Regelungs- und Systemtechnik 1

Kapitel 4: Analyse von Regelungssystemen im Zeitbereich

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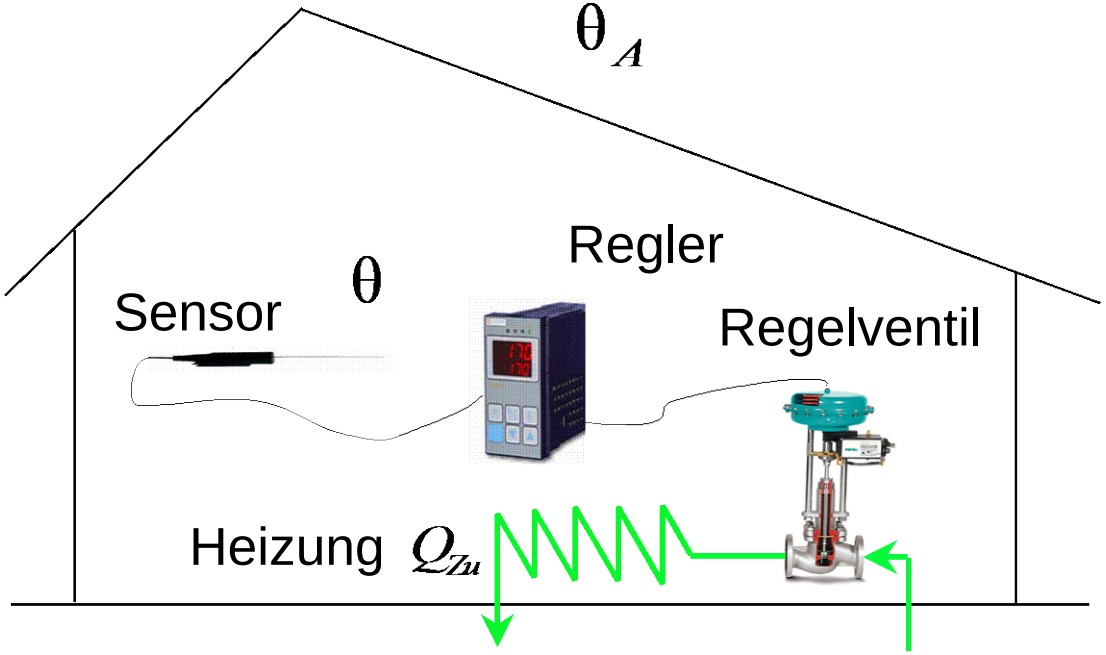
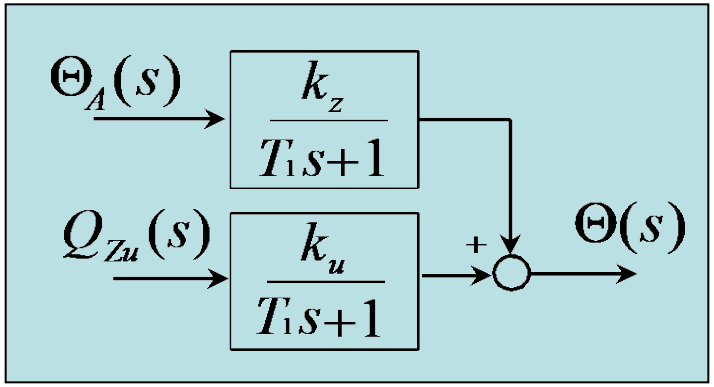
Fachgebiet Prozessoptimierung

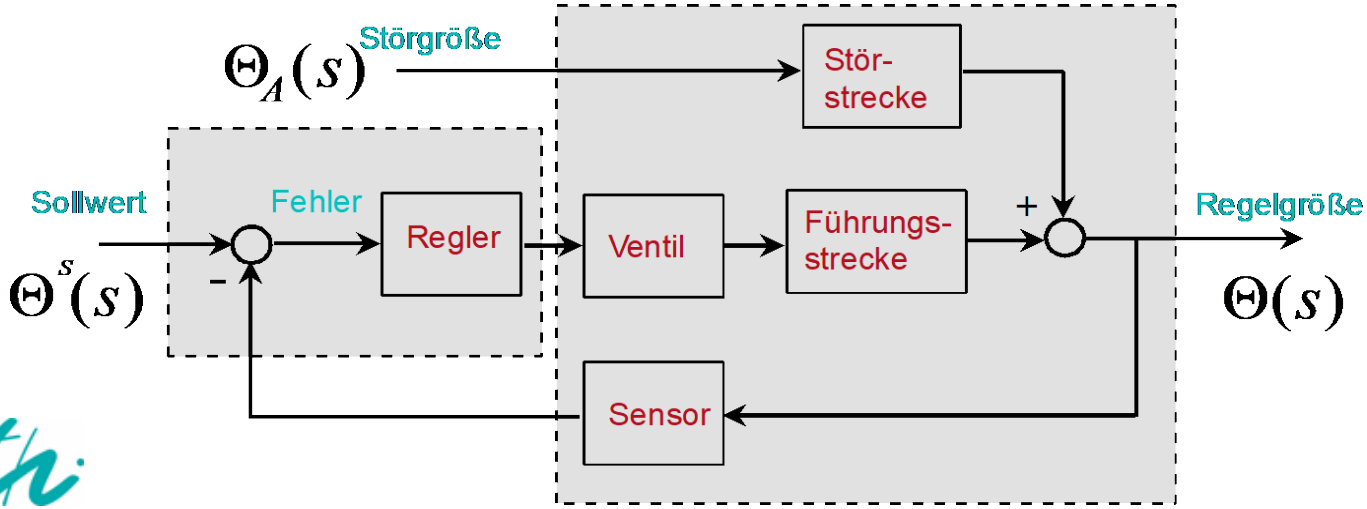
Problemdarstellung:



Das offene System



Das geschlossene System



Typische Eingangssignale

Einheitssprung: $x(t) = \sigma(t)$ $\quad X(s) = \frac{1}{s}$

Rampenfunktion: $x(t) = at$ $\quad X(s) = \frac{a}{s^2}$

Expotentialfunktion: $x(t) = e^{-at}$ $\quad X(s) = \frac{1}{s+a}$

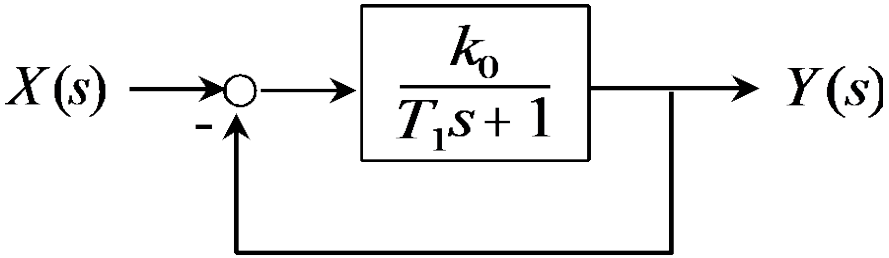
Impulsfunktion: $x(t) = \delta(t)$ $\quad X(s) = 1$

Sinusfunktion: $x(t) = a \sin \omega t$ $\quad X(s) = \frac{a\omega}{s^2 + \omega^2}$

Wie reagiert das geschlossene System, wenn ein Eingangssignal anliegt?

Antworten von Systemen 1. Ordnung

Blockschaltbild:



Die offene Kette: $G_0(s) = \dfrac{k_0}{T_1 s + 1}$

Das geschlossene System: $G_B(s) = \dfrac{Y(s)}{X(s)} = \dfrac{\dfrac{k_0}{k_0 + 1}}{\dfrac{T_1}{k_0 + 1} s + 1} = \dfrac{k_B}{T_B s + 1}$

Durch die Rückführung werden die Verstärkung und die Zeitkonstante verringert!

Die Auswirkung einer Parameteränderung wird verringert, d.h. das geschlossene System ist robuster.

Die Polstelle wird nach links verschoben, d.h. das geschlossene System ist schneller (der Pol ist weiter von der Stabilitätsgrenze entfernt).

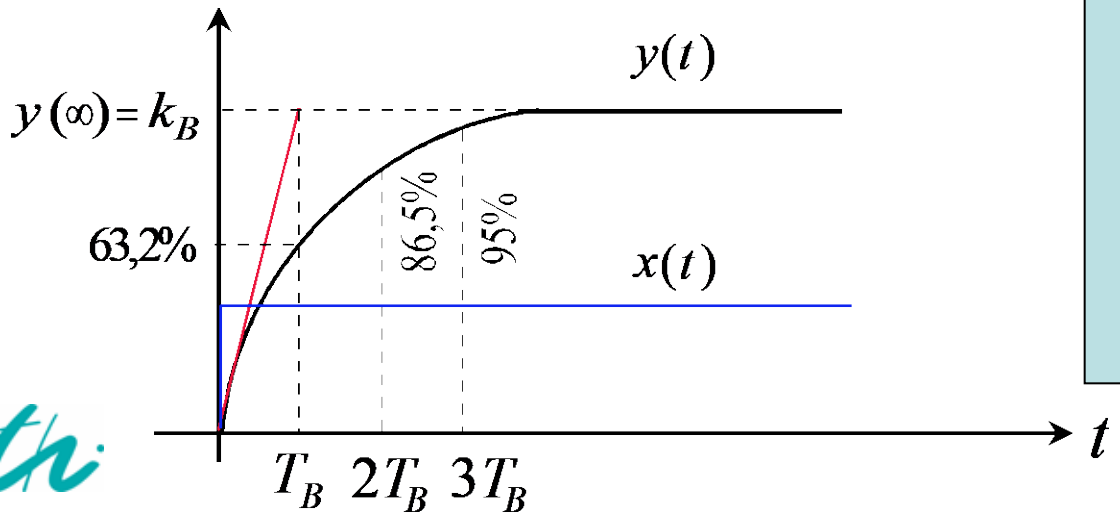
Antworten von Systemen 1. Ordnung

Das geschlossene System: $$\frac{Y(s)}{X(s)} = \frac{k_B}{T_B s + 1}$$

Auf einen Einheitssprung: $$Y(s) = \frac{k_B}{T_B s + 1} X(s) = \frac{k_B}{T_B s + 1} \frac{1}{s}$$

also $$y(t) = L^{-1}\left\{ \frac{k_B}{T_B s + 1} \frac{1}{s} \right\} = k_B (1 - e^{-\frac{t}{T_B}}) \qquad (1)$$

mit $$y(0) = 0, \quad y(\infty) = k_B, \quad e(t) = x(t) - y(t) = (1 - k_B) + k_B e^{-\frac{t}{T_B}}$$



Im Fall X(s) = W(s) soll

$$k_B \to 1, \quad T_B \to 0$$

Im Fall X(s) = Z(s) soll

$$k_B \to 0, \quad T_B \to 0$$

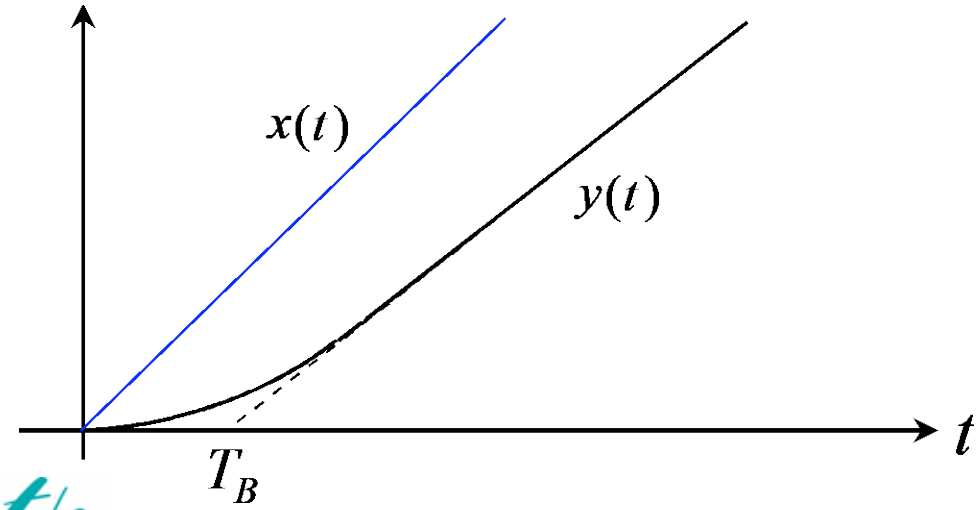
Antworten von Systemen 1. Ordnung

Das geschlossene System: $$\frac{Y(s)}{X(s)} = \frac{k_B}{T_B s + 1}$$

Auf eine Einheitsrampe: $$Y(s) = \frac{k_B}{T_B s + 1} X(s) = \frac{k_B}{T_B s + 1} \frac{1}{s^2}$$

also $$y(t) = L^{-1}\left\{ \frac{k_B}{T_B s + 1} \frac{1}{s^2} \right\} = k_B (t - T_B + T_B e^{-\frac{t}{T_B}}) \qquad (2)$$

mit $$y(0) = 0, \quad y(\infty) = \infty, \quad e(t) = x(t) - y(t) = (1 - k_B)t + k_B T_B (1 - e^{-\frac{t}{T_B}})$$



Im Fall X(s) = W(s) soll

$$k_B \to 1, \quad T_B \to 0$$

Im Fall X(s) = Z(s) soll

$$k_B \to 0, \quad T_B \to 0$$

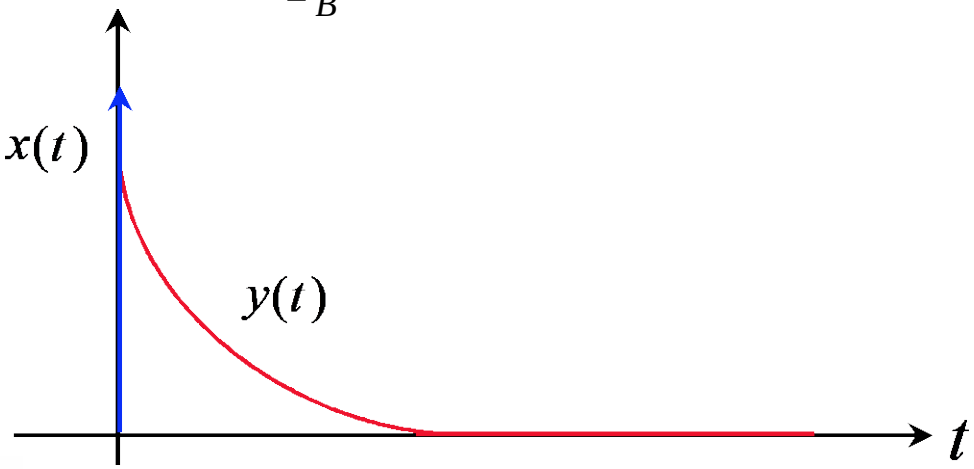
Antworten von Systemen 1. Ordnung

Das geschlossene System: $$\frac{Y(s)}{X(s)} = \frac{k_B}{T_B s + 1}$$

Auf einen Einheitsimpuls: $$Y(s) = \frac{k_B}{T_B s + 1} X(s) = \frac{k_B}{T_B s + 1}$$

also $$y(t) = L^{-1}\left\{\frac{k_B}{T_B s + 1}\right\} = \frac{k_B}{T_B} e^{-\frac{t}{T_B}} \qquad (3)$$

mit $y(0+) = \frac{k_B}{T_B}, \quad y(\infty) = 0$



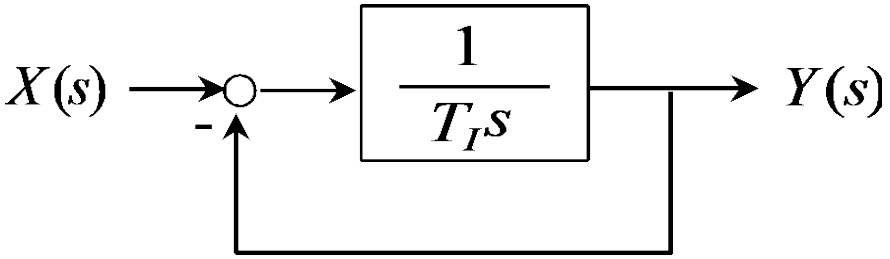
Es gilt:

$$\frac{d(1)}{dt} = (3)$$

$$\frac{d(2)}{dt} = (1)$$

Antworten von Systemen 1. Ordnung

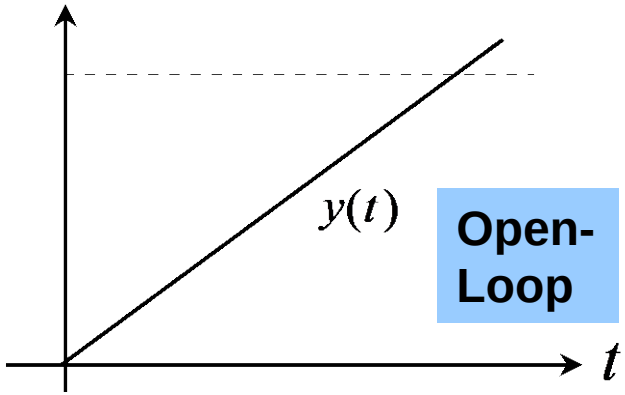
Blockschaltbild:

