Rational and Recognizable Sets in the Queue Monoid

Highlights of Games, Logic and Automata 2018, Berlin

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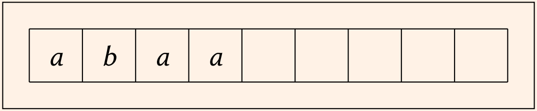
Queues

- Let A be an alphabet ($|A| \geq 2$).
- Two actions for each $a \in A$:
  - write letter $a \rightsquigarrow a$
  - read letter $a \rightsquigarrow \overline{a}$
- $\overline{A} := \{\overline{a} \mid a \in A\}$
- $\Sigma := A \uplus \overline{A}$

Example

$q = abaa$ $t = b\overline{a}b\overline{b}$



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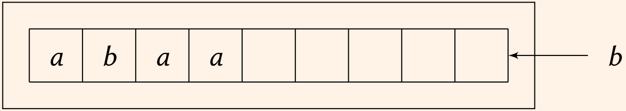

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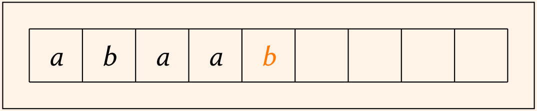

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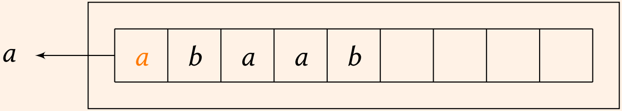

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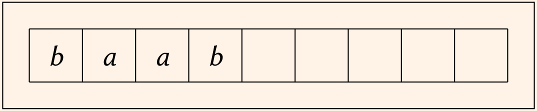

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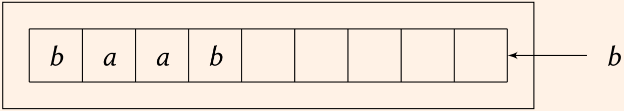

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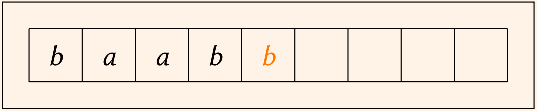

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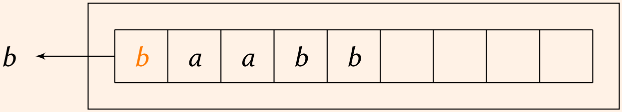

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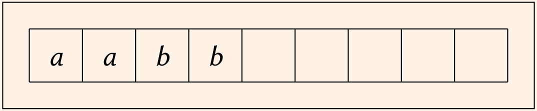

Queue Monoids

Definition

$s, t \in \Sigma^*$ act equally (in symbols $s \equiv t$) if, and only if,

$$\forall p, q \in A^*\colon p \xrightarrow{s} q \iff p \xrightarrow{t} q$$

Remark

$\equiv$ is the least congruence on Σ^* satisfying certain commutations of write and read actions, e.g., $a\overline{a}\overline{b} \equiv \overline{a}a\overline{b}$ for $a, b \in A$.

Definition

- $\mathcal{Q} := \Sigma^*/_{\equiv}$ ... queue monoid
- $\eta\colon \Sigma^* \to \mathcal{Q}\colon t \mapsto [t]_{\equiv}$ ... natural homomorphism

Definition

Let $S \subseteq \mathcal{Q}$.

1. S is rational if there is a regular language $L \subseteq \Sigma^*$ with $\eta(L) = S$.
   - closure properties: $\cup$, $\cdot$, *
   - generalizes regular expressions
2. S is recognizable if $\eta^{-1}(S)$ is regular.
   - closure properties: $\cup$, $\cap$, $\setminus$
   - generalizes acceptance by finite automata

Theorem (Kleene 1951)

In the free monoid, a set is rational if, and only if, it is recognizable.

Q-Rational Subsets

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- Here: There are rational sets that are not recognizable!
- But: Each recognizable set is rational [McKnight 1964].
- Restrict the rational sets in an appropriate way

  $\rightsquigarrow$ q-rational sets
  - start from $\eta(\overline{A}^* a \overline{A}^*)$ and $\eta(A^* \overline{a} A^*)$ for $a \in A$
  - closure under union and complementation
  - restricted closure under product and iteration

Main Theorem

Theorem

Let $S \subseteq \mathcal{Q}$. Then the following are equivalent:

1. *S is recognizable*
2. *S is q-rational*

Structures for Queue Transformations

- Let $a, b \in A$ be distinct. Consider $t = [\overline{b}\overline{a}baaa\overline{a}]_{\equiv}$.
- We model t as a structure $\widetilde{t}$ with infinitely many relations:

$$\overline{b} \longrightarrow \overline{a} \longrightarrow b \longrightarrow a \longrightarrow a \longrightarrow a \longrightarrow \overline{a}$$

- $\overline{b}\overline{a}baaa\overline{a} \equiv ba\overline{b}\overline{a}\overline{a}aa$

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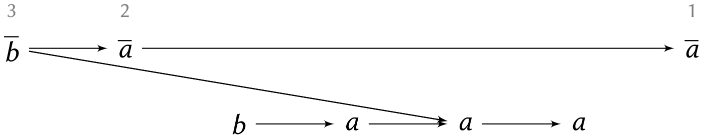
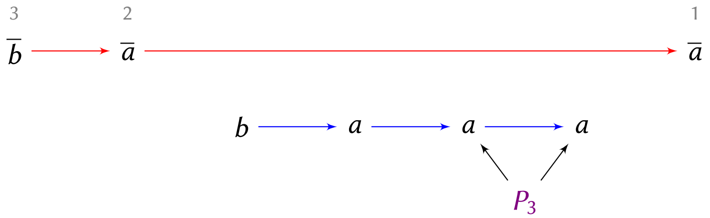


- Let $a, b \in A$ be distinct. Consider $t = [\overline{b}\overline{a}baaa\overline{a}]_\equiv$.
- We model t as a structure $\widetilde{t}$ with infinitely many relations:
  - $\leq_-$, $\leq_+$, P_n for any $n \in \mathbb{N}$



Theorem

Let $S \subseteq \mathcal{Q}$. Then the following are equivalent:

1. *S is recognizable*
2. *S is q-rational*
3. *$S = \{t \in \mathcal{Q} \mid \widetilde{t} \models \phi\}$ for some $\phi \in \mathrm{MSO}$*

- Similar results for aperiodic sets and for (partially) lossy queues

Thank you!