k+1} = 0$). Since we also have $b_\ell \dots b_1 \sqsubseteq_{\mathcal{L}} \mathrm{wrt}(w)$ there is another factorization $w'_\ell b_\ell \dots w'_1 b_1 w'_0 = \mathrm{wrt}(w)$ with $w'_i \in (A \setminus (U \cup \{b_i\}))^*$ for any $1 \leq i \leq \ell$ and $w'_0 \in A^*$. Let $q'_\ell, \dots, q'_1$ be the positions of $b_\ell, \dots, b_1$ with $\mathrm{wrt}(w[q'_{i+1} + 1, q_i]) = w'_i b_i$ for each $1 \leq i \leq \ell$ (where $q'_{\ell+1} = 0$). Hence, by $\ell \leq k$ we can infer $q'_i \leq q_i < p_i$ for each $1 \leq i \leq \ell$. This finally implies

$$(\underline{w}, p_1, \dots, p_\ell, q'_1, \dots, q'_\ell) \models \phi_1 \wedge \phi_2 \wedge \phi_3 \wedge \phi_4 \wedge \phi_5 \wedge \phi_6 ,$$

i.e., $w \in L(\mathsf{overlap}_\ell)$.

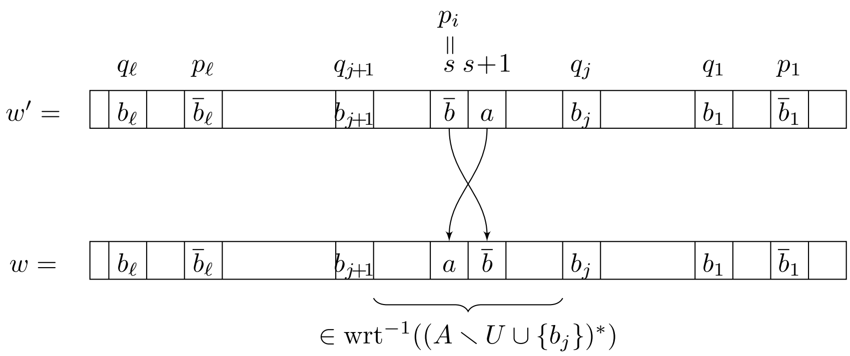

Fig. 1. Illustration of the induction step

Now, we assume $n > 0$. Then there is $w' \in \Sigma^*$ with $w \rightarrow_{\mathfrak{R}_{\mathcal{L}}} w' \rightarrow^{n-1}_{\mathfrak{R}_{\mathcal{L}}} \mathrm{nf}_{\mathcal{L}}(w)$. By definition of $\mathfrak{R}_{\mathcal{L}}$ there are words $x, y \in \Sigma^*$ and $a, b \in A$ with $w = xa\bar{b}y$ and $w' = x\bar{b}ay$. By induction hypothesis we can assume that

$w' \in L(\mathsf{overlap}_\ell)$ and we have to show that also $w \in L(\mathsf{overlap}_\ell)$. Let $p_1, \ldots, p_\ell, q_1, \ldots, q_\ell$ be the positions in w' satisfying ϕ_1-ϕ_6. Furthermore, let $s := |x| + 1$ be the position of $\overline{b}$ in w', i.e., a has position $s+1$ in w'.

It is obvious that $w \in L(\mathsf{overlap}_\ell)$ if $s < p_\ell$ or $s + 1 > q_1$. Hence, there are $1 \leq i, j \leq \ell$ such that $s = p_i$ and $q_{j+1} < s + 1 \leq q_j$ (where $q_{\ell+1} = 0$). We have $i > j$ since the case $i \leq j$ implies $s = p_i \geq p_j > q_j \geq s + 1$ by satisfaction of ϕ_1 and ϕ_6 (cf. Fig. 1). We know that $w'[q_{j+1}+1, q_j-1] \in \mathrm{wrt}^{-1}((A \setminus (U \cup \{b_j\}))^*)$ holds by satisfaction of ϕ_4 and ϕ_5. Since we only transpose a read action with a write action, we still have $w[q_{j+1}+1, q_j-1] \in \mathrm{wrt}^{-1}((A \setminus (U \cup \{b_j\}))^*)$ implying the satisfaction of ϕ_4 and ϕ_5. Hence, we have

$$(\underline{w}, p_1, \ldots, p_i + 1, \ldots, p_\ell, q_1, \ldots, q_j - 1, \ldots, q_\ell) \models \phi_1 \wedge \phi_2 \wedge \phi_3 \wedge \phi_4 \wedge \phi_5 \wedge \phi_6 \,,$$

i.e., $w \in L(\mathsf{overlap}_\ell)$. □

Hence, we can infer the recognizability of the subsets Ω_ℓ:

Lemma 5.14. *Let $\ell \in \mathbb{N}$. Then Ω_ℓ is aperiodic and, hence, recognizable.*

Proof. Since $\eta_{\mathcal{L}}$ is surjective, it suffices to show that $\eta_{\mathcal{L}}^{-1}(\Omega_\ell)$ is aperiodic in Σ^*. By Observation 5.5 and Lemma 5.13 we have

$$\eta_{\mathcal{L}}^{-1}(\Omega_\ell) = L\left(\bigwedge_{k=1}^{\ell} (\mathsf{prefix}_k \to \mathsf{overlap}_k)\right) .$$

Then by [20] $\eta_{\mathcal{L}}^{-1}(\Omega_\ell)$ is aperiodic since the given formula is contained in FO. □

Finally, this lemma implies the implication "(iii)⇒(i)" in Theorem 5.6 and, hence, finishes the proof of Theorem 5.6.

Proposition 5.15. *Let $\mathcal{L} = (A, U)$ be a lossiness alphabet and $T \subseteq \mathcal{Q}(\mathcal{L})$ be a Boolean combination of sets of the form $\mathrm{wrt}^{-1}(R)$ or $\mathrm{rd}^{-1}(R)$ for some regular $R \subseteq A^*$ and the sets Ω_ℓ for $\ell \in \mathbb{N}$. Then T is recognizable.*

Proof. Since the sets $\mathrm{wrt}^{-1}(R)$ and $\mathrm{rd}^{-1}(R)$ for any regular $R \subseteq A^*$ are recognizable and, since Ω_ℓ is recognizable for any $\ell \in \mathbb{N}$ due to Lemma 5.14, T is recognizable by closure properties of the class of recognizable subsets in $\mathcal{Q}(\mathcal{L})$. □

5.2. *From recognizability to q-rationality*

In this subsection we prove that each recognizable subset in the plq monoid is q-rational. To this end, we first need to define this notion which is a restriction to the rational expressions. We need this restriction since we cannot translate Kleene's Theorem [13] to plq monoids due to Theorem 4.5. Though, we can use Ochmański's approach from [22] to generate the recognizable subsets. Concretely, we restrict the

Kleene star and the concatenation of the plq monoid in an appropriate way. We call the sets generated by those operations *q-rational* and prove that these are exactly the recognizable subsets in the plq monoid.

At first, we prove that the class of recognizable subsets is not closed under iteration:

Remark 5.16. Let $T = \{[a\overline{a}]_{\equiv_{\mathcal{L}}}\}$, which is trivially recognizable. Then $\eta_{\mathcal{L}}^{-1}(T^*) \cap A^*\overline{A}^* \subseteq \Sigma^*$ is the set of all words $a^n\overline{a}^n$ with $n \in \mathbb{N}$ by Rule (c) in Definition 3.1. This language is not regular. Hence, $\eta_{\mathcal{L}}^{-1}(T^*)$ is also not regular and therefore T^* is not recognizable.

This is a very similar situation as in trace monoids. Here, Ochmański proved in [22] that it suffices to restrict iteration to obtain some kind of rational expressions that are generating all the recognizable subsets in trace monoids. Unfortunately, the class of recognizable subsets in the plq monoid also is not closed under concatenation.

Remark 5.17. Let $S = \{[a]_{\equiv_{\mathcal{L}}}\}^*$ and $T = \{[\overline{a}]_{\equiv_{\mathcal{L}}}\}^*$, which are recognizable. Then $\eta_{\mathcal{L}}^{-1}(S \cdot T) \cap \overline{A}^* A^* \overline{A}^* \subseteq \Sigma^*$ is the set of all words $\overline{u_1} u_2 \overline{u_3}$ with $u_1, u_2, u_3 \in a^*$ and $u_1 = \varepsilon$ or $|u_2| \leq |u_3|$ by Rule (c) in Definition 3.1. Since this language is not regular, $S \cdot T$ is not recognizable.

Hence, we have to restrict the monoid's product as well.

According to the above remarks we will define now the so-called *q-rational* subsets of the plq monoid. Afterwards we prove that this is a suitable restriction of rationality to describe exactly the recognizable subsets.