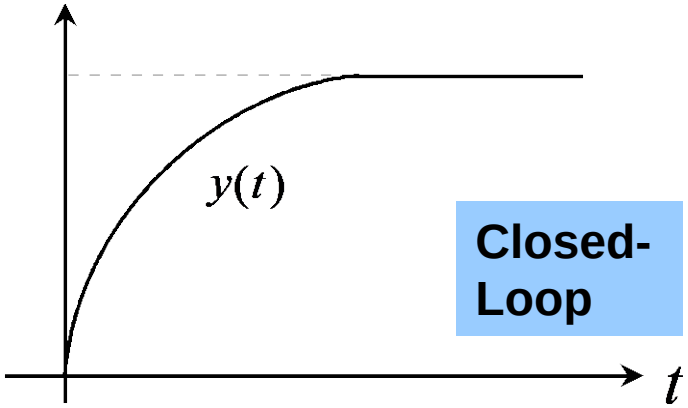


Die offene Kette (I-Glied): $$G_0(s) = \frac{1}{T_I s}$$

Das geschlossene System (PT_1-Glied): $$G_B(s) = \frac{Y(s)}{X(s)} = \frac{1}{T_I s + 1}$$

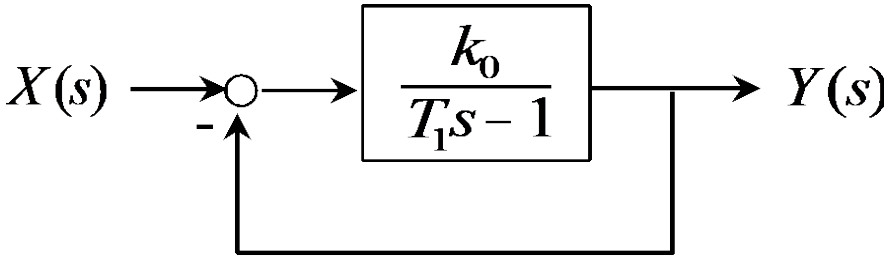
Durch die Rückführung wird die Eigenschaft des Systems verändert! Das System ist stabilisiert!





Antworten von Systemen 1. Ordnung

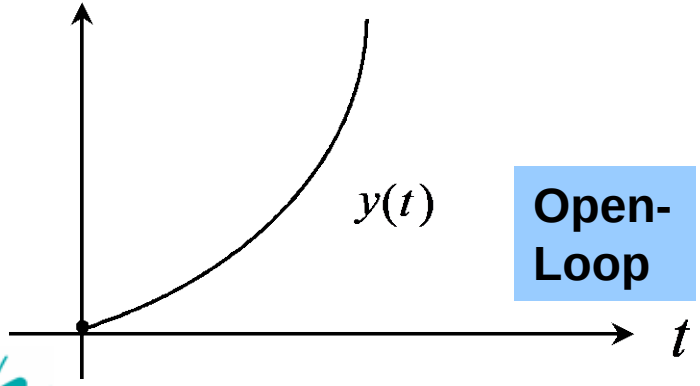
Blockschaltbild:



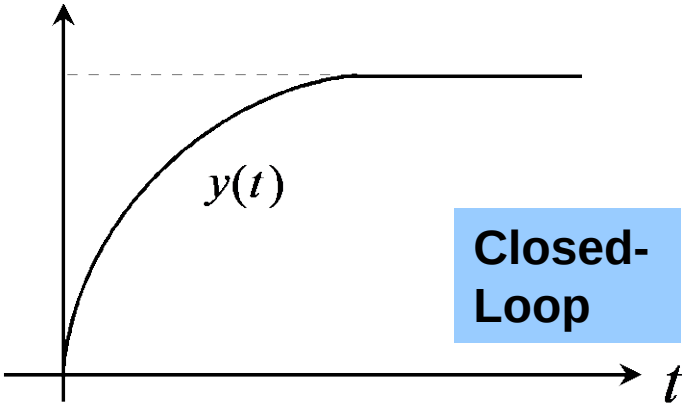
Die offene Kette: $$G_0(s) = \frac{k_0}{T_1 s - 1}$$

Das geschlossene System: $$G_B(s) = \frac{Y(s)}{X(s)} = \frac{\frac{k_0}{k_0 - 1}}{\frac{T_1}{k_0 - 1} s + 1} = \frac{k_B}{\tau_B s + 1}, \quad k_0 > 1$$

Das System ist stabilisiert!



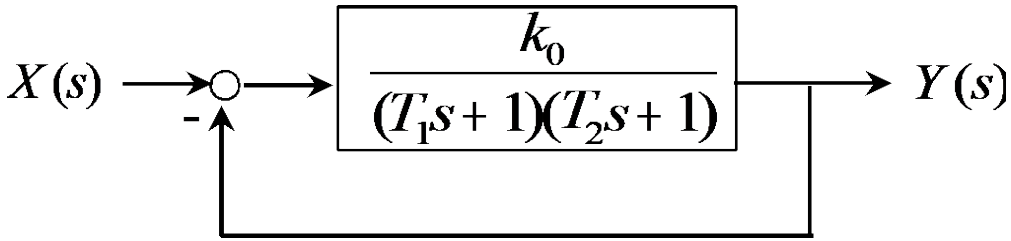
Open-Loop



Closed-Loop

Antworten von Systemen 2. Ordnung

Blockschaltbild:



Die offene Kette: $$G_0(s) = \frac{k_0}{(T_1 s + 1)(T_2 s + 1)}$$

Das geschlossene System:

$$G_B(s) = \frac{Y(s)}{X(s)} = \frac{k_0}{(T_1 s + 1)(T_2 s + 1) + k_0}$$

$$= \frac{k_0}{T_1 T_2 s^2 + (T_1 + T_2)s + 1 + k_0}$$

$$= \frac{\frac{k_0}{T_1 T_2}}{s^2 + \frac{T_1 + T_2}{T_1 T_2} s + \frac{1 + k_0}{T_1 T_2}} = \frac{\tilde{k}_0}{s^2 + 2 d \omega_0 s + \omega_0^2}$$

Eigen(kreis)frequenz:

$$\omega_0 = \sqrt{\frac{1 + k_0}{T_1 T_2}}$$

Dämpfungsfaktor:

$$d = \frac{1}{2} \frac{T_1 + T_2}{\sqrt{(1 + k_0) T_1 T_2}}$$

Fachgebiet Prozessoptimierung

Antworten von Systemen 2. Ordnung

Das geschlossene System: $$\frac{Y(s)}{X(s)} = \frac{\tilde{k}_0}{s^2 + 2d\omega_0 s + \omega_0^2}$$

Die Polstellen des geschlossenen Systems: $$s_{1,2} = \left(-d \pm \sqrt{d^2 - 1}\right)\omega_0$$

Auf einen Einheitssprung: $$Y(s) = \frac{\tilde{k}_0}{s^2 + 2d\omega_0 s + \omega_0^2}\frac{1}{s}$$

Wenn $d \geq 1$, da $\sqrt{d^2 - 1} < d$

dann $s_{1,2} = \left(-d \pm \sqrt{d^2 - 1}\right)\omega_0 = -a_1, -a_2$ mit $a_1 > 0, a_2 > 0$

daher

$$Y(s) = \frac{\tilde{k}_0}{(s + a_1)(s + a_2)}\frac{1}{s} = \tilde{k}_0\left(\frac{1}{a_1 a_2 s} + \frac{1}{a_1(a_1 - a_2)(s + a_1)} + \frac{1}{a_2(a_2 - a_1)(s + a_2)}\right)$$

Antworten von Systemen 2. Ordnung

$$Y(s) = \frac{\tilde{k}_0}{(s+a_1)(s+a_2)}\frac{1}{s} = \tilde{k}_0\left(\frac{1}{a_1 a_2 s} + \frac{1}{a_1(a_1-a_2)(s+a_1)} + \frac{1}{a_2(a_2-a_1)(s+a_2)}\right)$$

daher $$y(t) = \tilde{k}_0\left(\frac{1}{a_1 a_2} + \frac{1}{a_1(a_1-a_2)}e^{-a_1 t} + \frac{1}{a_2(a_2-a_1)}e^{-a_2 t}\right)$$

mit $$y(0) = 0, \quad y(\infty) = \frac{\tilde{k}_0}{a_1 a_2} = \frac{\tilde{k}_0}{\omega_0^2} = \frac{k_0}{k_0+1}$$

Anfangswertsatz: $$f(0) = \lim_{s\to\infty} sF(s)$$

$$y(0) = \lim_{s\to\infty} sY(s) = \lim_{s\to\infty} s\frac{\tilde{k}_0}{(s+a_1)(s+a_2)}\frac{1}{s} = 0$$

Endwertsatz: $$f(\infty) = \lim_{s\to 0} sF(s)$$

$$y(\infty) = \lim_{s\to 0} sY(s) = \lim_{s\to 0} s\frac{\tilde{k}_0}{(s+a_1)(s+a_2)}\frac{1}{s} = \frac{\tilde{k}_0}{a_1 a_2} = \frac{k_0}{k_0+1}$$

Antworten von Systemen 2. Ordnung

Wenn $0 < d < 1$

$$s_{1,2} = \left(-d \pm \sqrt{d^2 - 1}\right)\omega_0$$

$$s_{1,2} = \left(-d \pm j\sqrt{1-d^2}\right)\omega_0 = -d\omega_0 \pm j\sqrt{1-d^2}\,\omega_0 = -\tilde{\sigma} \pm j\tilde{\omega}$$

mit $\tilde{\sigma} > 0,\ \tilde{\omega} > 0$

$$Y(s) = \frac{\tilde{k}_0}{(s+\tilde{\sigma})^2 + \tilde{\omega}^2}\frac{1}{s} = \tilde{k}_0\left[\frac{A + Bs}{(s+\tilde{\sigma})^2 + \tilde{\omega}^2} + \frac{C}{s}\right]$$

mit $A = -\dfrac{2\tilde{\sigma}}{\tilde{\sigma}^2 + \tilde{\omega}^2}, \quad B = -\dfrac{1}{\tilde{\sigma}^2 + \tilde{\omega}^2}, \quad C = \dfrac{1}{\tilde{\sigma}^2 + \tilde{\omega}^2}$

$$Y(s) = \frac{\tilde{k}_0}{\tilde{\sigma}^2 + \tilde{\omega}^2}\left[\frac{1}{s} - \frac{2\tilde{\sigma} + s}{(s+\tilde{\sigma})^2 + \tilde{\omega}^2}\right]$$

$$= \frac{\tilde{k}_0}{\tilde{\sigma}^2 + \tilde{\omega}^2}\left[\frac{1}{s} - \frac{\tilde{\sigma} + s}{(s+\tilde{\sigma})^2 + \tilde{\omega}^2} - \frac{\tilde{\sigma}}{\tilde{\omega}}\frac{\tilde{\omega}}{(s+\tilde{\sigma})^2 + \tilde{\omega}^2}\right]$$

daher $y(t) = \dfrac{\tilde{k}_0}{\tilde{\sigma}^2 + \tilde{\omega}^2}\left[1 - e^{-\tilde{\sigma}t}\cos\tilde{\omega}t - \dfrac{\tilde{\sigma}}{\tilde{\omega}}e^{-\tilde{\sigma}t}\sin\tilde{\omega}t\right]$

Antworten von Systemen 2. Ordnung

$$y(t) = \frac{\tilde{k}_0}{\tilde{\sigma}^2 + \tilde{\omega}^2}\left[1 - e^{-\tilde{\sigma}t}\cos\tilde{\omega}t - \frac{\tilde{\sigma}}{\tilde{\omega}}e^{-\tilde{\sigma}t}\sin\tilde{\omega}t\right]$$

$$= \frac{\tilde{k}_0}{\omega_0^2}\left[1 - e^{-d\omega_0 t}\cos\left(\sqrt{1-d^2}\,\omega_0 t\right) - \frac{d}{\sqrt{1-d^2}}e^{-\omega_0 d t}\sin\left(\sqrt{1-d^2}\,\omega_0 t\right)\right]$$

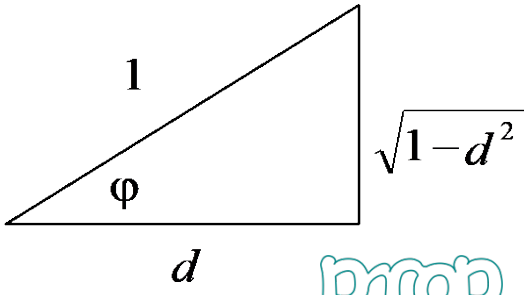
mit $\quad y(0) = 0, \quad y(\infty) = \dfrac{\tilde{k}_0}{a_1 a_2} = \dfrac{\tilde{k}_0}{\omega_0^2} = \dfrac{k_0}{k_0 + 1}$

Beziehen der Sprungantwort auf den Endwert:

$$\frac{y(t)}{y(\infty)} = 1 - e^{-d\omega_0 t}\cos\left(\sqrt{1-d^2}\,\omega_0 t\right) - \frac{d}{\sqrt{1-d^2}}e^{-\omega_0 d t}\sin\left(\sqrt{1-d^2}\,\omega_0 t\right)$$

$$= 1 - \frac{e^{-d\omega_0 t}}{\sqrt{1-d^2}}\left[\sqrt{1-d^2}\cos\left(\sqrt{1-d^2}\,\omega_0 t\right) + d\sin\left(\sqrt{1-d^2}\,\omega_0 t\right)\right]$$

