Introduction to set- and manifold-oriented analysis

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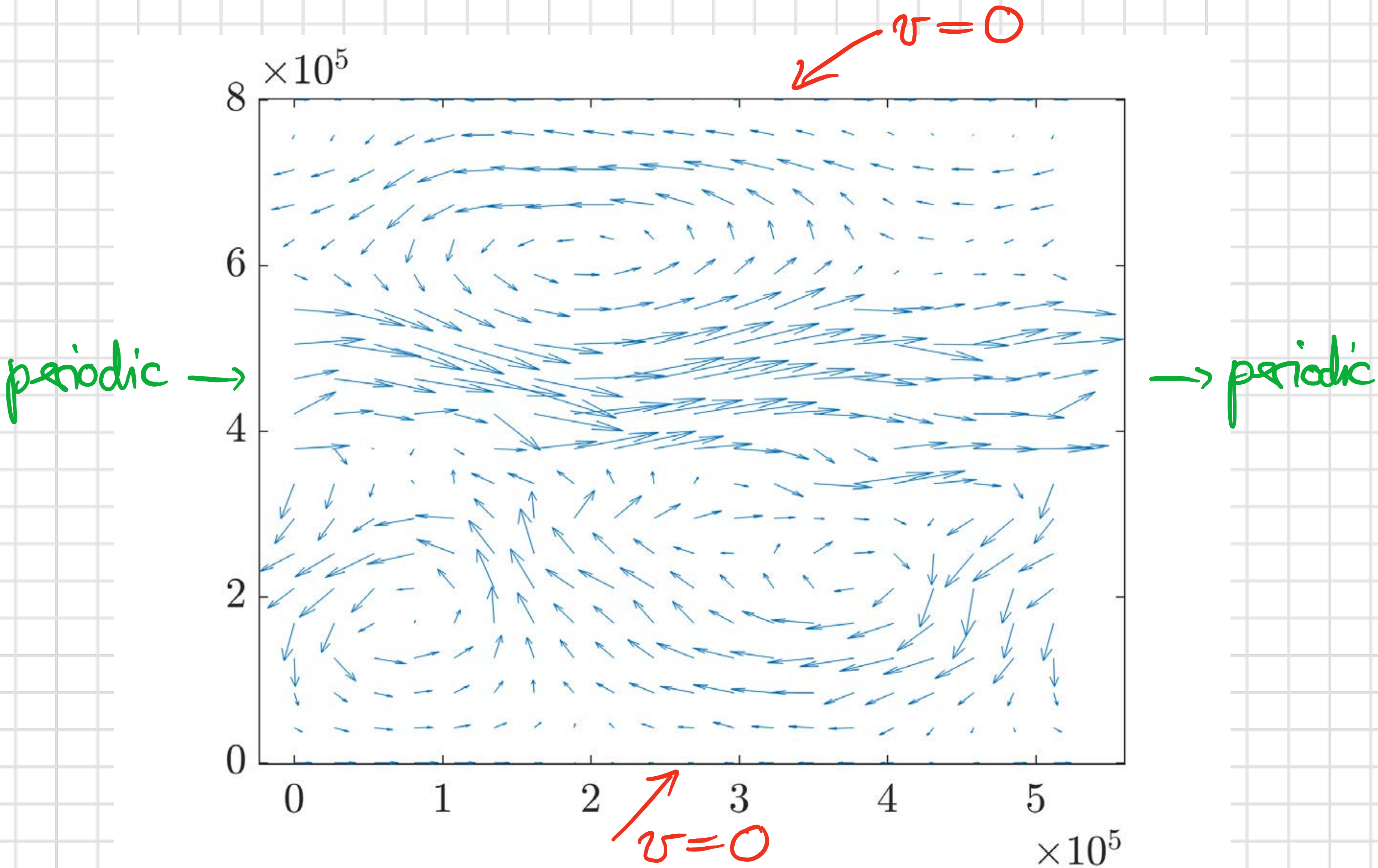
FLOW ON A CYLINDER



SETTING / NOTATION

X: domain / state space

$v(t,x)$: velocity field on X, $\operatorname{div} v = 0$

consider

$$\dot{x} = v(t,x) \quad \text{on } X$$

$\varphi^{t_0,t}: X \to X$ flow map, i.e.

solution

$$\varphi^{t_0,t}(x_0) = x(t; t_0, x_0)$$

Let us look at $v(t,x) = v(x)$ first:

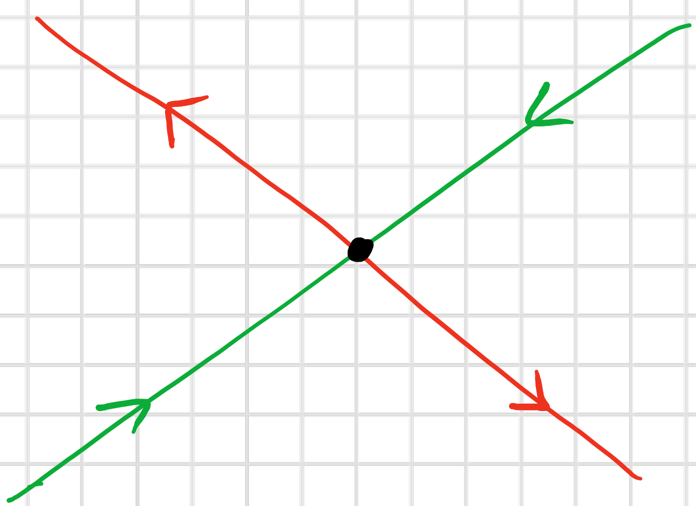
Local Picture

$\bar{x}$ equilibrium, if $v(\bar{x}) = 0$

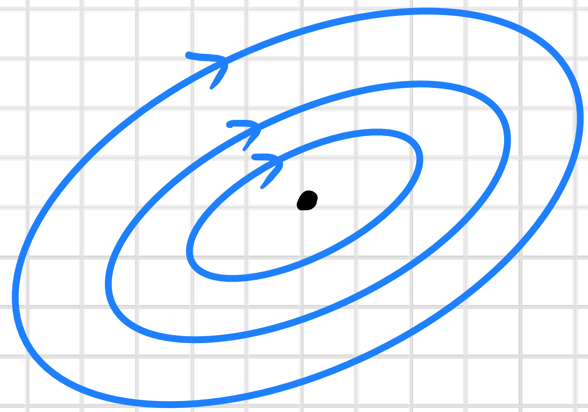
linearization at $\bar{x}$: $\quad \Delta\dot{x} = Dv(\bar{x})\Delta x$

dynamics of the linearized system:

$\bar{x}$ saddle

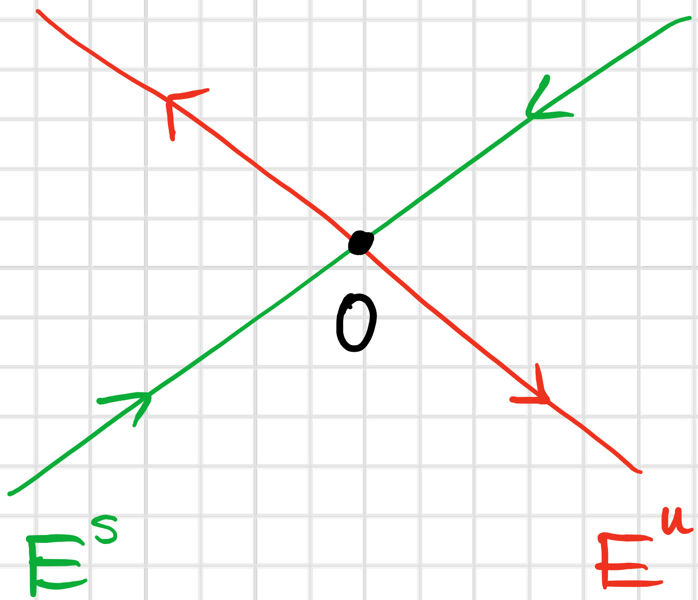


$\bar{x}$ center

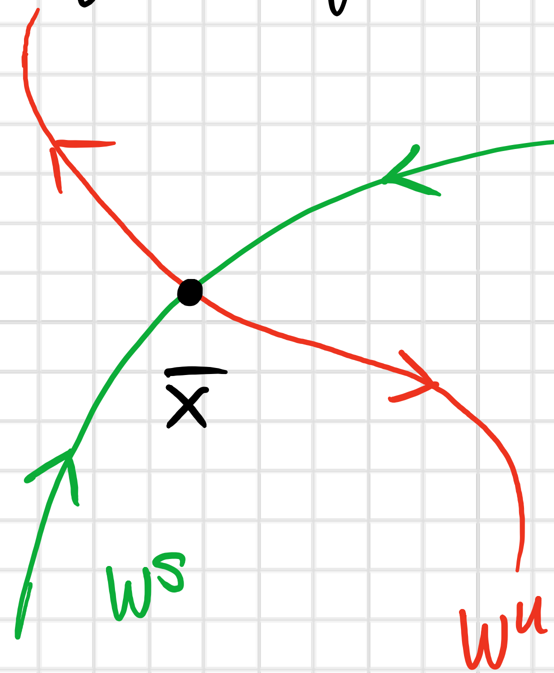


INVARIANT MANIFOLDS

linearized system



original system

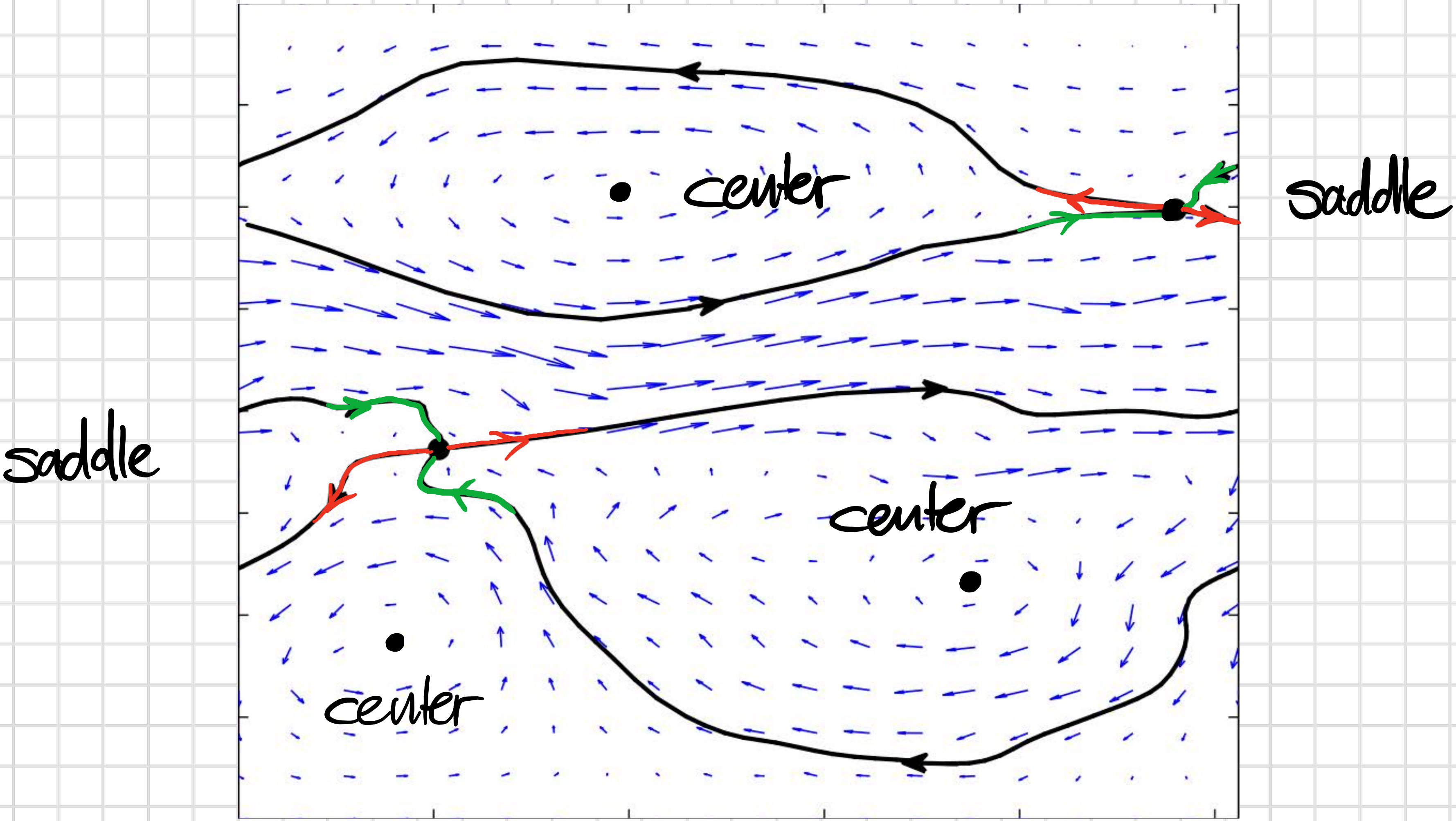


asymptotic concept:

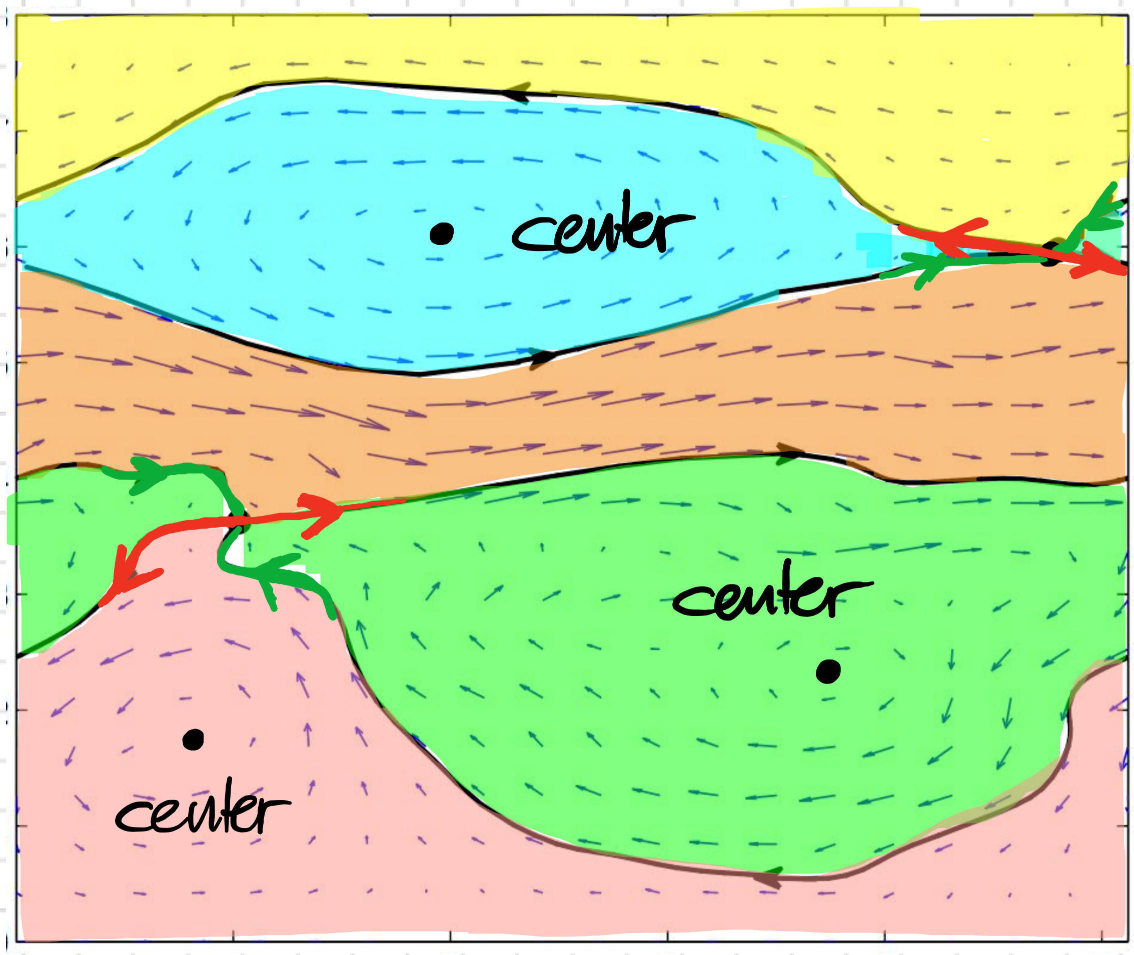
$$W^s(\bar{x}) = \{ x_0 : \varphi^t(x_0) \to \bar{x} \}$$

$$W^u(\bar{x}) = \{ x_0 \ \varphi^{-t}(x_0) \to \bar{x} \}$$

Global Picture



GLOBAL PICTURE



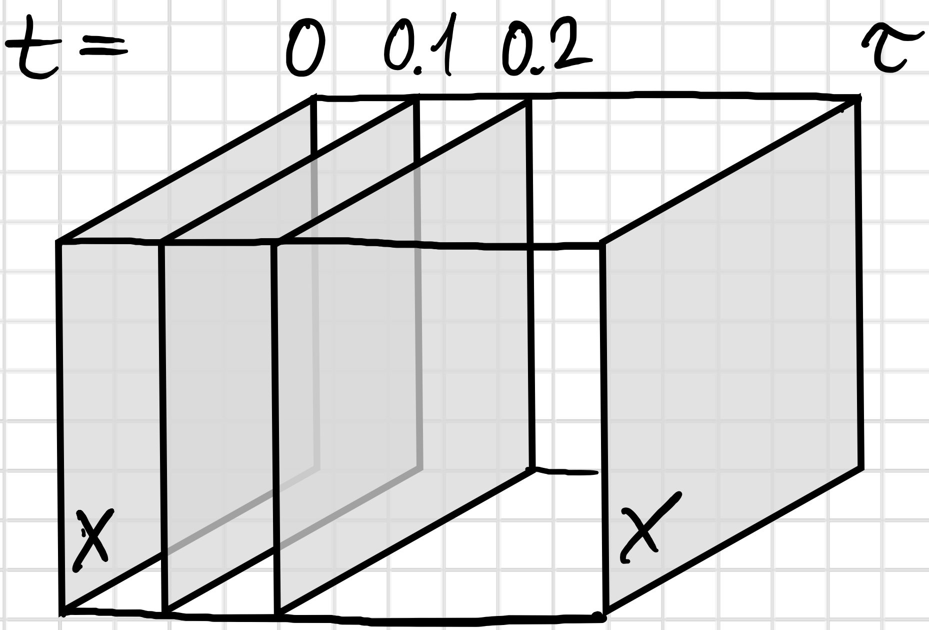
stable / unstable manifolds are transport barriers