At first, we say that a subset of $\mathcal{Q}(\mathcal{L})$ is *q^+-rational* if it can be obtained by the following rules:

(1^+) $\mathrm{wrt}^{-1}(\varepsilon)$, $\mathrm{wrt}^{-1}(\emptyset) = \emptyset$, and $\mathrm{wrt}^{-1}(a)$ for any $a \in A$ are q$^+$-rational
(2^+) if $S, T \subseteq \mathcal{Q}(\mathcal{L})$ are q$^+$-rational then $S \cup T$, $S \cdot T$, and S^* are q$^+$-rational

Similarly, by replacing wrt^{-1} by rd^{-1} in the rules above, we define the class of *q^--rational* subsets of $\mathcal{Q}(\mathcal{L})$.

Observation 5.18. *Let $T \subseteq \mathcal{Q}(\mathcal{L})$. Then the following statements hold:*

(i) T is q^+-rational if, and only if, there is some regular $R \subseteq A^*$ with $T = \mathrm{wrt}^{-1}(R)$.
(ii) T is q^--rational if, and only if, there is some regular $R \subseteq A^*$ with $T = \mathrm{rd}^{-1}(R)$.

Proof. Both equivalences hold since $\mathsf{wrt}^{-1}, \mathsf{rd}^{-1} \colon 2^{A^*} \to 2^{\mathcal{Q}(\mathcal{L})}$ are homomorphisms wrt. rational operations. □

Finally, a subset of $\mathcal{Q}(\mathcal{L})$ is *q-rational* if it can be constructed from the following rules:

(1) if $T \subseteq \mathcal{Q}(\mathcal{L})$ is q$^+$- or q$^-$-rational it also is q-rational
(2) if $S, T \subseteq \mathcal{Q}(\mathcal{L})$ are q-rational then $S \cup T$ and $\mathcal{Q}(\mathcal{L}) \setminus S$ are q-rational
(3) if $S \subseteq \mathcal{Q}(\mathcal{L})$ is q$^+$-rational and $T \subseteq \mathcal{Q}(\mathcal{L})$ is q$^-$-rational such that $\mathrm{rd}(T)$ is finite (i.e., T is obtained without usage of the *-operator) then $S \cdot \mathcal{Q}(\mathcal{L}) \cdot T$ is q-rational

Remark 5.19. Recall the classical definition of rational sets. This class is the closure of the finite sets under union, product, and iteration. All of these operations are monotonic. Here, we also need the closure under complementation which is not monotonic. However, we do not know a characterization of the recognizable sets in the plq monoid with monotic operations, only.

Example 5.20. Let $T = \{t \in \mathcal{Q}(\mathcal{L}) \mid \mathrm{wrt}(t) \in (ab)^*, \mathrm{rd}(t) = b\}$. Then T is q-rational since we have

$$T = \mathrm{rd}^{-1}(b) \cap (\mathrm{wrt}^{-1}(a) \cdot \mathrm{wrt}^{-1}(b))^* \,.$$

Note that the class of q-rational subsets also is closed under intersection due to Rule (2), i.e., this class is a Boolean algebra.

At first sight, the choice of Rule (3) seems to be some kind of random. But we can remove neither the factor "$\mathcal{Q}(\mathcal{L})$", which appears as separator in this product, nor the finiteness of $\mathrm{rd}(T)$. Additionally, we cannot simply remove this rule since the recognizable set $\{[a\overline{a}]_{\equiv_\mathcal{L}}\}$ cannot be built by application of the Rules (1) and (2), only.

Remark 5.21. Let $a, b \in A$ be distinct letters. Then the set $\mathrm{wrt}^{-1}((ab)^*a) \cdot \mathrm{rd}^{-1}(\varepsilon)$ is not recognizable, since $\eta_\mathcal{L}^{-1}(\mathrm{wrt}^{-1}((ab)^*a)\,\mathrm{rd}^{-1}(\varepsilon)) \cap (ab)^*\overline{(ab)}^*$ contains exactly those words $(ab)^m\overline{(ab)}^n$ with $m > n$, which is not regular.

Now let $a, b, c \in A$ be distinct letters with $a, c \in U$. Then the set

$$\mathrm{wrt}^{-1}(aA^*c) \cdot \mathcal{Q}(\mathcal{L}) \cdot \mathrm{rd}^{-1}(aA^*c)$$

is not recognizable since

$$L := \eta_\mathcal{L}^{-1}(\mathrm{wrt}^{-1}(aA^*c) \cdot \mathcal{Q}(\mathcal{L}) \cdot \mathrm{rd}^{-1}(aA^*c)) \cap \overline{ab^*c}ab^*c$$

contains exactly those words $\overline{ab^mc}ab^nc$ satisfying $m \neq n$ if $b \in U$ or $m > n$ otherwise. In both cases L is not regular and therefore $\mathrm{wrt}^{-1}(aA^*c) \cdot \mathcal{Q}(\mathcal{L}) \cdot \mathrm{rd}^{-1}(aA^*c)$ is not recognizable.

Now we can prove the implication "(A)$\Rightarrow$(B)" in Theorem 5.1. To do this, we utilize Theorem 5.6(iii). Concretely, we understand a recognizable set by such a Boolean combination of sets $\mathrm{wrt}^{-1}(R)$, $\mathrm{rd}^{-1}(R)$, and Ω_ℓ (where $R \subseteq A^*$ is regular and $\ell \in \mathbb{N}$). Then by induction on the syntax tree of such expression we prove q-rationality. The most complicated case in this proof is to show that Ω_ℓ is q-rational. For this proof we need the following lemma:

Lemma 5.22. *Let $\ell \in \mathbb{N}$, $t \in \mathcal{Q}(\mathcal{L})$, and $u = a_1 \dots a_\ell \in A^*$. Then we have*

$u \sqsubseteq_{\mathcal{L}} \mathrm{wrt}(t)$ *and* $\mathrm{rd}_2(t) \in A^*u$ *if, and only if,*

$$t \in \mathrm{wrt}^{-1}\left(\prod_{i=1}^{\ell}(A \setminus U)^* a_i\right) \cdot \mathcal{Q}(\mathcal{L}) \cdot \mathrm{rd}^{-1}(u) \ .$$

Proof. First, let $u \sqsubseteq_{\mathcal{L}} \mathrm{wrt}(q)$ and $\mathrm{rd}_2(t) \in A^*u$. Then we have $t = [\overline{\mathrm{rd}_1(t)}\mathrm{wrt}(t)\overline{\mathrm{rd}_2(t)}]_{\equiv_{\mathcal{L}}}$ by Lemma 3.6. By assumption we obtain

$$t = \left[\overline{\mathrm{rd}_1(t)}\mathrm{wrt}(t)\overline{\mathrm{rd}_2(t)}\right] \in \mathrm{wrt}^{-1}\left(\prod_{i=1}^{\ell}(A \setminus U)^* a_i\right) \cdot \mathcal{Q}(\mathcal{L}) \cdot \mathrm{rd}^{-1}(u) \ .$$

Now let $t \in \mathrm{wrt}^{-1}\left(\prod_{i=1}^{\ell}(A \setminus U)^* a_i\right) \cdot \mathcal{Q}(\mathcal{L}) \cdot \mathrm{rd}^{-1}(u)$. Then we have $u \sqsubseteq_{\mathcal{L}} \mathrm{wrt}(t)$ and $\mathrm{rd}(t) \in A^*u$. Furthermore, there are $w_1, w_2, w_3 \in \Sigma^*$ with $t = [w_1w_2w_3]$, $\mathrm{wrt}(w_1) \in \prod_{i=1}^{\ell}(A \setminus U)^* a_i$ and $\mathrm{rd}(w_3) = u$. Then the letters $\overline{a_1}, \ldots, \overline{a_k}$ appear to the right of $a_1, \ldots, a_\ell$ in $w_1w_2w_3$, i.e., $w_1w_2w_3 \in L(\mathsf{overlap}_\ell)$. Hence, by Lemma 5.13(ii), we obtain $\mathrm{rd}_2(w_1w_2w_3) \in A^*u$, i.e., $\mathrm{rd}_2(t) \in A^*u$ holds. □

Finally, we can state the following implication:

Proposition 5.23. *Let* $\mathcal{L} = (A, U)$ *be a lossiness alphabet and* $T \subseteq \mathcal{Q}(\mathcal{L})$ *be recognizable. Then* T *is q-rational.*

Proof. By Theorem 5.6(iii) T is a Boolean combination of sets $\mathrm{wrt}^{-1}(R)$, $\mathrm{rd}^{-1}(R)$, and $\Omega_{\ell+1}$ for some regular $R \subseteq A^*$ and $\ell \in \mathbb{N}$. We prove the claim by induction on the syntax tree.

At first, assume $T = \mathrm{wrt}^{-1}(R)$ where $R \subseteq A^*$ is regular. Then, T is q$^+$-rational by Observation 5.18(i) and, hence, q-rational. Similarly, $T = \mathrm{rd}^{-1}(R)$ is q$^-$-rational and hence q-rational if $R \subseteq A^*$ is regular.