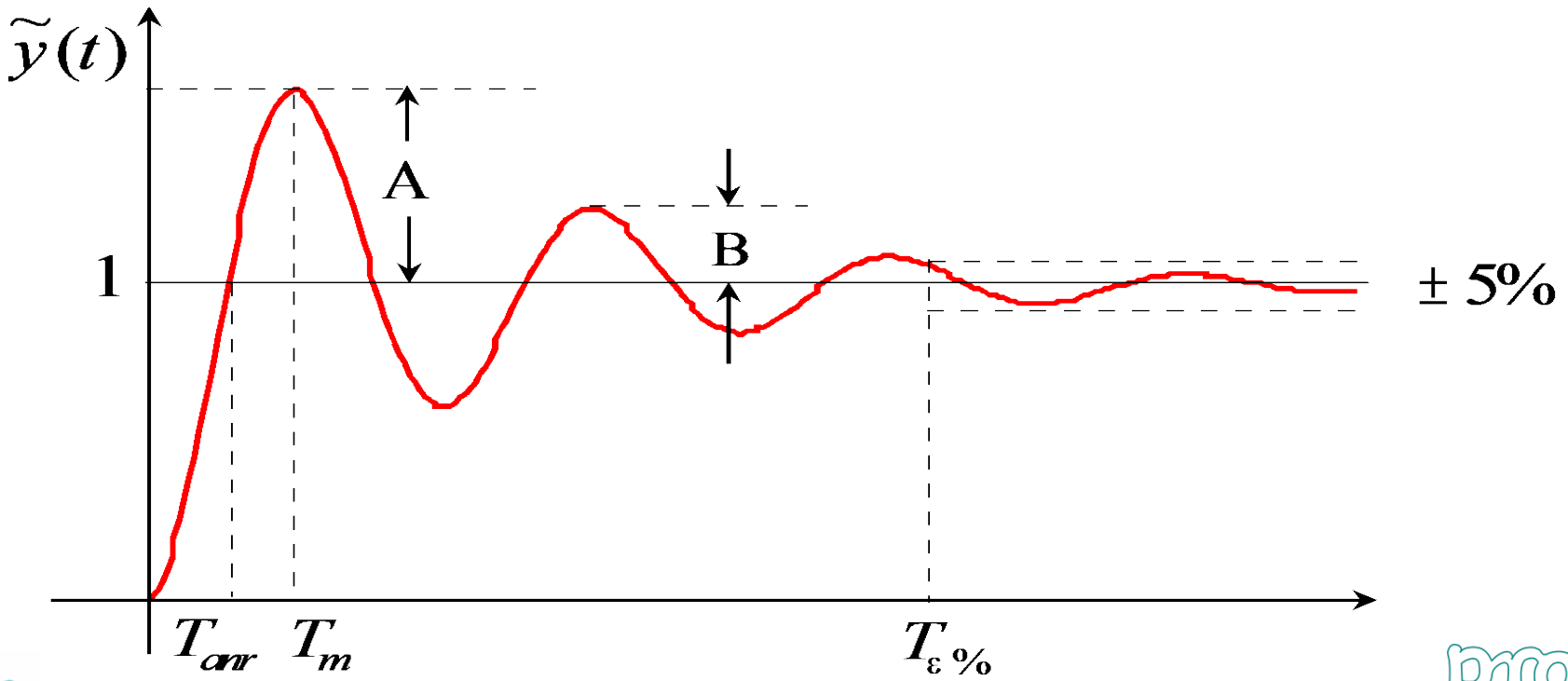
$$= 1 - \frac{e^{-d\omega_0 t}}{\sqrt{1-d^2}}\left[\sin\varphi\cos\left(\sqrt{1-d^2}\,\omega_0 t\right) + \cos\varphi\sin\left(\sqrt{1-d^2}\,\omega_0 t\right)\right]$$



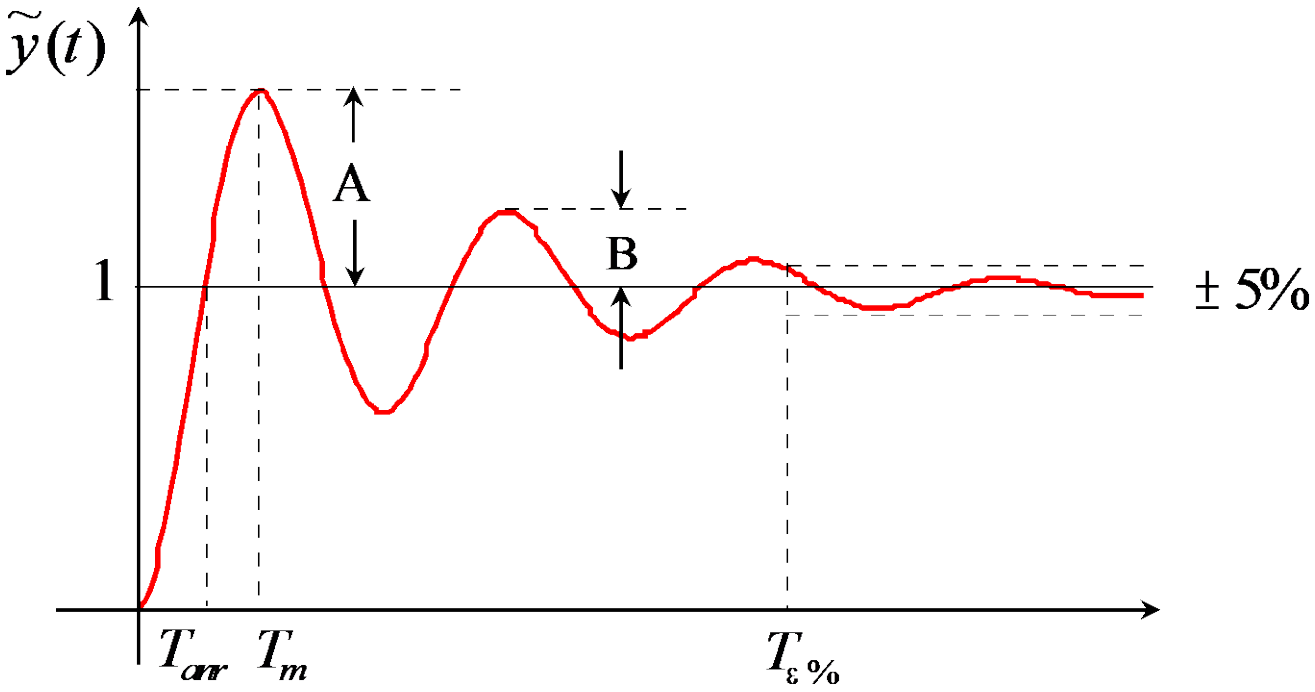
Antworten von Systemen 2. Ordnung

$$\tilde{y}(t) = 1 - \frac{e^{-d\omega_0 t}}{\sqrt{1-d^2}} \sin\left(\sqrt{1-d^2}\,\omega_0 t + \varphi\right)$$

Standardkurve der Sprungantwort:



Antworten von Systemen 2. Ordnung



Qualität der Regelung:

- Anregelzeit T_{anr}
- Überschwingzeit T_m
- Überschwingweite $\Delta h = A$
- Abnahme des Überschwingens A / B
- Beruhigungszeit $T_{\varepsilon\%}$

Antworten von Systemen 2. Ordnung

$$\tilde{y}(t) = 1 - \frac{e^{-d\omega_0 t}}{\sqrt{1-d^2}} \sin\left(\sqrt{1-d^2}\,\omega_0 t + \varphi\right)$$

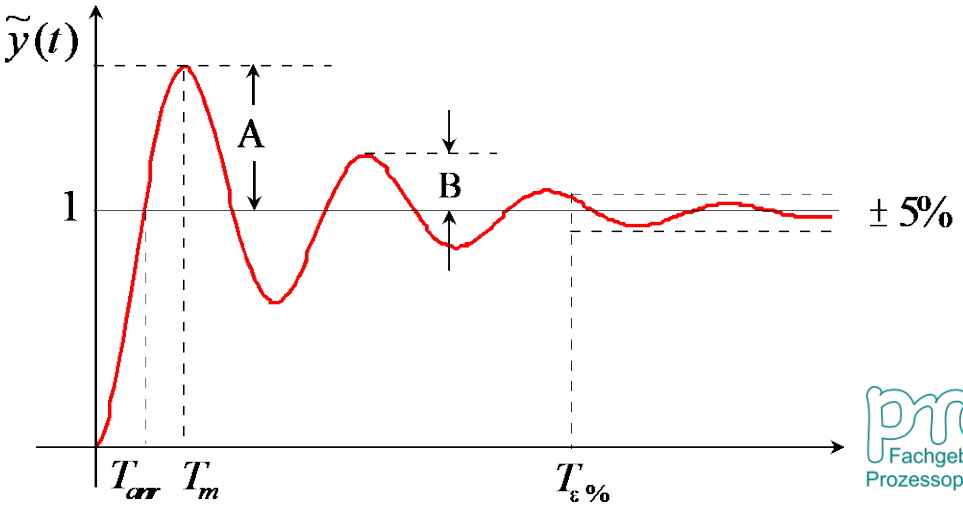
1) Anregelzeit: T_{anr}

$$\tilde{y}(T_{anr}) = 1 = 1 - \frac{e^{-d\omega_0 t_{anr}}}{\sqrt{1-d^2}} \sin\left(\sqrt{1-d^2}\,\omega_0 T_{anr} + \varphi\right)$$

also $$\sin\left(\sqrt{1-d^2}\,\omega_0 T_{anr} + \varphi\right) = 0$$

$$\sqrt{1-d^2}\,\omega_0 T_{anr} + \varphi = \pi \quad \text{mit} \quad \varphi = \arctan\frac{\sqrt{1-d^2}}{d}$$

daher $$\boxed{T_{anr} = \frac{\pi - \varphi}{\sqrt{1-d^2}\,\omega_0}}$$



Antworten von Systemen 2. Ordnung

2) Überschwingzeit: T_m

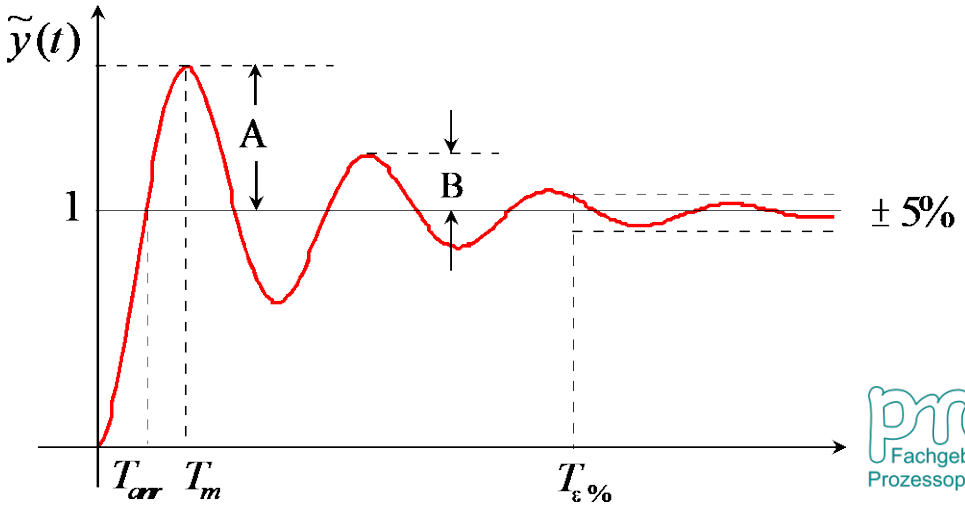
$$\frac{d\tilde{y}}{dt} = \frac{d\omega_0 e^{-d\omega_0 t}}{\sqrt{1-d^2}} \sin\left(\sqrt{1-d^2}\,\omega_0 t + \varphi\right) - \omega_0 e^{-d\omega_0 t} \cos\left(\sqrt{1-d^2}\,\omega_0 t + \varphi\right) = 0$$

$$\frac{\sin\left(\sqrt{1-d^2}\,\omega_0 t + \varphi\right)}{\cos\left(\sqrt{1-d^2}\,\omega_0 t + \varphi\right)} = \frac{\sqrt{1-d^2}}{d} \quad \Rightarrow \quad \tan\left(\sqrt{1-d^2}\,\omega_0 t + \varphi\right) = \tan\varphi$$

$$\Rightarrow \quad \sqrt{1-d^2}\,\omega_0 t = m\pi, \quad m = 1, 2, \cdots$$

daher

$$\boxed{T_m = \frac{\pi}{\sqrt{1-d^2}\,\omega_0}}$$



Antworten von Systemen 2. Ordnung

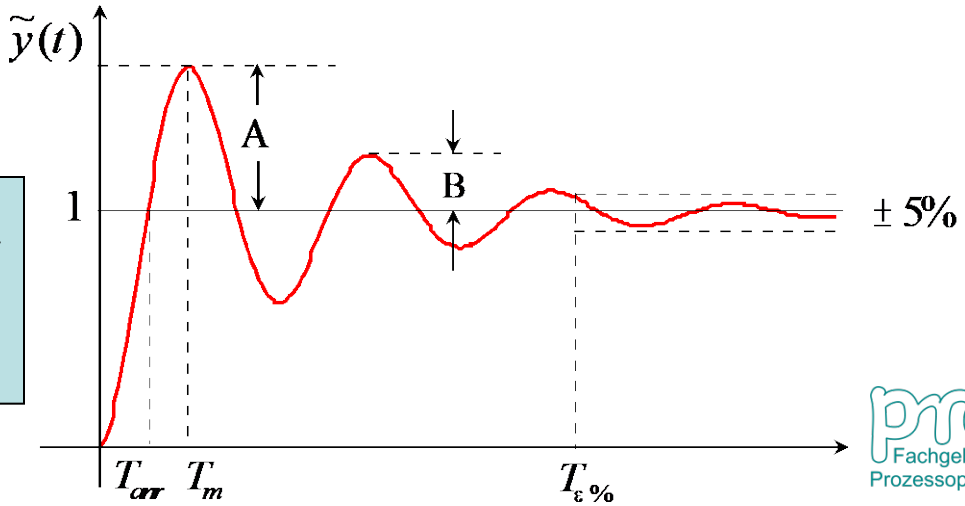
3) Überschwingweite: $\Delta h = A$

da $$T_m = \frac{\pi}{\sqrt{1-d^2}\,\omega_0}$$ und $$\tilde{y}(t) = 1 - \frac{e^{-d\omega_0 t}}{\sqrt{1-d^2}} \sin\left(\sqrt{1-d^2}\,\omega_0 t + \varphi\right)$$

$$\tilde{y}(T_m) = 1 - \frac{e^{-\frac{d\pi}{\sqrt{1-d^2}}}}{\sqrt{1-d^2}} \sin(\pi + \varphi) = 1 + \frac{e^{-\frac{d\pi}{\sqrt{1-d^2}}}}{\sqrt{1-d^2}} \sin\varphi$$

$$\sin\varphi = \sqrt{1-d^2} \quad \Rightarrow \quad \tilde{y}(T_m) = 1 + e^{-\frac{d\pi}{\sqrt{1-d^2}}}$$

daher $$\boxed{\Delta h = A = e^{-\frac{d\pi}{\sqrt{1-d^2}}}}$$



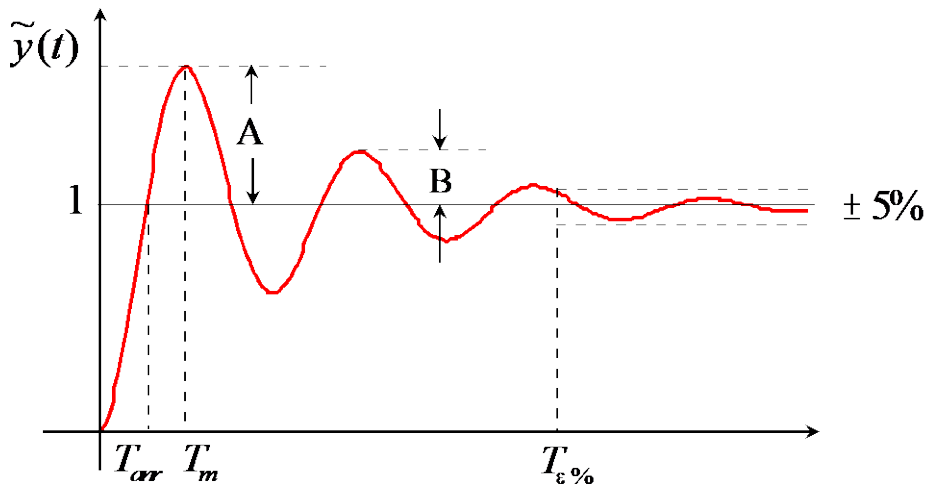
Antworten von Systemen 2. Ordnung

4) Abnahme des Überschwingens: A/B

$$A = e^{-\frac{d\pi}{\sqrt{1-d^2}}} \quad \Rightarrow \quad B = e^{-\frac{3d\pi}{\sqrt{1-d^2}}}$$

daher

$$\boxed{A/B = e^{\frac{2d\pi}{\sqrt{1-d^2}}}}$$



5) Beruhigungszeit: $T_{\varepsilon\%}$

da $\tilde{y}(t) = 1 - \frac{e^{-d\omega_0 t}}{\sqrt{1-d^2}} \sin\left(\sqrt{1-d^2}\,\omega_0 t + \varphi\right)$ und $-1 \le \sin\left(\sqrt{1-d^2}\,\omega_0 t + \varphi\right) \le 1$

