Regelungs- und Systemtechnik 1

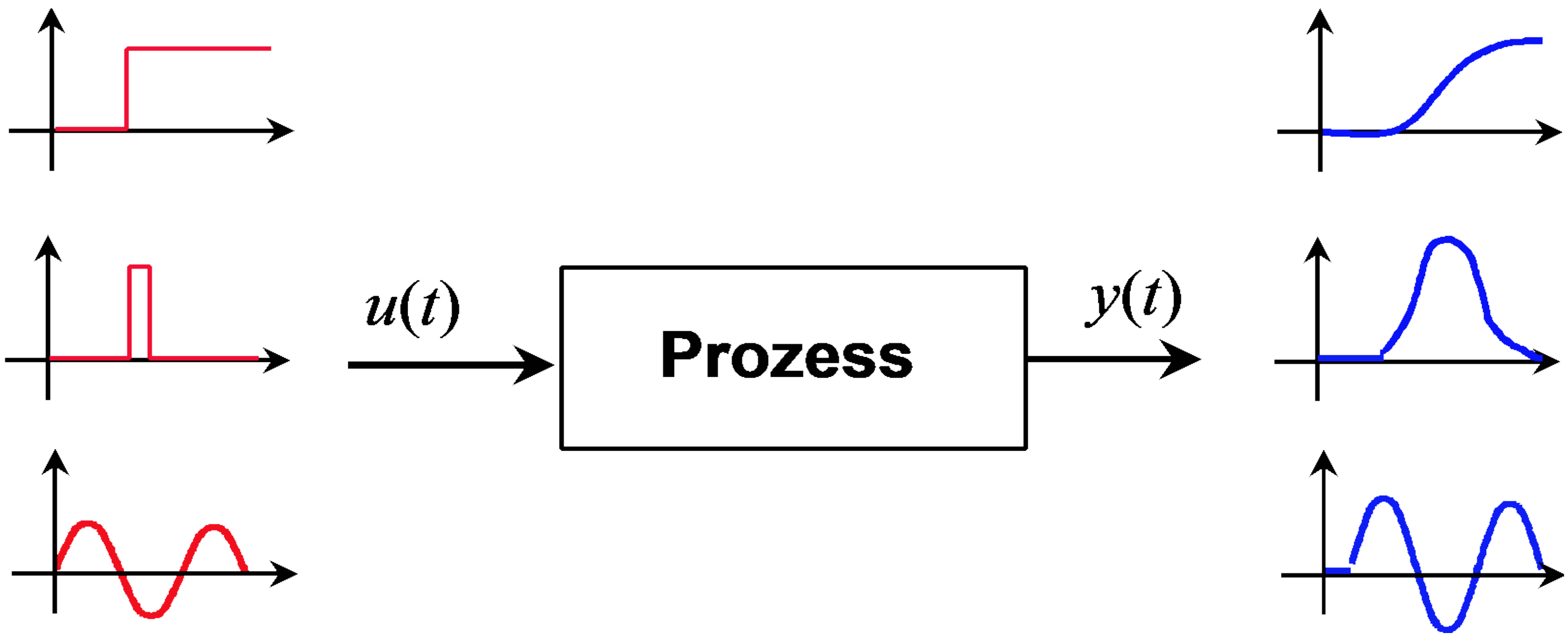
Kapitel 3: Laplace-Transformation

Prof. Dr.-Ing. habil. Pu Li

Fachgebiet Prozessoptimierung

Fachgebiet
Prozessoptimierung

Problemdarstellung:



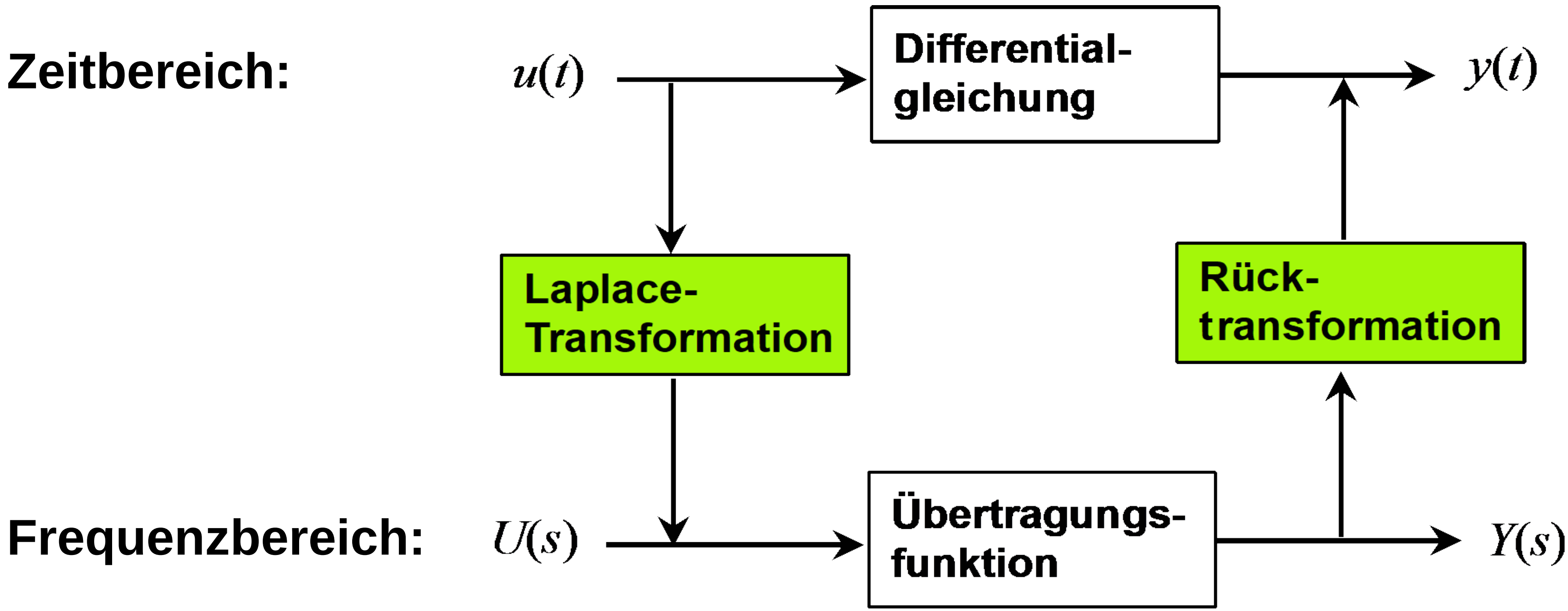
$$\frac{d^n y}{dt^n} + a_1 \frac{d^{n-1} y}{dt^{n-1}} + \cdots + a_{n-1} \frac{dy}{dt} + a_n y = k_u u$$

Die Eigenschaften des Systems sind schwer zu analysieren!

Man möchte die Differentialgleichung in eine algebraische Gleichung umwandeln.

Fachgebiet Prozessoptimierung

Laplace-Transformation typischer Funktionen



Definition:

$$F(s) = L\{f(t)\} = \int_0^{\infty} f(t) e^{-st} dt$$

mit

$$f(t) = \begin{cases} 0 & t < 0 \\ \neq 0 & t \geq 0 \end{cases}, \qquad s = \sigma + j\omega$$

Fachgebiet Prozessoptimierung

Laplace-Transformation typischer Funktionen

Einheitssprung:

$$f(t) = \sigma(t) = \begin{cases} 0 & t < 0 \\ 1 & t \geq 0 \end{cases}$$

$$F(s) = L\{\sigma(t)\} = \int_0^\infty e^{-st} dt = -\frac{1}{s} e^{-st} \Big|_0^\infty = \frac{1}{s}$$

Rampenfunktion: $f(t) = t$

$$F(s) = \int_0^\infty t e^{-st} dt = \int_0^\infty (-\frac{1}{s}) d e^{-st} = -\frac{1}{s}\left[t e^{-st} \Big|_0^\infty - \int_0^\infty e^{-st} dt \right] = \frac{1}{s^2}$$

Exponentialfunktion: $f(t) = e^{-at}$

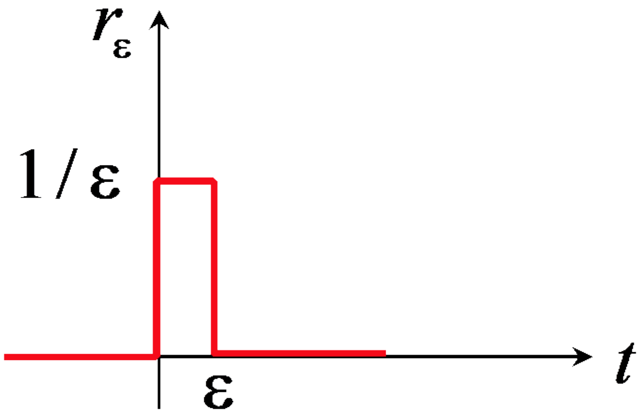
$$F(s) = \int_0^\infty e^{-at} e^{-st} dt = \int_0^\infty e^{-(s+a)t} dt = -\frac{1}{s+a} e^{-(s+a)t} \Big|_0^\infty = \frac{1}{s+a}$$

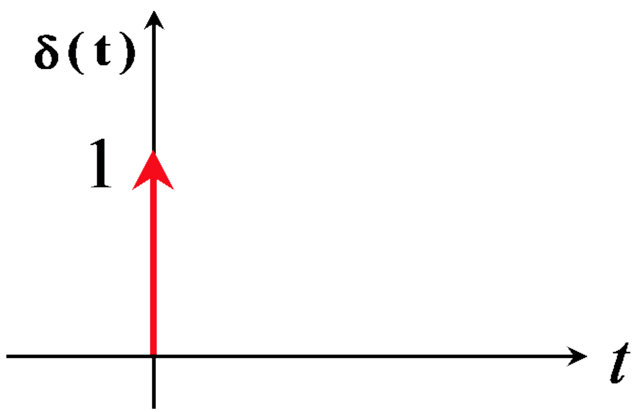
Laplace-Transformation typischer Funktionen

Impulsfunktion: $f(t) = \delta(t) = \lim_{\varepsilon \to 0} r_\varepsilon, \qquad r_\varepsilon = \begin{cases} 1/\varepsilon & 0 \le t \le \varepsilon \\ 0 & \text{sonst} \end{cases}$

also $\delta(t) = 0 \quad \text{für} \quad t \neq 0$

und $\int_{-\infty}^{\infty} \delta(t)dt = \lim_{\varepsilon \to 0} \int_0^{\varepsilon} \frac{1}{\varepsilon} dt = 1$





$$F(s) = L\{\delta(t)\} = \int_0^{\infty} \delta(t) e^{-st} dt = \lim_{\varepsilon \to 0} \int_0^{\varepsilon} \frac{1}{\varepsilon} e^{-st} dt = 1$$

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Laplace-Transformation typischer Operatoren