Next, let $\ell \in \mathbb{N}$ and $T = \Omega_\ell$. Then by Observation 5.5 and Lemma 5.22 we have

$$\Omega_\ell = \bigcap_{u \in A^{\leq \ell}} \Big(\mathcal{Q}(\mathcal{L}) \setminus \big(\mathrm{wrt}^{-1}(W_uA^*) \cap \mathrm{rd}^{-1}(A^*u)\big) \cup \mathrm{wrt}^{-1}(W_u) \cdot \mathcal{Q}(\mathcal{L}) \cdot \mathrm{rd}^{-1}(u)\Big) \,,$$

where $W_u = \prod_{i=1}^{k}(A \setminus U)^* a_i$ with $u = a_1 \ldots a_k$. Since the sets $\mathrm{wrt}^{-1}(W_uA^*)$, $\mathrm{rd}^{-1}(A^*u)$, $\mathrm{wrt}^{-1}(W_u)$, and $\mathrm{rd}^{-1}(u)$ are q-rational by the first case of this proof, Ω_ℓ is q-rational due to Rules (2) and (3).

Finally, let T be a Boolean combination of sets of the form $\mathrm{wrt}^{-1}(R)$ or $\mathrm{rd}^{-1}(R)$ for some regular $R \subseteq A^*$ and the sets Ω_ℓ for $\ell \in \mathbb{N}$. Then by the two cases above and by Rule (2) T is q-rational. □

5.3. *From q-rationality to logic*

The second implication from Theorem 5.1 states that each q-rational subset is definable in a special monadic second-order logic which we call $\mathsf{MSO_q}$. Here, we try to exhibit the knowledge from the preceding subsection such that this logic defines exactly the recognizable subsets. In fact, we have to add some modifications to Büchi's MSO-logic from [4]. At first, we should understand $p \leq^w q$ as follows: the rules from the semi-Thue system $\mathfrak{R}_{\mathcal{L}}$ do not allow to move the letter a on position p in w to the right of the letter b on position q, i.e., for any $v \in [w]_{\equiv_{\mathcal{L}}}$ the letter a appears left from b in v. Additionally, we have to restrict comparisons of write and read operations:

Remark 5.24. It is not possible to compare arbitrary letters in w without any restrictions. For example, let

$$\phi = \exists x, y\colon \begin{pmatrix} \Lambda_A(x) \wedge \forall z\colon (\Lambda_A(z) \to z \leq x) \wedge \\ \Lambda_{\overline{A}}(y) \wedge \forall z\colon (\Lambda_{\overline{A}}(z) \to y \leq z) \wedge \neg y \leq x \end{pmatrix},$$

i.e., $[w]_{\equiv_{\mathcal{L}}}$ satisfies ϕ if, and only if, in w the first read action can be moved to the right of the last write action. Then we have

$$\{w \in \Sigma^* \mid [w]_{\equiv_{\mathcal{L}}} \models \phi\} \cap \overline{a}^* a^* \overline{a}^* = \{\overline{a}^k a^\ell \overline{a}^m \mid k = 0 \text{ or } m \geq \ell\}.$$

Since the right-hand side of the equation above is not regular, the set $T(\phi)$ of all $t \in \mathcal{Q}(\mathcal{L})$ satisfying ϕ is not recognizable either.

Now, let $w = a_1 \dots a_n \in \Sigma^*$. The *plq model* for w is the relational structure

$$\widetilde{w} := (\mathrm{dom}(w), <_+^w, <_-^w, (P_\ell^w)_{\ell \in \mathbb{N}_+}, (\Lambda_a^w)_{a \in \Sigma})$$

where $\mathrm{dom}(w) := \{1, \dots, n\}$, $\Lambda_a^w := \{i \in \mathrm{dom}(w) \mid a_i = a\}$, $<_+^w$ and $<_-^w$ are the natural orderings on $\Lambda_A^w := \bigcup_{a \in A} \Lambda_a^w$ and $\Lambda_{\overline{A}}^w$, respectively, and

$$P_\ell^w := \left\{ i \in \Lambda_A^w \,\middle|\, \begin{matrix} \forall v_1, v_2 \in \Sigma^*, w \equiv_{\mathcal{L}} v_1 v_2, \mathrm{wrt}(v_1) = \mathrm{wrt}(w[1, i])\colon \\ |\mathrm{rd}(v_2)| < \ell \end{matrix} \right\},$$

i.e., we have $i \in P_\ell^w$ iff $a_i \in A$ and the ℓ^{th} last read action in w is left from a_i and cannot be moved to the right of a_i according to the rules from $\mathfrak{R}_{\mathcal{L}}$. This is conform to the approaches known from [4,7] since the relations $<_+^w$, $<_-^w$, and P_ℓ^w specify which letter have to appear to the left of another one in any word equivalent to w. Hence, we can infer that $\widetilde{w}$ identifies the equivalence class $[w]_{\equiv_{\mathcal{L}}}$:

Lemma 5.25. *Let $v, w \in \Sigma^*$. Then we have $v \equiv_{\mathcal{L}} w$ if, and only if, $\widetilde{v} \cong \widetilde{w}$.*

Proof. At first, we assume $v \equiv_{\mathcal{L}} w$. Then by Proposition 3.5 there is a permutation σ of $\mathrm{dom}(v) = \mathrm{dom}(w)$ such that σ is compatible to $<_+$, $<_-$, and Λ_a for any $a \in A$. Additionally, by definition σ is compatible to P_ℓ for any $\ell \in \mathbb{N}$. Hence, σ is an isomorphism from $\widetilde{v}$ into $\widetilde{w}$, i.e., $\widetilde{v} \cong \widetilde{w}$.

Now assume $v \not\equiv_{\mathcal{L}} w$. If $\mathrm{wrt}(v) \neq \mathrm{wrt}(w)$ or $\mathrm{rd}(v) \neq \mathrm{rd}(w)$ then $\widetilde{v} \ncong \widetilde{w}$ since any possible bijection from $\mathrm{dom}(v)$ into $\mathrm{dom}(w)$ (if there is one) cannot be compatible to $<_+$, $<_-$, and Λ_a at the same time by definition of $\equiv_{\mathcal{L}}$. So, assume $\mathrm{wrt}(v) = \mathrm{wrt}(w)$ and $\mathrm{rd}(v) = \mathrm{rd}(w)$. Let σ be the uniquely determined permutation of $\mathrm{dom}(v) = \mathrm{dom}(w)$ which is compatible to $<_+$, $<_-$, and Λ_a for any $a \in A$. W.l.o.g., we have $|\mathrm{rd}_2(v)| > |\mathrm{rd}_2(w)|$. Let $i \in \mathrm{dom}(v)$ be the position of the last letter of $\mathrm{wrt}(v)$ in v.

From Lemma 3.6 we know that $v \equiv_{\mathcal{L}} \overline{\mathrm{rd}_1(v)}\mathrm{wrt}(v)\overline{\mathrm{rd}_2(v)}$ and hence $i \notin P^v_{|\mathrm{rd}_2(v)|}$. Additionally, by Lemma 3.6 we have $w \not\equiv_{\mathcal{L}} \overline{x}\mathrm{wrt}(w)\overline{y}$ for any $xy = \mathrm{rd}(w)$ with $|y| \geq |\mathrm{rd}_2(v)|$. Hence, we have $\sigma(i) \in P^w_{|\mathrm{rd}_2(v)|}$, i.e., σ is no isomorphism from $\widetilde{v}$ into $\widetilde{w}$. Since σ is unique we have $\widetilde{v} \not\cong \widetilde{w}$. □

Therefore, we can define the *plq model* of some transformation sequence $t \in \mathcal{Q}(\mathcal{L})$ by $\widetilde{t} := \widetilde{\mathrm{nf}_{\mathcal{L}}(t)}$.

By definition the signature of our plq models $\widetilde{w}$ is infinite. However, we can represent these structures finitely, since we can split $\mathbb{N}_+$ into at most three intervals I_1, I_2, and I_3, where the sets P^w_ℓ and P^w_k are equal if, and only if, ℓ and k belong to the same interval I_i. These intervals are closely related to $\mathrm{obw}(w)$, $\mathrm{wrt}(w)$, and $\mathrm{rd}(w)$: if $|\mathrm{wrt}(w)| = 0$ we have $w \in \overline{A}^*$ and, hence, $P^w_i = \emptyset$ for each $i \in \mathbb{N}$. The case $|\mathrm{wrt}(w)| > 0$ is considered in the following observation:

Observation 5.26. *Let $w \in \Sigma^*$ with $|\mathrm{wrt}(w)| > 0$. Then the following statements hold:*

(i) For each $1 \leq \ell \leq \min\{\mathrm{obw}(w) - 1, |\mathrm{rd}(w)|\}$ we have $P^w_\ell = \emptyset$.
(ii) For each $\mathrm{obw}(w) \leq \ell \leq |\mathrm{rd}(w)|$ we have $P^w_\ell = P^w_{\mathrm{obw}(w)} \neq \emptyset$.
(iii) For each $\ell > |\mathrm{rd}(w)|$ we have $P^w_\ell = \Lambda^w_A$.