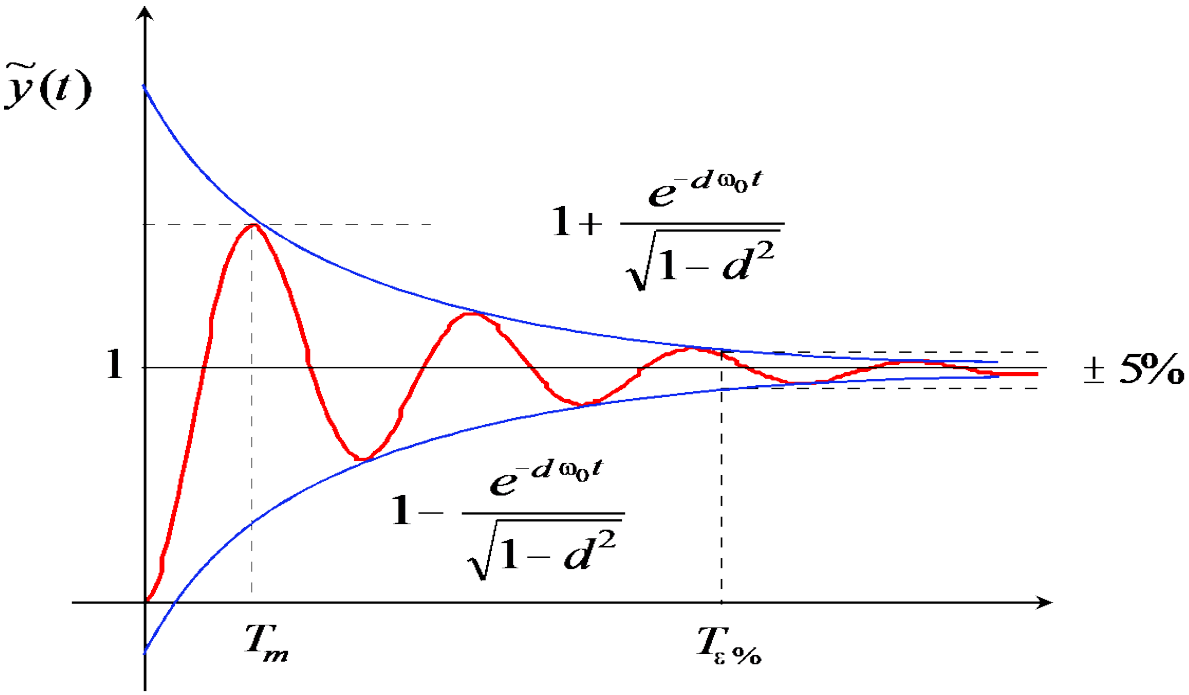
daher $$1 - \frac{e^{-d\omega_0 t}}{\sqrt{1-d^2}} \le \tilde{y}(t) \le 1 + \frac{e^{-d\omega_0 t}}{\sqrt{1-d^2}}$$

Fachgebiet Prozessoptimierung

Antworten von Systemen 2. Ordnung

5) Beruhigungszeit: $T_{\varepsilon\%}$

daher $$1-\frac{e^{-d\omega_0 t}}{\sqrt{1-d^2}} \le \tilde{y}(t) \le 1+\frac{e^{-d\omega_0 t}}{\sqrt{1-d^2}}$$



bei $t = T_{\varepsilon\%},\ \varepsilon\% = 5\%$

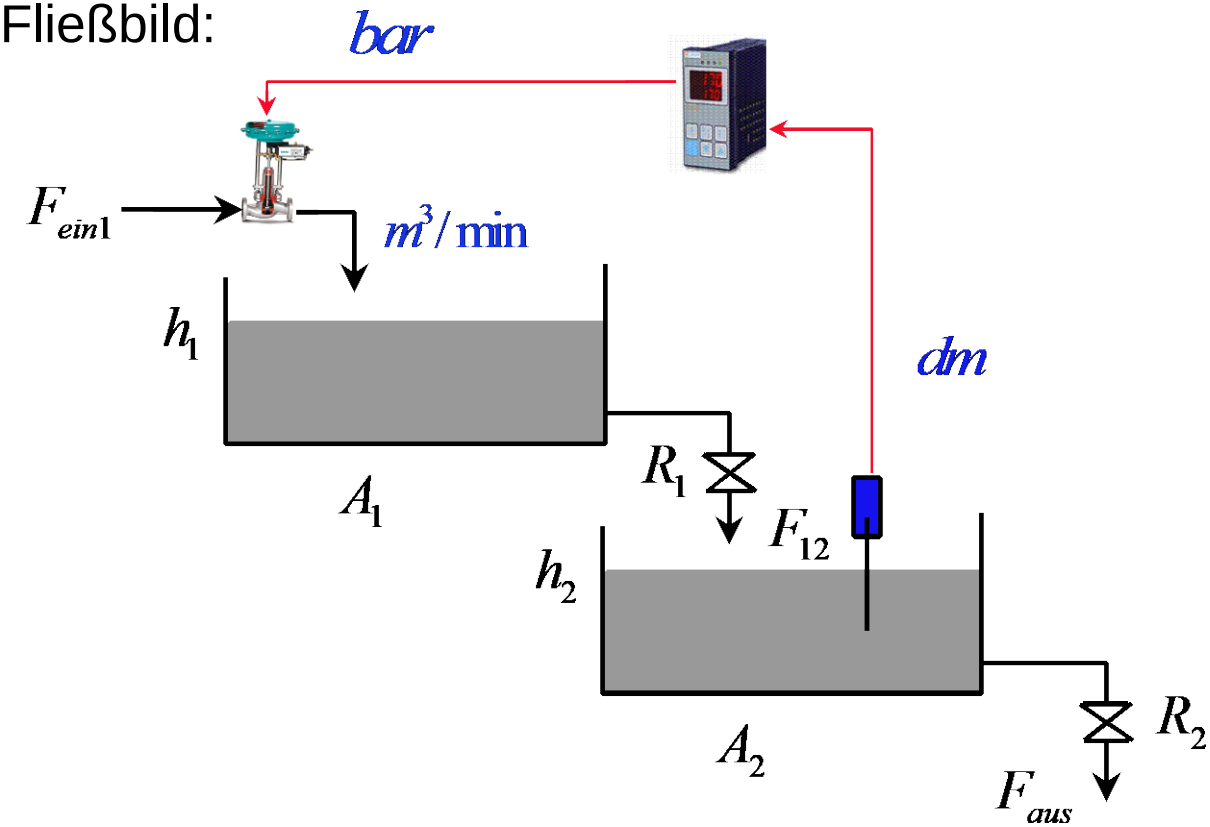
$$\frac{e^{-d\omega_0 T_{\varepsilon\%}}}{\sqrt{1-d^2}} = 5\% = 0{,}05$$

$$T_{5\%} = -\frac{\ln(0{,}05\sqrt{1-d^2})}{d\omega_0}$$

$$\boxed{T_{5\%} \approx \frac{3}{d\omega_0}}$$

Beispiel: Behälterkaskade

Fließbild:



$$T_1 = 2{,}5\,\min, \quad T_2 = 5\,\min$$

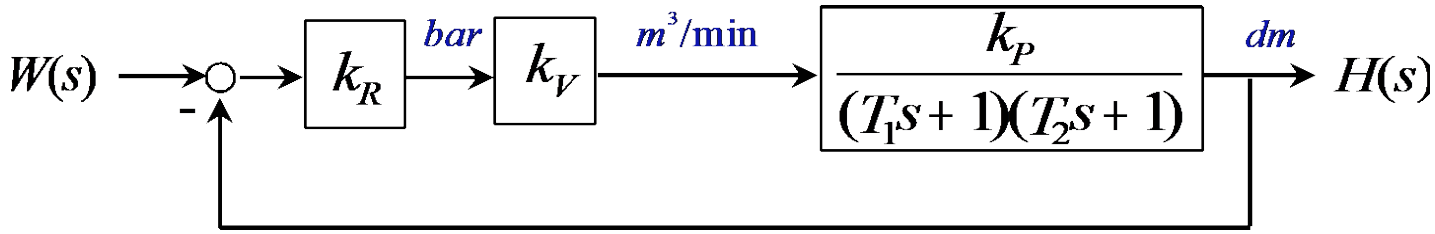
$$k_P = 162\,\frac{dm}{m^3/\min}$$

$$k_V = 1\,\frac{m^3/\min}{bar}$$

$$k_R = 0{,}08\,\frac{bar}{dm}$$

$$W(s) = \frac{1}{s}\,dm$$

Blockschaltbild:



Beispiel: Behälterkaskade

Das geschlossene System:

$$G_B(s) = \frac{H(s)}{W(s)} = \frac{k_0}{(T_1 s + 1)(T_2 s + 1) + k_0} = \frac{k_0}{T_1 T_2 s^2 + (T_1 + T_2)s + 1 + k_0}$$

$$= \frac{\dfrac{k_0}{T_1 T_2}}{s^2 + \dfrac{T_1 + T_2}{T_1 T_2} s + \dfrac{1 + k_0}{T_1 T_2}} = \frac{\tilde{k}_0}{s^2 + 2d\omega_0 s + \omega_0^2} = \frac{1{,}037}{s^2 + 0{,}6s + 1{,}117}$$

mit $\omega_0 = \sqrt{1{,}117} = 1{,}057, \quad d = \dfrac{0{,}6}{2 \cdot 1{,}057} = 0{,}284, \quad \varphi = 1{,}283$

Polstellen des geschlossenen Systems: $s_{1,2} = -0{,}3 \pm j1{,}013$

$$H(s) = \frac{1{,}037}{s^2 + 0{,}6s + 1{,}117} W(s) = \frac{1{,}037}{s(s^2 + 0{,}6s + 1{,}117)}$$

$$\Delta h(t) = 1{,}037 \left[1 - \frac{e^{-d\omega_0 t}}{\sqrt{1 - d^2}} \sin\left(\sqrt{1 - d^2}\, \omega_0 t + \varphi \right) \right]$$

Fachgebiet Prozessoptimierung

Beispiel: Behälterkaskade

1) Anregelzeit

$$T_{anr} = \frac{\pi - \varphi}{\sqrt{1-d^2}\,\omega_0} = \frac{3{,}14 - 1{,}283}{\sqrt{1-0{,}284^2}\cdot 1{,}057} = 1{,}83\,\text{min}$$

2) Überschwingzeit

$$T_m = \frac{\pi}{\sqrt{1-d^2}\,\omega_0} = \frac{3{,}14}{\sqrt{1-0{,}284^2}\cdot 1{,}057} = 3{,}1\,\text{min}$$

3) Überschwingweite

$$\Delta h = A = e^{-\frac{d\pi}{\sqrt{1-d^2}}} = e^{-\frac{3{,}14\cdot 0{,}284}{\sqrt{1-0{,}284^2}}} = 0{,}415\ dm$$

4) Abnahme des Überschwingens

$$A/B = e^{\frac{2d\pi}{\sqrt{1-d^2}}} = e^{\frac{2\cdot 3{,}14\cdot 0{,}284}{\sqrt{1-0{,}284^2}}} = 5{,}82$$

5) Beruhigungszeit (5%)

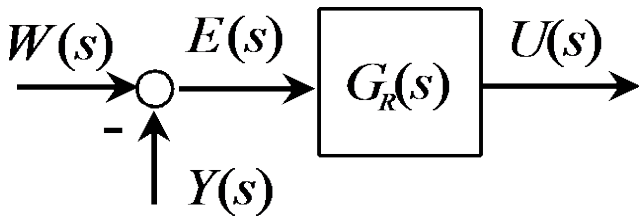
$$T_{5\%} = \frac{3}{d\omega_0} = \frac{3}{0{,}284\cdot 1{,}057} = 10\,\text{min}$$

6) Stationärer Fehler

$$e(\infty) = 1 - \Delta h(\infty) = 1 - \lim_{s\to 0} s\,\frac{1{,}037}{s(s^2 + 0{,}6s + 1{,}117)} = 0{,}072\ dm$$

Konventionelle Regler

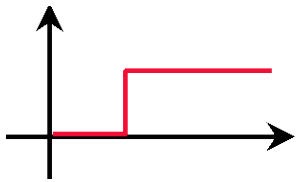
Der Reglertyp bzw. die Übertragungsfunktion des Reglers:

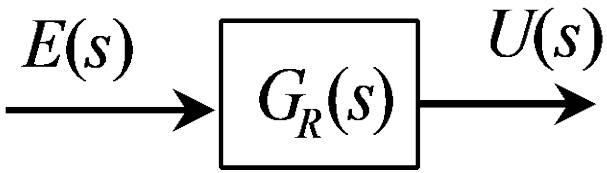


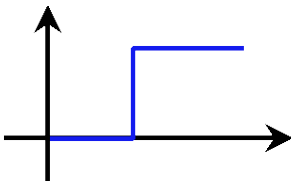
$$\Rightarrow \quad U(s) = G_R(s)E(s) = G_R(s)\left[W(s) - Y(s)\right]$$

Ein Reglertyp wird anhand der Eigenschaft des Prozesses ausgewählt, um die gewünschte Qualität der Regelung zu realisieren.