Differential:

$$L\left\{\frac{df}{dt}\right\}=\int_0^\infty\left(\frac{df}{dt}\right)e^{-st}dt=\int_0^\infty e^{-st}df=\left[f(t)e^{-st}\Big|_0^\infty-\int_0^\infty f(t)(-s)e^{-st}dt\right]$$

$$=-f(0)+s\int_0^\infty f(t)e^{-st}dt=sF(s)-f(0)$$

Integral:

$$L\left\{\int_0^t f(z)dz\right\}=\int_0^\infty\left(\int_0^t f(z)dz\right)e^{-st}dt=-\frac{1}{s}\int_0^\infty\left(\int_0^t f(z)dz\right)de^{-st}$$

$$=-\frac{1}{s}\left[\left(\int_0^t f(z)dz\right)e^{-st}\Bigg|_0^\infty-\int_0^\infty f(t)e^{-st}dt\right]$$

$$=\frac{1}{s}\int_0^\infty f(t)e^{-st}dt=\frac{1}{s}F(s)$$

Eigenschaften der Laplace-Transformation

Zeitverschiebung (Totzeit):

$$L\{f(t-\tau)\} = \int_0^\infty f(t-\tau)e^{-st}dt = e^{-\tau s}\int_0^\infty f(t-\tau)e^{-s(t-\tau)}d(t-\tau)$$

$$= e^{-\tau s}\int_0^\infty f(\omega)e^{-s\omega}d\omega = e^{-\tau s}F(s)$$

Überlagerung: $$L\{a_1 f_1(t) + a_2 f_2(t)\} = a_1 F_1(s) + a_2 F_2(s)$$

Achtung: $$L\{f_1(t) f_2(t)\} \neq F_1(s) F_2(s)$$

Ähnlichkeit: $$L\{f(at)\} = \int_0^\infty f(at)e^{-st}dt = \frac{1}{a}\int_0^\infty f(at)e^{-s(at)/a}d(at)$$

$$= \frac{1}{a}\int_0^\infty f(\omega)e^{-s\omega/a}d\omega = \frac{1}{a}\int_0^\infty f(\omega)e^{-\left(\frac{s}{a}\right)\omega}d\omega = \frac{1}{a}F(\frac{s}{a})$$

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Laplace-Transformation typischer Funktionen

$f(t)$	$F(s)$		$f(t)$	$F(s)$
$\sigma(t)$	$\frac{1}{s}$		$\sin \omega t$	$\frac{\omega}{s^2+\omega^2}$
t	$\frac{1}{s^2}$		$\cos \omega t$	$\frac{s}{s^2+\omega^2}$
t^n	$\frac{n!}{s^{n+1}}$		$e^{-at}\sin \omega t$	$\frac{\omega}{(s+a)^2+\omega^2}$
e^{-at}	$\frac{1}{s+a}$		$e^{-at}\cos \omega t$	$\frac{s}{(s+a)^2+\omega^2}$
$1-e^{-at}$	$\frac{a}{s(s+a)}$		$\delta(t)$	1
$\frac{dx}{dt}$	$sX(s)-x(0)$		$\frac{d^2x}{dt^2}$	$s^2X(s)-sx(0)-\dot{x}(0)$

Laplace-Transformation: $f(t) \rightarrow F(s)$ $\quad F(s)=L\{f(t)\}$

Inverse Laplace-Transformation: $F(s) \rightarrow f(t)$ $\quad f(t)=L^{-1}\{F(s)\}$

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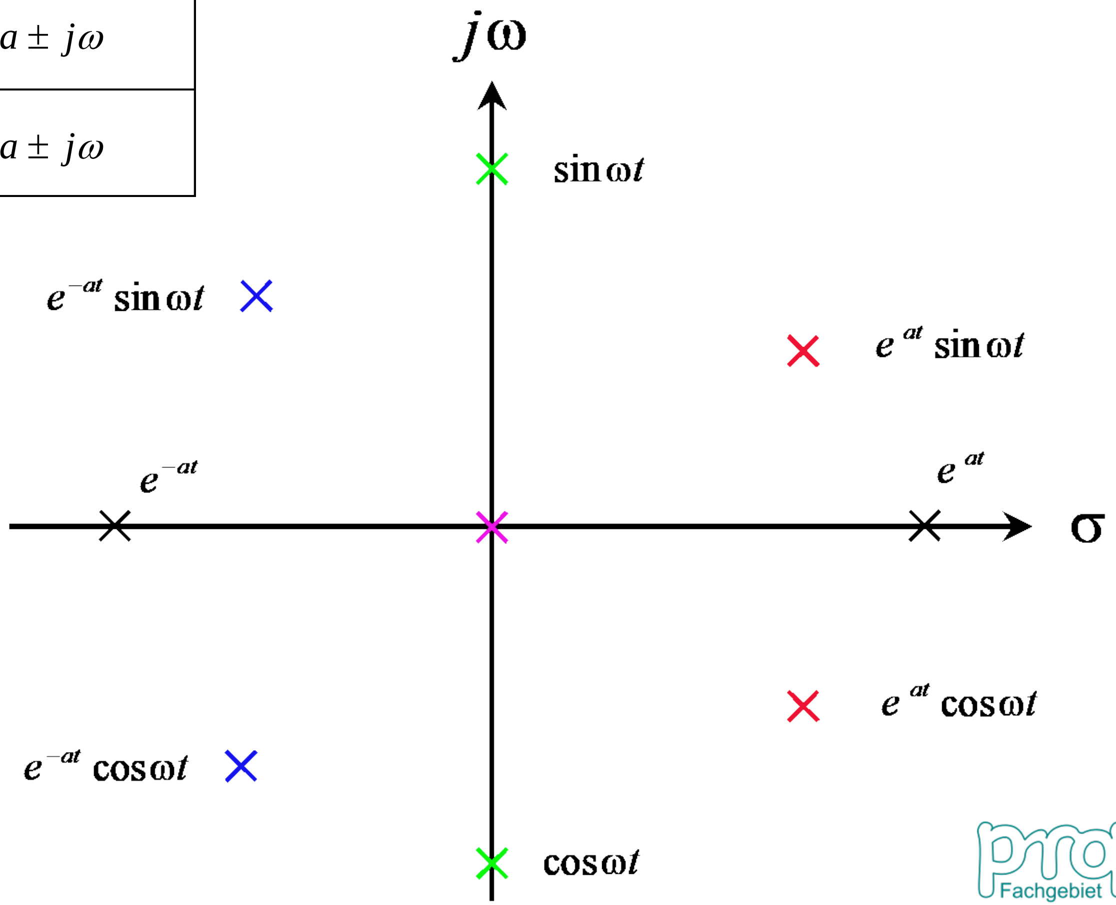
Wirkungen der Polstellen

$f(t)$	$F(s)$	p
$\sin \omega t$	$\dfrac{\omega}{s^2+\omega^2}$	$\pm j\omega$
$\cos \omega t$	$\dfrac{s}{s^2+\omega^2}$	$\pm j\omega$
$e^{-at} \sin \omega t$	$\dfrac{\omega}{(s+a)^2+\omega^2}$	$-a \pm j\omega$
$e^{-at} \cos \omega t$	$\dfrac{s+a}{(s+a)^2+\omega^2}$	$-a \pm j\omega$
t	$\dfrac{1}{s^2}$	
t^n	$\dfrac{n!}{s^{n+1}}$	
e^{-at}	$\dfrac{1}{s+a}$	
$1-e^{-at}$	$\dfrac{a}{s(s+a)}$	



Eigenschaften der Laplace-Transformation

Satz vom Anfangswert: $f(0)$

da $$L\left\{\frac{df}{dt}\right\} = \int_0^\infty \left(\frac{df}{dt}\right) e^{-st}dt = sF(s) - f(0)$$

$$\lim_{s\to\infty} \int_0^\infty \left(\frac{df}{dt}\right) e^{-st}dt = \lim_{s\to\infty} \left[sF(s) - f(0)\right]$$

Weil $$\lim_{s\to\infty} \left(\frac{df}{dt}\right) e^{-st} = 0$$

$$\boxed{f(0) = \lim_{s\to\infty} sF(s)}$$

Beispiel: $$F(s) = \frac{1}{s(s+1)(s+2)}$$

$$f(0) = \lim_{s\to\infty} sF(s) = \lim_{s\to\infty} \frac{1}{(s+1)(s+2)} = 0$$

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Eigenschaften der Laplace-Transformation

Satz vom Endwert: $f(\infty)$

da $$L\left\{\frac{df}{dt}\right\} = \int_0^\infty \left(\frac{df}{dt}\right) e^{-st} dt = sF(s) - f(0)$$

$$\lim_{s \to 0} \int_0^\infty \left(\frac{df}{dt}\right) e^{-st} dt = \lim_{s \to 0} \left[sF(s) - f(0)\right]$$

Weil $$\lim_{s \to 0} \int_0^\infty \left(\frac{df}{dt}\right) e^{-st} dt = \int_0^\infty df = f(\infty) - f(0) = \lim_{s \to 0} sF(s) - f(0)$$

$$f(\infty) = \lim_{s \to 0} sF(s)$$

Beispiel: $$F(s) = \frac{1}{s(s+1)(s+2)}$$

$$f(\infty) = \lim_{s \to 0} sF(s) = \lim_{s \to 0} \frac{1}{(s+1)(s+2)} = \frac{1}{2}$$

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Inverse Laplace-Transformation

$$F(s) \rightarrow f(t) \quad \Longrightarrow \quad f(t) = L^{-1}\{F(s)\}$$

Umformung der Funktion zu elementaren Funktionen:

$$F(s) = F_1(s) + F_2(s) + \cdots + F_n(s)$$

damit

$$\begin{aligned} f(t) &= L^{-1}\{F(s)\} = L^{-1}\{F_1(s)\} + L^{-1}\{F_2(s)\} + \cdots + L^{-1}\{F_n(s)\} \\ &= f_1(t) + f_2(t) + \cdots + f_n(t) \end{aligned}$$

Für die Funktion ($m<n$)

$$F(s) = \frac{b_0 s^m + b_1 s^{m-1} + \cdots + b_{m-1} s + b_m}{s^n + a_1 s^{n-1} + \cdots + a_{n-1} s + a_n} = \frac{Z(s)}{N(s)}$$

mit

$$N(s) = (s + s_1)(s + s_2)\cdots(s + s_n)$$

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Inverse Laplace-Transformation

$$N(s) = (s+s_1)(s+s_2)\cdots(s+s_n) = 0 \quad \Rightarrow \quad p_k = -s_k, \quad k = 1,\cdots,n$$

dann
$$F(s) = \sum_{k=1}^{n} \frac{c_k}{s+s_k}$$

damit
$$f(t) = L^{-1}\{F(s)\} = L^{-1}\left\{\sum_{k=1}^{n} \frac{c_k}{s+s_k}\right\} = \sum_{k=1}^{n} c_k e^{-s_k t}$$

Beispiel 1:
$$F(s) = \frac{s+3}{(s+1)(s+2)} \qquad p_1 = -1, \quad p_2 = -2$$

$$F(s) = \frac{A}{s+1} + \frac{B}{s+2} = \frac{(A+B)s + (2A+B)}{(s+1)(s+2)}$$

$$A+B=1, \quad 2A+B=3 \quad \Rightarrow \quad A=2, \;\; B=-1$$

Daher
$$f(t) = 2e^{-t} - e^{-2t}$$

Inverse Laplace-Transformation

Beispiel 2: $F(s) = \dfrac{1}{(s+2)(s^2+2s+2)}$ $\qquad p_1 = -2, \quad p_{2,3} = -1 \pm j$

$$F(s) = \frac{A}{s+2} + \frac{Bs+C}{s^2+2s+2} = \frac{(A+B)s^2 + (2A+2B+C)s + 2(A+C)}{(s+2)(s^2+2s+2)}$$

$$A+B=0, \quad 2A+2B+C=0, \quad 2A+2C=1 \quad \Rightarrow \quad A=\frac{1}{2}, B=-\frac{1}{2}, C=0$$