TIME-VARIANT FLOWS

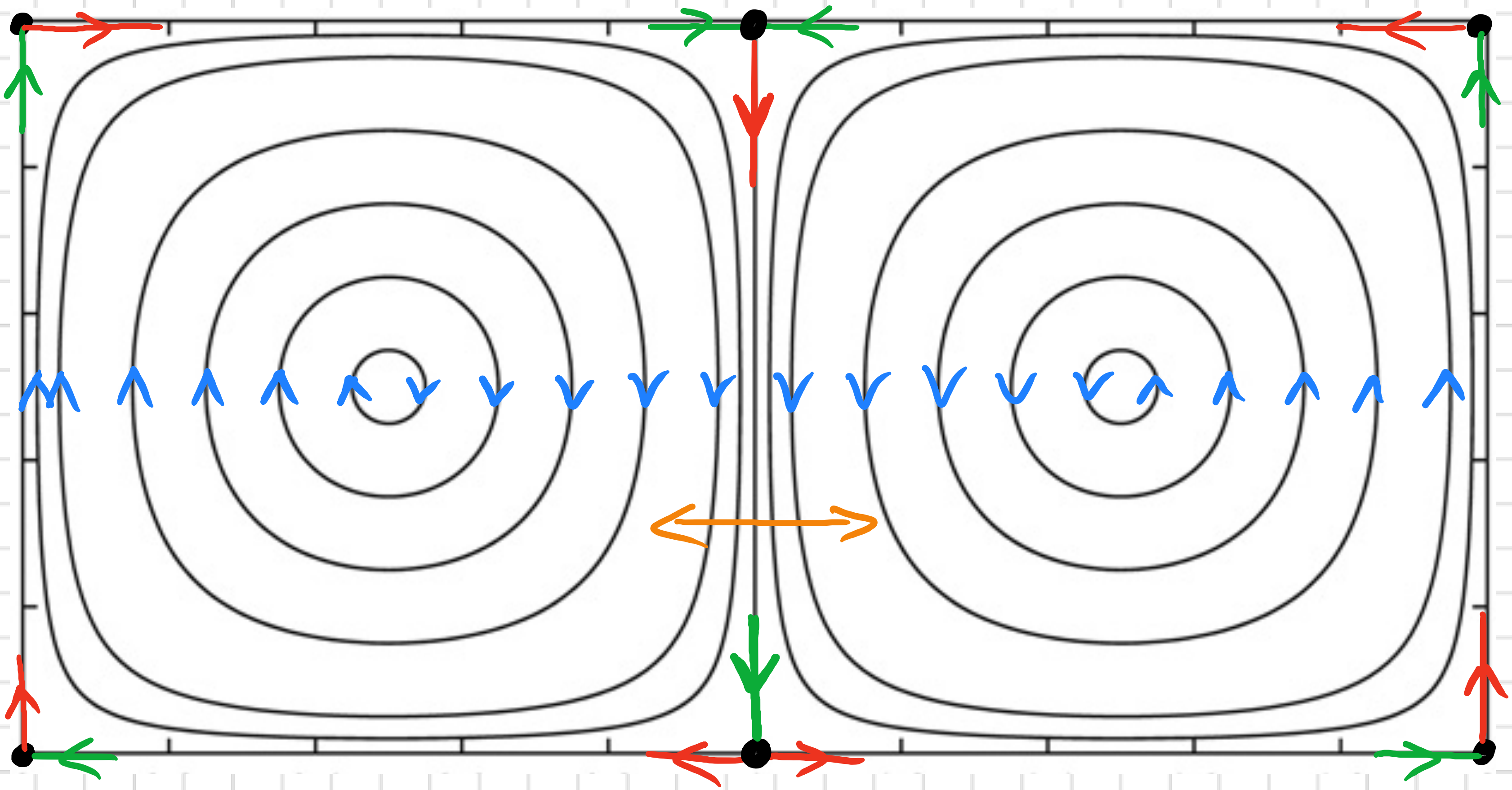
Now : $v(t, x)$

periodic : $v(t, x) = v(t + \tau, x)$

extended phase space : $\quad X \times [0, \tau]$



PROTOTYPE THE DOUBLE GYRE



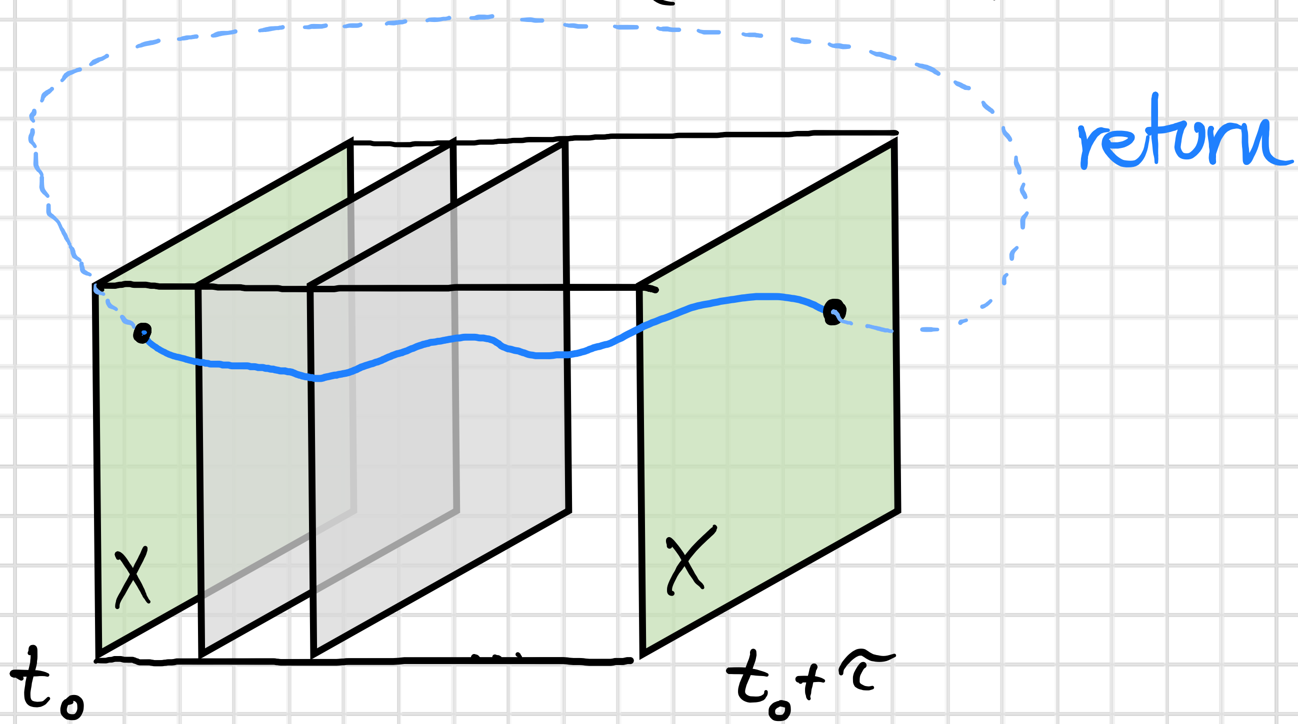
instantaneous velocity field at t_0

POINCARÉ / RETURN MAP

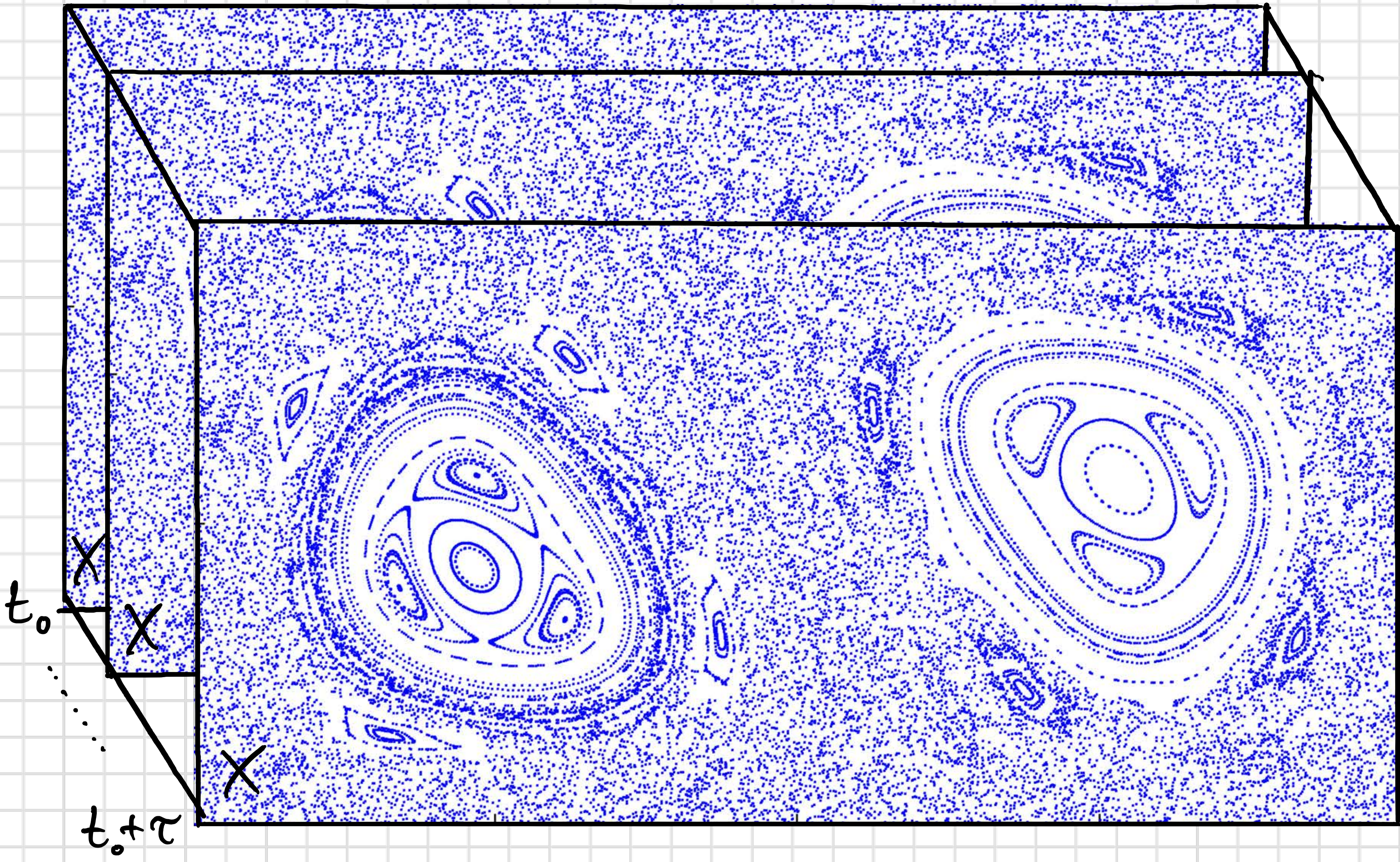
reduce analysis to the map

$$\Phi(x) := \varphi^{t_0, t_0+\tau}(x)$$

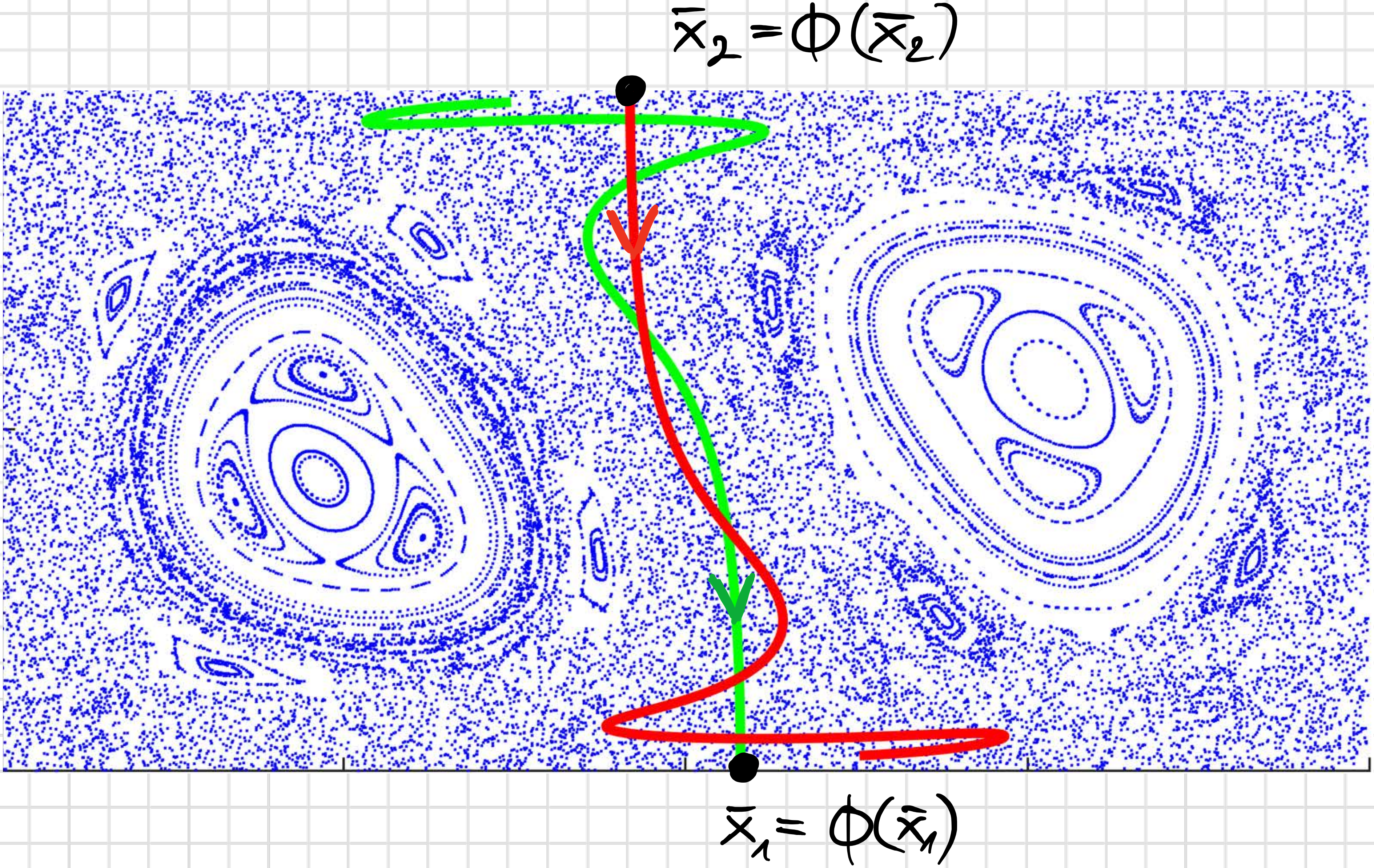
$$\Phi : X \to X \qquad (\text{„at } t_0\text{“})$$



