Rational, Recognizable, and Aperiodic Sets in the Partially Lossy Queue Monoid

27. Theorietag "Automaten und Formale Sprachen", Bonn

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What is a Partially Lossy Queue?

- there are two types of fifo-queues:
  - Reliable Queues
    - nothing can be forgotten or injected
    - applications: software and algorithms engineering
  - Lossy Queues
    - everything can be forgotten, nothing can be injected
    - applications: verification and telematics
- natural combination of both: Partially Lossy Queues (PLQs)
  - some parts can be forgotten
  - nothing can be injected

PLQs: Basics

- Let A be an alphabet ($|A| \geq 2$) and $U \subseteq A$.
  - U ... unforgettable letters
  - $A \setminus U$ ... forgettable letters
- two controllable operations for each $a \in A$:
  - write letter $a \rightsquigarrow a$
  - read letter $a \rightsquigarrow \overline{a}$
- $\Sigma := \{a, \overline{a} \mid a \in A\}$
- non-controllable operation: forgetting letters from $A \setminus U$

Example

$$A = \{a, b\},\ U = \{b\}$$

$q = aaba$ $\qquad$ $v = bb\overline{a}\overline{b}$

a	a	b	a					

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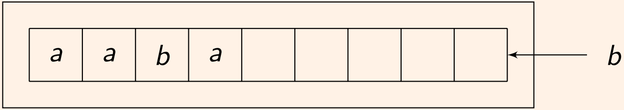

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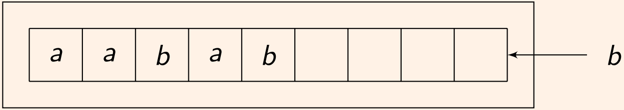

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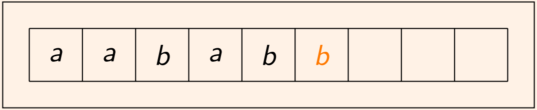

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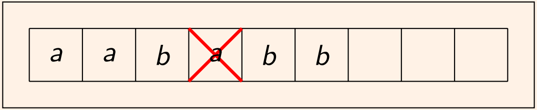

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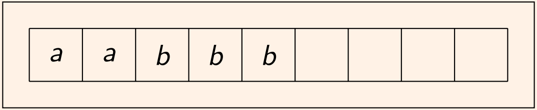

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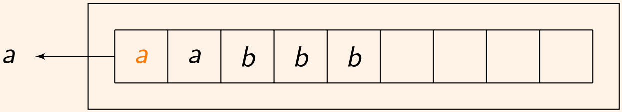

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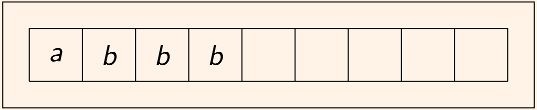

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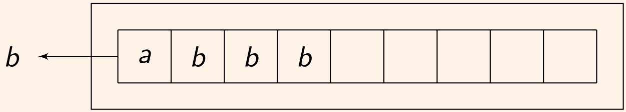

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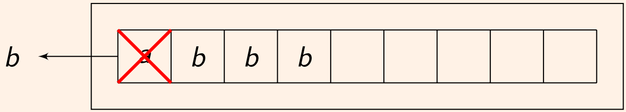

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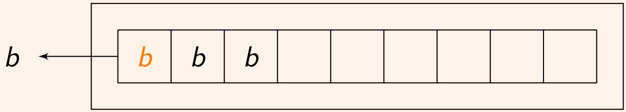

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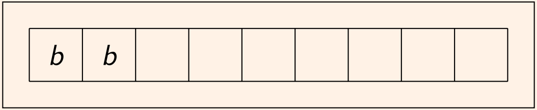


- we model this behavior as a monoid

Theorem (K., Kuske 2017, cf. CSR 2017)

Two sequences of actions $v, w \in \Sigma^*$ act equally (in symbols $v \equiv w$) if, and only if, they can be equated by application of the following commutations:

- $a\overline{b} \equiv \overline{b}a$ *if* $a \neq b$
- $a\overline{a}\overline{b} \equiv \overline{a}a\overline{b}$
- $xwa\overline{a} \equiv xw\overline{a}a$ *if* $x \in U \cup \{a\}$ *and* $w \in A^*$

for any $a, b \in A$.

Definition

- $\mathcal{Q}(A, U) := \Sigma^*/_{\equiv}$ ... the plq monoid

Definition

Let $\mathcal{M}$ be a monoid and $S \subseteq \mathcal{M}$.