Daher $F(s) = \dfrac{1}{2}\left(\dfrac{1}{s+2} - \dfrac{s}{s^2+2s+2}\right) = \dfrac{1}{2}\left(\dfrac{1}{s+2} - \dfrac{s}{(s+1)^2+1}\right)$

$$= \frac{1}{2}\left(\frac{1}{s+2} - \frac{s+1-1}{(s+1)^2+1}\right) = \frac{1}{2}\left(\frac{1}{s+2} - \frac{s+1}{(s+1)^2+1} + \frac{1}{(s+1)^2+1}\right)$$

dann $f(t) = \dfrac{1}{2}\left(e^{-2t} - e^{-t}\cos t + e^{-t}\sin t\right)$

Inverse Laplace-Transformation

Beispiel 3: $$F(s) = \frac{1}{(s+2)(s+1)^2} \qquad p_1 = -2, \quad p_{2,3} = -1$$

$$F(s) = \frac{A}{s+2} + \frac{Bs+C}{s^2+2s+1} = \frac{(A+B)s^2 + (2A+2B+C)s + (A+2C)}{(s+2)(s^2+2s+1)}$$

$$A+B=0, \quad 2A+2B+C=0, \quad A+2C=1 \quad \Rightarrow \quad A=1, B=-1, C=0$$

Daher $$F(s) = \frac{1}{s+2} - \frac{s}{s^2+2s+1} = \frac{1}{s+2} - \frac{s}{(s+1)^2}$$

$$= \frac{1}{s+2} - \frac{s+1-1}{(s+1)^2} = \frac{1}{s+2} - \frac{1}{(s+1)} + \frac{1}{(s+1)^2}$$

dann $$f(t) = e^{-2t} - e^{-t} + te^{-t}$$

Fachgebiet Prozessoptimierung

Lösung linearer Differentialgleichungen

$$\frac{d^n y}{dt^n} + a_1 \frac{d^{n-1} y}{dt^{n-1}} + \cdots + a_{n-1} \frac{dy}{dt} + a_n y = z(t)$$

Laplace-Transformation:

$$N(s)Y(s) = Z(s) \quad \Rightarrow \quad Y(s) = \frac{Z(s)}{N(s)} \quad \Rightarrow \quad y(t) = L^{-1}\left\{\frac{Z(s)}{N(s)}\right\}$$

Beispiel: $$\frac{d^2 y}{dt^2} + 5\frac{dy}{dt} + 6y = e^{-t}, \quad y(0) = \dot{y}(0) = 0$$

Laplace-Transformation: $$(s^2 + 5s + 6)Y(s) = \frac{1}{s+1}$$

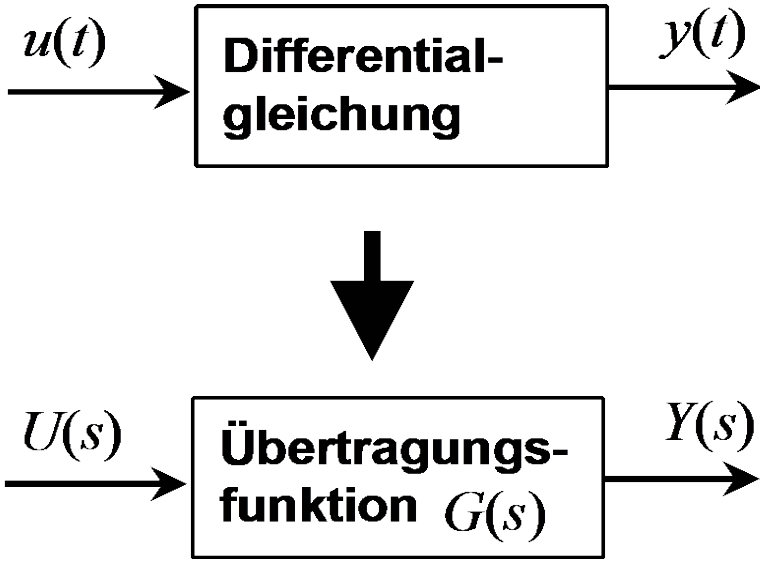
$$Y(s) = \frac{1}{(s+1)(s^2+5s+6)} = \frac{1}{(s+1)(s+2)(s+3)} = \frac{A}{s+1} + \frac{B}{s+2} + \frac{C}{s+3}$$

Daher $$y(t) = Ae^{-t} + Be^{-2t} + Ce^{-3t}$$

Übertragungsfunktion

Zeitbereich:

Frequenzbereich:



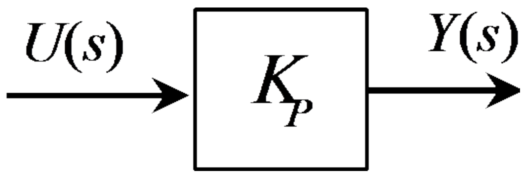
Ausgang: $Y(s) = G(s)U(s)$

Übertragungsfunktion: $G(s) = \dfrac{Y(s)}{U(s)}$

Übertragungsfunktion elementarer Glieder

P-Glied (Proportional-Glied): $y(t) = K_P\, u(t)$

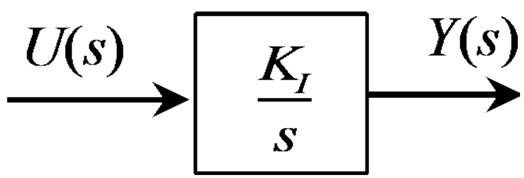
$$Y(s) = K_P\, U(s) \quad \Rightarrow \quad G(s) = \frac{Y(s)}{U(s)} = K_P$$



Sprungantwort?

I-Glied (Integrierglied): $y(t) = K_I \int_0^t u(\tau)\, d\tau$

$$Y(s) = \frac{K_I}{s} U(s) \quad \Rightarrow \quad G(s) = \frac{Y(s)}{U(s)} = \frac{K_I}{s}$$



Sprungantwort?

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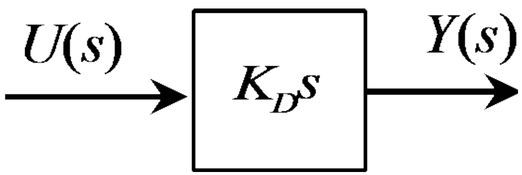
Übertragungsfunktion elementarer Glieder

D-Glied (Differenzierglied):

$$y(t) = K_D \frac{du}{dt}, \quad u(0) = 0$$

$$Y(s) = K_D s U(s) \quad \Rightarrow \quad G(s) = \frac{Y(s)}{U(s)} = K_D s$$

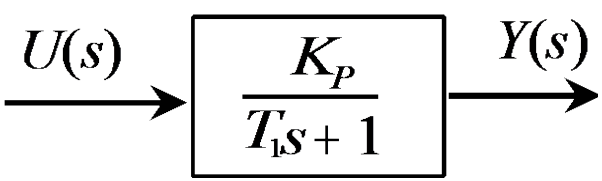


Sprungantwort?

PT$_1$-Glied (Verzögerungs-, Trägheitsglied):

$$T_1 \frac{dy}{dt} + y(t) = K_P u, \quad y(0) = 0$$

$$T_1 s Y(s) + Y(s) = K_P U(s) \quad \Rightarrow \quad G(s) = \frac{Y(s)}{U(s)} = \frac{K_P}{T_1 s + 1}$$



Sprungantwort?

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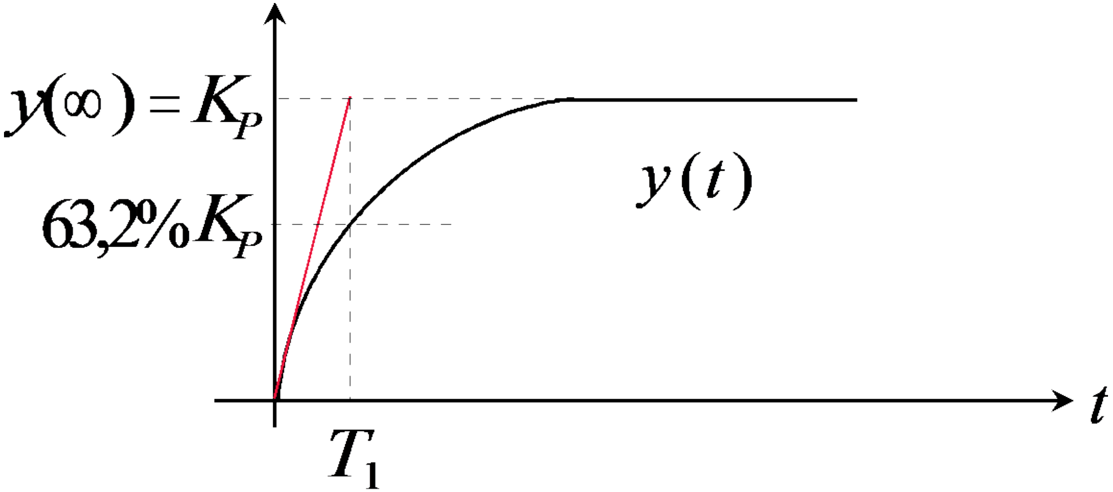
Übertragungsfunktion elementarer Glieder

Die Sprungantwort: $y(t) = K_P(1 - e^{-\frac{t}{T_1}})$

mit $y(0) = 0, \quad y(\infty) = K_P$

Da $\dot{y}(t) = \frac{K_P}{T_1} e^{-\frac{t}{T_1}}$ dann $\dot{y}(0) = \frac{K_P}{T_1}$

Wenn $t = T_1$ dann $y(T_1) = K_P(1 - e^{-1}) = 0{,}632 K_P$



Übertragungsfunktion elementarer Glieder

PT_1T_t (Verzögerung + Totzeit):

$$T_1 \frac{dy}{dt} + y(t) = K_P u(t - T_t), \quad y(0) = 0$$

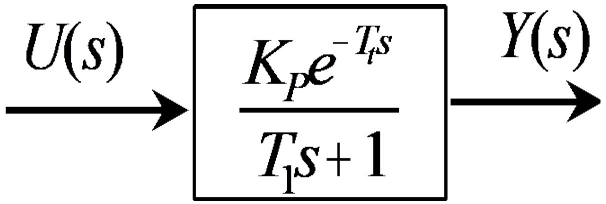
$$G(s) = \frac{Y(s)}{U(s)} = \frac{K_P e^{-T_t s}}{T_1 s + 1}$$

Die Parameter:

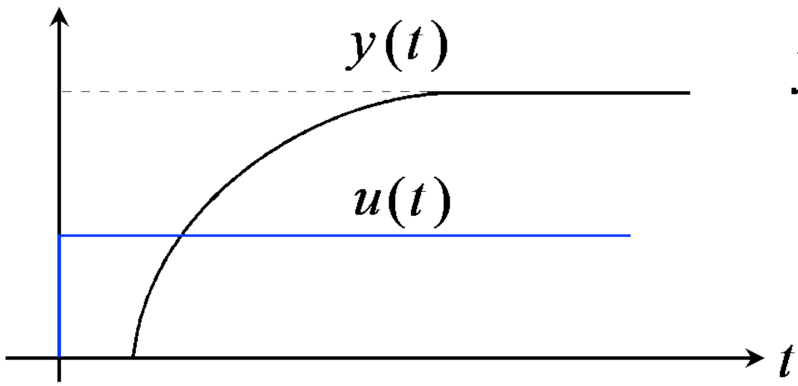
K_P : Verstärkung

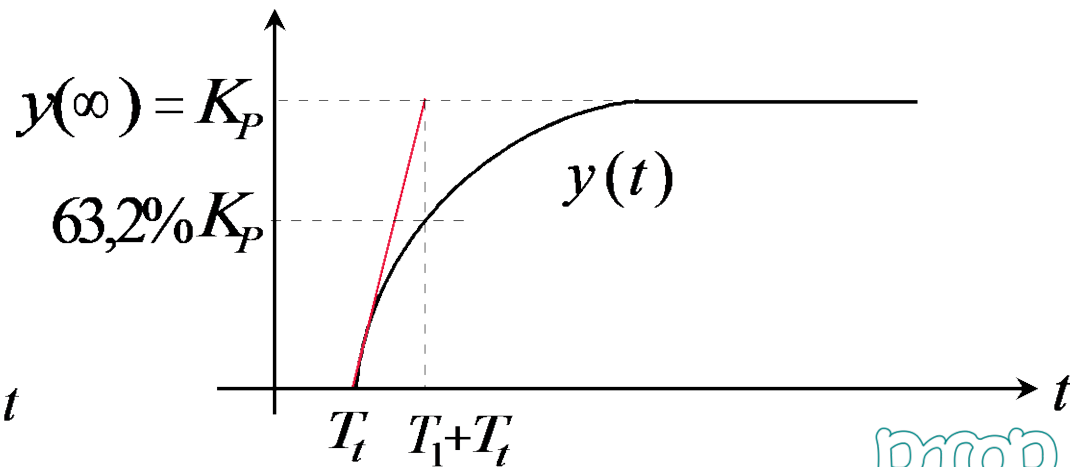
T_1 : Trägheitszeitkonstante

T_t : Totzeit



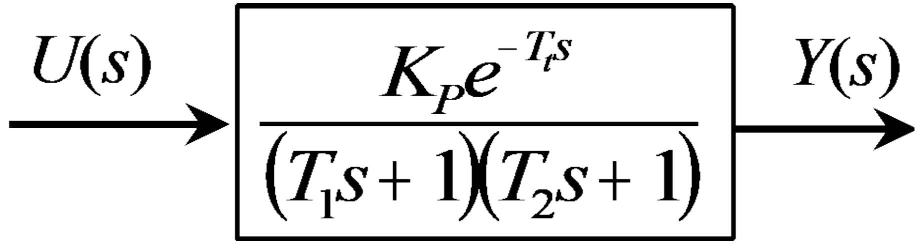
Die Sprungantwort:



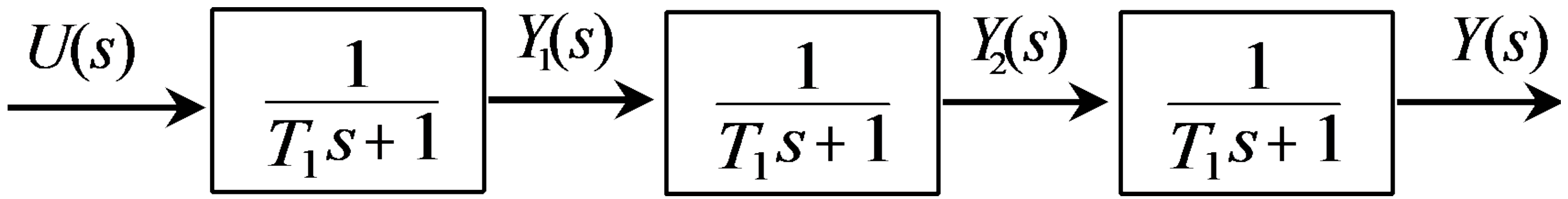


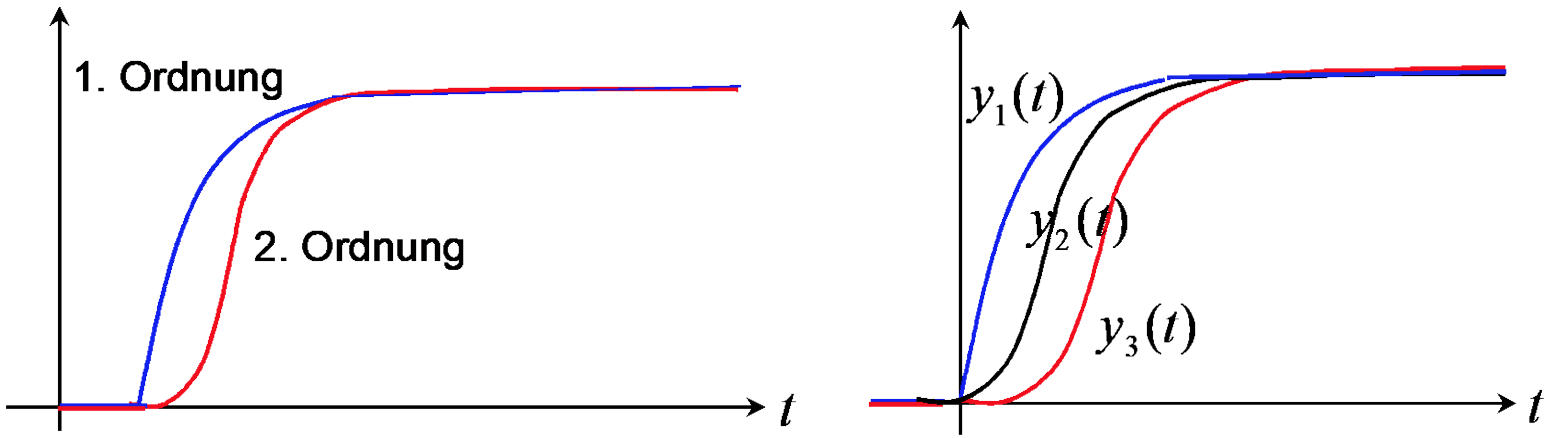
Übertragungsfunktion elementarer Glieder

PT_2T_t-Glied:



PT_3-Glied (z.B. Behälterkaskade):





Übertragungsfunktion allgemeiner Systeme:

$$\frac{d^n y}{dt^n} + a_1 \frac{d^{n-1} y}{dt^{n-1}} + \cdots + a_{n-1} \frac{dy}{dt} + a_n y = k_u u + k_z z$$

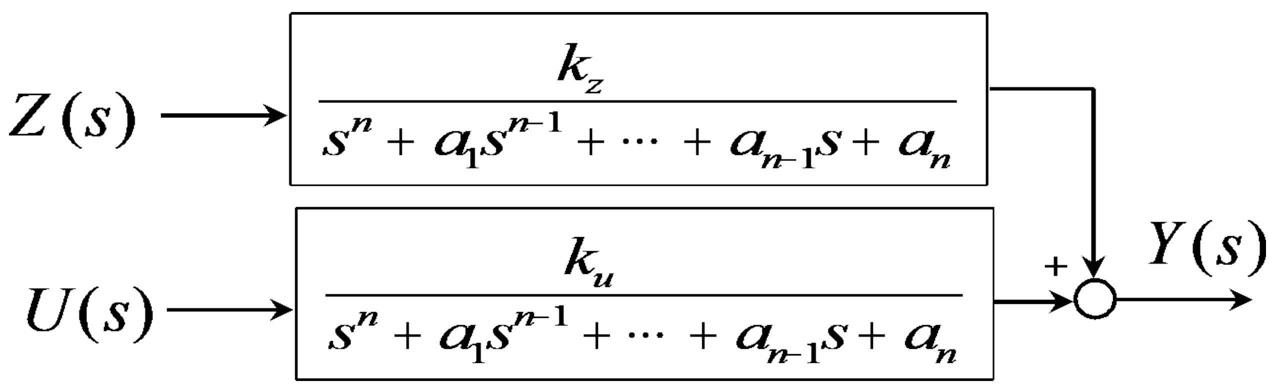
Die Eigenschaften des Systems sind schwer zu analysieren!

Man möchte die Differentialgleichung in eine algebraische Gleichung umwandeln.

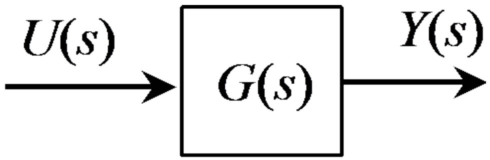
$$s^n Y(s) + a_1 s^{n-1} Y(s) + \cdots + a_{n-1} s Y(s) + a_n Y(s) = k_u U(s) + k_z Z(s)$$

$$Y(s) = \frac{k_u U(s)}{s^n + a_1 s^{n-1} + \cdots + a_{n-1} s + a_n} + \frac{k_z Z(s)}{s^n + a_1 s^{n-1} + \cdots + a_{n-1} s + a_n}$$

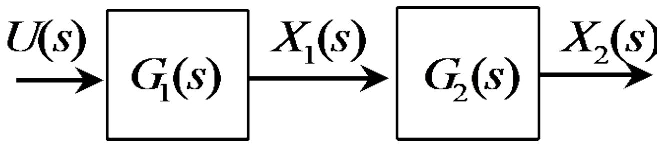
Physikalische Bedeutung:



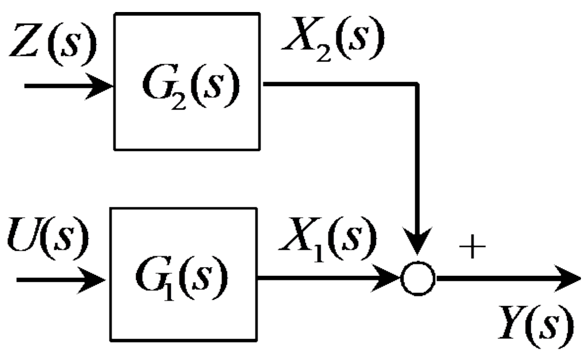
Blockschaltbild (Strukturbild):