P-Regler: $G_R(s) = k_R \quad \Rightarrow \quad u(t) = k_R e(t)$

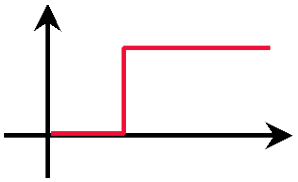


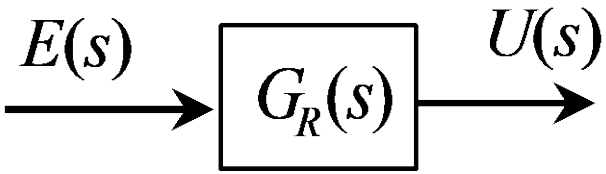


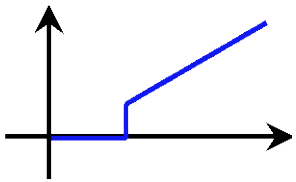


PID-Regler

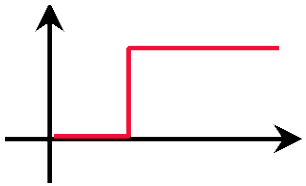
PI-Regler: $$G_R(s) = k_R\left(1+\frac{1}{T_N s}\right) \quad \Rightarrow \quad u(t) = k_R\left[e(t)+\frac{1}{T_N}\int_0^t e(\tau)d\tau\right]$$

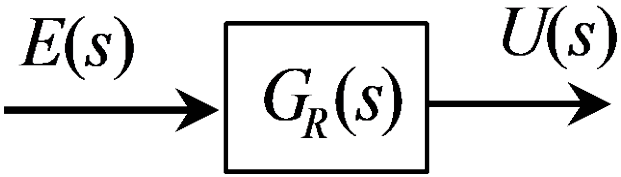


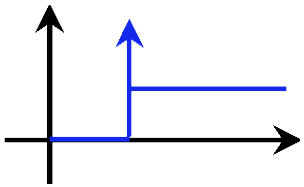




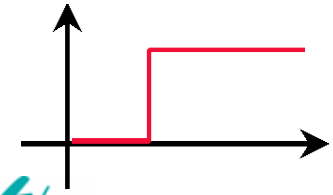
Idealer PD-Regler: $$G_R(s) = k_R(1+T_V s) \quad \Rightarrow \quad u(t) = k_R\left(e(t)+T_V\frac{de}{dt}\right)$$

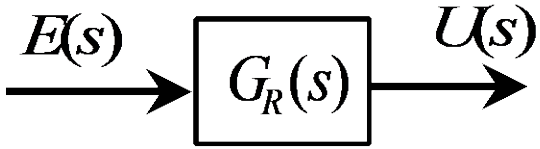


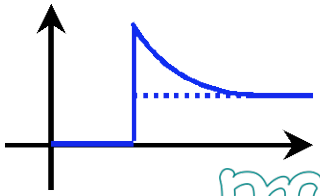




Realer PD-Regler: $$G_R(s) = k_R\left(1+\frac{T_V s}{T_1 s+1}\right), \quad T_1 << T_V$$

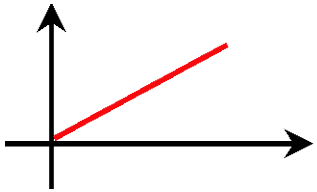


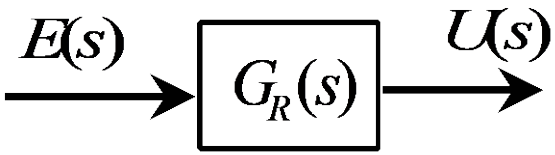


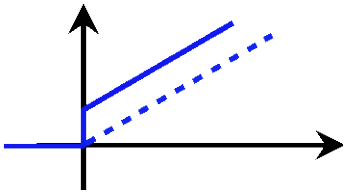


PID-Regler

Idealer PD-Regler: $G_R(s) = k_R(1 + T_V s) \quad \Rightarrow \quad u(t) = k_R\left(e(t) + T_V \frac{de}{dt}\right)$



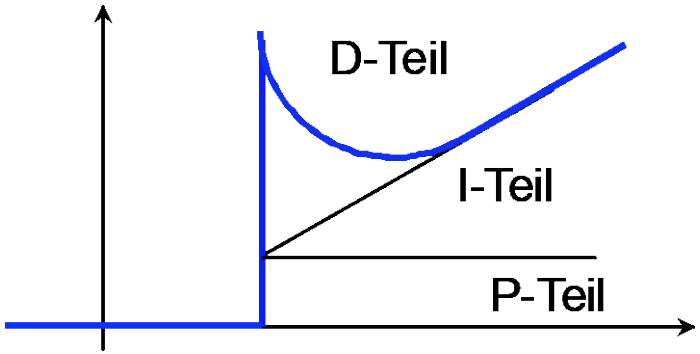


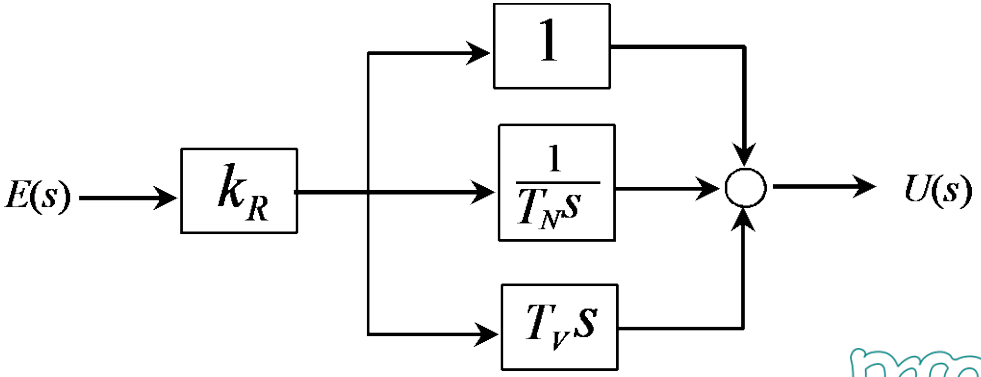


Idealer PID-Regler:

$$G_R(s) = k_R\left(1 + \frac{1}{T_N s} + T_V s\right) \quad \Rightarrow \quad u(t) = k_R\left[e(t) + \frac{1}{T_N}\int_0^t e(\tau)d\tau + T_V \frac{de}{dt}\right]$$

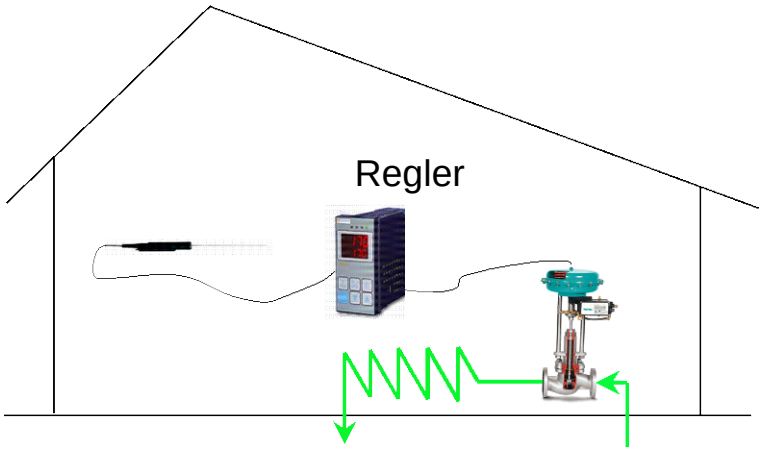
Tuning-Parameter (Einstellparameter): $k_R, \quad T_N, \quad T_V$

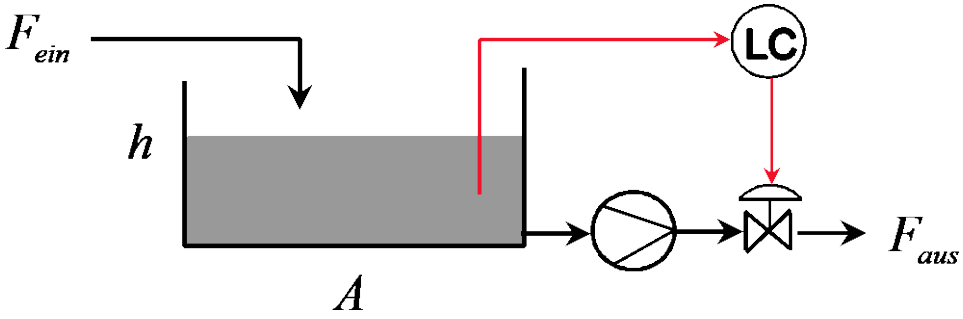




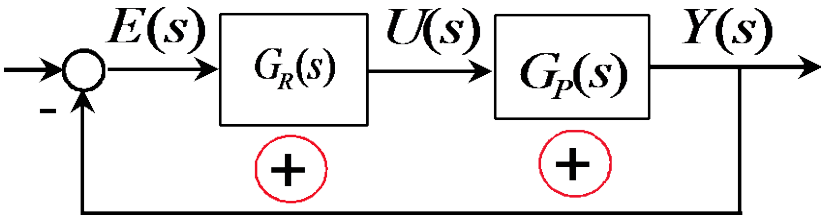
Vorzeichen von Reglern:

Das Vorzeichen des Reglers muss so ausgewählt werden, dass für das geschlossene System eine negative Rückführung gewährleistet wird.

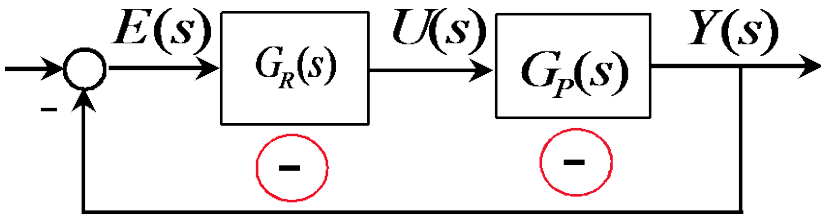




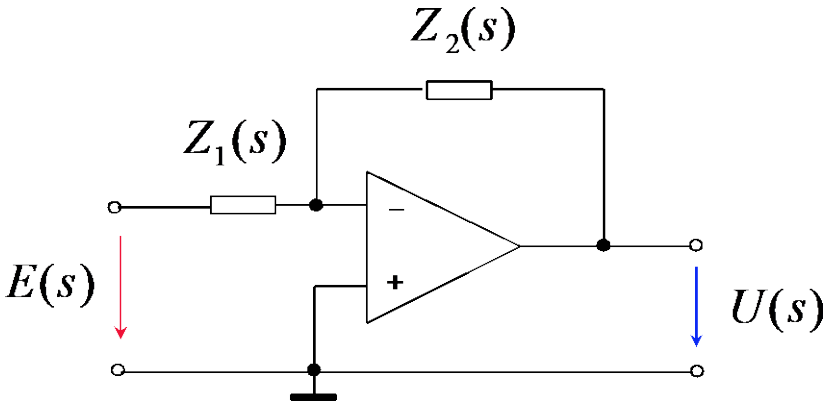
Im Fall Prozessverstärkung „+“:



Im Fall Prozessverstärkung „-“:



Elektronische Regler mittels Operationsverstärker

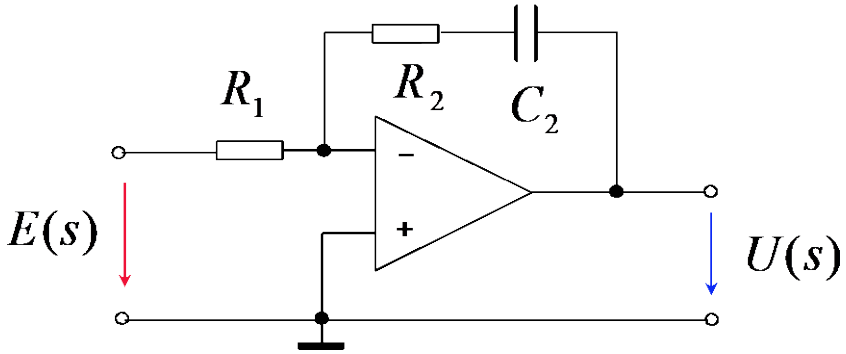


Es gilt:

$$\frac{E(s)}{Z_1(s)} = -\frac{U(s)}{Z_2(s)}$$

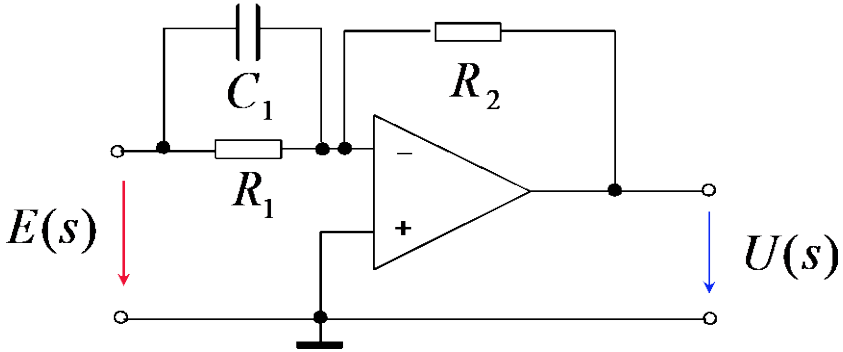
$$\frac{U(s)}{E(s)} = -\frac{Z_2(s)}{Z_1(s)}$$

PI-Regler



$$G_R(s) = -\frac{R_2}{R_1}\left(1 + \frac{1}{sR_2C_2}\right)$$

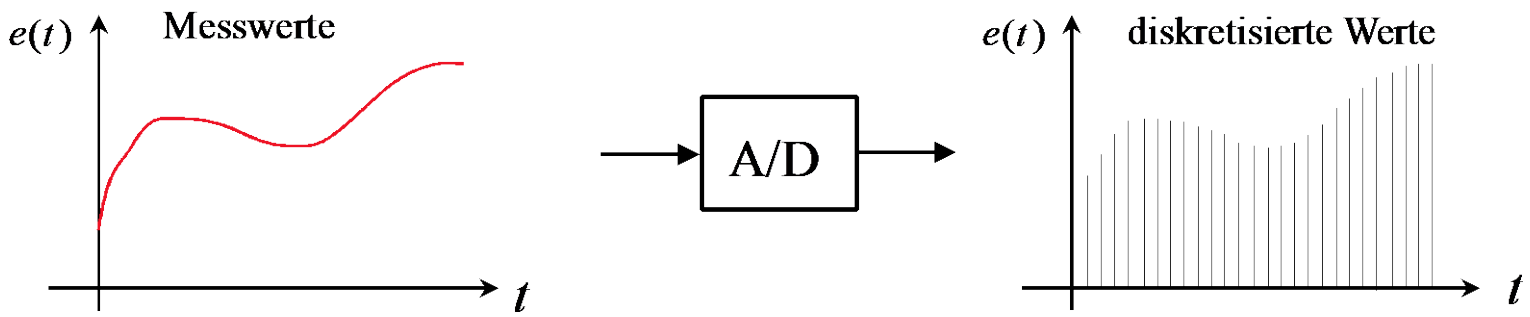
PD-Regler



$$G_R(s) = -\frac{R_2}{R_1}\left(1 + sR_1C_1\right)$$

Fachgebiet Prozessoptimierung

Digitalregler (Diskrete Regler):



PID-Regler (kontinuierlich):

$$u(t) = k_R \left[e(t) + \frac{1}{T_N} \int_0^t e(\tau) d\tau + T_V \frac{de}{dt} \right]$$

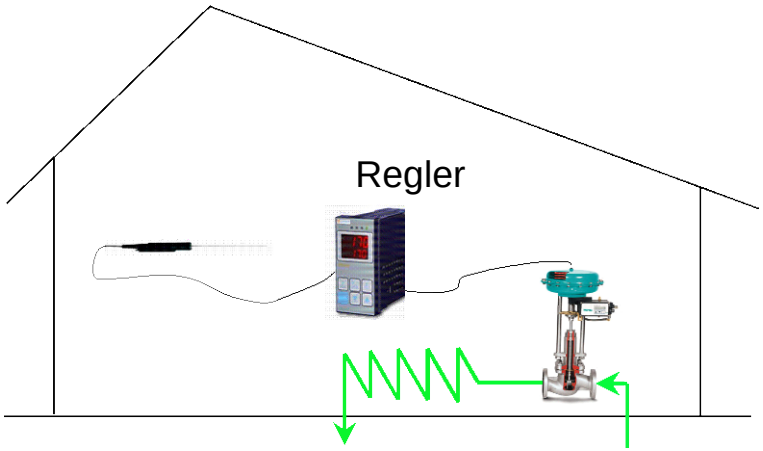
PID-Regler (diskret):