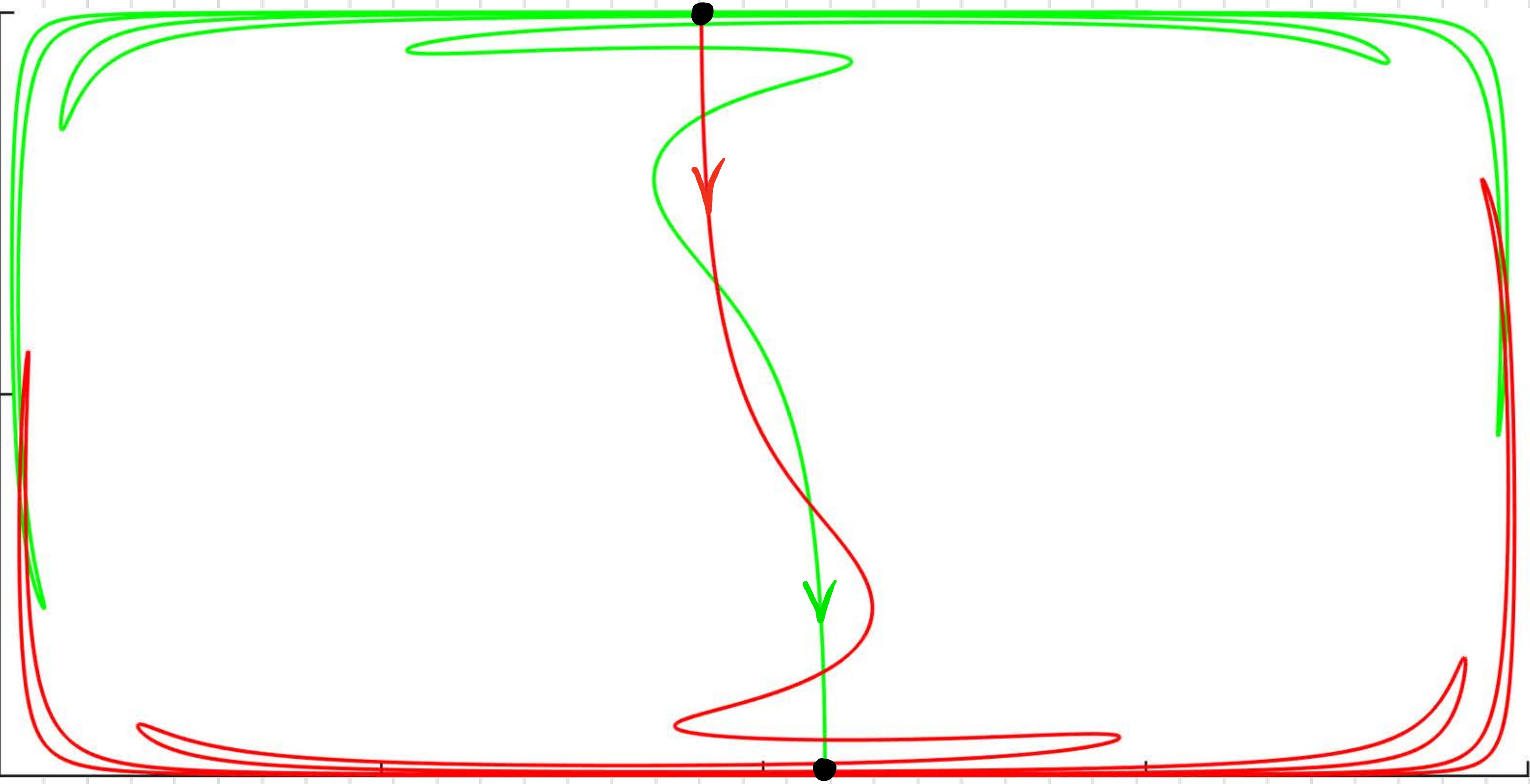
RETURN MAP FOR THE DOUBLE GYRE



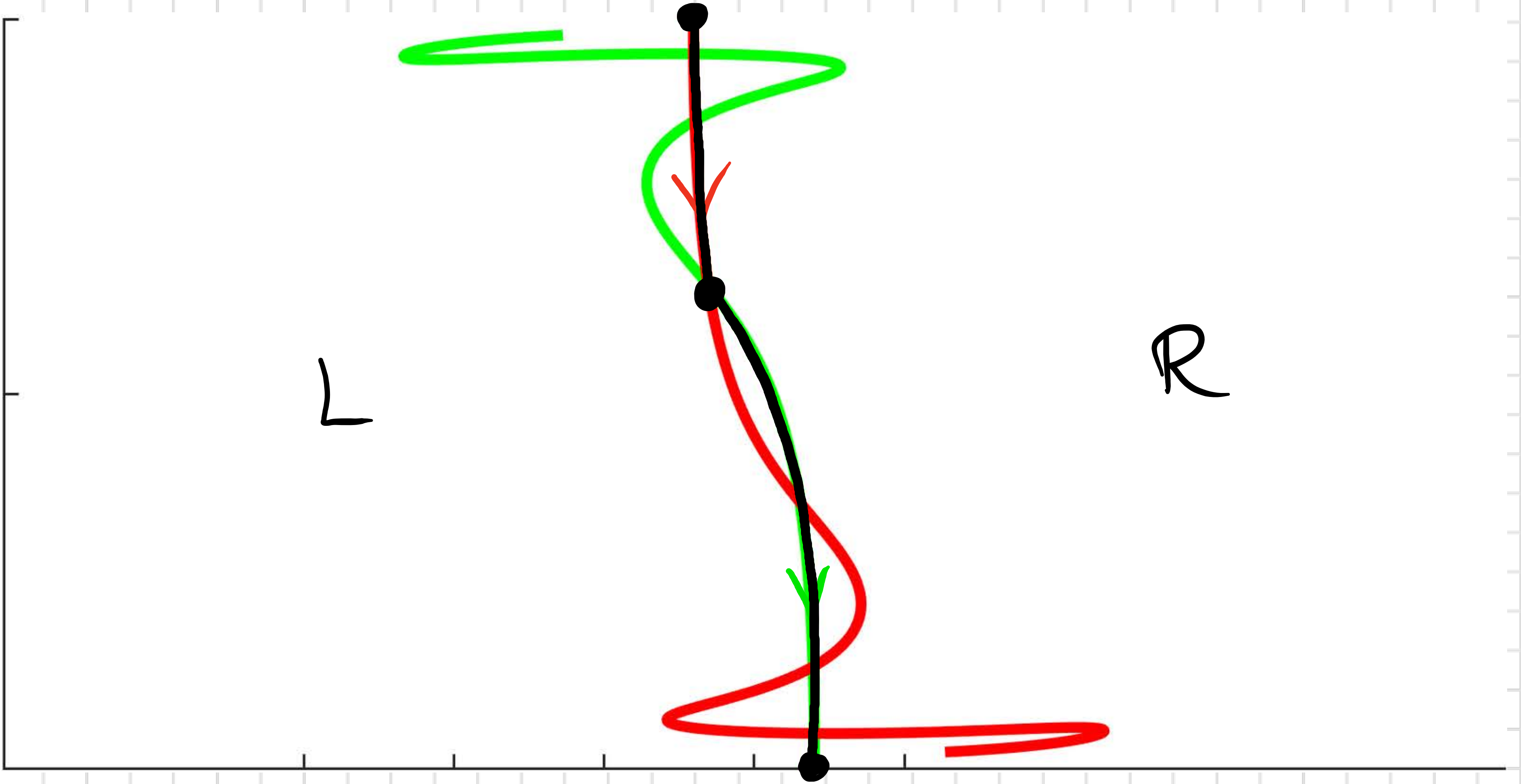
Dynamics of the Return Map



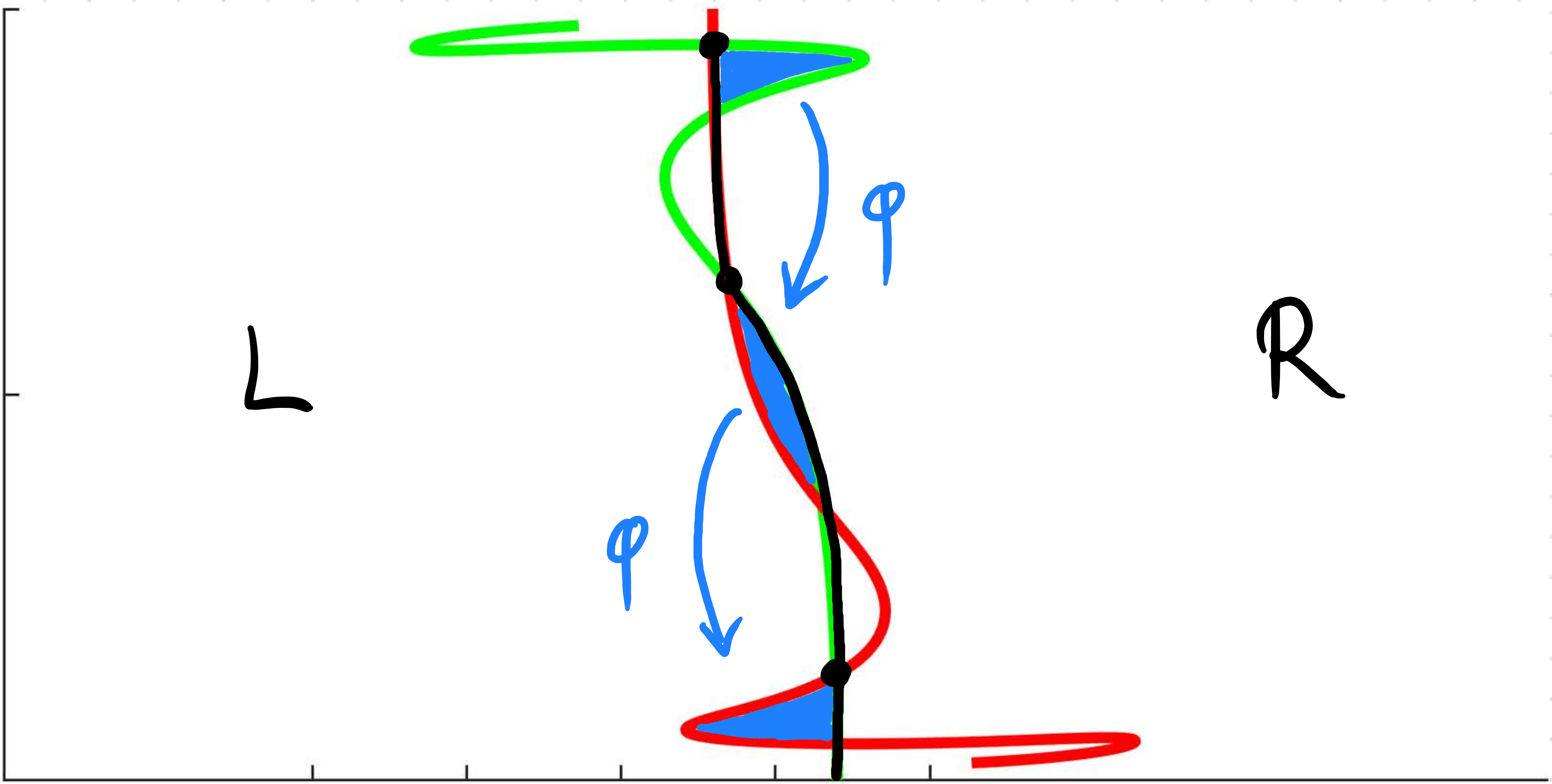
MANIFOLDS GLOBALLY



HETEROCLINIC TANGLE

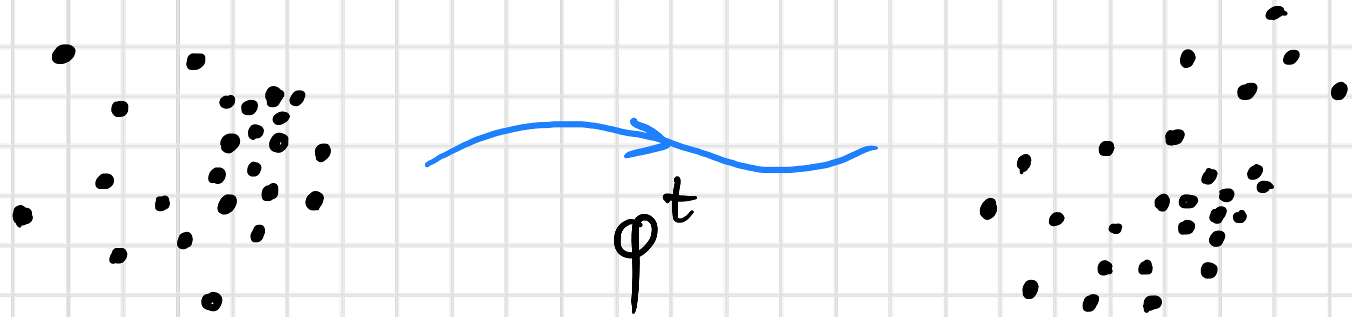


TRANSPORT R → L



„lobe dynamics": Rom-Kedar, Wiggins, Haller, Ross, Marsden, ...

Evolution of Point Distributions



assume $x_i \sim \mu$ $\qquad\qquad$ $\varphi^t(x_i) \sim \mu_* = ?$

probability measure

$$\mathrm{Prob}(x_i \in A) = \mu(A) \qquad \mathrm{Prob}(\varphi^t(x_i) \in A) = \mu_*(A)$$

$$\mu^*(A) = \mathrm{Prob}(\varphi^t(x_i) \in A) = \mathrm{Prob}(x_i \in \varphi^{-t}(A)) = \mu(\varphi^{-t}(A))$$

THE TRANSFER OPERATOR

$\mathcal{M}$ = measures on X

$$\varphi_*^t : \mathcal{M} \to \mathcal{M}$$
$$\mu \mapsto \mu \circ \varphi^{-t}$$

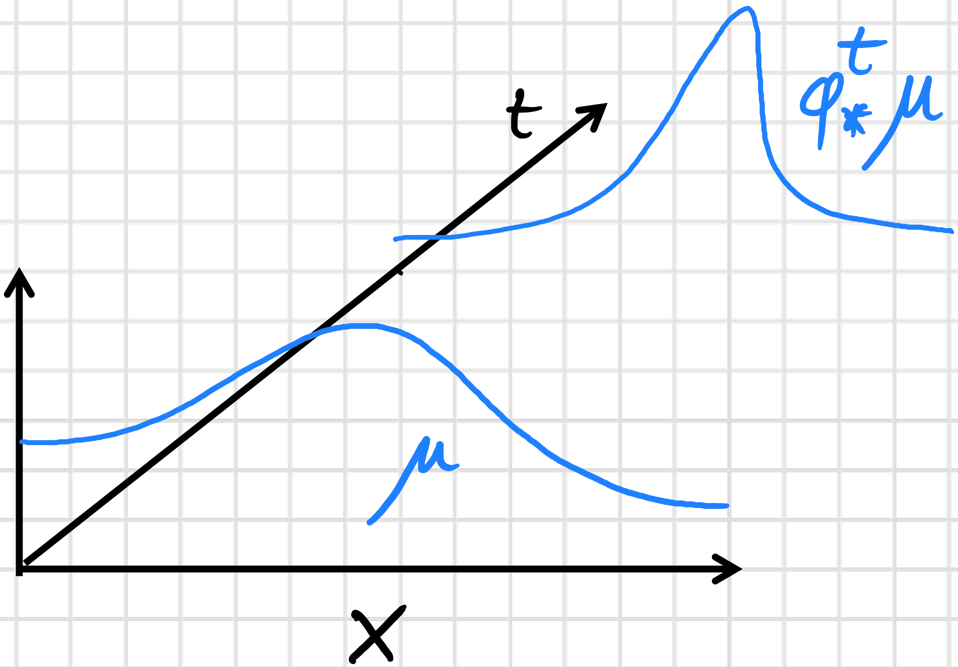
linear!

Frobenius-Perron operator

transfer operator

push forward op.

...



THE TRANSFER OPERATOR

if μ has a density (w.r.t to volume)

$$\mu(A) = \int_A u \; dm$$

then φ^t_* acts on densities like ($\operatorname{div} v = 0$)

$$\varphi^t_* : u \mapsto u \circ \varphi^{-t}$$

$\varphi^t_* u_0$ solves

$$u_t = -\operatorname{div}(uv) \; , \quad u(t,\cdot) = u_0$$

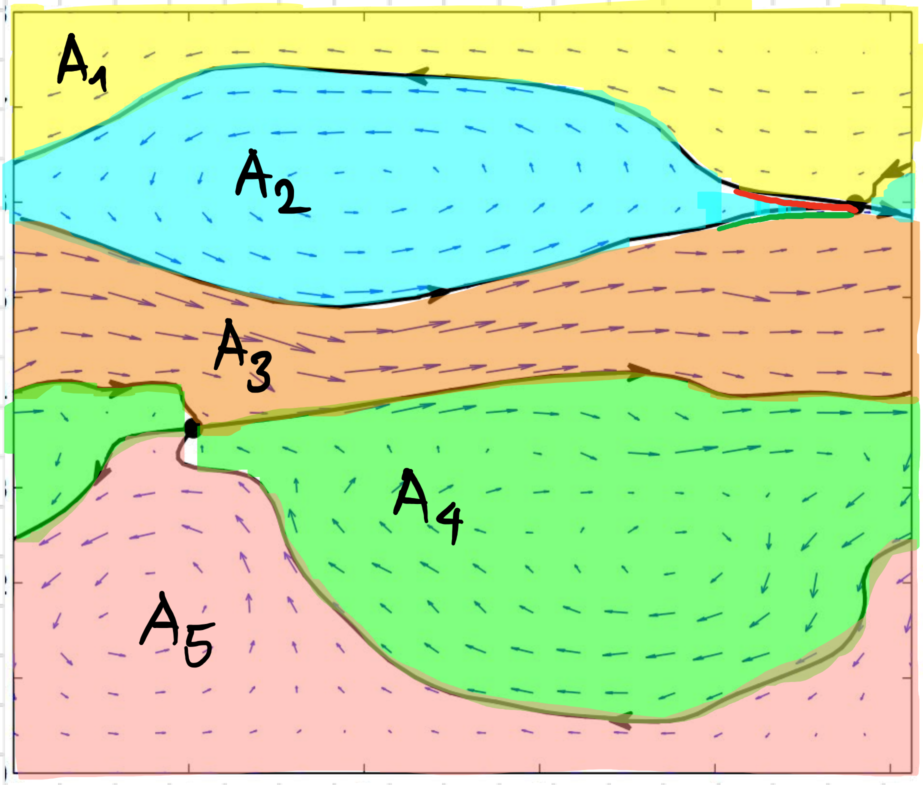
Liouville / advection equation

INVARIANT SETS AND EIGENVECTORS OF φ_*^t

flow on a cylinder: $A_1, \dots, A_5$ invariant

$$\Rightarrow \quad \varphi_*^t \mathbb{1}_{A_i} = \mathbb{1}_{A_i} \quad , \quad i = 1, \dots, 5$$