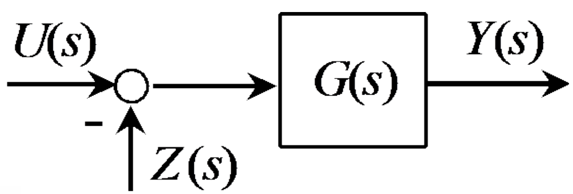
$$\Rightarrow \quad Y(s) = G(s)U(s)$$



$$\Rightarrow \quad Y(s) = G_1(s)G_2(s)U(s)$$



$$\Rightarrow \quad Y(s) = G_1(s)U(s) + G_2(s)Z(s)$$



$$\Rightarrow \quad Y(s) = G(s)\left[U(s) - Z(s)\right]$$

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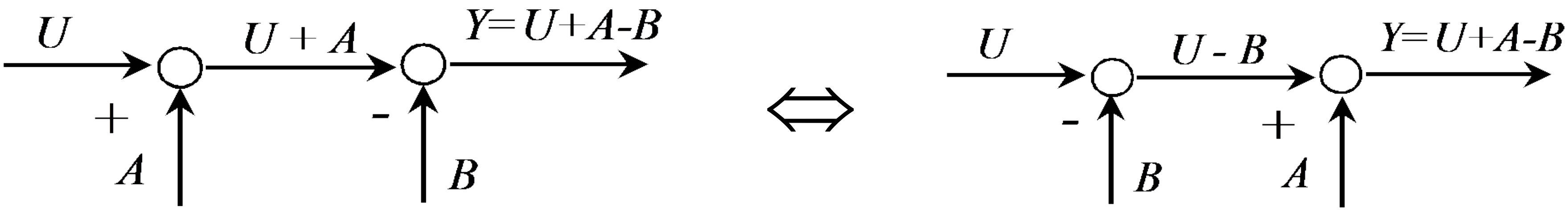
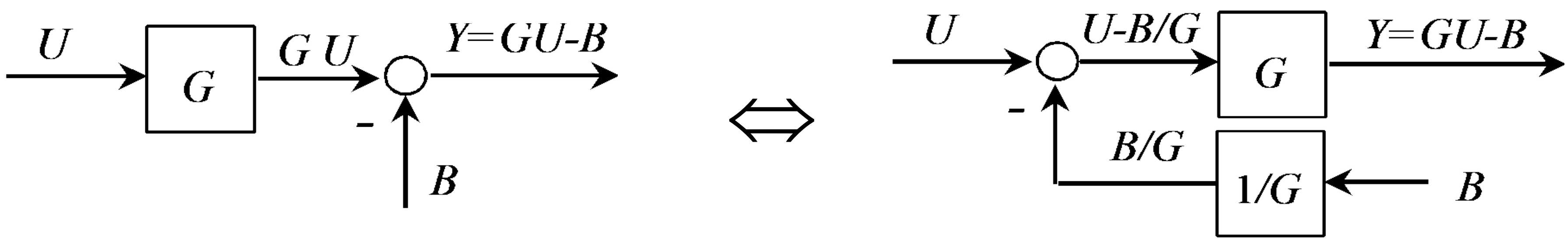
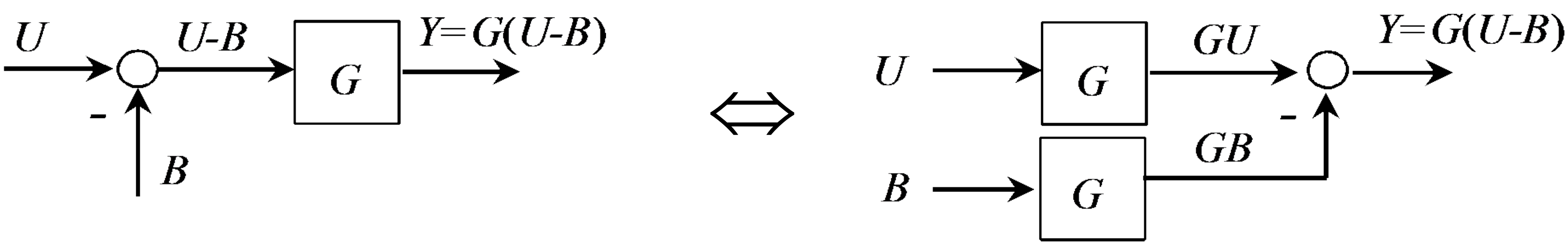
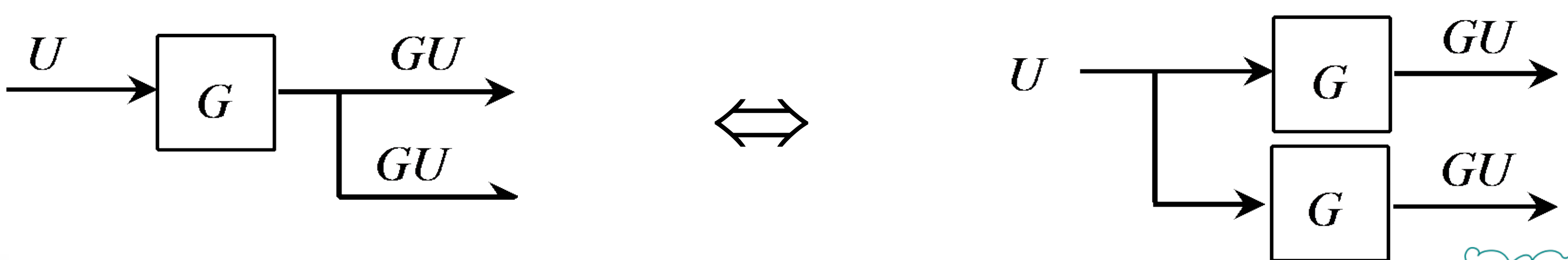
Blockschaltbild:



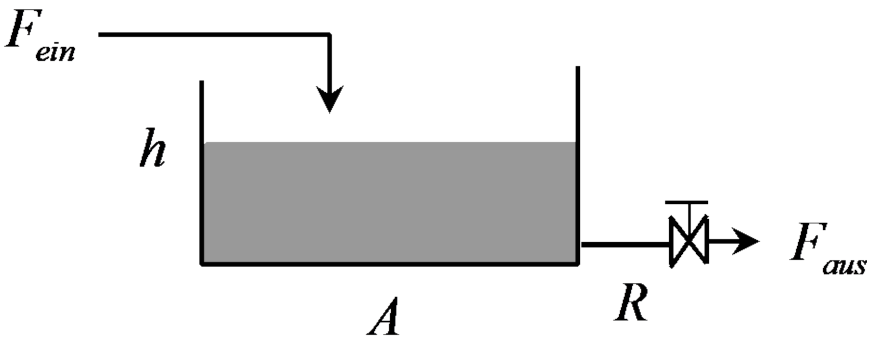






Fachgebiet Prozessoptimierung

Dynamik eines Behälters:

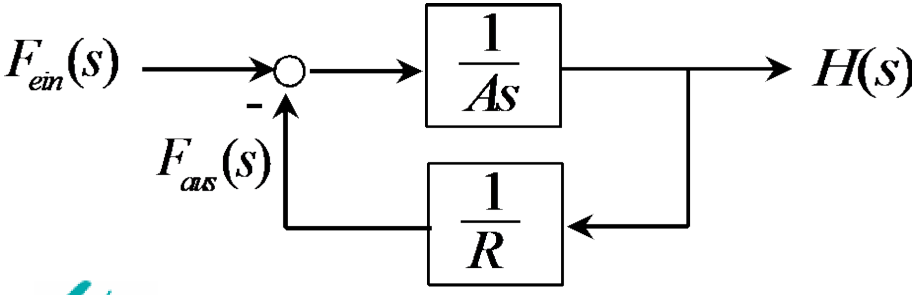


Bilanzgleichungen:

$$A\frac{dh}{dt} = F_{ein} - F_{aus}, \quad h(0) = h_0$$

$$A\frac{d\Delta h}{dt} = \Delta F_{ein} - \Delta F_{aus} \quad \Rightarrow \quad H(s) = \frac{1}{As}\left[F_{ein}(s) - F_{aus}(s)\right]$$

$$\Delta F_{aus} = \frac{\Delta h}{R} \quad \Rightarrow \quad F_{aus}(s) = \frac{1}{R}H(s)$$

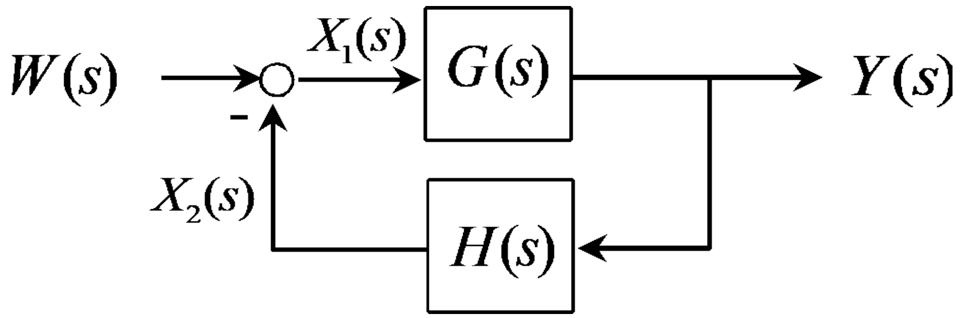


$$\Rightarrow \quad \frac{H(s)}{F_{ein}(s)} = \frac{R}{RAs+1}$$

Negative Rückführung!

Fachgebiet Prozessoptimierung

Blockschaltbild:



Wie lautet die Übertragungsfunktion zwischen W und Y?

$$X_1(s) = W(s) - X_2(s)$$
$$Y(s) = G(s)X_1(s)$$
$$X_2(s) = H(s)Y(s)$$

$$\Rightarrow$$

$$X_1(s) = \frac{Y(s)}{G(s)}$$
$$X_2(s) = H(s)Y(s)$$
$$\frac{Y(s)}{G(s)} = W(s) - H(s)Y(s)$$

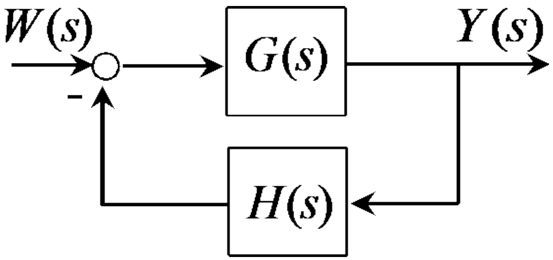
Daher

$$\frac{Y(s)}{G(s)} + H(s)Y(s) = W(s) \quad \Rightarrow \quad \left[\frac{1}{G(s)} + H(s)\right]Y(s) = W(s)$$

$$\left[\frac{1+G(s)H(s)}{G(s)}\right]Y(s) = W(s) \quad \Rightarrow \quad \frac{Y(s)}{W(s)} = \frac{G(s)}{1+G(s)H(s)}$$

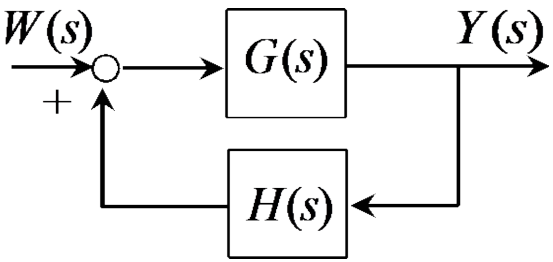
prop Fachgebiet Prozessoptimierung

Blockschaltbild:



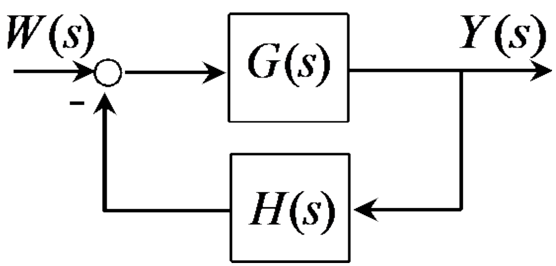
$\Rightarrow$

$$\frac{Y(s)}{W(s)} = \frac{G(s)}{1 + G(s)H(s)}$$

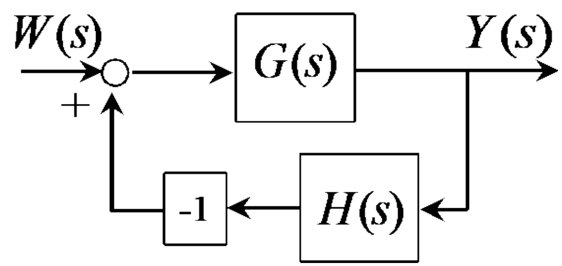


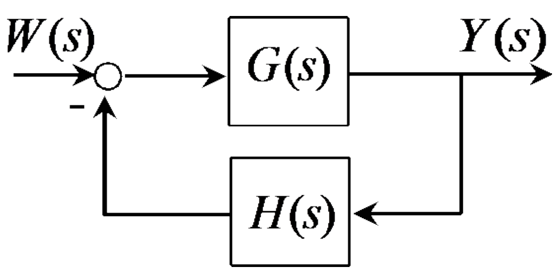
$\Rightarrow$

$$\frac{Y(s)}{W(s)} = \frac{G(s)}{1 - G(s)H(s)}$$

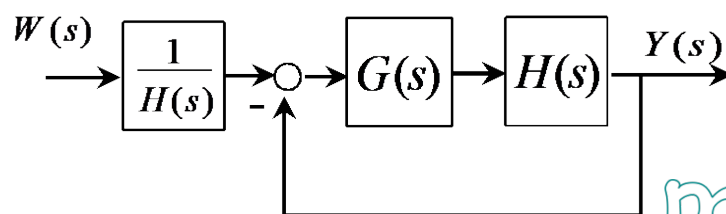


$\Rightarrow$

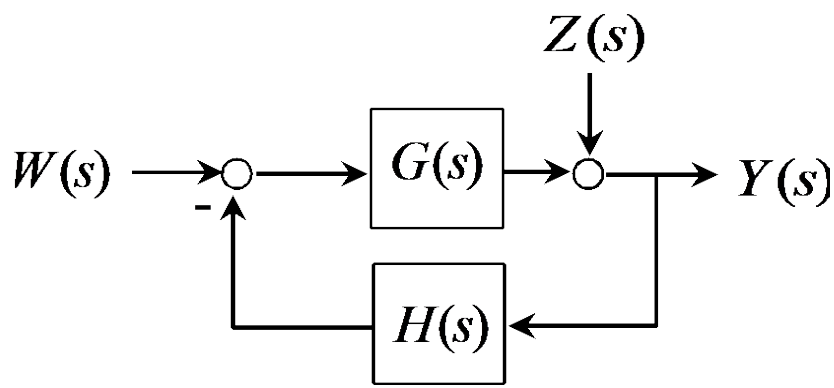




$\Rightarrow$



Blockschaltbild:

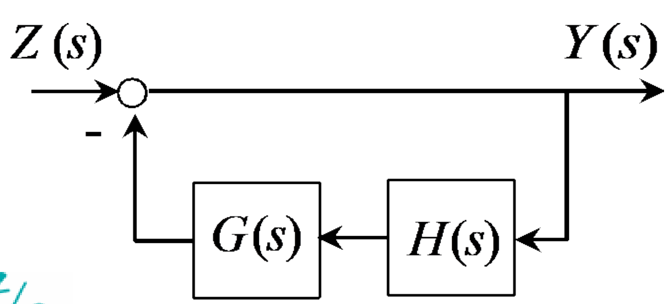


Wenn $Z(s) = 0$

$$\frac{Y(s)}{W(s)} = \frac{G(s)}{1+G(s)H(s)}$$

Wenn $W(s) = 0$

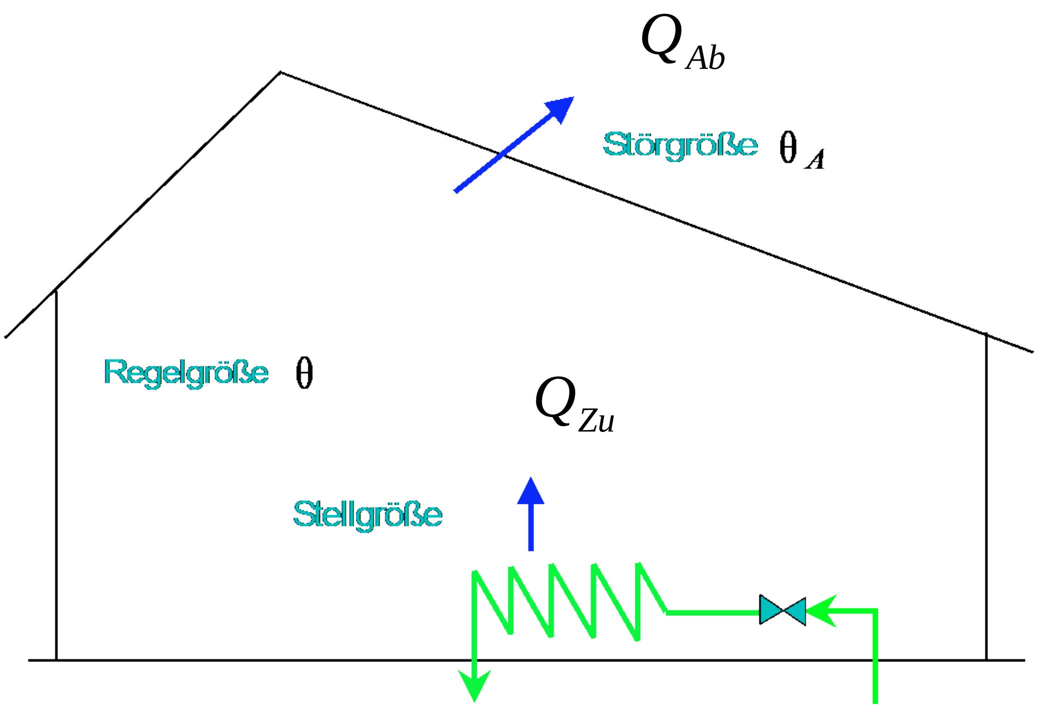
$$\frac{Y(s)}{Z(s)} = ?$$



$$\Rightarrow \qquad \frac{Y(s)}{Z(s)} = \frac{1}{1+G(s)H(s)}$$

Fachgebiet Prozessoptimierung

Beispiel: Temperatur im Gewächshaus



Übertragungsfunktion:

$$\theta(s) = \frac{k_u}{T_1 s + 1} Q_{Zu}(s) + \frac{k_z}{T_1 s + 1} \theta_A(s)$$

Führungsstrecke:

$$\theta(s) = \frac{k_u}{T_1 s + 1} Q_{Zu}(s)$$

Störstrecke:

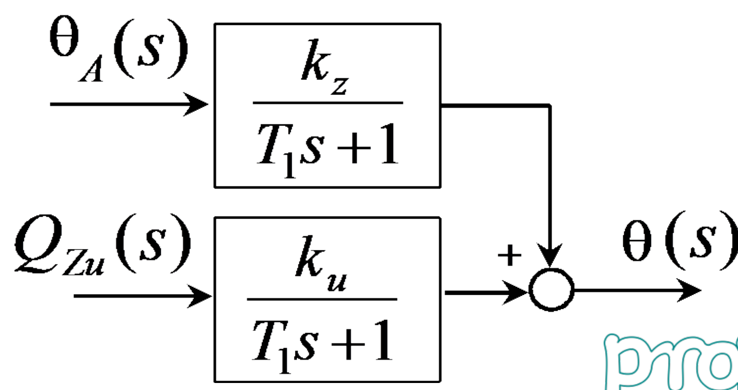
$$\theta(s) = \frac{k_z}{T_1 s + 1} \theta_A(s)$$

Die Gleichung durch die Änderung:

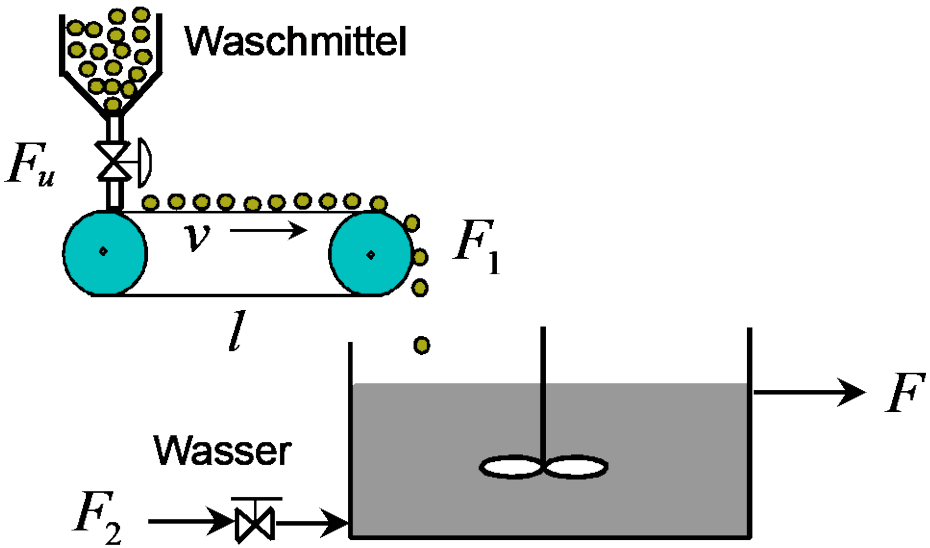
$$T_1 \frac{d\Delta\theta}{dt} + \Delta\theta = k_u \Delta Q_{Zu} + k_z \, \Delta\theta_A$$

Laplace-Transformation:

$$(T_1 s + 1)\theta(s) = k_u Q_{Zu}(s) + k_z \, \theta_A(s)$$



Beispiel: Ein Mischungsprozess

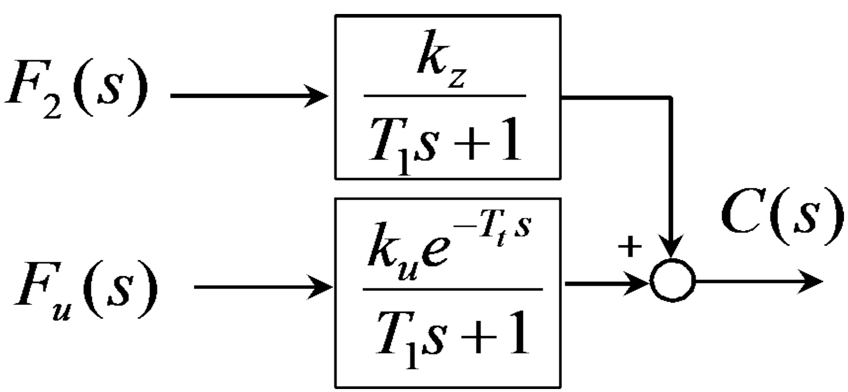


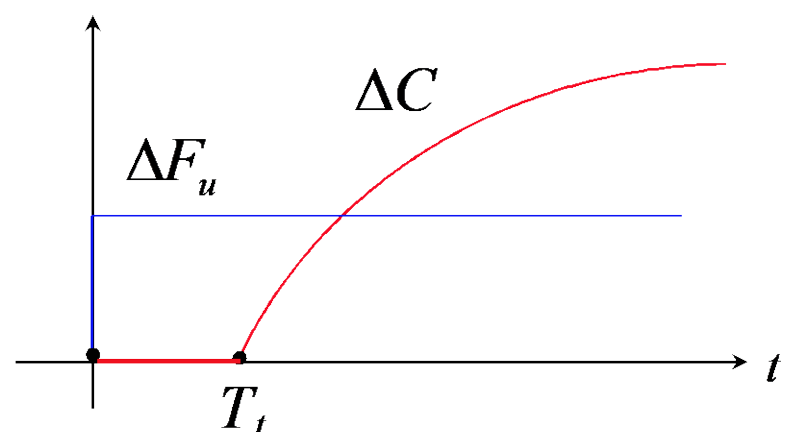
Modellgleichung:

$$T_1 \frac{d\Delta C}{dt} + \Delta C = k_u \Delta F_u (t - T_t) + k_z \Delta F_2$$

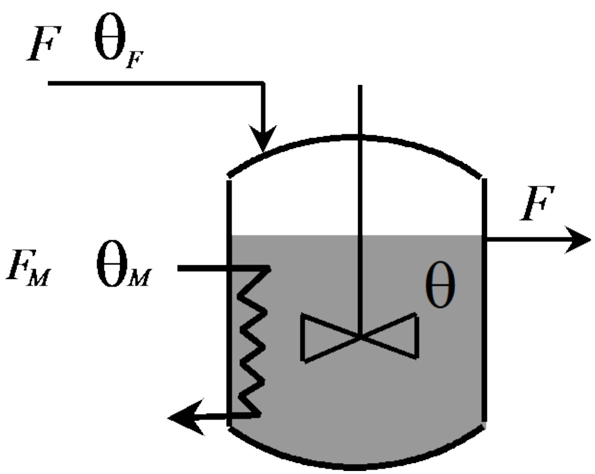
Laplace-Transformation:

$$C(s) = \frac{k_u e^{-T_t s}}{T_1 s + 1} F_u(s) + \frac{k_z}{T_1 s + 1} F_2(s)$$





Beispiel: Reaktor mit exothermer Reaktion



Energiebilanz: $$\frac{dH}{dt} = Q_F - Q_M + Q_R$$

$$C_P \rho V \frac{d\theta}{dt} = F C_P \rho (\theta_F - \theta) - Q_M + Q_R$$

wobei

$$Q_R = rV = k_0 e^{-\frac{E}{R\theta}} V$$

Daher

$$C_P \rho V \frac{d\theta}{dt} = F C_P \rho (\theta_F - \theta) - Q_M + k_0 e^{-\frac{E}{R\theta}} V$$

$$\frac{V}{F}\frac{d\theta}{dt} = \theta_F - \theta - \frac{Q_M}{C_P \rho F} + \frac{k_0 V}{C_P \rho F} e^{-\frac{E}{R\theta}}$$

also

$$T_1 \frac{d\theta}{dt} + \theta = \theta_F - k_u Q_M + k_R e^{-\frac{E}{R\theta}}$$

Der Prozess ist nichtlinear!