Proof.

(i) By definition of ℓ we have $w \in \Omega_\ell$. Let $r_1, r_2 \in A^*$ with $\mathrm{rd}(w) = r_1 r_2$ and $|r_2| = \ell$. Then, by Lemma 5.9 there is $v \in \Sigma^*$ such that $w \equiv_{\mathcal{L}} v\overline{r_2}$ holds. This finally implies $P^w_\ell = \emptyset$.

(ii) First, suppose $P^w_{\mathrm{obw}(w)} = \emptyset$. Set $p := \max \Lambda^w_A$. Then, there are $v_1, v_2 \in A^*$ with $w \equiv_{\mathcal{L}} v_1 v_2$, $\mathrm{wrt}(v_1) = \mathrm{wrt}(w[1,p]) = \mathrm{wrt}(w)$, and $|\mathrm{rd}(v_2)| \geq \mathrm{obw}(w)$, i.e., $v_2 \in \overline{A}^{\geq \mathrm{obw}(w)}$. Due to $\mathrm{obw}(w) \leq |\mathrm{rd}(w)| < \infty$ we have $|\mathrm{rd}_2(v_1 v_2)| \geq \mathrm{obw}(w) > |\mathrm{rd}_2(w)|$ which contradicts to Proposition 3.5.

Next, we prove $P^w_\ell = P^w_{\mathrm{obw}(w)}$. By definition, the inclusion "$\supseteq$" is trivial. We prove the converse implication "$\subseteq$" by induction on ℓ. If $\ell = \mathrm{obw}(w)$, we are done. So, assume $\ell > \mathrm{obw}(w)$. Let $p \in \Lambda^w_A \setminus P^w_{\ell-1} = \Lambda^w_A \setminus P^w_{\mathrm{obw}(w)}$. Then there are $v_1, v_2 \in \Sigma^*$ with $w \equiv_{\mathcal{L}} v_1 v_2$, $\mathrm{wrt}(v_1) = \mathrm{wrt}(w[1,p])$, and $|\mathrm{rd}(v_2)| \geq \ell - 1$. We are done, if $|\mathrm{rd}(v_2)| \geq \ell$. Hence, we assume $|\mathrm{rd}(v_2)| = \ell - 1$. Let $r_1, r_2 \in A^*$ with $\mathrm{rd}(w) = r_1 r_2$ and $|r_2| = \ell - 1$, i.e., $\mathrm{rd}(v_1) = r_1$ and $\mathrm{rd}(v_2) = r_2$. Additionally, let $r_1 = r_1' a$ with $a \in A$ and let $v_1' \in \Sigma^*$ be the word which arises from v_1 by removing the rightmost $\overline{a}$.

Suppose $ar_2 \not\sqsubseteq_{\mathcal{L}} \mathrm{wrt}(w)$. Using the rules from the semi-Thue system $\mathfrak{R}_{\mathcal{L}}$, we can move the letter $\overline{a}$ in $v_1 v_2$ to the right-hand side of the write action on position p in w. Then we have $v_1' \overline{a} v_2 \equiv_{\mathcal{L}} v_1 v_2 \equiv_{\mathcal{L}} w$ (since $\mathrm{rd}_2(v_1' \overline{a} v_2) = \mathrm{rd}_2(v_1 v_2)$), $\mathrm{wrt}(v_1') = \mathrm{wrt}(v_1) = \mathrm{wrt}(w[1,p])$, and $|\mathrm{rd}(\overline{a} v_2)| = \ell$ implying $p \in \Lambda^w_A \setminus P^w_\ell$.

Now, assume $ar_2 \sqsubseteq_{\mathcal{L}} \mathrm{wrt}(w)$. Let $r_2 = b_{\ell-1} \dots b_1$ with $b_1, \dots, b_{\ell-1} \in A$. We know $|\mathrm{rd}_2(v_1v_2)| = |\mathrm{rd}_2(w)| < \mathrm{obw}(w) \leq \ell - 1$. By definition of $\mathrm{obw}(w)$, there is $u \in A^{\mathrm{obw}(w)}$ with $\mathrm{rd}(v_1v_2) \in A^*u$, $u \sqsubseteq_{\mathcal{L}} \mathrm{wrt}(v_1v_2)$, and $|\mathrm{rd}_2(v_1v_w)| < |u| = \mathrm{obw}(w)$. Let $p_1, \dots, p_{\mathrm{obw}(w)} \in \mathrm{dom}(w) = \mathrm{dom}(v_1v_2)$ be the positions of $\overline{b_1}, \dots, \overline{b_{\mathrm{obw}(w)}}$ in v_1v_2. Additionally, let $q_1, \dots, q_{\mathrm{obw}(w)} \in \mathrm{dom}(w)$ be the positions of the corresponding write actions b_i in v_1v_2, i.e., $\mathrm{wrt}(v_1v_2[q_{i+1}+1, q_i - 1]) \in (A \setminus U \cup \{b_i\})^*$ holds for each $1 \leq i \leq \mathrm{obw}(w)$ (where $q_{\mathrm{obw}(w)+1} = 0$). By Lemma 5.13 there is $1 \leq i \leq \mathrm{obw}(w)$ such that $q_i > p_i$. Consider the positions $p'_1, \dots, p'_\ell \in \mathrm{dom}(w)$ of the ℓ rightmost read actions (in descending order) and $q'_1, \dots, q'_\ell \in \mathrm{dom}(w)$ be the positions of the corresponding write actions in v_1v_2. Then, we have $q'_i \leq q_i < p_i = p'_i$ implying, by application of Lemma 5.13, $|\mathrm{rd}_2(v'_1\overline{a}v_2)| < \ell = |ar_2|$. Hence, we can infer $\mathrm{rd}_2(v_1v_2) = \mathrm{rd}_2(v'_1\overline{a}v_2)$. But, this finally implies $w \equiv_{\mathcal{L}} v_1v_2 \equiv_{\mathcal{L}} v'_1\overline{a}v_2$ and, therefore, $p \in \Lambda_A^w \setminus P_\ell^w$.

(iii) This is a simple consequence of the definition of P_ℓ^w. □

Now, we are able to define our logics: by $\mathsf{FO_q}$ we denote the set of all first-order formulas build up from the atomic formulas of the form

$$x = y, \quad x <_+ y, \quad x <_- y, \quad P_\ell(x) \text{ for } \ell \in \mathbb{N}_+, \quad \Lambda_a(x) \text{ for } a \in A$$

where x and y are variables. Additionally, by $\mathsf{MSO_q}$ we denote the monadic second-order extension of $\mathsf{FO_q}$.

Let $\phi \in \mathsf{MSO_q}$. The set defined by ϕ is $T(\phi) := \{t \in \mathcal{Q}(\mathcal{L}) \mid \widetilde{t} \models \phi\}$. We say that $T \subseteq \mathcal{Q}(\mathcal{L})$ is $\mathsf{MSO_q}$-*definable* ($\mathsf{FO_q}$-*definable*) if there is $\phi \in \mathsf{MSO_q}$ ($\phi \in \mathsf{FO_q}$, respectively) with $T = T(\phi)$.

Remark 5.27. The sets P_ℓ^w also are conform to the special product in the definition of q-rational subsets. In particular, we have

$$T(\exists x \colon \neg P_\ell(x) \wedge \Lambda_A(x)) = \mathrm{wrt}^{-1}\big(A^+\big) \cdot \mathcal{Q}(\mathcal{L}) \cdot \mathrm{rd}^{-1}\big(A^\ell\big) \ .$$

In the proof of implication "(B)⇒(C)" in Theorem 5.1 we need the following notion of restricted quantification: Let $\phi(\vec{x}), \xi(x, \vec{y}) \in \mathsf{MSO}$. Then the *restriction* $\phi|_\xi$ *of* ϕ *on* ξ is defined by

$$\phi|_\xi(\vec{x}, \vec{y}) := \begin{cases} \phi(\vec{x}) & \text{if } \phi(\vec{x}) \text{ is atomic} \\ \psi|_\xi(\vec{x}, \vec{y}) \vee \chi|_\xi(\vec{x}, \vec{y}) & \text{if } \phi(\vec{x}) = \psi(\vec{x}) \vee \chi(\vec{x}) \\ \neg\psi|_\xi(\vec{x}, \vec{y}) & \text{if } \phi(\vec{x}) = \neg\psi(\vec{x}) \\ \exists x \colon (\psi|_\xi(x, \vec{x}, \vec{y}) \wedge \xi(x, \vec{y})) & \text{if } \phi(\vec{x}) = \exists x \colon \psi(x, \vec{x}) \\ \exists X \colon (\psi|_\xi(X, \vec{x}, \vec{y}) \wedge \forall x \colon X(x) \to \xi(x, \vec{y})) & \text{if } \phi(\vec{x}) = \exists X \colon \psi(X, \vec{x}) \,. \end{cases}$$

In other words, we restrict the quantifiers in ϕ to the values satisfying ξ. Obviously, we have $\phi|_\xi \in \mathsf{FO}$ if $\phi, \xi \in \mathsf{FO}$ holds.