i.e. $\mathbb{1}_{A_i}$ are eigenvectors of φ_*^t at eigenvalue 1

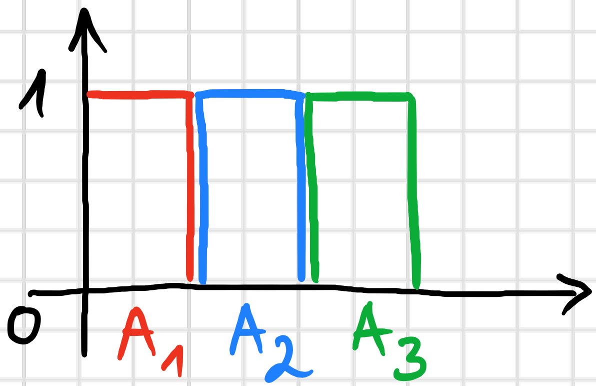


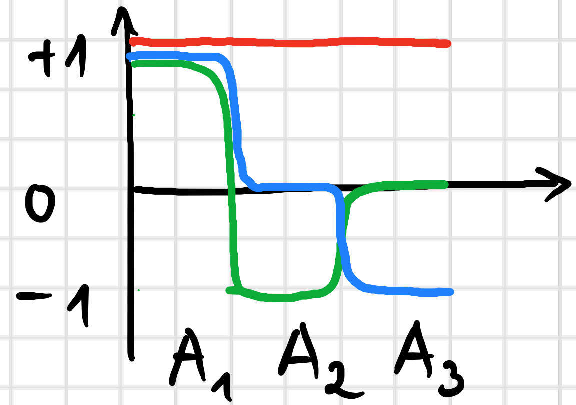
Almost Invariant Sets

φ_*^t has k-fold eigenvalue 1 $\xrightarrow{\text{perturbation}}$ φ_*^t has k eigenvalues close to 1 (including 1)

basis: $1_{A_1}, 1_{A_2}, \dots, 1_{A_{k-1}}$

basis: $u_1, u_2, \dots, u_{k-1}$

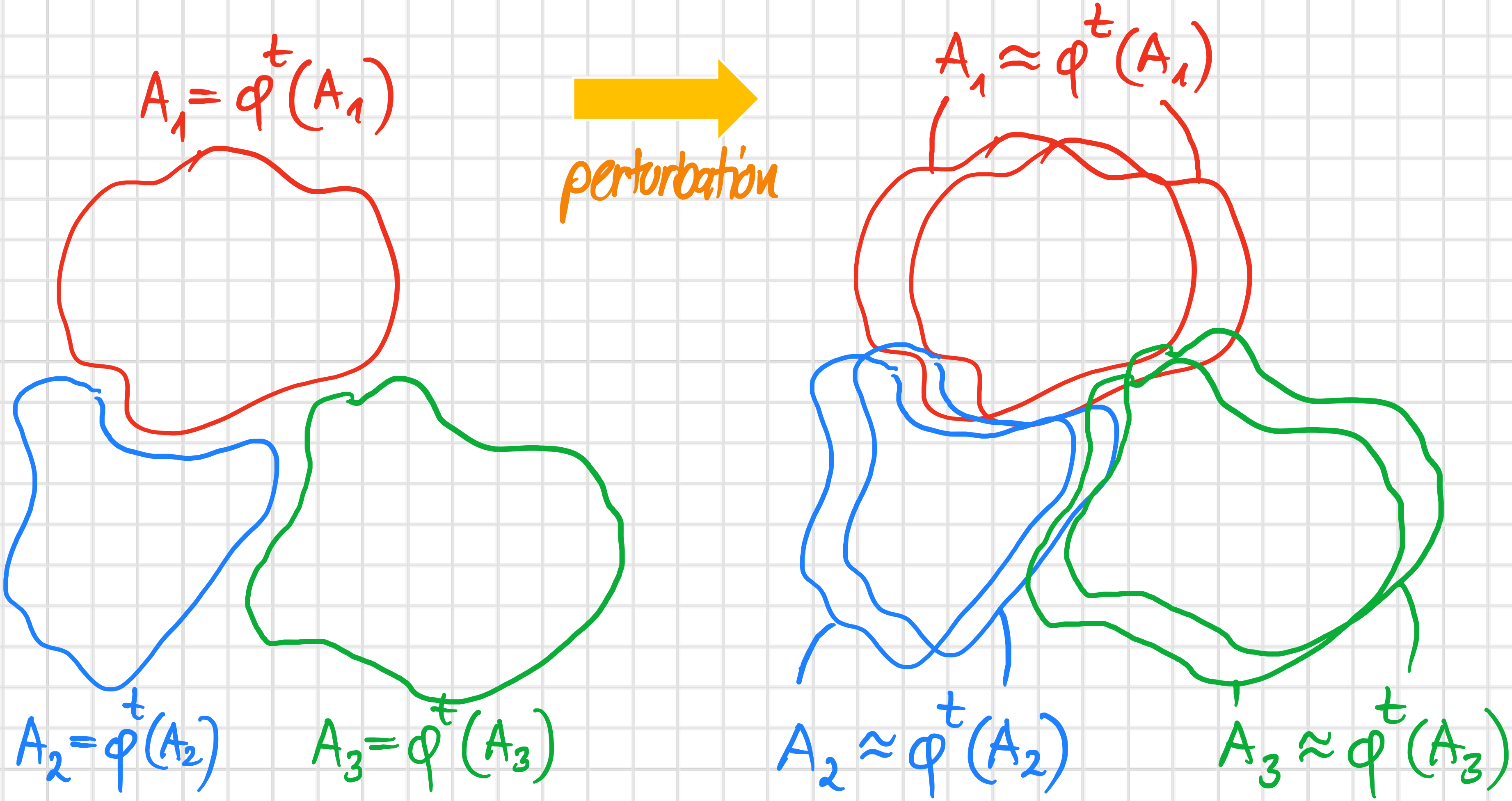




$$1_{A_1} \approx \frac{1}{3}(u_1 + u_2 + u_3)$$

Dellnitz, Junge, 99

ALMOST INVARIANT SETS



A_1, A_2, A_3 invariant

A_1, A_2, A_3 almost invariant

Dellnitz, Junge, 99

ALMOST INVARIANT SETS

$$p(A,B) = \frac{\langle \varphi_*^t 1_A, 1_B \rangle}{\langle 1_A, 1_A \rangle}$$

transition probability $A \to B$

Under certain assumptions on φ_*^t,

$$p(A_1,A_1) + \dots + p(A_n,A_n) \leq 1 + \lambda_2 + \dots + \lambda_n$$

$$1 + c_2\lambda_2 + \dots + c_n\lambda_n \leq p(A_1,A_1) + \dots + p(A_n,A_n)$$

where $c_j \approx 1$, if $u_i|_{A_i} \approx$ constant.

Dellnitz, Junge, Huisinga, Schmidt, Deuflhard, Schütte, ...

COMPUTATION

eigenproblem

$$\varphi^t_* u = \lambda u \quad , \; u \in V$$

Galerkin: choose $\tilde{V} \subset V$, finite dimensional

require $$\langle \varphi^t_* \tilde{u}, \tilde{v} \rangle = \lambda \langle \tilde{u}, \tilde{v} \rangle \quad \forall \tilde{v} \in \tilde{V}$$

special case: $\tilde{V} = \mathrm{span}\left(1_{B_1}, \dots, 1_{B_n}\right)$

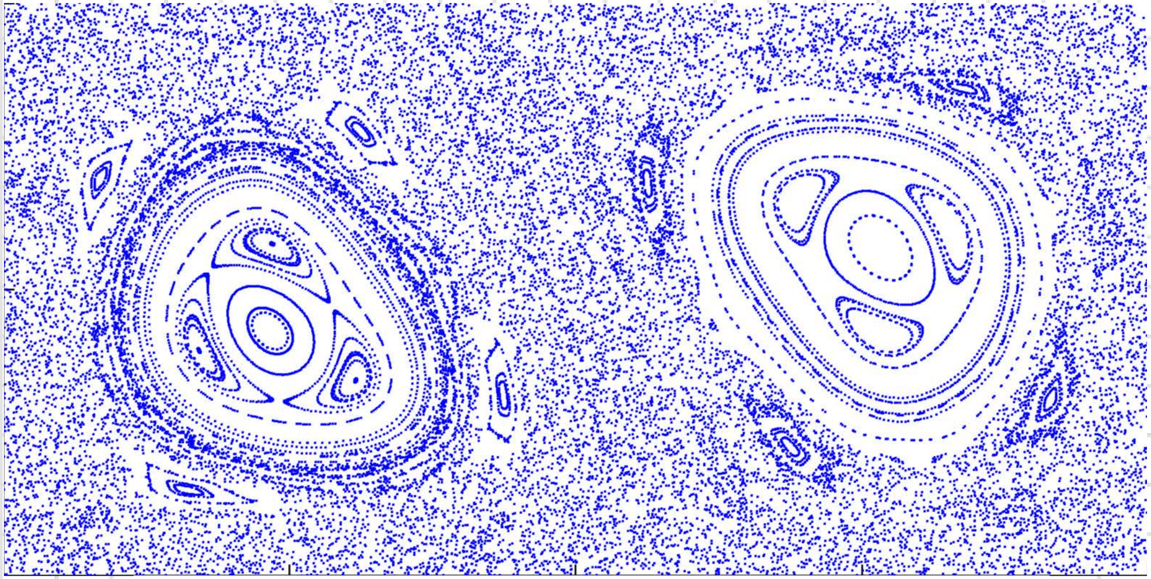
$\rightarrow$ $$P^t \tilde{u} = \lambda \tilde{u}$$

where $(P^t_{ij}) = p(B_j, B_i)$

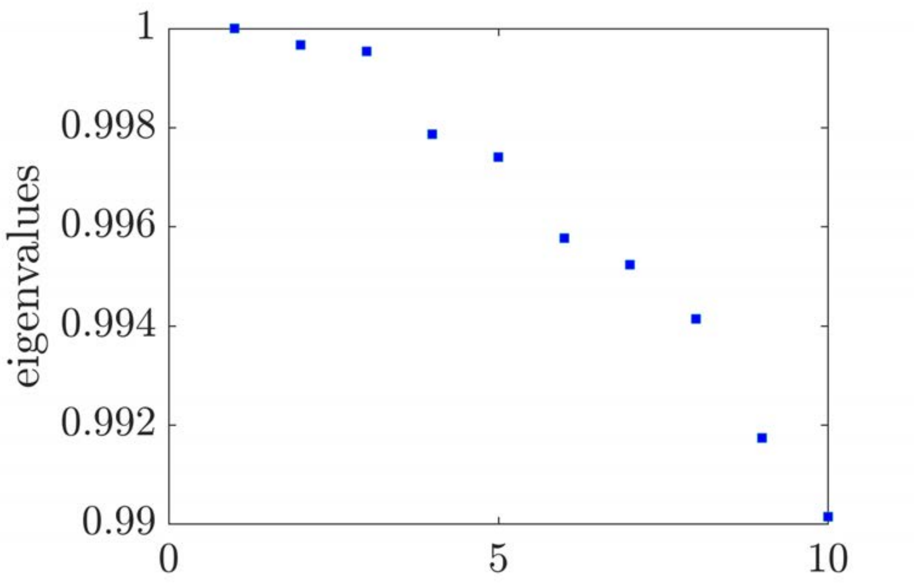
Ulam's method

Ulam, Li, Ding, Du, Boyarsky, ...

