Network Algorithms

Chapter 6
Biconnectivity Augmentation

Thomas Böhme
Technische Universität Ilmenau

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Let $G = (V, E)$ be a finite undirected graph.

- G is *connected* if for any two distinct vertices $u, v \in V$ there is a path connecting u and v in G.
- A maximal connected subgraph of G is called a *component* of G.
- G is *biconnected* if it is connected and for any vertex $v \in V$ the graph $G - v$ obtained from G by deleting v is connected too.
- A maximal biconnected subgraph of G is called a *block* of G.
- A block B of G is called *isolated* if it is also a component of G. B is called an *end block* if it contains exactly one articulation.
- A vertex $v \in V$ is said to be an *articulation* if $G - v$ has more components than G.

Let $G = (V, E)$ be a finite undirected graph and $v \in V$.

- $c = c(G)$ ... number of components of G
- $p = p(G)$ ... number of end blocks of G
- $q = q(G)$ ... number of isolated blocks of G
- $d(v : G)$ ... number of components of $G - v$
- $d = d(G) = \max\{d(v : G) \mid v \in V\}$

Remark. Note that $p = 0$ implies $d = 1$, and $p > 0$ implies $p \geq 2$.

Proposition (1)

Let $= (V, E)$ be a finite undirected graph with c components. The minimal number of additional edges needed to make G connected is $c - 1$.

Proposition (2)

Let $G = (V, E)$ be a finite undirected graph and $E' \subseteq \binom{V}{2}$ such that G is not biconnected but $G' = (V, E \cup E')$ is biconnected. Then $|E'| \geq \max\{d - 1, q + \lceil \frac{p}{2} \rceil\}$.

Proof.

(i) Let $v \in V$ be a vertex of G and $A_1, \ldots, A_{d(v:G)}$ the components of $G - v$. It follows from Proposition (1) that E' contains $d(v : G) - 1$ edges connecting the components of $G - v$. Hence, $|E'| \geq d - 1$.

(ii) Any isolated block of G is incident with at least two distinct edges in E' and any end block is incident with at least one edge in E'. Thus $2|E'| \geq 2q + p$ and so, $|E'| \geq q + \lceil p/2 \rceil$. □

Proposition (3)

Let $G = (V, E)$ be a finite undirected graph that is not biconnected. Then there is a set $E' \subseteq \binom{V}{2}$ such that $G' = (V, E \cup E')$ is biconnected and $|E'| = h(G) = \max\{d - 1, q + \lceil p/2 \rceil\}$.

Proof.

The proof is by induction on $h(G)$. We let $p' = p(G')$, $q' = q(G')$, $d' = d(G')$, and $h' = h(G')$.

If $h(G) = 1$, then $q + \lceil p/2 \rceil \leq 1$. If $p = 0$, then $d = 1$, and $q = 1$. This contradicts the assumption that G is not biconnected. Hence, $p = 2$ and $q = 0$. Let A, B be the two end blocks of G. Adding an edge that connects a non articulation vertex in A with one in B results in a biconnected graph. This establishes the base case.

For the inductive step we prove that if $h(G) \geq 2$, then there is an edge $e \in \binom{V}{2}$ such that $h(G') = h(G) - 1$ for $G' = (V, E \cup \{e\})$.

Case 1: G is not connected.

Let A, B be two distinct components of G and $a \in V(A), b \in V(B)$ such that

- a and b are not articulations,
- if A is not an isolated block, then a is contained in an end block, and
- if B is not an isolated block, then b is contained in an end block.

Case 1.1: $p = 0$

- $d = q$ unless G is edgeless, in which case $d = q - 1$. Consequently, $d - 1 < q + \lceil p/2 \rceil$.
- $q' \leq q$
- If $q' = q - 1$, then $p' = p = 0$.
- If $q' = q - 2$, then $p' = p + 2 = 2$.

In either case $q' + \lceil p'/2 \rceil = q + \lceil p/2 \rceil - 1$. Hence $h' = h - 1$.