Finally, we can state:

Proposition 5.28. *Let $\mathcal{L} = (A, U)$ be a lossiness alphabet and $T \subseteq \mathcal{Q}(\mathcal{L})$ be q-rational. Then T is $\mathsf{MSO_q}$-definable.*

Proof. If T is q$^+$-rational then we have $T = \mathrm{wrt}^{-1}(R)$ for some regular $R \subseteq A^*$. By [4] there is an MSO-formula ϕ with $L(\phi) = R$. Then by replacing of all occurrences of $<$ in ϕ by $<_+$ we obtain an $\mathsf{MSO_q}$-formula ϕ' with $T(\phi'|_{\Lambda_A(x)}) = \mathrm{wrt}^{-1}(L(\phi)) = T$.

Similarly, we can prove that T is $\mathsf{MSO_q}$-definable if T is q$^-$-rational (here, we replace $<$ by $<_-$ and restrict to $\Lambda_{\overline{A}}$).

If $T = S_1 \cup S_2$ or $T = \mathcal{Q}(\mathcal{L}) \setminus S_1$, where S_1, S_2 are q-rational, there are $\phi_1, \phi_2 \in \mathsf{MSO_q}$ with $T(\phi_1) = S_1$ and $T(\phi_2) = S_2$ by induction hypothesis. Then we have $T = T(\phi_1 \vee \phi_2)$ and $T = T(\neg\phi_1)$, respectively.

Finally, let $T = \mathrm{wrt}^{-1}(R) \cdot \mathcal{Q}(\mathcal{L}) \cdot \mathrm{rd}^{-1}(F)$ where $R \subseteq A^*$ is regular and $F \subseteq A^*$ is finite. W.l.o.g. we can assume that $F = \{w\}$ holds. Then there are $\mathsf{MSO_q}$-formulas ϕ_R and ϕ_F defining $\mathrm{wrt}^{-1}(R)$ and $\mathrm{rd}^{-1}(F)$, respectively. Set

$$\phi := \exists x_1, x_2 \colon \phi_R|_{x \leq_+ x_1} \wedge \phi_F|_{x_2 \leq_- x} \wedge \neg P_{|w|}(x_1)\,.$$

Then we have $T = T(\phi)$. $\square$

5.4. *From logic to recognizability*

Finally, we have to prove that each $\mathsf{MSO_q}$-definable subset is recognizable. Concretely, we have to translate a formula $\phi \in \mathsf{MSO_q}$ to some formula $\psi \in \mathsf{MSO}$ such that $\eta_{\mathcal{L}}^{-1}(T(\phi)) = L(\psi)$ holds. In this case, the right-hand side of this equation is regular by [4] implying that $T(\phi)$ is recognizable in $\mathcal{Q}(\mathcal{L})$.

The most complicated case in our construction is the translation of the atomic formula $P_\ell(x)$ since write and read actions are commutative in certain contexts given in Definition 3.1. For this translation, we will utilize the connection between P_ℓ^w and $\mathrm{obw}(w)$ as described in Observation 5.26.

So, let $w \in \Sigma^*$ be some word, $p \in \mathrm{dom}(w)$ by some position in p (which will be represented by x in our formula), and $\ell \geq 1$. Then we express $p \in P_\ell^w$ as follows:

Due to $P_\ell^w \subseteq \Lambda_A^w$ we know that $(\underline{w}, p) \models \Lambda_A(x)$ holds. Additionally, from Observation 5.26 we can infer two cases: if $|\mathrm{rd}(w)| < \ell$ then we are done. Obviously, there is some FO-formula short_ℓ satisfying $L(\mathsf{short}_\ell) = \eta_{\mathcal{L}}^{-1}(\mathrm{rd}^{-1}(A^{<\ell}))$. So, we can assume $|\mathrm{rd}(w)| \geq \ell$ from now on. Then, by Observation 5.26 we have $\mathrm{obw}(w) \leq \ell$, i.e., $[w]_{\equiv_{\mathcal{L}}} \in \mathcal{Q}(\mathcal{L}) \setminus \Omega_\ell$. By Lemma 5.14 the set Ω_ℓ is aperiodic implying the existence of some FO-formula Omega_ℓ satisfying $\eta_{\mathcal{L}}^{-1}(\Omega_\ell) = L(\mathsf{Omega}_\ell)$.

Next, we want to determine the values $k := \mathrm{obw}(w)$ and $m := |\mathrm{rd}_2(w)|$ as well as the positions of the last k resp. m read actions and their corresponding write actions in w. To this end, we utilize Lemma 5.13 with some small modifications to the formulas prefix_i and $\mathsf{overlap}_i$:

$$\mathsf{prefix}'_i(\vec{x}, \vec{y}) := \phi_1 \wedge \phi_2 \wedge \phi_3 \wedge \phi_4 \wedge \phi_5$$
$$\mathsf{overlap}'_i(\vec{x}, \vec{y}) := \phi_1 \wedge \phi_2 \wedge \phi_3 \wedge \phi_4 \wedge \phi_5 \wedge \phi_6\,.$$

Then $(\underline{w}, \vec{p}, \vec{q}) \models \mathsf{prefix}'_i(\vec{x}, \vec{y})$ if, and only if, there is $u \in A^i$ with $u \sqsubseteq_{\mathcal{L}} \mathrm{wrt}(w)$, $\mathrm{rd}(w) \in A^*u$, $p_1, \ldots, p_i$ are the positions of the last i read actions in w (in descending order), and $q_1, \ldots, q_i$ are the positions of the corresponding write actions in w. Moreover, we have $(\underline{w}, \vec{p}, \vec{q}) \models \mathsf{overlap}'_i(\vec{x}, \vec{y})$ if, and only if, in addition to the conditions above, we have $\mathrm{rd}_2(w) \in A^*u$. Hence, we can express "$k = \mathrm{obw}(w)$" as follows:

$$\mathsf{obw}_k(\vec{x}, \vec{y}) := \mathsf{prefix}'_k(\vec{x}, \vec{y}) \wedge \neg\mathsf{overlap}'_k(\vec{x}, \vec{y}) \wedge \bigwedge_{i=1}^{k-1} \mathsf{prefix}_i \to \mathsf{overlap}_i \,.$$

Additionally, the property "$m = |\mathrm{rd}_2(w)| < k$" can be expressed by the following FO-formula:

$$\mathsf{shuffle}_{m,k}(\vec{x}, \vec{z}) := \mathsf{overlap}'_m(\vec{x}, \vec{z}) \wedge \bigwedge_{i=m+1}^{k} \neg\mathsf{overlap}_i \,.$$

Now, let $\vec{p}, \vec{q}, \vec{s} \in \mathrm{dom}(w)^\ell$ be positions in w with $(\underline{w}, \vec{p}, \vec{q}) \models \mathsf{shuffle}_{m,k}(\vec{x}, \vec{y})$ and $(\underline{w}, \vec{p}, \vec{s}) \models \mathsf{obw}_k(\vec{x}, \vec{z})$. It is easy to see, that $q_i \leq s_i$ for each $1 \leq i \leq m$ holds. In particular, if $q_i = s_i$ holds, then we have $q_j = s_j$ for each $1 \leq j < i$. So, let $1 \leq i \leq m$ be minimal such that $q_{i+1} < s_{i+1}$ holds (where $q_{m+1} := 0$). This value can be determined with the following formula:

$$\mathsf{diff}_i(\vec{y}, \vec{z}) := \bigwedge_{j=1}^{i} y_j = z_j \wedge y_{i+1} < z_{i+1} \,,$$

where "$y_{m+1} < z_{m+1}$" means "true". Then, by utilization of the equations from Definition 3.1 we can move the read actions on positions $p_{i+1}, \ldots, p_k$ in w to the direct left-hand side of the write action on position s_{i+1}. But it is impossible to transpose the read action on position p_{i+1} with its corresponding write action on position s_{i+1}. In other words, we have $P_\ell^w = P_k^w = \{p' \in \Lambda_A^w \mid s_{i+1} \leq p'\}$.