Beispiel: Reaktor mit exothermer Reaktion

Linearisierung am Arbeitspunkt:

$$T_1 \frac{d\Delta\theta}{dt} + \Delta\theta = \Delta\theta_F - k_u \Delta Q_M + k_R \frac{E}{R\theta_0^2} e^{-\frac{E}{R\theta_0}} \Delta\theta$$

Damit $$T_1 \frac{d\Delta\theta}{dt} + (1 - \tilde{k}_R)\Delta\theta = \Delta\theta_F - k_u \Delta Q_M$$

Normalerweise $\tilde{k}_R >> 1$

D.h. $$T_1 \frac{d\Delta\theta}{dt} - \tilde{k}_R \Delta\theta = \Delta\theta_F - k_u \Delta Q_M$$

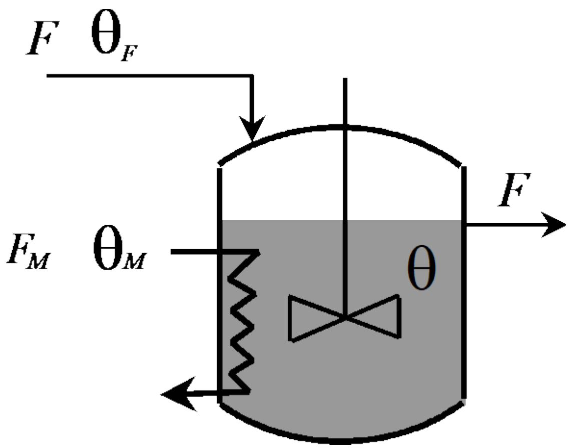
D.h. $$\frac{T_1}{\tilde{k}_R} \frac{d\Delta\theta}{dt} - \Delta\theta = \frac{1}{\tilde{k}_R}\Delta\theta_F - \frac{k_u}{\tilde{k}_R}\Delta Q_M$$

$$\tilde{T}_1 \frac{d\Delta\theta}{dt} - \Delta\theta = k_z \Delta\theta_F - \tilde{k}_u \Delta Q_M$$

Wie verhält sich die Temperatur?

prop Fachgebiet Prozessoptimierung

Beispiel: Reaktor mit exothermer Reaktion



Die linearisierte Modellgleichung:

$$\tilde{T}_1 \frac{d\Delta\theta}{dt} - \Delta\theta = k_z \Delta\theta_F - \tilde{k}_u \Delta Q_M$$

Laplace-Transformation:

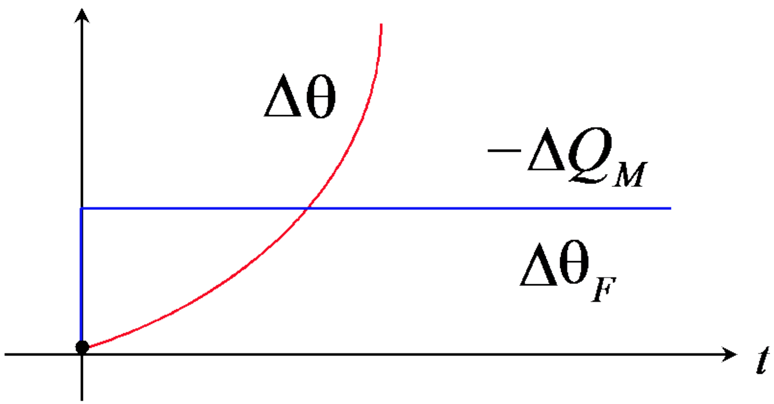
$$(\tilde{T}_1 s - 1)\theta(s) = k_z \theta_F(s) - \tilde{k}_u Q_M(s)$$

Führungsstrecke:

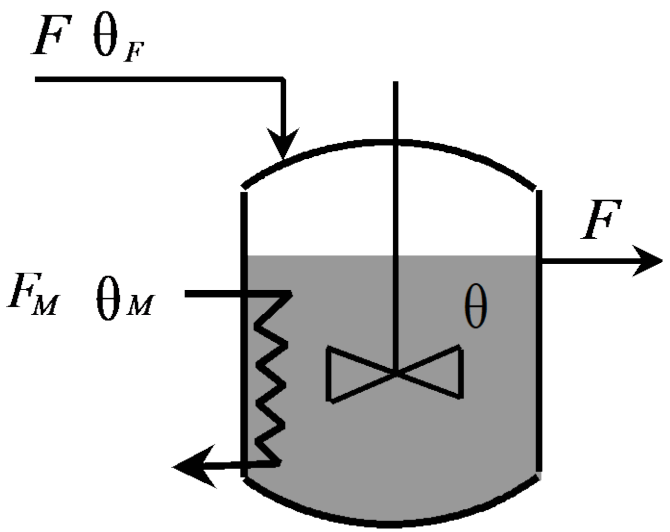
$$\theta(s) = -\frac{\tilde{k}_u}{\tilde{T}_1 s - 1} Q_M(s)$$

Störstrecke:

$$\theta(s) = \frac{k_z}{\tilde{T}_1 s - 1} \theta_F(s)$$



Beispiel: Reaktor mit exothermer Reaktion



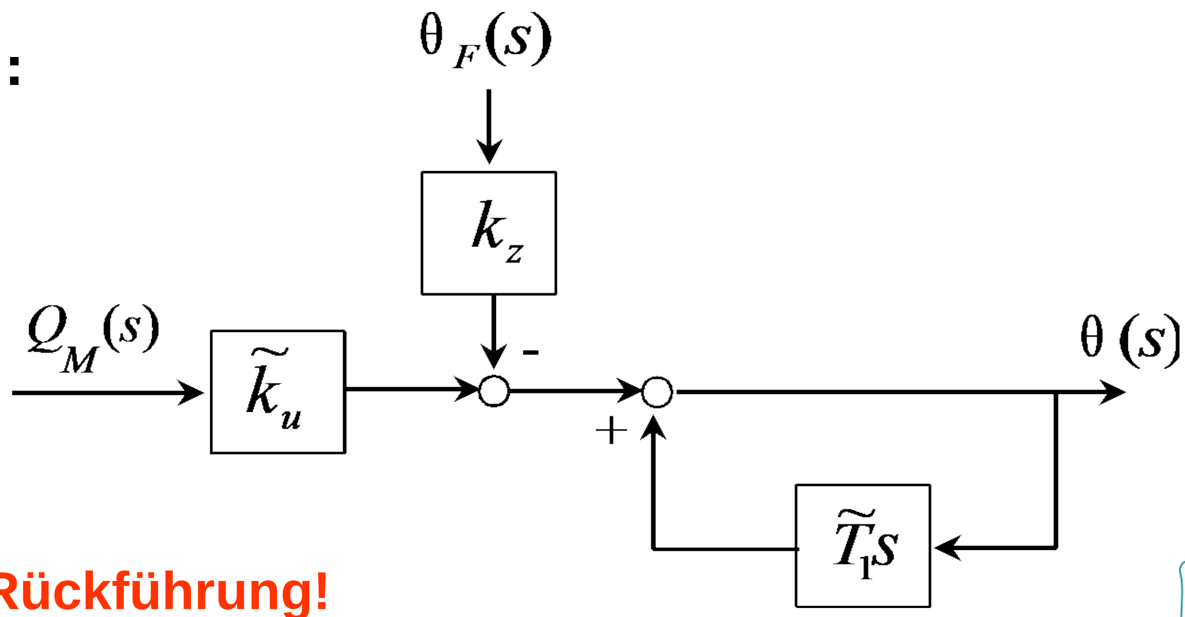
Laplace-Transformation:

$$(\tilde{T}_1 s - 1)\theta(s) = k_z \theta_F(s) - \tilde{k}_u Q_M(s)$$

und zwar

$$\theta(s) = -k_z \theta_F(s) + \tilde{k}_u Q_M(s) + \tilde{T}_1 s \theta(s)$$

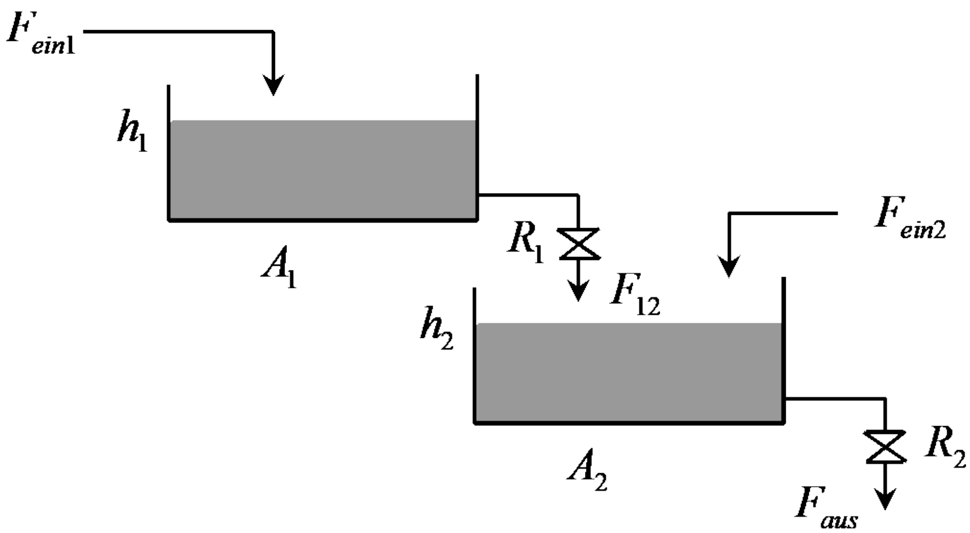
Blockschaltbild:



Positive Rückführung!