DOUBLE GYRE

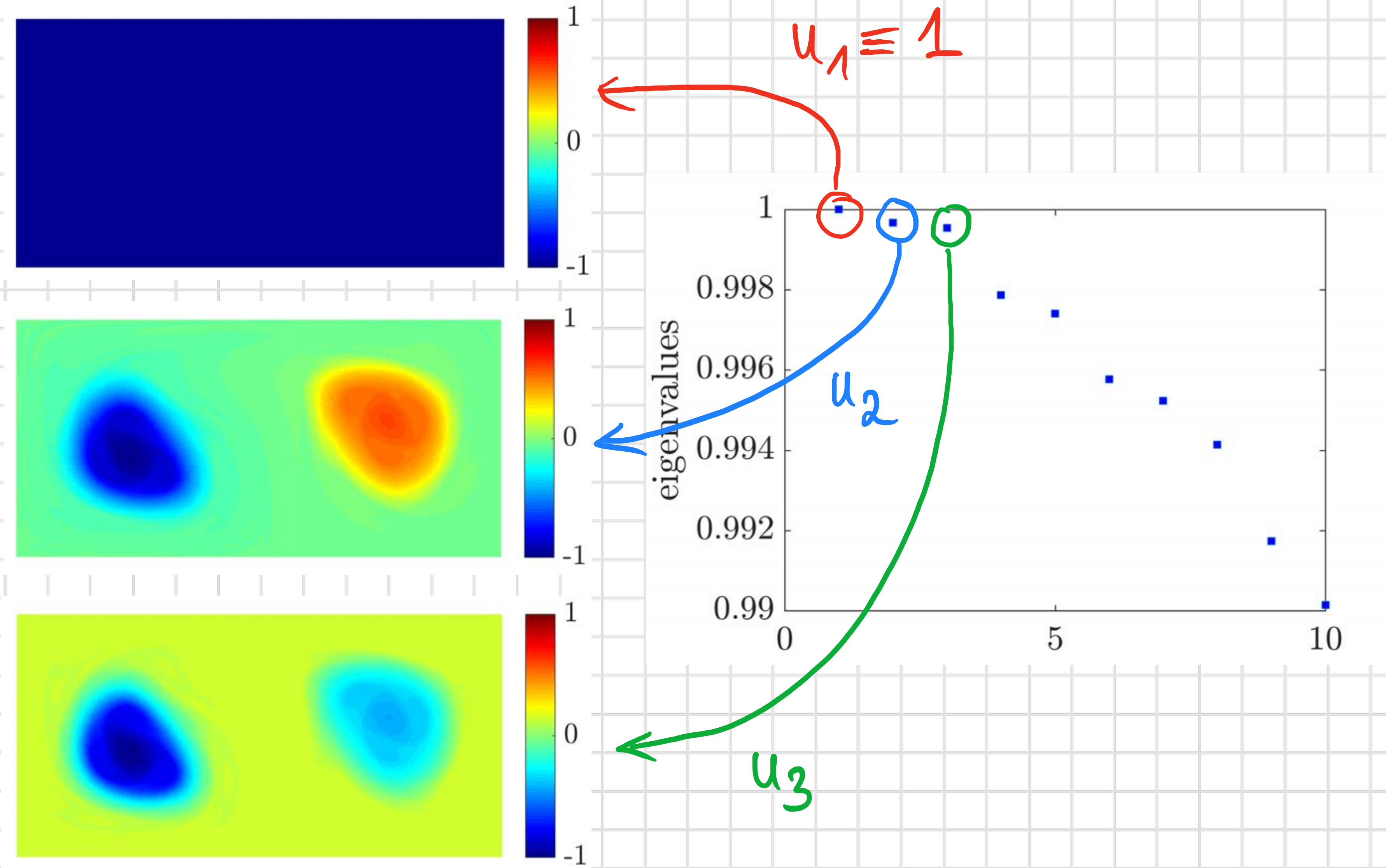


return map



spectrum of φ_*^t

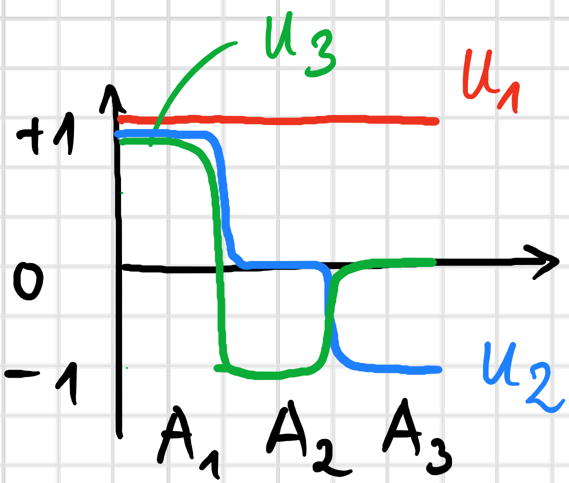
DOUBLE GYRE



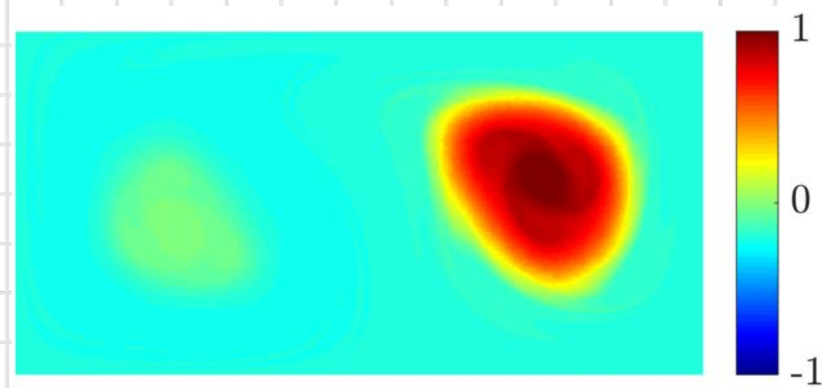
$u_1 \equiv 1$
u_2
u_3
1
0
-1
eigenvalues
1
0.998
0.996
0.994
0.992
0.99
0
5
10

EXTRACTION OF (ALMOST) INVARIANT SETS

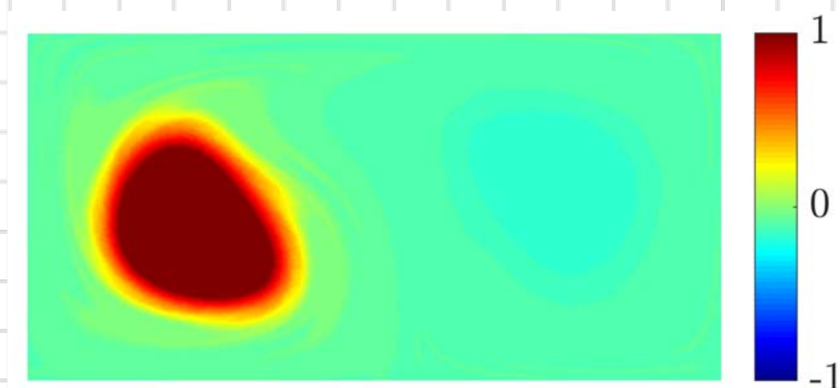
$$\varphi_*^t u = \lambda u$$



$u_2 + u_3$

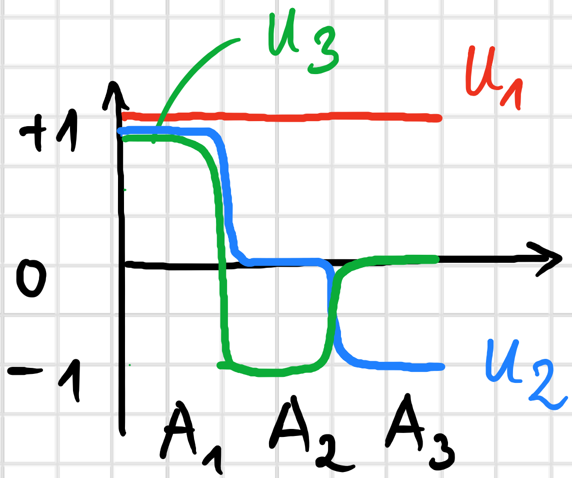


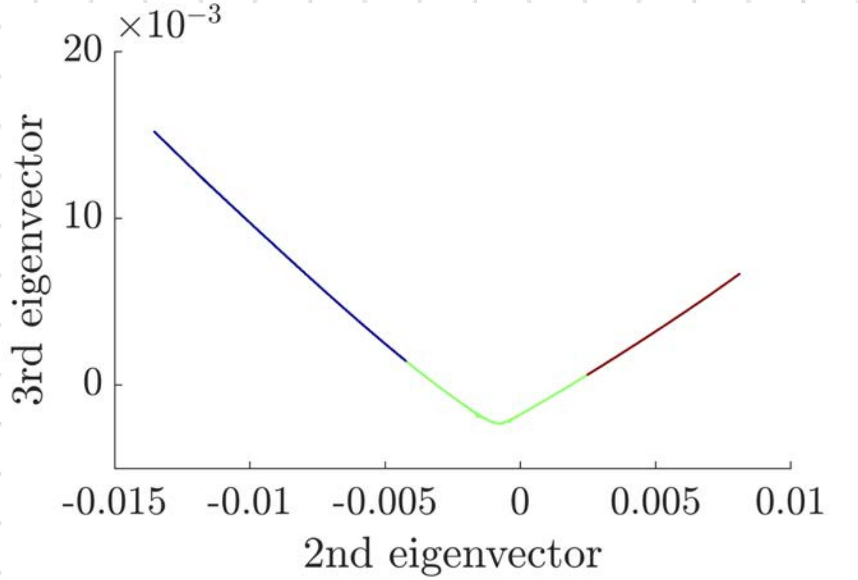
$u_2 - u_3$



Dellnitz, Junge, 99

EXTRACTION OF (ALMOST) INVARIANT SETS

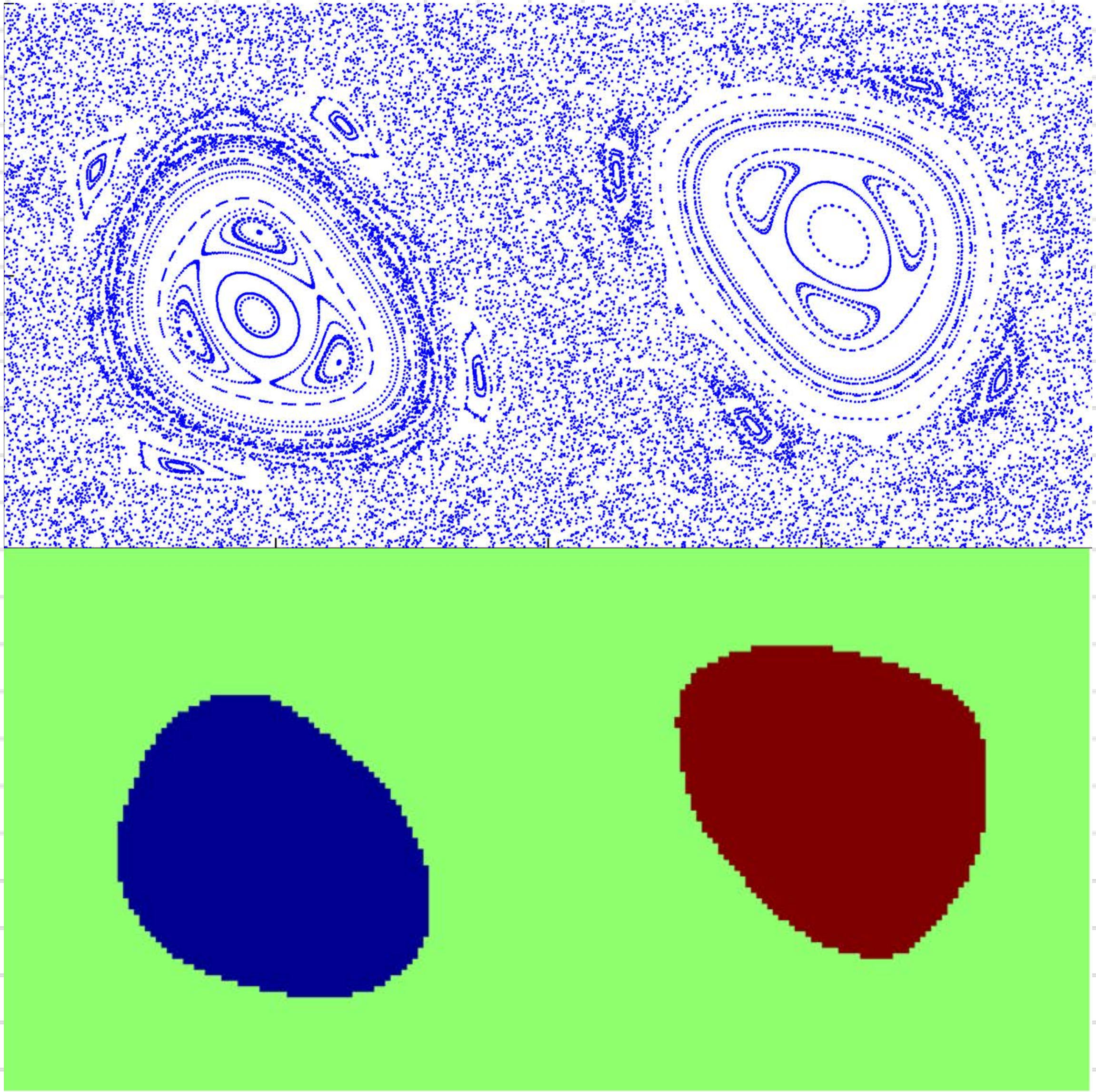




algorithms (heuristics): linear algebra
graph partitioning / spectral clustering
(e.g. k-means)

Deuflhard, Huisinga, Fischer, Schütte, 00; Dellnitz, Junge, ..., 05, ...