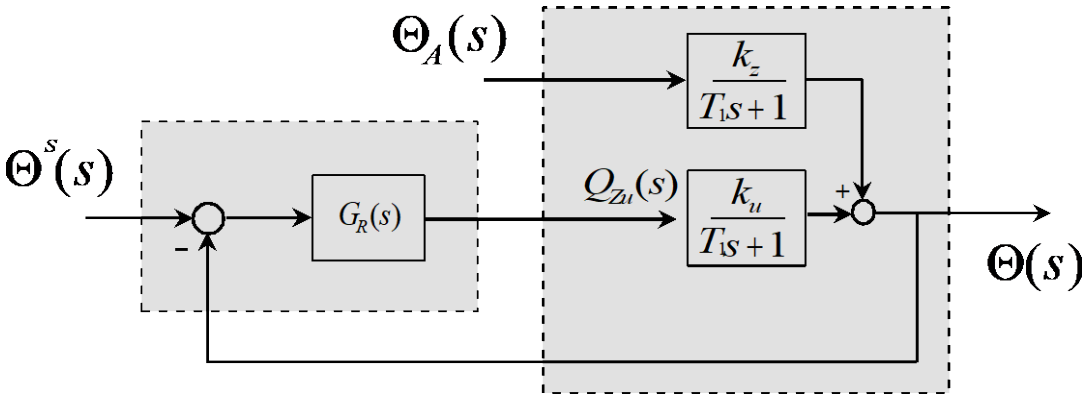
$$u(k) = k_R \left\{ e(k) + \frac{\Delta t}{T_N} \sum_{i=1}^{k} e(i) + \frac{T_V}{\Delta t} \left[(e(k) - e(k-1) \right] \right\}$$

oder

$$u(k) = u(k-1) + k_R \left\{ \left[e(k) - e(k-1) \right] + \frac{\Delta t}{T_N} e(k) + \frac{T_V}{\Delta t} \left[(e(k) - 2e(k-1) + e(k-2) \right] \right\}$$

Beispiel: Temperaturregelung:





Bei einer Veränderung der Außentemperatur:

$$\theta(s) = \frac{k_z}{T_1 s + 1 + G_R(s) k_u} \theta_A(s), \quad k_z = 1$$

Mit einem P-Regler: $G_R(s) = k_R$ und angenommen: $k_u = 1$

$$\theta(s) = \frac{1}{(T_1 s + 1 + k_R)} \frac{1}{s} = \frac{1}{1+k_R} \left(\frac{1}{s} - \frac{1}{s + \frac{1+k_R}{T_1}} \right)$$

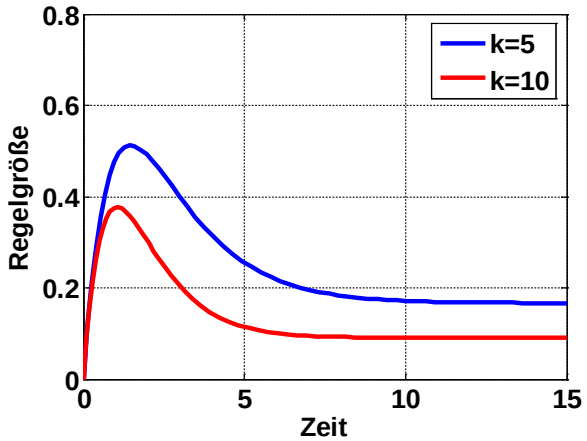
$$\Delta\theta(t) = \frac{1}{1+k_R} \left(1 - e^{-\frac{(1+k_R)t}{T_1}} \right)$$

Beispiel: Temperaturregelung:

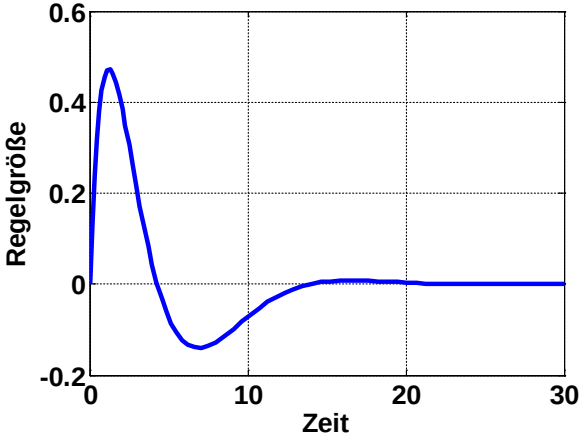
$$\theta_A(t) = 1 \cdot \sigma(t),\ k_z = 1,\ T_{1z} = 1,$$
$$k_u = 1,\ T_1 = 10$$

Regelungsqualität mit verschiedenen Reglern:

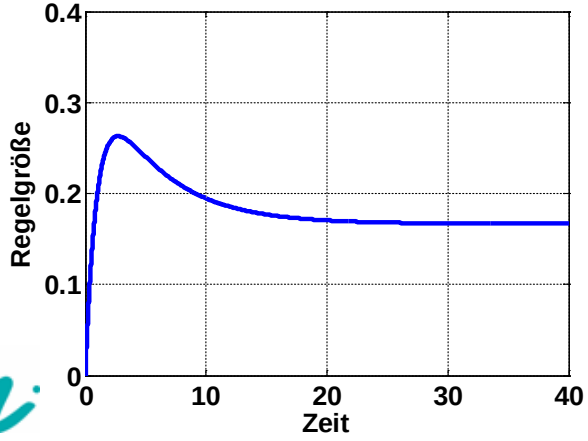
P-Regler: $k_R = 5, \quad 10$



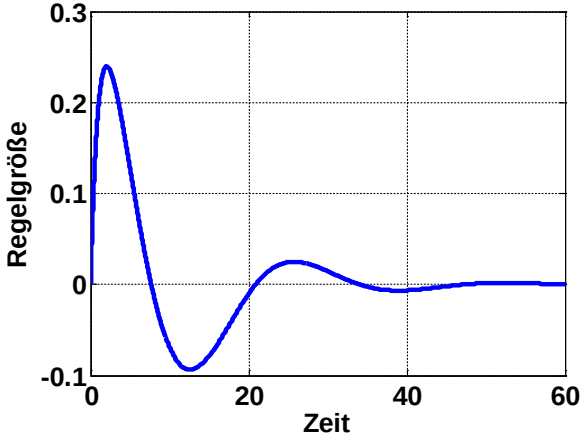
PI-Regler: $k_R = 5, T_N = 10\,\text{min}$



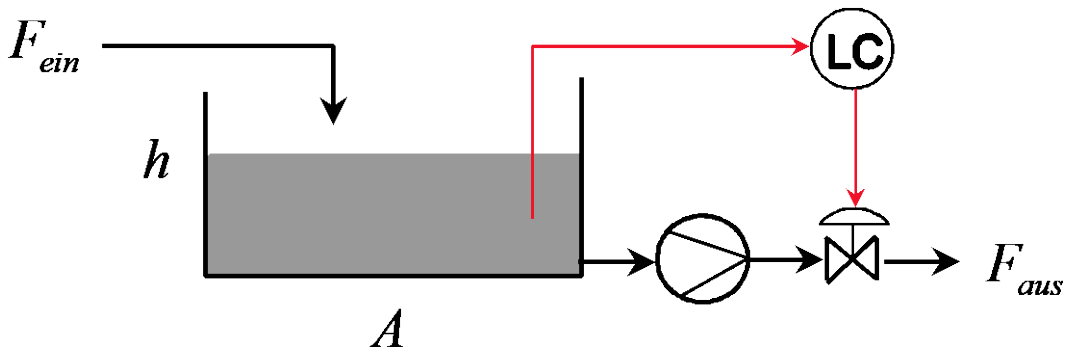
PD-Regler: $k_R = 5, T_V = 4\,\text{min}$



PID-Regler: $k_R = 5, T_N = 10, T_V = 4$



Beispiel: Füllstandregelung eines Behälters



Regelgröße: h

Stellgröße: F_{aus}

Störgröße: F_{ein}

Prozess: $$A\frac{dh}{dt} = F_{ein} - F_{aus}, \quad h(0) = h_0$$

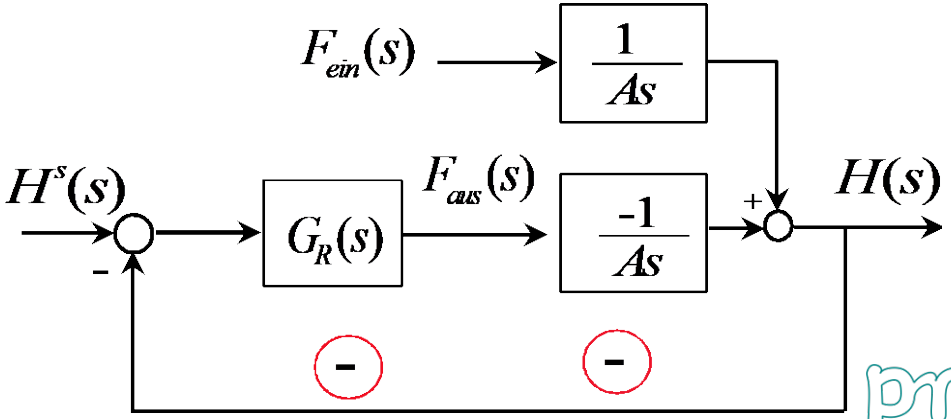
$$A\frac{d\Delta h}{dt} = \Delta F_{ein} - \Delta F_{aus} \quad \Rightarrow \quad AsH(s) = F_{ein}(s) - F_{aus}(s)$$

Regelstrecke:

$$\frac{H(s)}{F_{aus}(s)} = \frac{-1}{As}$$

Störstrecke:

$$\frac{H(s)}{F_{ein}(s)} = \frac{1}{As}$$



Beispiel: Füllstandregelung eines Behälters

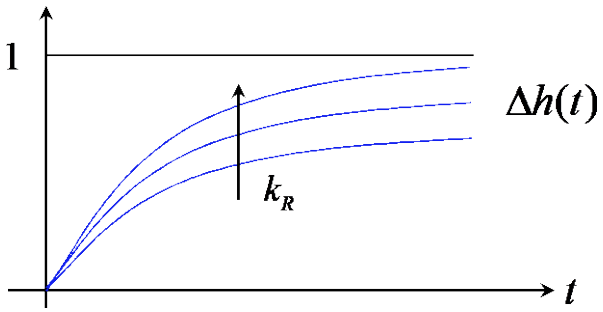
Regelungsqualität mit einem P-Regler: $G_R(s) = -k_R$

Führungsverhalten des geschlossenen Systems:

$$\frac{H(s)}{H^S(s)} = \frac{k_R}{As + k_R} \quad \Rightarrow \quad H(s) = \frac{k_R}{As + k_R} H^S(s) = \frac{k_R}{As + k_R} \frac{1}{s}$$

Dann

$$\Delta h(t) = 1 - e^{-\frac{k_R}{A} t}, \quad t \geq 0$$

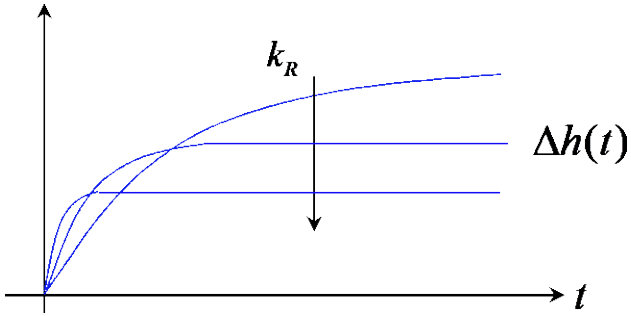


Störverhalten des geschlossenen Systems:

$$\frac{H(s)}{F_{ein}(s)} = \frac{1}{As + k_C} \quad \Rightarrow \quad H(s) = \frac{1}{As + k_R} F_{ein}(s) = \frac{1}{As + k_R} \frac{1}{s}$$

Dann

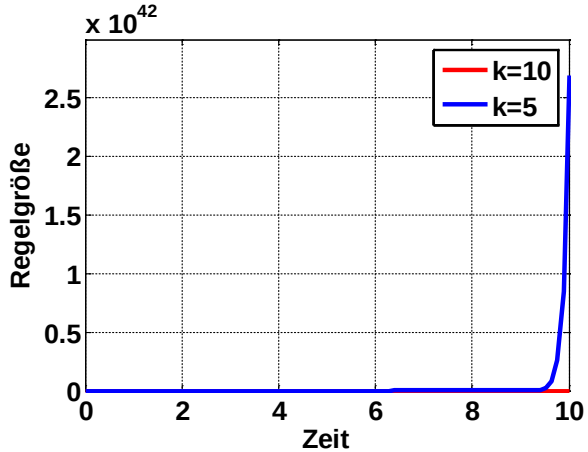
$$\Delta h(t) = \frac{1}{k_R} \left(1 - e^{-\frac{k_R}{A} t} \right)$$



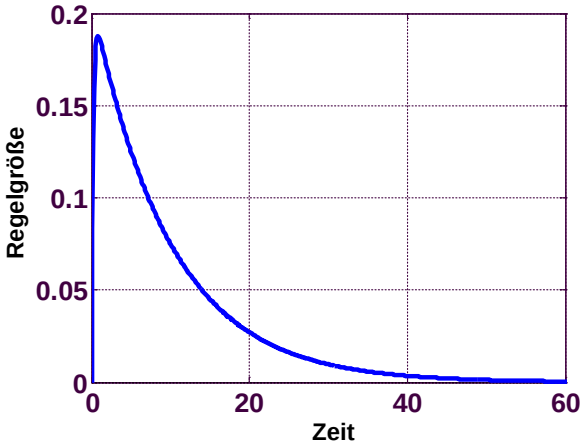
Beispiel: Füllstandregelung eines Behälters

Regelungsqualität mit verschiedenen Reglern: $F_{ein}(t) = 1 \cdot \sigma(t),\ A = 1$