All in all, we have:

$$\begin{aligned}\mathsf{P}_\ell(x) := (\Lambda_A(x) \wedge \mathsf{short}_\ell) \vee \Big(&\Lambda_A(x) \wedge \neg\mathsf{Omega}_\ell \wedge \forall \vec{x}, \vec{y}, \vec{z}\colon \\ &\bigwedge_{0 \leq i \leq m < k \leq \ell} \big(\mathsf{shuffle}_{m,k}(\vec{x}, \vec{y}) \wedge \mathsf{obw}_k(\vec{x}, \vec{z}) \wedge \mathsf{diff}_i(\vec{y}, \vec{z})\big) \to z_{i+1} \leq x\Big)\end{aligned}$$

The next two lemmas prove the correctness and completeness of this formula $\mathsf{P}_\ell(x)$:

Lemma 5.29. *Let $\ell \in \mathbb{N}$, $w \in \Sigma^*$, and $p \in \mathrm{dom}(w)$ with $(\underline{w}, p) \not\models \mathsf{P}_\ell(x)$. Then we have $p \notin P_\ell^w$.*

Proof. If we have $p \notin \Lambda_A^w$ we are done since $P_\ell^w \subseteq \Lambda_A^w$. So, we can assume $p \in \Lambda_A^w$ from now on. Then, we have $\underline{w} \not\models \mathsf{short}_\ell$, i.e., $|\mathrm{rd}(w)| \geq \ell$.

If $\underline{w} \models \mathsf{Omega}_\ell$, we have $\mathrm{obw}(w) > \ell$ and, hence, $P_\ell^w = \emptyset$ by Observation 5.26. In this case, we are done. Now, we assume $\underline{w} \models \neg\mathsf{Omega}_\ell$, i.e., $\mathrm{obw}(w) \leq \ell$. Then there are $\vec{p}, \vec{q}, \vec{s} \in \mathrm{dom}(w)^\ell$ and $0 \leq i \leq m < k \leq \ell$ such that

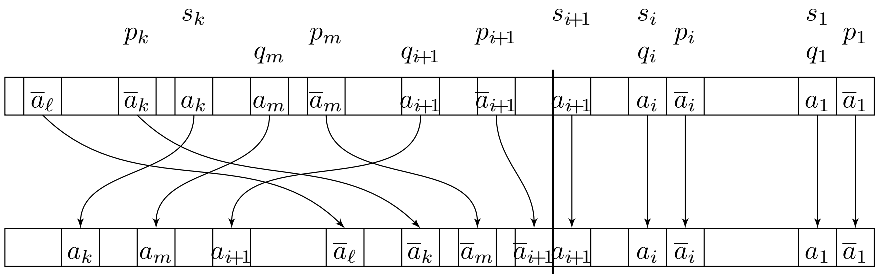

Fig. 2. Visualization of some possible movement of read and write actions. In both words, we have $q_j < p_j$ for all $1 \le j \le m$ and $s_{i+1} \ge p_{i+1}$. Hence, both words have some overlap of length m. The write actions on the left-hand side of the bold line are contained in P_ℓ^w, the ones on the right-hand side are not contained in P_ℓ^w.

- $(\underline{w}, \vec{p}, \vec{q}, \vec{s}) \models \mathsf{shuffle}_{m,k}(\vec{x}, \vec{y}) \wedge \mathsf{obw}_k(\vec{x}, \vec{z}) \wedge \mathsf{diff}_i(\vec{y}, \vec{z})$ and
- $(\underline{w}, p, \vec{s}) \not\models z_{i+1} \le x$.

As we have argued above we have $m = |\mathrm{rd}_2(w)|$ and $k = \mathrm{obw}(w)$. Additionally, we have:

- $p_1, \ldots, p_k$ are the positions of the last k read actions in w (in descending order),
- $q_1, \ldots, q_m$ are the write actions corresponding to $p_1, \ldots, p_m$, and
- similarly, $s_1, \ldots, s_k$ are the write actions corresponding to $p_1, \ldots, p_k$.

By definition of diff_i the number i is the minimal index such that $q_{i+1} < s_{i+1}$ holds (where $q_{m+1} = 0$). We have to prove that $p < s_{i+1}$ implies $p \notin P_\ell^w$. To this end, let $r_1, r_2, w_1, w_2 \in A^*$ such that $\mathrm{rd}(w) = r_1 r_2$, $|r_2| = i$, $\mathrm{wrt}(w) = w_1 w_2$, and $w_1 = \mathrm{wrt}(w[1, s_{i+1} - 1])$. We first show $w \equiv_{\mathcal{L}} w_1 \overline{r_1} w_2 \overline{r_2}$. By Proposition 3.5 it suffices to prove $|\mathrm{rd}_2(w)| = |\mathrm{rd}_2(w_1 \overline{r_1} w_2 \overline{r_2})|$. We have

- the last i read actions are on the right-hand side of each write action and, hence, on the right-hand side of their corresponding write actions and
- the read actions on positions $p_{i+1}, \ldots, p_k$ are on the right-hand side of their corresponding write actions since $q_{i+1} < s_{i+1}$, i.e., the write actions on positions $q_{i+1}, \ldots, q_k$ in w are contained in w_1 which is on the left-hand side of all read actions.

Hence, we have $\underline{w_1 \overline{r_1} w_2 \overline{r_2}} \models \mathsf{overlap}_m$ implying $|\mathrm{rd}_2(w)| = m \le |\mathrm{rd}_2(w_1 \overline{r_1} w_2 \overline{r_2})|$. However, we have $\underline{w_1 \overline{r_1} w_2 \overline{r_2}} \not\models \mathsf{overlap}_k$ since the read action on position p_{i+1} in w (which is the last letter in $\overline{r_2}$) is on the left-hand side of their corresponding write action s_{i+1} (which is the first letter in w_2). Hence, we have $|\mathrm{rd}_2(w_1 \overline{r_1} w_2 \overline{r_2})| < k = \mathrm{obw}(w)$. By definition of $\mathrm{obw}(w)$ we can infer that $|\mathrm{rd}_2(w)| \ge |\mathrm{rd}_2(w_1 \overline{r_1} w_2 \overline{r_2})|$ holds. Therefore, we have $w \equiv_{\mathcal{L}} w_1 \overline{r_1} w_2 \overline{r_2}$.

By $p < s_{i+1}$ we can split $w_1\overline{r_1}w_2\overline{r_2}$ as follows: let $v_1, v_2 \in A^*$ with $v_1 = \mathrm{wrt}(w[1,p])$, i.e., v_1 is a prefix of w_1, and v_2 is the complementary suffix of $w_1\overline{r_1}w_2\overline{r_2}$ wrt. v_1. Then, we have $w \equiv_{\mathcal{L}} w_1\overline{r_1}w_2\overline{r_2} = v_1v_2$ and $|\mathrm{rd}(v_2)| = |\mathrm{rd}(w)| \geq \ell$ implying $p \notin P_\ell^w$. □

Lemma 5.30. *Let $\ell \in \mathbb{N}$, $w \in \Sigma^*$, and $p \in \mathrm{dom}(w) \setminus P_\ell^w$. Then we have $(\underline{w}, p) \not\models \mathsf{P}_\ell(x)$.*

Proof. If we have $p \notin \Lambda_A^w$ we have obviously $(\underline{w}, p) \not\models \mathsf{P}_\ell(x)$. So, from now on, we assume $p \in \Lambda_A^w$.

By $p \in \mathrm{dom}(w) \setminus P_\ell^w$ there are $v_1, v_2 \in \Sigma^*$ with $\mathrm{wrt}(v_1) = \mathrm{wrt}(w[1,p])$, $|\mathrm{rd}(v_2)| \geq \ell$, and $w \equiv_{\mathcal{L}} v_1v_2$. Hence, we have $|\mathrm{rd}(w)| \geq |\mathrm{rd}(v_2)| \geq \ell$ implying $\underline{w} \not\models \mathsf{short}_\ell$.

Next, we consider $\mathrm{obw}(w)$. If $\mathrm{obw}(w) > \ell$, we have $[w]_{\equiv_{\mathcal{L}}} \in \Omega_\ell$ implying $\underline{w} \models \mathsf{Omega}_\ell$. Hence, we are done in this case. Now, assume $\mathrm{obw}(w) \leq \ell$. Then there are $0 \leq m < k \leq \ell$ with $\mathrm{obw}(w) = k$ and $|\mathrm{rd}_2(w)| = m$. Let $\vec{p}, \vec{q}, \vec{s} \in \mathrm{dom}(w)^\ell$ be the following positions in w:

- $p_1, \ldots, p_k$ are the positions of the last k read actions in w (in descending order),
- $q_1, \ldots, q_m$ are the write actions corresponding to $p_1, \ldots, p_m$, and
- similarly, $s_1, \ldots, s_k$ are the write actions corresponding to $p_1, \ldots, p_k$.

Since we have $q_i \leq s_i$ for each $1 \leq i \leq m$, there is a minimal $0 \leq i \leq m$ such that $q_{i+1} < s_{i+1}$ holds (where $q_{m+1} = 0$). Then we have

$$(\underline{w}, \vec{p}, \vec{q}, \vec{s}) \models \mathsf{shuffle}_{m,k}(\vec{x}, \vec{y}) \wedge \mathsf{obw}_k(\vec{x}, \vec{z}) \wedge \mathsf{diff}_i(\vec{y}, \vec{z})\,.$$

Now we prove, that $(\underline{w}, p, \vec{s}) \not\models z_{i+1} \leq x$, i.e., we prove $p < s_{i+1}$. Towards a contradiction, suppose that $s_{i+1} \leq p$ holds.