Fachgebiet Prozessoptimierung

Beispiel: Behälterkaskade (1)



Bilanzgleichungen:

$$A_1 \frac{dh_1}{dt} = F_{ein1} - F_{12}, \qquad h_1(0) = h_{10}$$

$$A_2 \frac{dh_2}{dt} = F_{12} + F_{ein2} - F_{aus}, \qquad h_2(0) = h_{20}$$

Das linearisierte Modell:

$$\frac{d\Delta h_1}{dt} = -\frac{1}{A_1 R_1}\Delta h_1 + \frac{1}{A_1}\Delta F_{ein1}$$

$$\frac{d\Delta h_2}{dt} = \frac{1}{A_2 R_1}\Delta h_1 - \frac{1}{A_2 R_2}\Delta h_2 + \frac{1}{A_2}\Delta F_{ein2}$$

Fachgebiet Prozessoptimierung

Beispiel: Behälterkaskade (1)

$$A_1 R_1 \frac{dx_1}{dt} + x_1 = R_1 u$$

$$A_2 R_2 \frac{dx_2}{dt} + x_2 = \frac{R_2}{R_1} x_1 + R_2 z$$

$$\Rightarrow$$

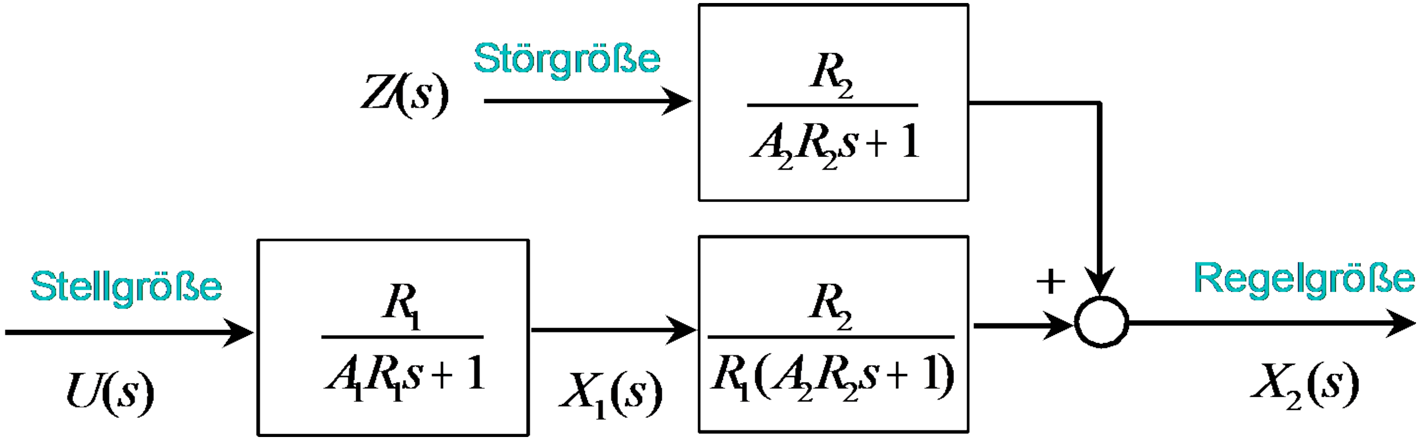
$$A_1 R_1 s X_1(s) + X_1(s) = R_1 U(s)$$

$$A_2 R_2 X_2(s) + X_2(s) = \frac{R_2}{R_1} X_1(s) + R_2 Z(s)$$

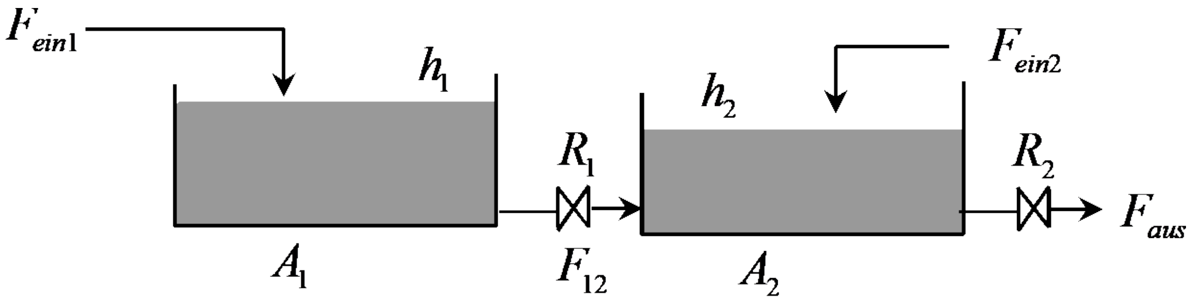
dann

$$X_1(s) = \frac{R_1}{A_1 R_1 s + 1} U(s), \quad X_2(s) = \frac{R_2}{R_1(A_2 R_2 s + 1)} X_1(s) + \frac{R_2}{A_2 R_2 s + 1} Z(s)$$

Physikalische Bedeutung:



Beispiel: Behälterkaskade (2)



Bilanzgleichungen:

$$A_1 \frac{dh_1}{dt} = F_{ein1} - F_{12}, \qquad h_1(0) = h_{10}$$

$$A_2 \frac{dh_2}{dt} = F_{12} + F_{ein2} - F_{aus}, \qquad h_2(0) = h_{20}$$

Das linearisierte Modell:

$$A_1 \frac{d\Delta h_1}{dt} = \Delta F_{ein1} - \Delta F_{12}$$

$$A_2 \frac{d\Delta h_2}{dt} = \Delta F_{12} + \Delta F_{ein2} - \Delta F_{aus}$$

$$\Delta F_{12} = \frac{\Delta h_1 - \Delta h_2}{R_1}$$

$$\Delta F_{aus} = \frac{\Delta h_2}{R_2}$$

Beispiel: Behälterkaskade (2)

$$A_1 \frac{d\Delta h_1}{dt} = \Delta F_{ein1} - \Delta F_{12}$$

$$A_2 \frac{d\Delta h_2}{dt} = \Delta F_{12} + \Delta F_{ein2} - \Delta F_{aus}$$

$$\Delta F_{12} = \frac{\Delta h_1 - \Delta h_2}{R_1}$$

$$\Delta F_{aus} = \frac{\Delta h_2}{R_2}$$

$$\Rightarrow$$

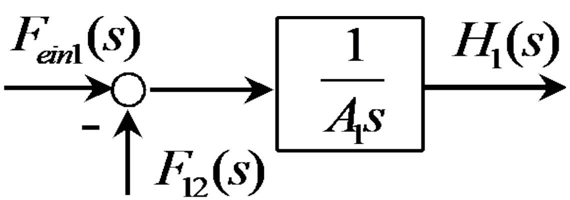
$$H_1(s) = \frac{1}{A_1 s}\left[F_{ein1}(s) - F_{12}(s)\right]$$

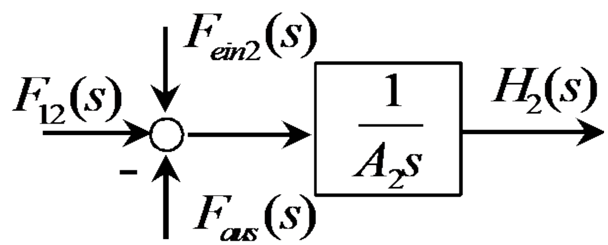
$$H_2(s) = \frac{1}{A_2 s}\left[F_{12}(s) + F_{ein2}(s) - F_{aus}(s)\right]$$

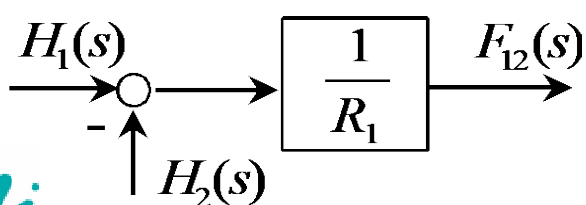
$$F_{12}(s) = \frac{1}{R_1}\left[H_1(s) - H_2(s)\right]$$

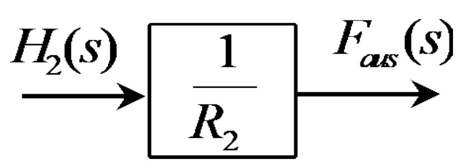
$$F_{aus}(s) = \frac{1}{R_2} H_2(s)$$

Blockschaltbilder:



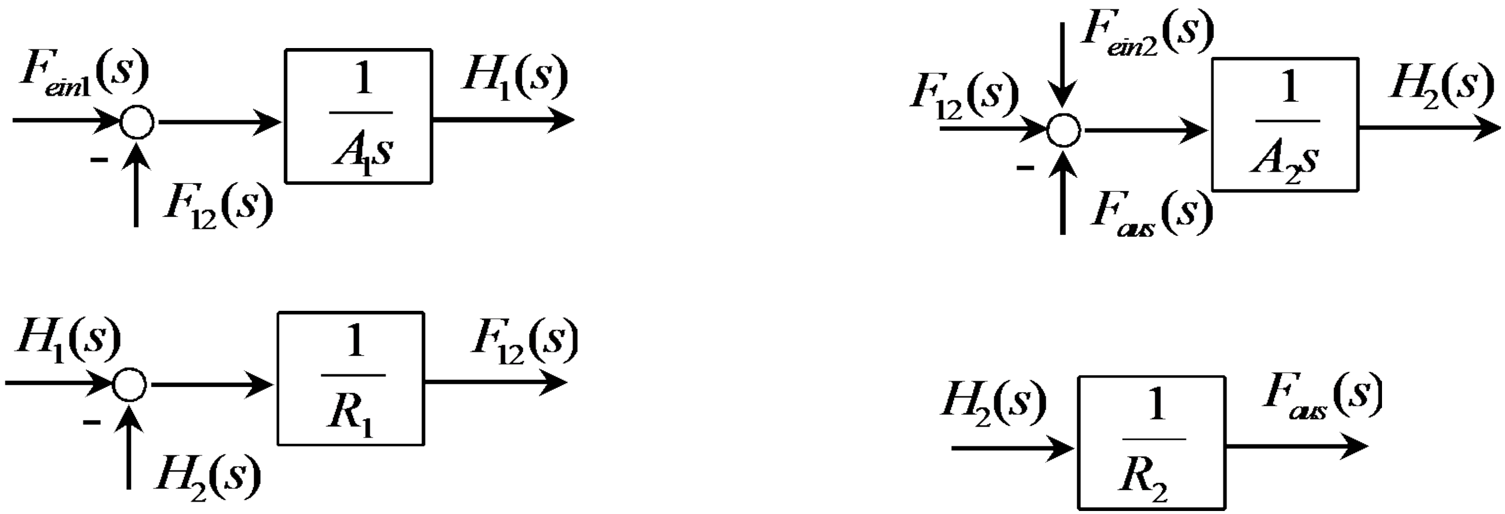




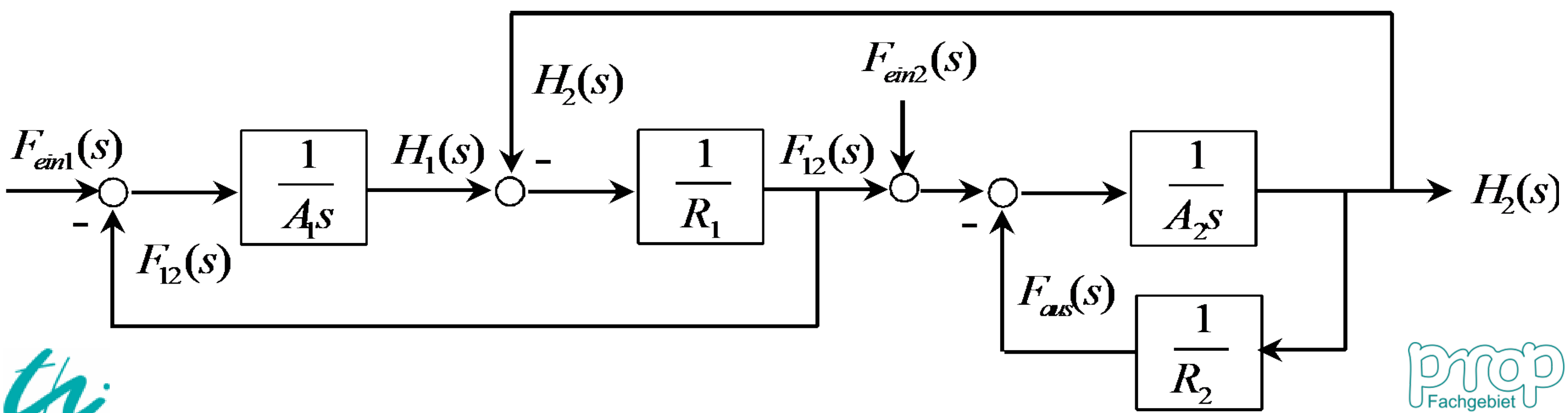


Beispiel: Behälterkaskade (2)

Blockschaltbilder:



Das Gesamtsystem:



Beispiel: Behälterkaskade (2)