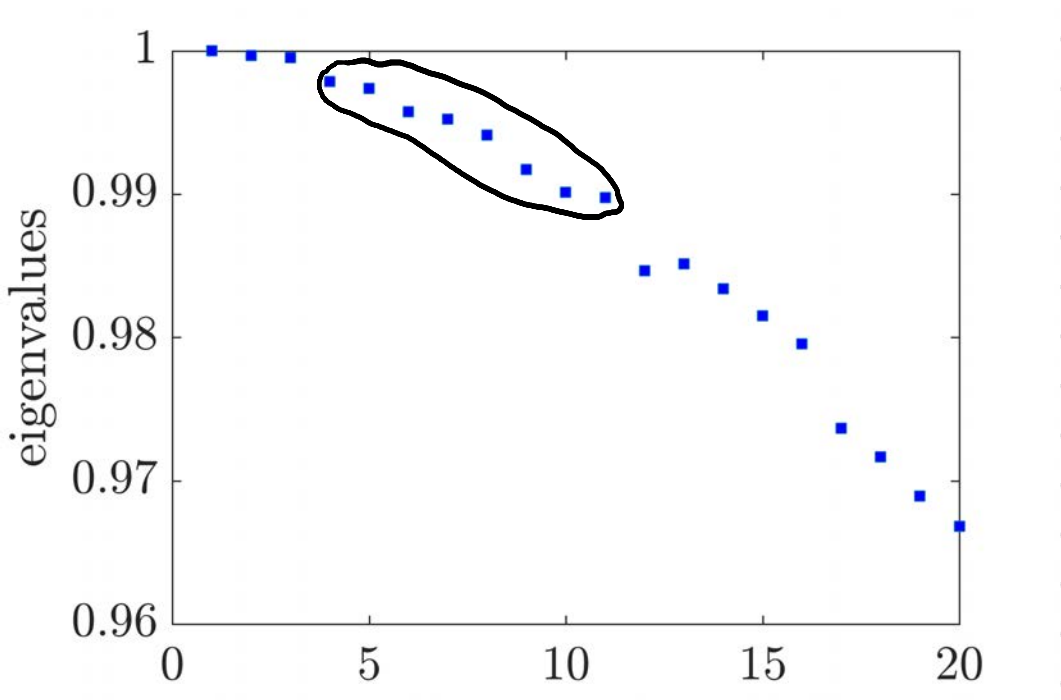
DOUBLE GYRE

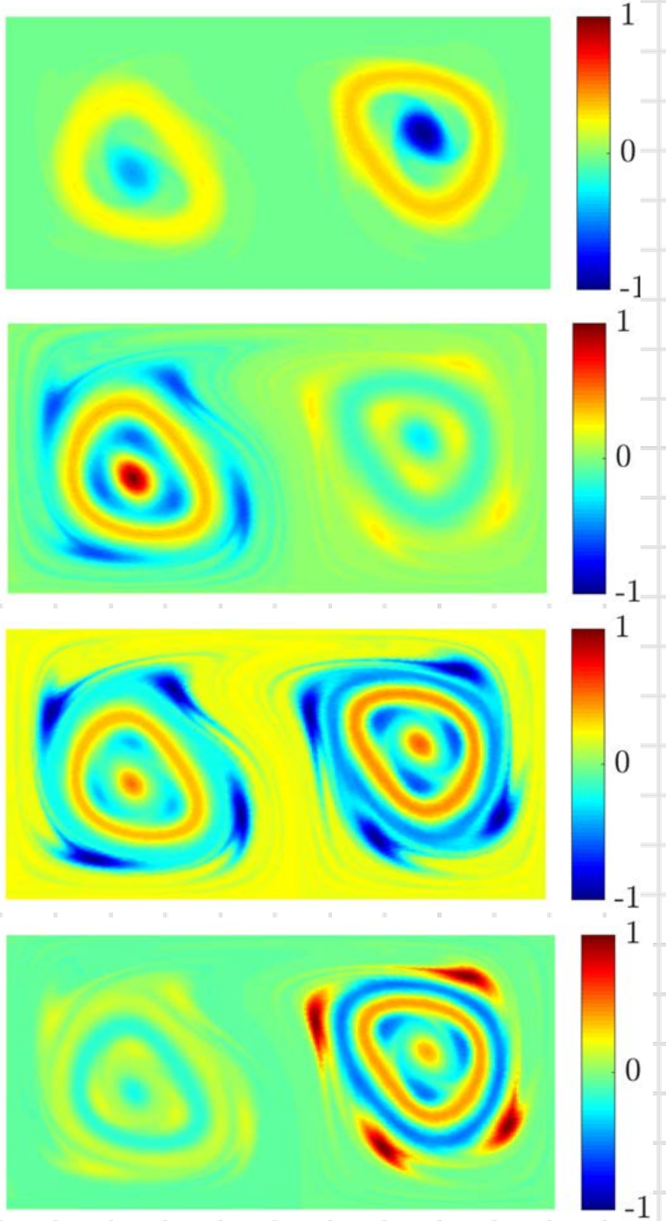


return map

almost invariant sets

The Rest of the Spectrum





THE REST OF THE SPECTRUM

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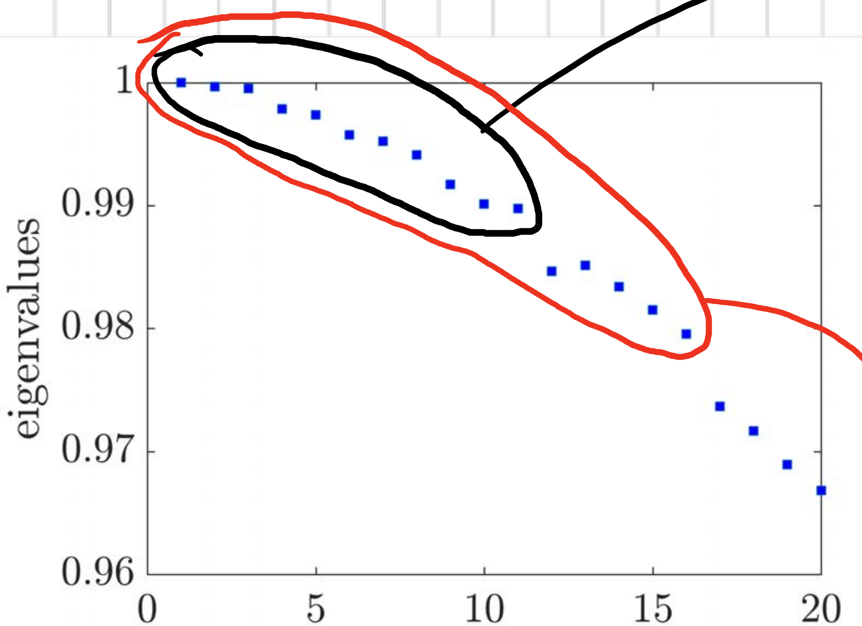

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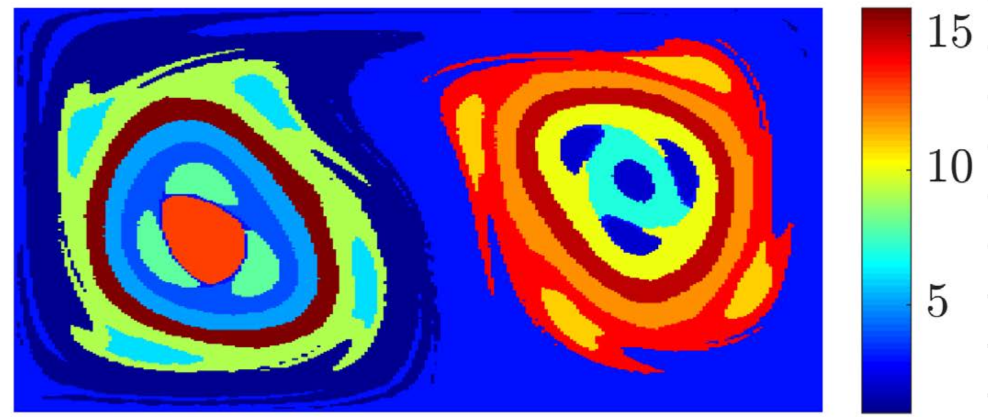
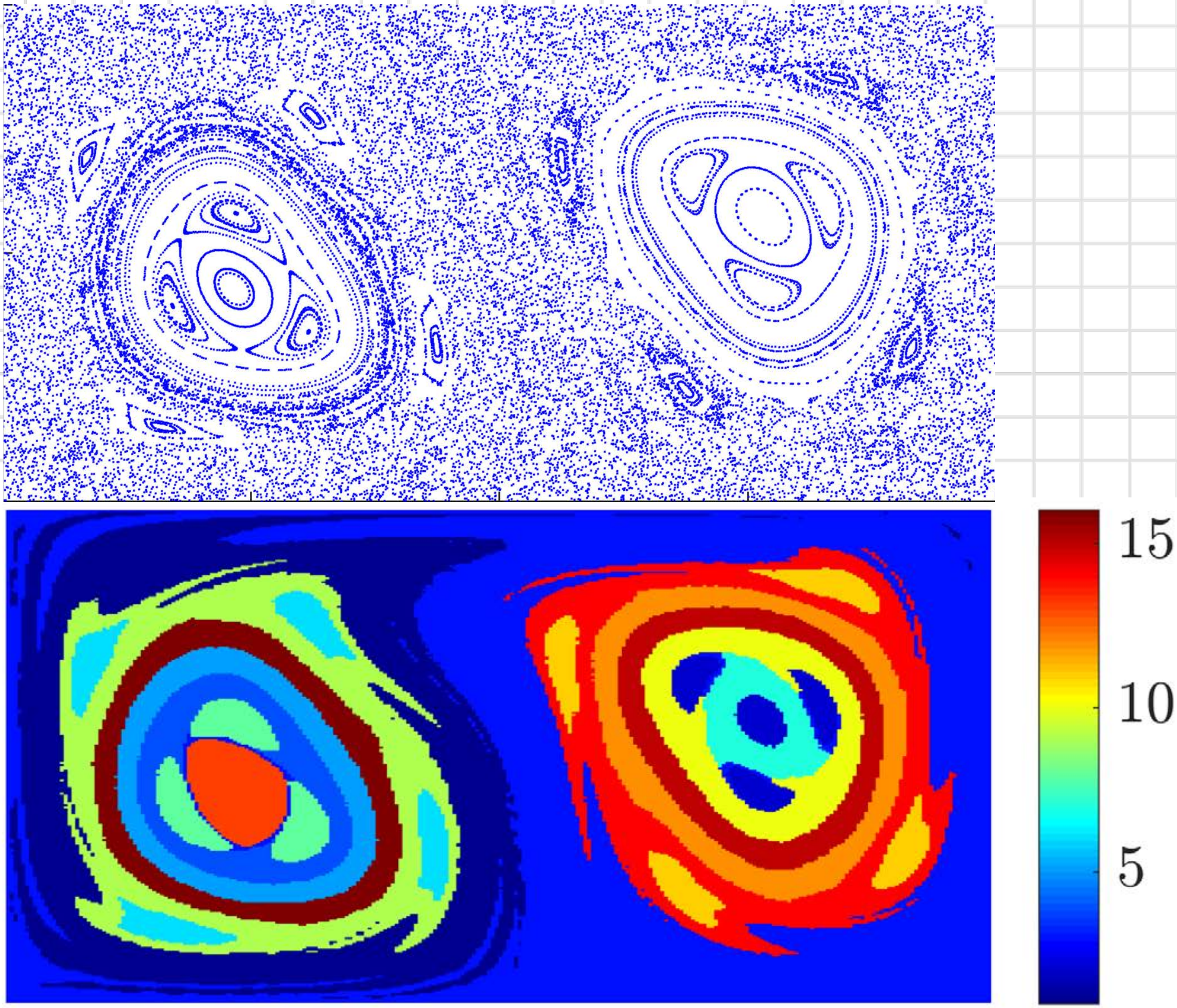
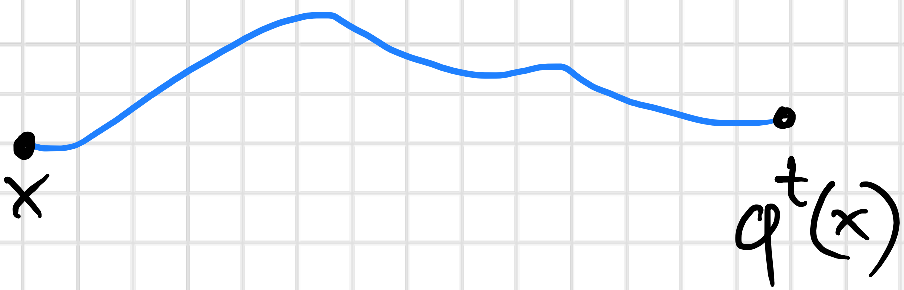