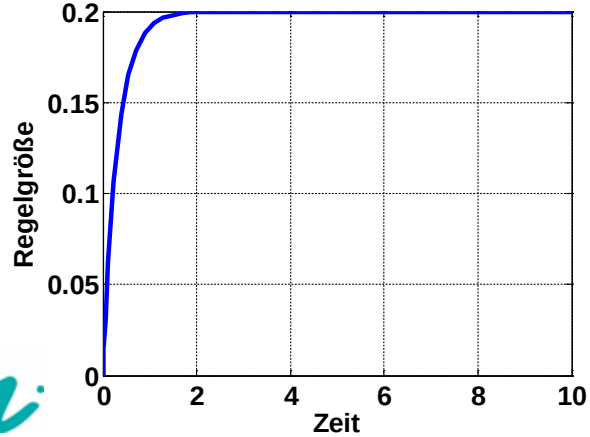
P-Regler: $k_R = 5,\ 10$



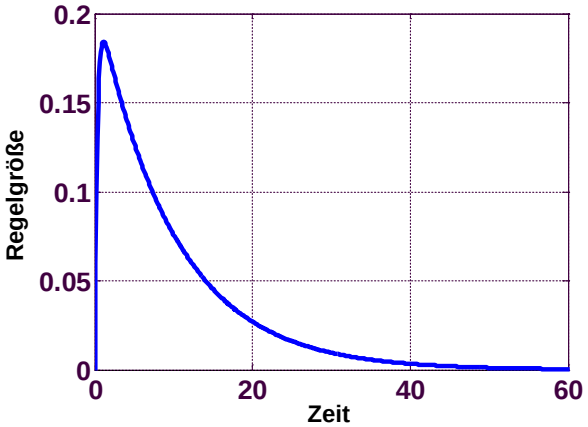
PI-Regler: $k_R = -5, T_N = 10$



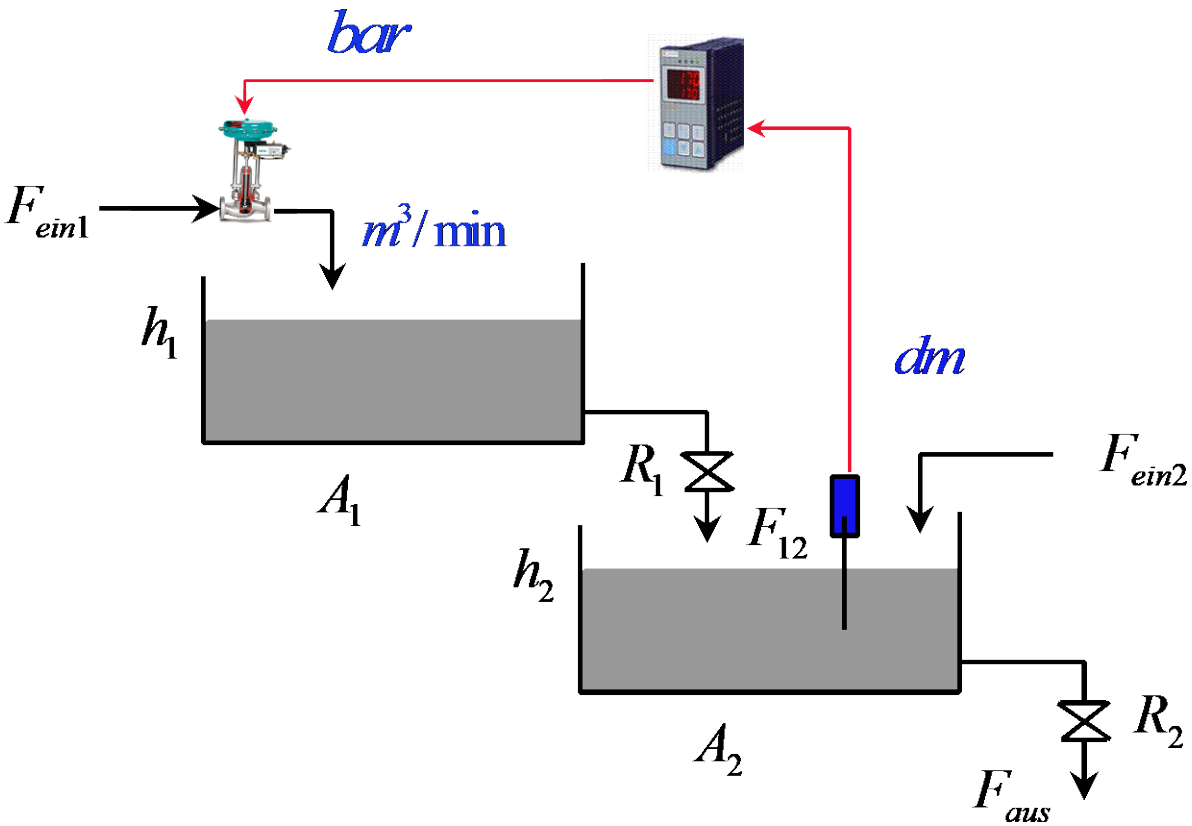
PD-Regler: $k_R = -5, T_V = 0{,}1$



PID-Regler: $k_R = -5, T_N = 10, T_V = 0{,}1$



Beispiel: Behälterkaskade



$$T_1 = 2{,}5\,\min \qquad T_2 = 5\,\min$$

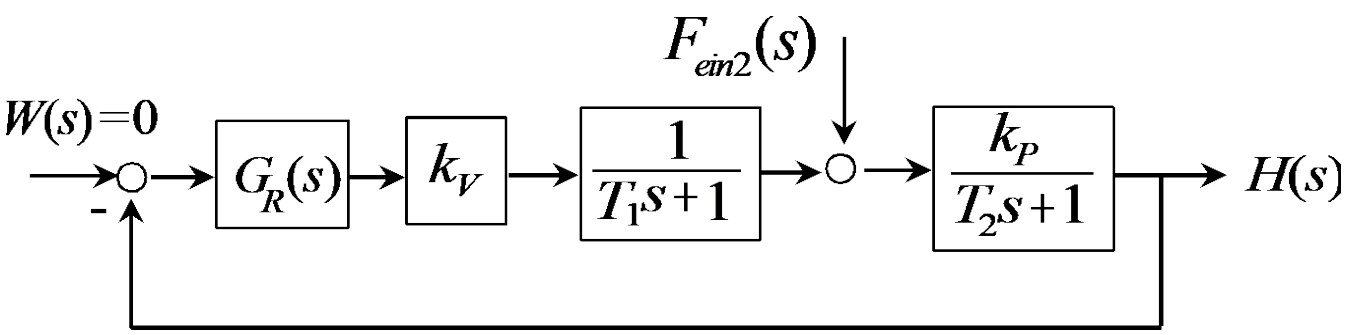
$$k_P = 162\,\frac{dm}{m^3/\min}$$

$$k_V = 1\,\frac{m^3/\min}{bar}$$

$$k_R = 0{,}08\,\frac{bar}{dm}$$

$$F_{ein2}(s) = \frac{1}{s}\,m^3$$

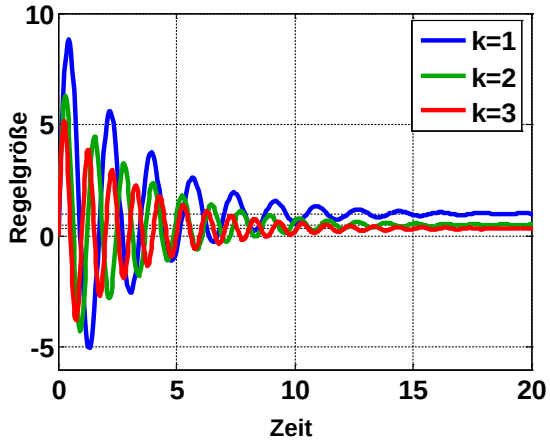
Blockschaltbild:



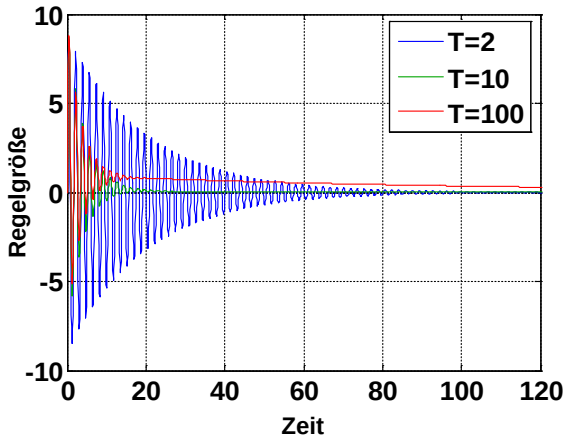
Beispiel: Behälterkaskade

Regelungsqualität mit verschiedenen Reglern: $F_{ein2}(t) = 1 \cdot \sigma(t),\ k_V = 1,\ T_1 = 2{,}5,$ $k_P = 162,\ T_2 = 5$

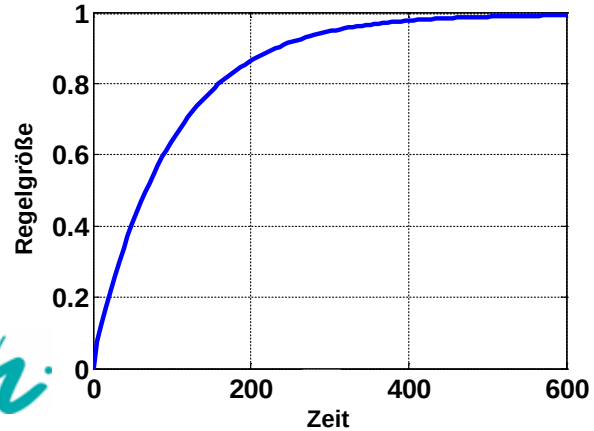
P-Regler: $k_R = 1,\ 2,\ 3$



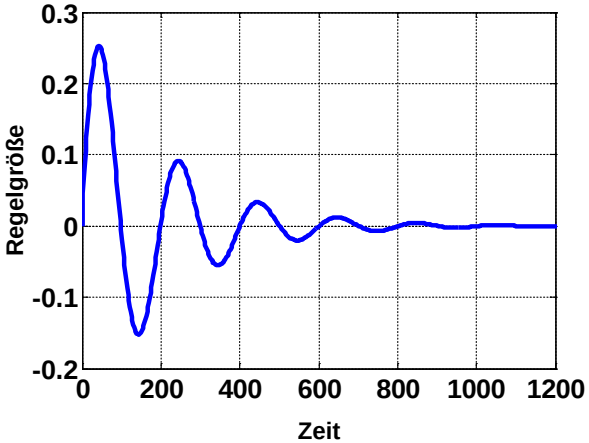
PI-Regler: $k_R = 1, T_N = 2,\ 10,\ 100$



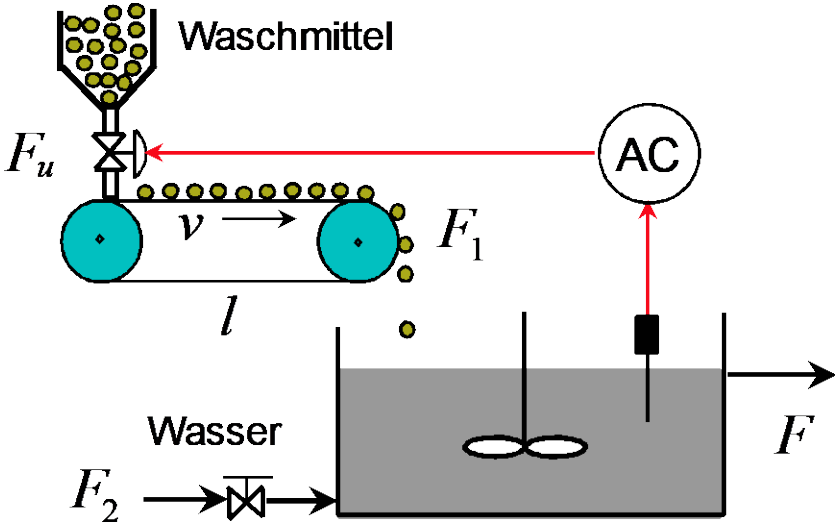
PD-Regler: $k_R = 1, T_V = 100$

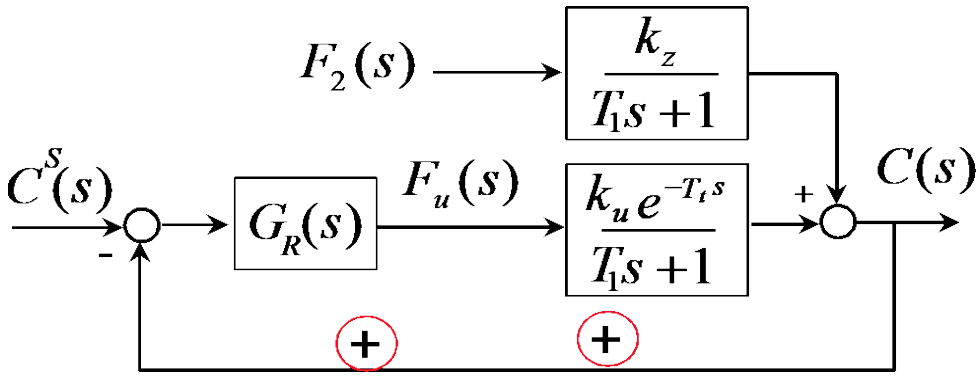


PID-Regler: $k_R = 1, T_N = 10, T_V = 100$



Beispiel: Regelung des Mischungsprozesses



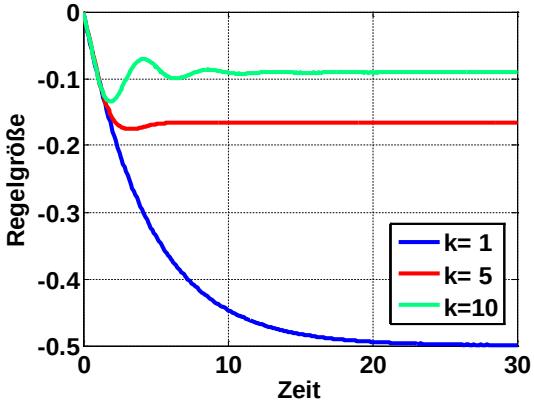


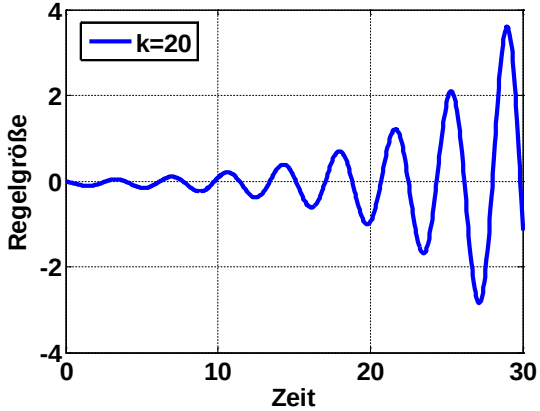
$$T_1 = 3\text{min},\ T_t = 1\text{min},\ k_u = 1,\ k_z = -1$$

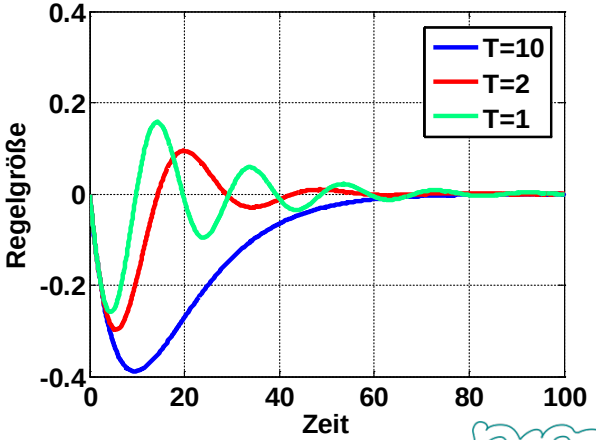
Aufgrund der Totzeit ist das geschlossene System instabil!