Recall that $v_1, v_2 \in \Sigma^*$ are defined such that $w \equiv_{\mathcal{L}} v_1v_2$, $\mathrm{wrt}(v_1) = \mathrm{wrt}(w[1,p])$, and $\mathrm{rd}(v_2) \geq \ell$. Consider the positions in v_1v_2 of the letters on positions p, $\vec{p}$, $\vec{q}$, and $\vec{s}$. To this end, let σ be the permutation of $\mathrm{dom}(w)$ mapping the positions in $\underline{w}$ to the ones in $\underline{v_1v_2}$. Then we have $\sigma(q_j) < \sigma(p_j)$ for each $1 \leq j \leq m$ since $\mathrm{rd}_2(w) = \mathrm{rd}_2(v_1v_2)$ and, therefore, $\underline{v_1v_2} \models \mathsf{overlap}_m$ holds. By $\mathrm{wrt}(v_1) = \mathrm{wrt}(w[1,p])$ we have $\sigma(p) \leq |v_1|$ and, by $\mathrm{rd}(v_2) \geq \ell \geq k$ we have $|v_1| < \sigma(p_j)$ for each $1 \leq j \leq k$. From $w \equiv_{\mathcal{L}} v_1v_2$ we know that $\mathrm{wrt}(w) = \mathrm{wrt}(v_1v_2)$ and $\mathrm{rd}(w) = \mathrm{rd}(v_1v_2)$ hold implying

$$\sigma(s_k) < \cdots < \sigma(s_{i+1}) \leq \sigma(p) \leq |v_1| < \sigma(p_k) < \cdots < \sigma(p_1)\,.$$

All in all, we have $\sigma(s_j) < \sigma(p_j)$ for each $1 \leq j \leq k$, i.e., $\underline{v_1v_2} \models \mathsf{overlap}_k$. We can infer that $|\mathrm{rd}_2(v_1v_2)| \geq k > m = |\mathrm{rd}_2(w)|$ holds - in contradiction to $\mathrm{rd}_2(v_1v_2) = \mathrm{rd}_2(w)$ by $w \equiv_{\mathcal{L}} v_1v_2$. Hence, we have $p < s_{i+1}$ implying $(\underline{w}, p) \not\models \mathsf{P}_\ell(x)$. □

Finally, with the help of the formula P_ℓ we can prove the last implication of

Theorem 5.1:

Proposition 5.31. *Let $\mathcal{L} = (A, U)$ be a lossiness alphabet and $T \subseteq \mathcal{Q}(\mathcal{L})$ be $\mathsf{MSO_q}$-definable. Then T is recognizable.*

Proof. Let $T \subseteq \mathcal{Q}(\mathcal{L})$ be $\mathsf{MSO_q}$-definable. Then there is $\phi \in \mathsf{MSO_q}$ with $T = T(\phi)$. We construct $\phi' \in \mathsf{MSO}$ such that for each $w \in \Sigma^*$ we have $\widetilde{w} \models \phi$ if, and only if, $\underline{w} \models \phi'$.

The formula ϕ' is inductively defined as follows:

$$\phi' = \begin{cases} x = y & \text{if } \phi = (x = y) \\ x < y \wedge \Lambda_A(x) \wedge \Lambda_A(y) & \text{if } \phi = x <_+ y \\ x < y \wedge \Lambda_{\overline{A}}(x) \wedge \Lambda_{\overline{A}}(y) & \text{if } \phi = x <_- y \\ \mathsf{P}_\ell(x) & \text{if } \phi = P_\ell(x) \\ \Lambda_a(x) & \text{if } \phi = \Lambda_a(x) \\ X(x) & \text{if } \phi = X(x) \\ \psi' \vee \xi' & \text{if } \phi = \psi \vee \xi \\ \neg\psi' & \text{if } \phi = \neg\psi \\ \exists x \colon \psi' & \text{if } \phi = \exists x \colon \psi \\ \exists X \colon \psi' & \text{if } \phi = \exists X \colon \psi \end{cases}$$

Then by Lemmas 5.29 and 5.30 we have $\widetilde{w} \models \phi$ if, and only if, $\underline{w} \models \phi'$ for any $w \in \Sigma^*$. Hence, by Büchi's Theorem [4] $\eta_{\mathcal{L}}^{-1}(T)$ is regular implying that $T = T(\phi)$ is recognizable. □

6. Characterizations of the Aperiodic Subsets

In the previous section we have seen a Kleene- and Büchi-type characterization of the recognizable subsets in the plq monoid. Another more involved task is to describe the aperiodic subsets in the plq monoid. Schützenberger has proven in [27] that the aperiodic subsets in the free monoid are exactly the star-free languages. This result gives us a decision procedure to decide whether a given regular language is star-free. Another similar result for trace monoids can be found in [10]. We cannot translate these two results to plq monoids since the class of aperiodic subsets is not closed under product. For example, the two sets S and T in Remark 5.17 are aperiodic and their product is not even recognizable. Though, we will see that we can restrict the monoid's product to describe exactly the aperiodic subsets of the plq monoid.

Another characterization of the aperiodic languages was proven by [20]: similar to Büchi's Theorem [4], McNaughton and Papert proved that these are exactly the FO-definable languages. Here, we will see that analogously the aperiodic subsets in the plq monoid are the $\mathsf{FO_q}$-definable subsets.

Before we start to prove these equivalences we have to define the restriction of star-freeness: to this end, let $\mathcal{L} = (A, U)$ be a lossiness alphabet. We say that a subset of $\mathcal{Q}(\mathcal{L})$ is *q^+-star-free* if it can be constructed by the following rules:

(1^+) $\mathrm{wrt}^{-1}(\varepsilon)$ and $\mathrm{wrt}^{-1}(a)$ for any $a \in A$ are q$^+$-star-free
(2^+) if $S, T \subseteq \mathcal{Q}(\mathcal{L})$ are q$^+$-star-free then $S \cup T$, $S \cdot T$, and $\mathcal{Q}(\mathcal{L}) \setminus S$ are q$^+$-star-free

Similarly, by replacing wrt^{-1} by rd^{-1} in the rules above, we define the class of *q$^-$-star-free* subsets of $\mathcal{Q}(\mathcal{L})$. Finally, a subset of $\mathcal{Q}(\mathcal{L})$ is *q-star-free* if it can be constructed from the following rules:

(1) if $T \subseteq \mathcal{Q}(\mathcal{L})$ is q$^+$- or q$^-$-star-free it also is q-star-free
(2) if $S, T \subseteq \mathcal{Q}(\mathcal{L})$ are q-star-free then $S \cup T$ and $\mathcal{Q}(\mathcal{L}) \setminus S$ are q-star-free
(3) if $S \subseteq \mathcal{Q}(\mathcal{L})$ is q$^+$-star-free and $T \subseteq \mathcal{Q}(\mathcal{L})$ is q$^-$-star-free such that $\mathrm{rd}(T)$ is finite (i.e., T is obtained without usage of the $\setminus$-operator) then $S \cdot \mathcal{Q}(\mathcal{L}) \cdot T$ is q-star-free

Hence, the only difference of the rules above in comparison to the rules constructing the q-rational subsets is the missing iteration. Therefore, $T \subseteq \mathcal{Q}(\mathcal{L})$ is q$^+$-star-free (q$^-$-star-free) if, and only if, there is an aperiodic language $R \subseteq A^*$ with $T = \mathrm{wrt}^{-1}(R)$ ($T = \mathrm{rd}^{-1}(R)$, respectively).

Similarly, to Theorems 5.1 and 5.6 we can state the following result:

Theorem 6.1. *Let $\mathcal{L} = (A, U)$ be a lossiness alphabet and $T \subseteq \mathcal{Q}(\mathcal{L})$. Then the following are equivalent:*

(A) T is aperiodic.
(B) $\eta_{\mathcal{L}}^{-1}(T) \cap \overline{A}^* A^* \overline{A}^*$ is aperiodic.
(C) T is a Boolean combination of sets of the form $\mathrm{wrt}^{-1}(R)$ or $\mathrm{rd}^{-1}(R)$ for some aperiodic $R \subseteq A^*$ and the sets Ω_ℓ for $\ell \in \mathbb{N}$.
(D) T is FO_q-definable.
(E) T is q-star-free.

Proof. To prove this theorem we recall the proof of Theorem 5.1.

At first we show the implication "(A)⇒(B)". So, let $T \subseteq \mathcal{Q}(\mathcal{L})$ be aperiodic. Then $\eta_{\mathcal{L}}^{-1}(T)$ is aperiodic. Since $\overline{A}^* A^* \overline{A}^*$ is aperiodic and the class of aperiodic languages is closed under intersection we can infer that $\eta_{\mathcal{L}}^{-1}(T) \cap \overline{A}^* A^* \overline{A}^*$ also is aperiodic.