- S is rational if it can be constructed from finite subsets of $\mathcal{M}$ using $\cup$, $\cdot$, and *
  - i.e., generalizes regular expressions
- S is recognizable if $\eta^{-1}(S) \subseteq \Gamma^*$ is regular where $\eta\colon \Gamma^* \to \mathcal{M}$ is a homomorphism and Γ is an alphabet.
  - closure properties: $\cup$, $\cap$, $\setminus$
  - i.e., generalizes acceptance of finite automata

Theorem (Kleene 1951)

$S \subseteq \Sigma^*$ is rational if, and only if, it is recognizable.

Question

Is $S \subseteq \mathcal{Q}(A, U)$ rational if, and only if, it is recognizable? NO!

- class of rational sets is not closed under intersection
- class of recognizable sets is not closed under $\cdot$ and *
- BUT: each recognizable set is rational due to [McKnight 1964]

Question

When is a rational set recognizable?

- recognizability of rational sets is undecidable

Q-Rational Subsets

Definition

- $S \subseteq \mathcal{Q}(A, U)$ is q$^+$-rational if $S = \text{write}^{-1}(R)$ where $R \subseteq A^*$ is regular, i.e., if it can be constructed from $\text{write}^{-1}(a)$ for $a \in A$, $\text{write}^{-1}(\varepsilon)$, and $\emptyset$ using $\cup$, $\cdot$, and *.
- Similar: $S \subseteq \mathcal{Q}(A, U)$ is q$^-$-rational if $S = \text{read}^{-1}(R)$ where $R \subseteq (\Sigma \setminus A)^*$ is regular.
- $S \subseteq \mathcal{Q}(A, U)$ is q-rational if
  - S is q$^+$- or q$^-$-rational
  - $S = S_1 \cup S_2$ if S_1, S_2 q-rational
  - $S = S_1 \cdot \mathcal{Q}(A, U) \cdot S_2$ if S_1 q$^+$-rational, S_2 q$^-$-rational, and $\text{read}(S_2)$ finite
  - $S = \mathcal{Q}(A, U) \setminus S_1$ if S_1 q-rational

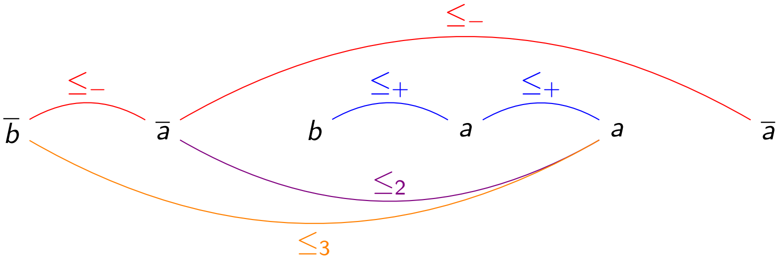
Main Theorem

Theorem

Let $S \subseteq \mathcal{Q}(A, U)$. Then the following are equivalent:

1. *S is recognizable*
2. *S is q-rational*

- Let $a, b \in A$, $b \notin U$. Consider $w = \overline{b}\overline{a}baa\overline{a}$.
- We model w as a graph $\widetilde{w}$:



- FO … first-order logic on these graphs
- MSO … FO + quantification of sets

Main Theorem

Theorem

Let $S \subseteq \mathcal{Q}(A, U)$. *Then the following are equivalent:*

1. S *is recognizable*
2. S *is q-rational*
3. *there is a sentence* $\phi \in \mathrm{MSO}$ *with* $S = \{[w] \mid \widetilde{w} \models \phi\}$

Aperiodic and Star-free Sets

Definition

Let $\mathcal{M}$ be a monoid and $S \subseteq \mathcal{M}$.

- S is star-free if it can be constructed from finite subsets of $\mathcal{M}$ using $\cup$, $\cdot$, and $\setminus$
  - i.e., generalizes star-free expressions
- S is aperiodic if it is recognizable and there is $n \in \mathbb{N}$ s.t.

$$\forall x, y, z \in \mathcal{M}\colon xy^n z \in S \iff xy^{n+1} z \in S$$

  - closure properties: $\cup$, $\cap$, $\setminus$
  - i.e., generalizes acceptance of finite, counter-free automata

Theorem (Schützenberger 1965)

$S \subseteq \Sigma^*$ is aperiodic if, and only if, it is star-free.

Question

Is $S \subseteq \mathcal{Q}(A, U)$ aperiodic if, and only if, it is star-free? NO!

- class of aperiodic sets is not closed under $\cdot$
- define q-star-free sets similar to q-rational ones by replacing * by $\setminus$

Theorem

Let $S \subseteq \mathcal{Q}(A, U)$. Then the following are equivalent:

1. *S is aperiodic*
2. *S q-star-free*
3. *there is a sentence $\phi \in \mathsf{FO}$ with $S = \{[w] \mid \widetilde{w} \models \phi\}$*

Thank you!