Case 1.2: $p > 0$

- There is an articulation x such that $d(x : G) = d$.
- It follows $d' = d - 1$.

As in case 1.1, $q' + \lceil p'/2 \rceil = q + \lceil p/2 \rceil - 1$. Hence $h' = h - 1$.

Case 2: G is connected. Since G is not biconnected, $q = 0$. Thus $h = h(G) = \max\{d-1, \lceil p/2 \rceil\}$. Furthermore, if G' is obtained from G by adding an arbitrary edge, then $q(G') = q = 0$.

- If $d - 1 < \lceil p/2 \rceil$, then adding an edge connecting two non articulation vertices in two distinct end blocks of G results in a graph G' with $p' = p(G') = p - 2$. It follows $h' = h - 1$.

For the remaining case $d - 1 \geq \lceil p/2 \rceil$ some preparation is needed.

Let u be an articulation of G and $A_1, \ldots, A_k$ the components of $G - u$. If F is an end block of G, then there is exactly one component A_i of $G - u$ such that A_i contains $F - u$. Conversely, for any component A_i there is at least one end block F such that A_i contains $F - u$. Let a_i denote the number of end blocks F such that A_i contains $F - u$. Clearly, $a_1 + \cdots + a_k = p$ and $a_i \geq 1$ for all $i = 1, \ldots, k$.

Suppose that $d(u : G) - 1 = d - 1 \geq \lceil p/2 \rceil$. Since $a_1 + \cdots + a_d = p$ and $a_i \geq 1$ for all $i = 1, \ldots, d$, it follows that $\max\{a_i \mid i = 1, \ldots, d\} \leq p - (d - 1)$. This implies that

- $\max\{a_i \mid i = 1, \ldots, d\} \leq \lceil p/2 \rceil$, and that
- $\max\{a_i \mid i = 1, \ldots, d\} < \lceil p/2 \rceil$ if $d - 1 > \lceil p/2 \rceil$, or, if $d - 1 = \lceil p/2 \rceil$ and p is odd. Thus $\max\{a_i \mid i = 1, \ldots, d\} = \lceil p/1 \rceil$ if and only if $d - 1 = \lceil p/2 \rceil$ and p is even. We also conclude that in this case exactly one a_i is $p/2$ and all other a_i's are one.

Let y be an articulation of G different from u such that $d(y:G) = d = d(x:G)$. Assume w.l.o.g. that y is contained in A_1. Let $B_1, \ldots, B_d$ be the components of $G - y$, and let $b_1, \ldots, b_d$ be the number of end blocks F such that B_i contains $F - u$. Suppose that u is contained in B_1. Then $b_1 \geq a_2 + \cdots + a_d$ and $a_1 \geq b_2 + \cdots + b_d$. Hence, $d(y:G) \leq 1 + b_2 + \cdots + b_d \leq 1 + a_1$. Consequently, $d(y:G) - 1 = d(u:G) - 1 = d - 1 \geq \lceil p/2 \rceil$ implies that $a_1 = b_1 = \lceil p/2 \rceil$ and $d(y:G) = d(u:G) = d = \lceil p/2 \rceil$. In this case is $a_1 = b_2 + \cdots + b_d$. It follows that $B_1, \ldots, B_d$ are exactly the end blocks contained in A_1. Furthermore, it follows that $a_2 = \ldots = a_d = b_2 = \ldots = b_d = 1$ and no A_i with $i \neq 1$ can contain an articulation v with $d(v:G) - 1 = d - 1 = \lceil p/2 \rceil$. We conclude that adding an edge connecting a non articulation vertex in an end block that intersects A_1 with one in an end block intersecting A_2 (or any other A_i with $i \neq 1$) results in a graph G' with $d' = d(G') = d - 1$ and $p' = p(G') = p - 2$, i.e. $h' = h(G') = h - 1$. (Note that this is also true if u is the only articulation with $d(u:G) - 1 = d - 1 = \lceil p/2 \rceil$.) □