P-Regler: $k_R = 1,\quad 5,\quad 10,\quad 20$

PI-Regler: $k_R = 0{,}1,\ T_N = 10,\ 2,\ 1$

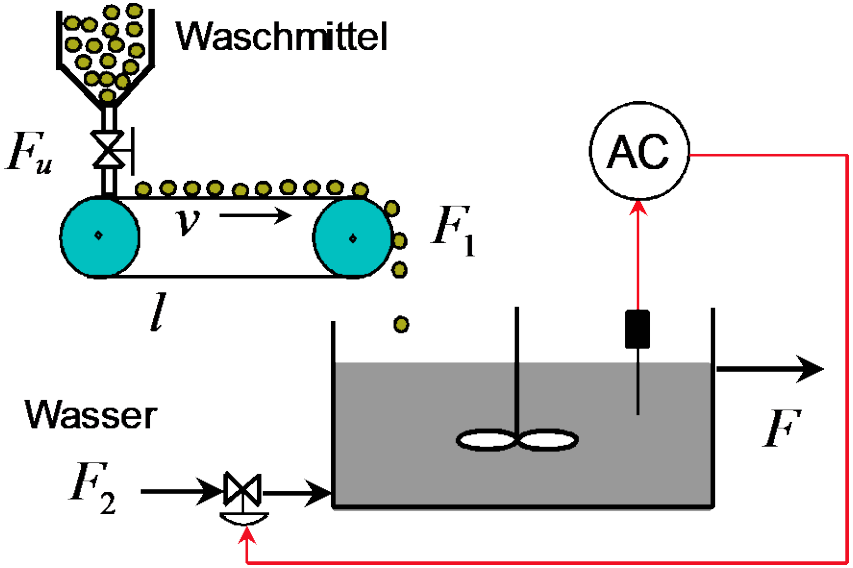


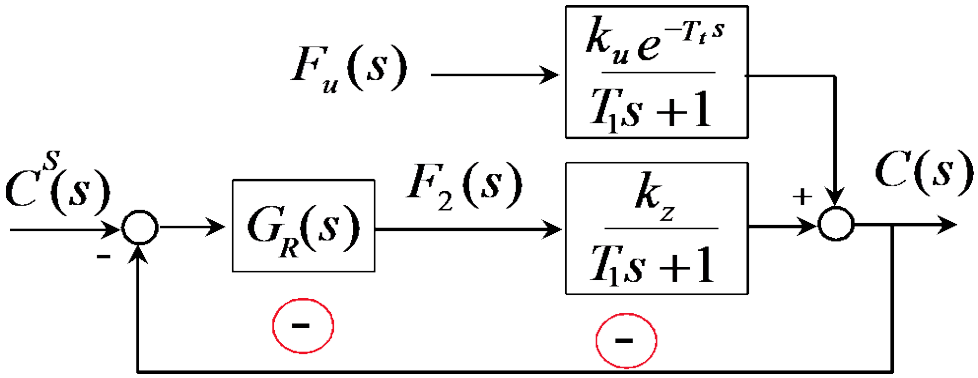




Fachgebiet Prozessoptimierung

Beispiel: Regelung des Mischungsprozesses

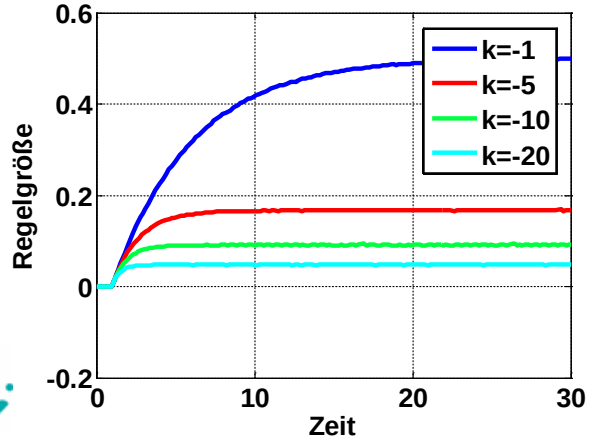




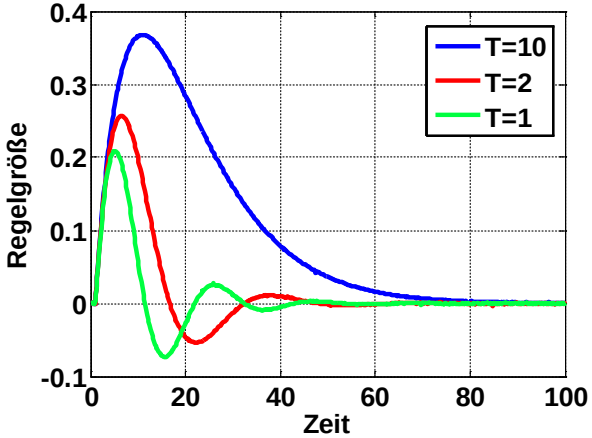
$$T_1 = 3\text{min},\ T_t = 1\text{min},\ k_u = 1,\ k_z = -1$$

Die Totzeit ist nicht innerhalb des Regelkreises. Das System ist stabil!

P-Regler $k_R = -1, -5, -10, -20$

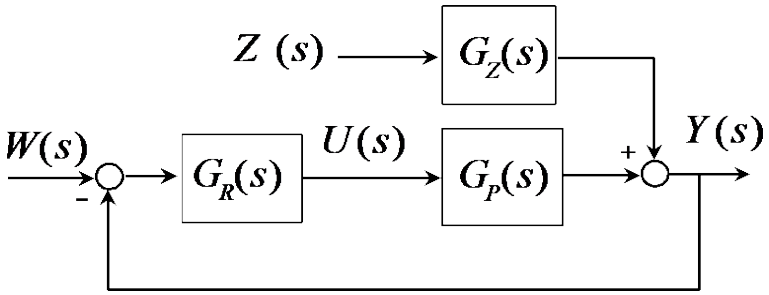


PI-Regler: $k_R = -1, T_N = 1, 2, 10\ \text{min}$



Fachgebiet Prozessoptimierung

Forderung an die stationäre Genauigkeit



$$E(s) = W(s) - Y(s)$$

$$e(\infty) = \lim_{t\to\infty} e(t) = \lim_{s\to 0} sE(s)$$

Regelabweichung bei Führung ($Z(s) = 0$):

$$\frac{E(s)}{W(s)} = 1 - \frac{Y(s)}{W(s)} = 1 - \frac{G_R(s)G_P(s)}{1 + G_R(s)G_P(s)} = \frac{1}{1 + G_R(s)G_P(s)}$$

$$E(s) = \frac{1}{1 + G_R(s)G_P(s)} W(s)$$

$$e(\infty) = \lim_{t\to\infty} e(t) = \lim_{s\to 0} sE(s) = \lim_{s\to 0} \frac{s}{1 + G_R(s)G_P(s)} W(s) = 0$$

Bei einem Sprung: $e(\infty) = \dfrac{1}{1 + G_R(0)G_P(0)}$

Ein Integrator wird benötigt, um den stationären Fehler zu beseitigen.

Forderung an die stationäre Genauigkeit

Regelabweichung bei Störung ($W(s)=0$):

$$E(s)=W(s)-Y(s)=-Y(s)=-\frac{G_z(s)}{1+G_R(s)G_P(s)}Z(s)$$

$$y(\infty)=\lim_{t\to\infty}y(t)=\lim_{s\to 0}sY(s)=\lim_{s\to 0}\frac{sG_z(s)}{1+G_R(s)G_P(s)}Z(s)=0$$

Bei einer Sprung:

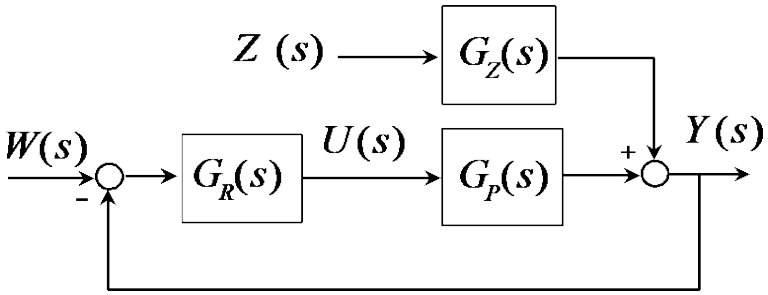
$$y(\infty)=\frac{G_z(0)}{1+G_R(0)G_P(0)}$$

Für den Fall $G_P(s)=G_z(s)=\frac{k_P}{s}$ ist ein P-Regler $G_R(s)=k_R$

nicht ausreichend, weil

$$y(\infty)=\lim_{s\to 0}\frac{G_z(s)}{1+G_R(s)G_P(s)}=\lim_{s\to 0}\frac{\frac{k_P}{s}}{1+k_R\frac{k_P}{s}}=\lim_{s\to 0}\frac{k_P}{s+k_Rk_P}=\frac{1}{k_R}\neq 0$$

Stabilität eines Regelkreises



Das charakteristische Polynom:

$$f(s) = 1 + G_R(s)G_P(s)$$

Die Nullstellen dieses Polynoms sind Polstellen des geschlossenen Systems:

$$f(s) = 1 + G_R(s)G_P(s) = 0$$

Es gibt 6 Möglichkeiten:

$$p = \sigma$$

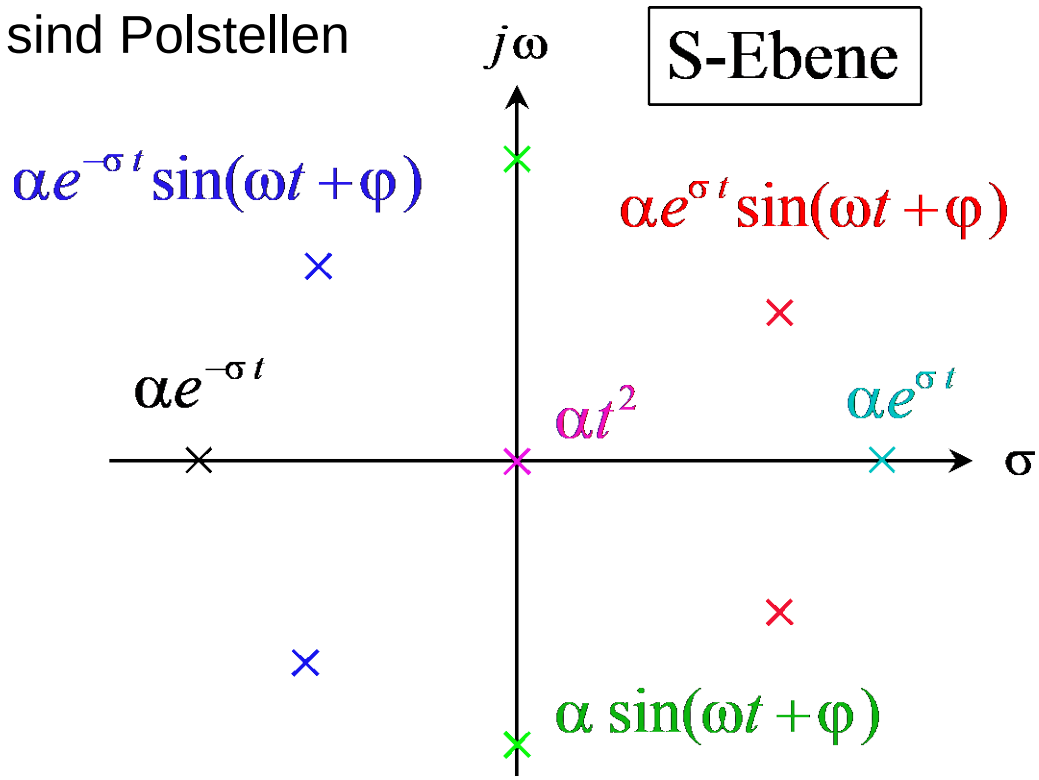
$$p = -\sigma$$

$$p = \sigma \pm j\omega$$

$$p = -\sigma \pm j\omega$$

$$p = 0$$

$$p = \pm j\omega$$



Stabilitätsprüfung nach dem Routh-Kriterium

Die charakteristische Gleichung des geschlossenen Systems:

$$f(s) = a_0 s^n + a_1 s^{n-1} + \cdots + a_{n-1} s + a_n = 0$$

Bei einem stabilen System liegen die Nullstellen der charakteristischen Gleichung bzw. die Polstellen des geschlossenen Systems in der linken s-Halbebene.

Welche Bedingungen muss das Polynom $f(s)$ haben?

J. C. Maxwell löste das Problem bis zur dritten Ordnung und schrieb 1875 einen Preis mit der Aufgabe einer allgemeinen Lösung aus.

E. J. Routh gewann den Preis 1877 mit dem folgenden Satz:

(1) Alle Koeffizienten $a_0, a_1, \cdots, a_{n-1}, a_n$ müssen positiv sein.

(2) Die Elemente der ersten Spalte der Routh-Matrix müssen positiv sein.

(3) Die Anzahl der Vorzeichenwechsel dieser Elemente ist gleich die Anzahl der Nullstellen des charakteristischen Polynoms in der rechten s-Halbebene.

E. J. Routh, *Dynamik der Systeme starrer Körper*, *Band 2: Die höhere Dynamik*, Teubner-Verlag, 1898.

Die Routh-Matrix

Die charakteristische Gleichung des geschlossenen Systems:

$$f(s) = a_0 s^n + a_1 s^{n-1} + \cdots + a_{n-1} s + a_n = 0$$

$$\begin{array}{c|cccccc}
s^n & a_0 & a_2 & a_4 & a_6 & \cdots \\
s^{n-1} & a_1 & a_3 & a_5 & a_7 & \cdots \\
s^{n-2} & b_1 & b_2 & b_3 & b_4 & \cdots \\
s^{n-3} & c_1 & c_2 & c_3 & c_4 & \cdots \\
s^{n-4} & d_1 & d_2 & d_3 & d_4 & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots
\end{array}$$

wobei

$$b_1 = \frac{a_1 a_2 - a_0 a_3}{a_1}, \quad b_2 = \frac{a_1 a_4 - a_0 a_5}{a_1}, \quad b_3 = \frac{a_1 a_6 - a_0 a_7}{a_1}, \quad \cdots$$

$$c_1 = \frac{b_1 a_3 - a_1 b_2}{b_1}, \quad c_2 = \frac{b_1 a_5 - a_1 b_3}{b_1}, \quad c_3 = \frac{b_1 a_7 - a_1 b_4}{b_1}, \quad \cdots$$

$$d_1 = \frac{c_1 b_2 - b_1 c_2}{c_1}, \quad d_2 = \frac{c_1 b_3 - b_1 c_3}{c_1}, \quad d_3 = \frac{c_1 b_4 - b_1 c_4}{c_1}, \quad \cdots$$

Beispiele:

(1) $$f(s) = s^4 + 7s^3 + 18s^2 + 21s + 10$$

s^4	1	18	10
s^3	7	21	
s^2	$\frac{105}{7}$	10	
s^1	$\frac{1715}{105}$		

Das System ist stabil.

$$s_1 = -2{,}00$$

$$s_2 = -2{,}75$$

$$s_{3,4} = -1{,}12 \pm j0{,}74$$

(2) $$f(s) = s^4 + 2s^3 + 3s^2 + 4s + 5$$

s^4	1	3	5
s^3	2	4	
s^2	1	5	
s^1	-6		
s^0	5		

Das System ist instabil. Es gibt zweimal einen Wechsel des Vorzeichens.

$$s_{1,2} = -1{,}28 \pm j0{,}86$$

$$s_{3,4} = 0{,}29 \pm j1{,}42$$

Beispiele:

(3) $f(s) = s^5 + 2s^4 + 3s^3 + 6s^2 + 10s + 15$

s^5	1	3	10
s^4	2	6	15
s^3	$0 = \varepsilon$	$5/2$	
s^2	$\dfrac{6\varepsilon - 5}{\varepsilon}$	15	
s^1	$\dfrac{30\varepsilon - 25 - 30\varepsilon^2}{12\varepsilon - 10}$	15	
s^0	15		

$\Rightarrow$

s^5	1
s^4	2
s^3	ε
s^2	$\dfrac{-5}{\varepsilon}$
s^1	$\dfrac{25}{10}$
s^0	15

Das System ist instabil.

$$s_{1,2} = 0{,}83 \pm j1{,}53$$

$$s_3 = -1{,}84$$

$$s_{3,4} = -0{,}91 \pm j1{,}37$$

(4) $f(s) = s^3 + 10s^2 + 16s + 160$

s^3	1	16
s^2	10	160
s^1	0	0

Dann gibt es

$$s^2 + 16 = 0$$

$$s_{1,2} = \pm j4$$

$$s_3 = -10$$

Beispiele:

(5) $$G_P(s) = \frac{1}{s(s-1)(s^2+4s+20)}$$

Stabilisierung des Systems mit einem PD-Regler:

$$G_0(s) = \frac{k_R(s+1)}{s(s-1)(s^2+4s+20)}$$

Die charakteristische Gleichung: $$s^4 + 3s^3 + 16s^2 + (k_R - 20)s + k_R = 0$$

s^4	1	16	k_R
s^3	3	$k_R - 20$	0
s^2	$\frac{3\cdot 16-(k_R-20)}{3}$	$\frac{3k_R}{3}$	
s	$\frac{(68-k_R)(k_R-20)-9k_R}{\frac{68-k_R}{3}}$		

$\Rightarrow$

$$3\cdot 16-(k_R-20)>0 \quad\Rightarrow\quad 68-k_R>0 \quad\Rightarrow\quad k_R<68$$

$$(68-k_R)(k_R-20)-9k_R>0 \quad\Rightarrow\quad k_R^2-79k_R+1360<0$$

$$(k_R-25{,}35)(k_R-53{,}65)<0$$

$$25{,}35<k_R<53{,}65$$