Next we prove "(B)⇒(C)". We recall the proof of Lemma 5.10 and assume that the recognizing monoid $\mathcal{M}$ is aperiodic. The arguments of the inverse projections wrt^{-1} or rd^{-1} are single words, $\mu^{-1}(m)$, $\overline{\mu}^{-1}(m)$ for any $m \in \mathcal{M}$ and products of those languages. These are all aperiodic since aperiodicity is preserved under inverse homomorphisms. (Note that the class of aperiodic subsets in a free monoid is closed under product according to Schützenberger's Theorem [27].) Hence, T is a Boolean combination of sets $\mathrm{wrt}^{-1}(R)$ and $\mathrm{rd}^{-1}(R)$ where $R \subseteq A^*$ is aperiodic and Ω_ℓ for any $\ell \in \mathbb{N}$.

In the proof of Proposition 5.23 we see that Ω_ℓ is even q-star-free and hence the implication "(C)⇒(D)" holds.

In the construction of the proof of Proposition 5.28 each of the used formulas can be expressed in first-order logic. Therefore, we have "(D)⇒(E)".

Finally we have to prove "(E)⇒(A)". Each translation of a $\mathsf{FO_q}$-formula as seen in Proposition 5.31 results in a formula in FO. Therefore, by [20] each $\mathsf{FO_q}$-definable subset is aperiodic. □

7. Conclusion and Open Problems

In this paper we investigated some generalizations of famous results from automata theory concerning rational, recognizable, star-free, and aperiodic languages. Concretely, we consider these classes in the so-called partially lossy queue monoid (plq monoid for short). This monoid models the behavior of a partially lossy queue and was first introduced in [16]. First, we proved that, in contrast to Kleene's theorem [13], the classes of rational and recognizable subsets in the plq monoid do not coincide. Hence, we defined some restrictions to concatenations and iterations of subsets of the plq monoid. Then by closure of these restricted operations and the Boolean operations we obtain the so-called q-rational subsets. We proved in this paper that a set is recognizable if, and only if, it is q-rational. Additionally, we defined a generalization of Büchi's MSO logic (cf. [4]) called $\mathsf{MSO_q}$ which describes exactly the recognizable subsets in the plq monoid.

Similarly, we defined so-called q-star-free expressions generating the aperiodic subsets (similar to Schützenberger's theorem [27]) and proved that the first-order fragment of our $\mathsf{MSO_q}$ logic describes the aperiodic subsets (similar to McNaughton-Papert's theorem [20]).

There are some open questions relating to the subsets of the plq monoid. So, we do not know a "rational" characterization of the recognizable subsets using monotonic operations, only (recall that complementation is not monotonic).

One may also consider the aperiodicity problem of rational or recognizable subsets, i.e., the question whether a given rational (or recognizable, resp.) subset of the plq monoid is aperiodic.

Additionally, one could study some sub-classes of the star-free and aperiodic sets. For example, maybe there is a simple generalization of the dot-depth hierarchy or the Straubing-Thérien hierarchy such that the nth level of this hierarchy corresponds to the $\mathcal{B}\Sigma_n$-fragment of the $\mathsf{MSO_q}$ logic and the $(n-1/2)^{\text{th}}$ level corresponds to the Σ_n-fragment of $\mathsf{MSO_q}$ (similar to Perrin's and Pin's result in [23]).

References

[1] Abdulla, P.A., Jonsson, B.: Verifying programs with unreliable channels. Information and Computation **127**(2), 91–101 (1996). DOI 10.1006/inco.1996.0053

[2] Berstel, J.: Transductions and Context-Free Languages. Teubner Studienbücher (1979). DOI 10.1007/978-3-663-09367-1

[3] Brand, D., Zafiropulo, P.: On Communicating Finite-State Machines. Journal of the ACM **30**(2) (1983). DOI 10.1145/322374.322380

[4] Büchi, J.R.: Weak Second-Order Arithmetic and Finite Automata. Mathematical Logic Quarterly **6**(1-6), 66–92 (1960). DOI 10.1002/malq.19600060105

[5] Cécé, G., Finkel, A., Purushotaman Iyer, S.: Unreliable channels are easier to verify than perfect channels. Information and Computation **124**(1), 20–31 (1996). DOI 10.1006/inco.1996.0003

[6] Chrobak, M., Rytter, W.: Unique decipherability for partially commutative alphabet. In: MFCS'86, *Lecture Notes in Computer Science*, vol. 233, pp. 256–263. Springer (1986). DOI 10.1007/BFb0016249

[7] Diekert, V., Rozenberg, G.: The Book of Traces. World scientific (1995). DOI 10.1142/9789814261456

[8] Finkel, A.: Decidability of the termination problem for completely specified protocols. Distributed Computing **7**(3), 129–135 (1994). DOI 10.1007/BF02277857

[9] Gibbons, A., Rytter, W.: On the decidability of some problems about rational subsets of free partially commutative monoids. Theoretical Computer Science **48**, 329–337 (1986). DOI 10.1016/0304-3975(86)90101-5

[10] Guaiana, G., Restivo, A., Salemi, S.: On aperiodic trace languages. In: STACS'91, *Lecture Notes in Computer Science*, vol. 480, pp. 76–88. Springer (1991). DOI 10.1007/BFb0020789

[11] Huschenbett, M., Kuske, D., Zetzsche, G.: The monoid of queue actions. Semigroup forum **95**(3), 475–508 (2017). DOI 10.1007/s00233-016-9835-4

[12] Kambites, M.: Formal languages and groups as memory. Communications in Algebra **37**(1), 193–208 (2009). DOI 10.1080/00927870802243580

[13] Kleene, S.C.: Representation of events in nerve nets and finite automata. Tech. rep., DTIC Document (1951)

[14] Köcher, C.: Einbettungen in das Transformationsmonoid einer vergesslichen Warteschlange. Master's Thesis, Technische Universität Ilmenau, Ilmenau (2016)

[15] Köcher, C.: Rational, Recognizable, and Aperiodic Sets in the Partially Lossy Queue Monoid. In: STACS 2018, *LIPIcs*, vol. 96, pp. 45:1–45:14. Dagstuhl Publishing (2018). DOI 10.4230/LIPIcs.STACS.2018.45

[16] Köcher, C., Kuske, D., Prianychnykova, O.: The Inclusion Structure of Partially Lossy Queue Monoids and their Trace Submonoids. RAIRO - Theoretical Informatics and Applications **52**(1), 55–86 (2018). DOI 10.1051/ita/2018003

[17] Lohrey, M.: The rational subset membership problem for groups: A survey. In: Groups St Andrews, vol. 422, pp. 368–389. Cambridge University Press (2013). DOI 10.1017/CBO9781316227343.024

[18] Masson, B., Schnoebelen, P.: On verifying fair lossy channel systems. In: MFCS'02, *Lecture Notes in Computer Science*, vol. 2420, pp. 543–555. Springer (2002). DOI 10.1007/3-540-45687-2_45

[19] McKnight, J.D.: Kleene quotient theorems. Pacific Journal of Mathematics **14**(4), 1343–1352 (1964)

[20] McNaughton, R., Papert, S.A.: Counter-Free Automata, vol. 65. The MIT Press (1971)

[21] Muscholl, A., Petersen, H.: A Note on the Commutative Closure of Star-Free Languages. Information Processing Letters **57**(2), 71–74 (1996). DOI 10.1016/0020-0190(95)00187-5

[22] Ochmański, E.: Regular behaviour of concurrent systems. Bulletin of the EATCS **27**, 56–67 (1985)

[23] Perrin, D., Pin, J.E.: First-order logic and star-free sets. Journal of Computer and System Sciences **32**(3), 393–406 (1986). DOI 10.1016/0022-0000(86)90037-1

[24] Pin, J.É.: Mathematical foundations of automata theory. Lecture notes LIAFA, Uni-

versité Paris **7** (2010)
[25] Post, E.L.: A variant of a recursively unsolvable problem. Bulletin of the American Mathematical Society **52**(4), 264–268 (1946)
[26] Render, E., Kambites, M.: Rational subsets of polycyclic monoids and valence automata. Information and Computation **207**(11), 1329–1339 (2009). DOI 10.1016/j.ic.2009.02.012
[27] Schützenberger, M.P.: On finite monoids having only trivial subgroups. Information and control **8**(2), 190–194 (1965). DOI 10.1016/S0019-9958(65)90108-7
[28] Thomas, W.: Languages, Automata, and Logic. In: G. Rozenberg, A. Salomaa (eds.) Beyond Words, *Handbook of Formal Languages*, vol. 3, pp. 389–455. Springer (1997). DOI 10.1007/978-3-642-59126-6_7
[29] Zetzsche, G.: Monoids as Storage Mechanisms. Ph.D. thesis, Technische Universität Kaiserslautern, Kaiserslautern (2016)