The Rest of the Spectrum

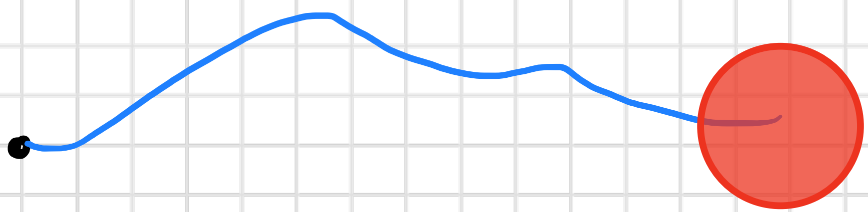


THE ROLE OF PERTURBATIONS

Ulam's method = „exact" solution of a **perturbed problem**



deterministic



ε-perturbation

$$\varphi^t(x) + \xi \,, \quad \xi \sim \mathcal{N}(0,\delta)$$

ADDING PERTURBATIONS

$$u_t = -\operatorname{div}(uv)$$

Liouville

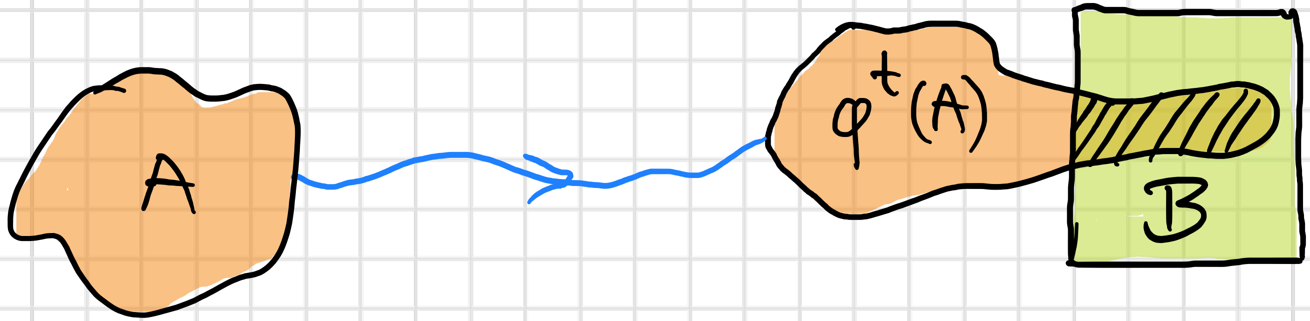
$$\Big\downarrow \ + \text{ small diffusion}$$

$$u_t = \varepsilon^2 \Delta u - \operatorname{div}(uv)$$

Fokker-Planck/

advection-diffusion

TRANSPORT RATES



$$m(\varphi^t(A) \cap B) = \int \mathbb{1}_{\varphi^t(A)} \cdot \mathbb{1}_B \, dm$$

$$= \int \mathbb{1}_A \circ \varphi^{-t} \cdot \mathbb{1}_B \, dm$$

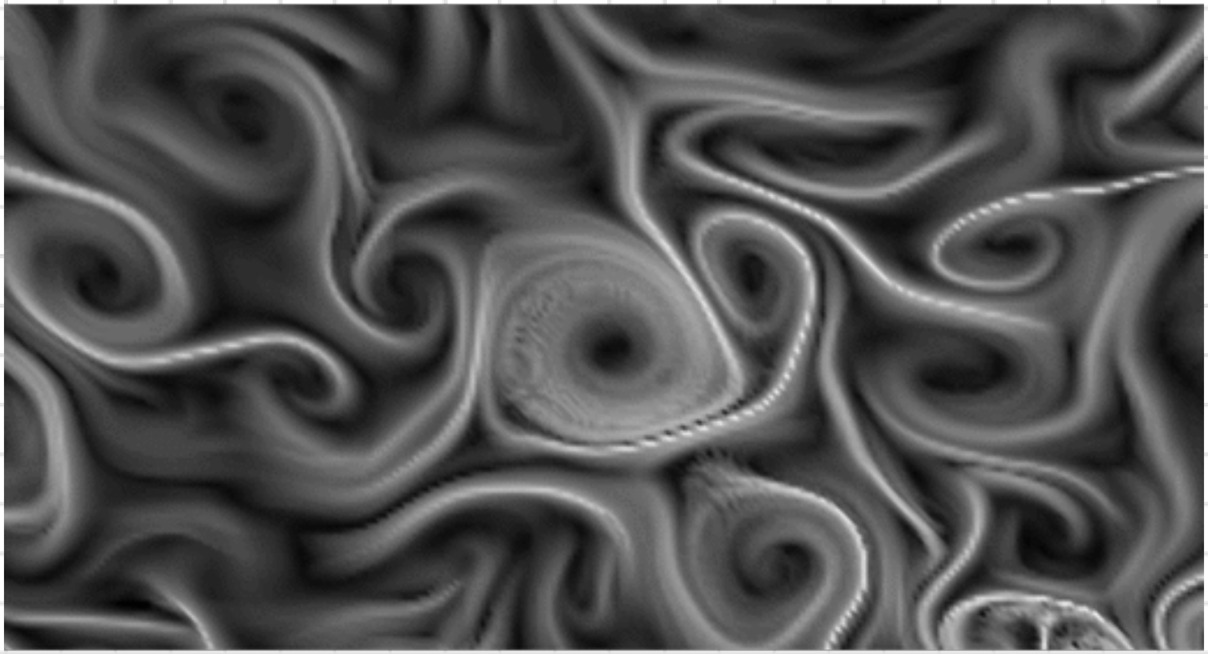
$$= \int \varphi^t_* \mathbb{1}_A \cdot \mathbb{1}_B \, dm$$

$$\approx \mathbb{1}_B^T P^t \mathbb{1}_A$$

Dellnitz, Junge, Padberg, et al 05

FINITE TIME SYSTEMS

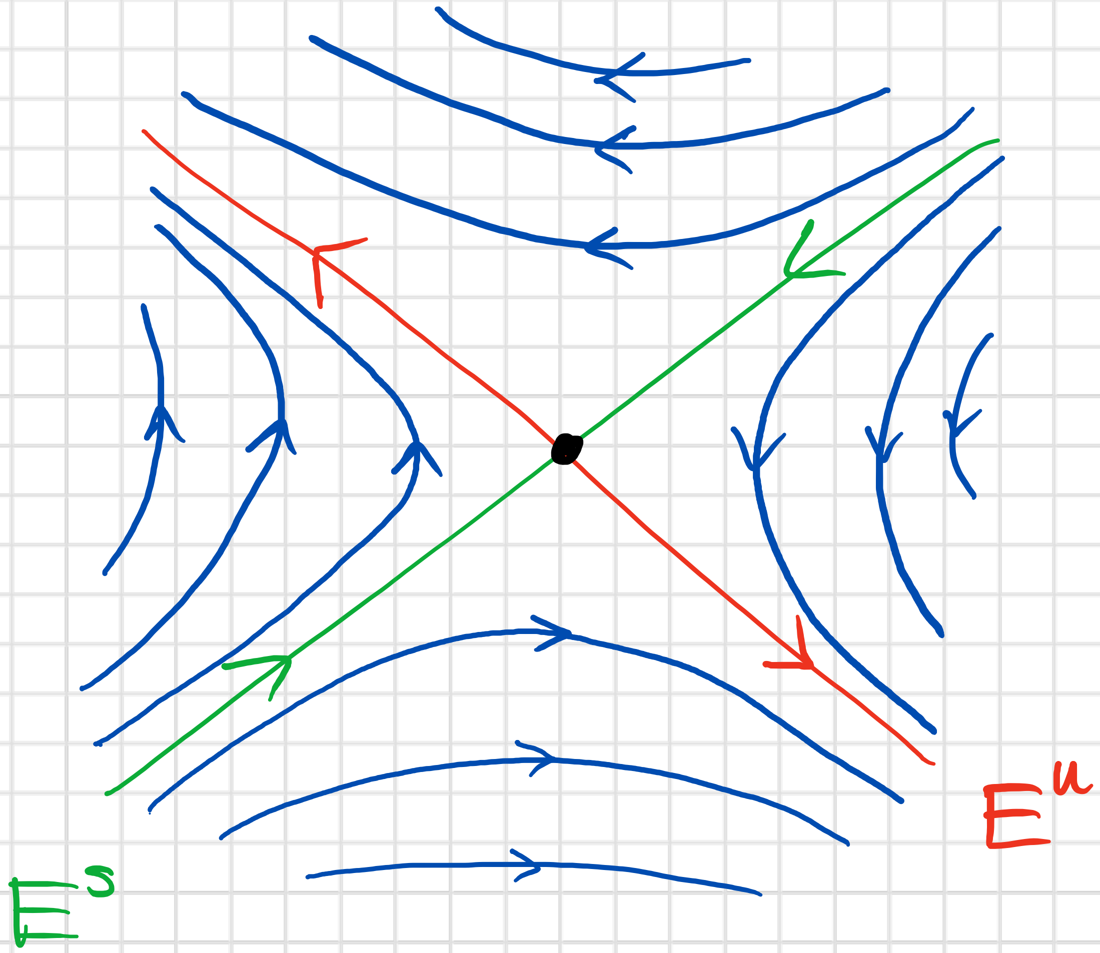
Now: $v(t, x)$, $t \in [t_0, t_f]$ (non-periodic)



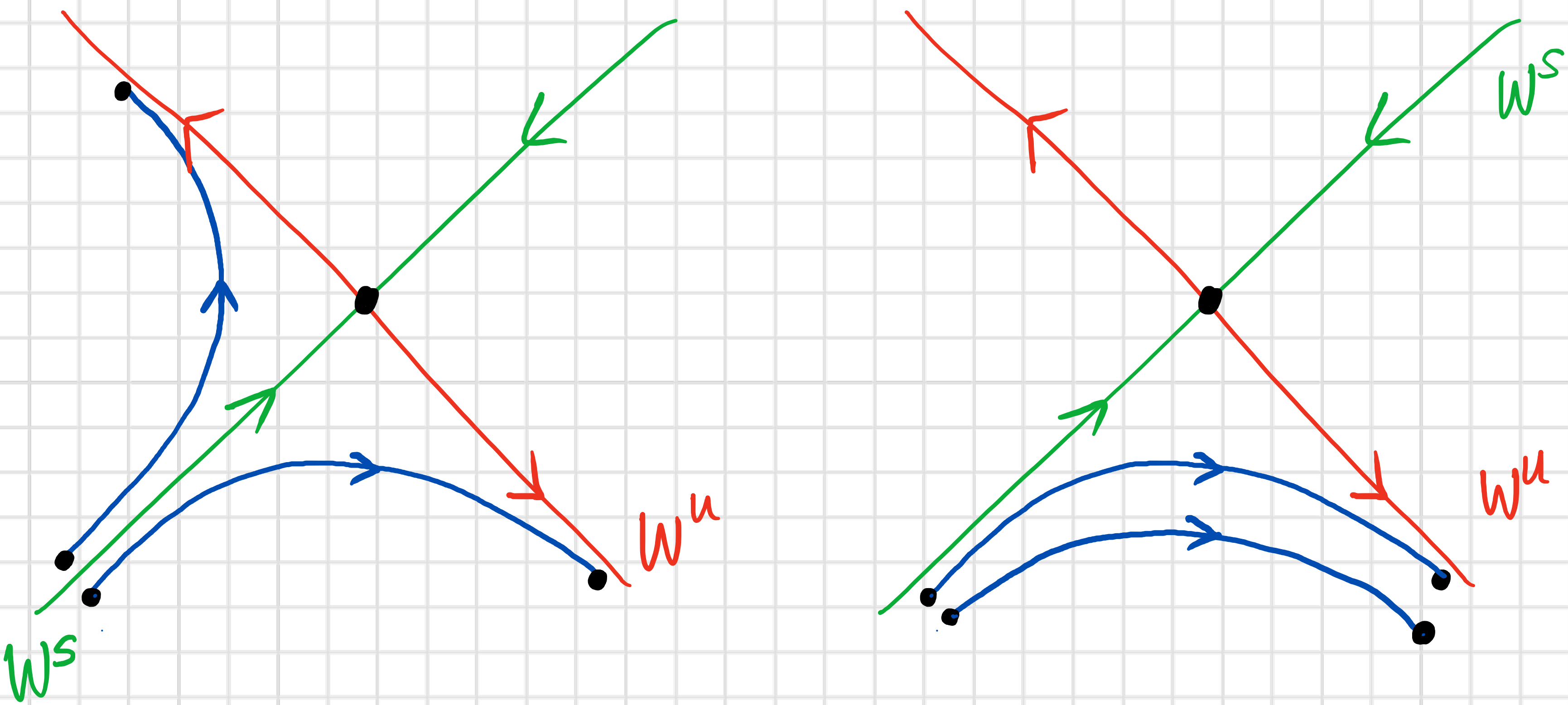
problem:

asymptotic concepts not applicable

DYNAMICS NEAR SADDLES



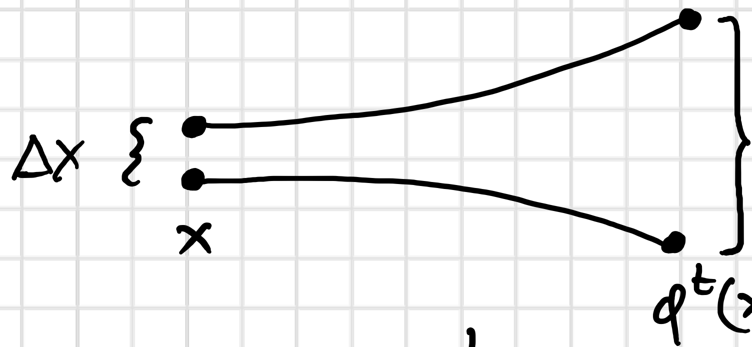
MANIFOLDS AND LOCAL STRETCHING



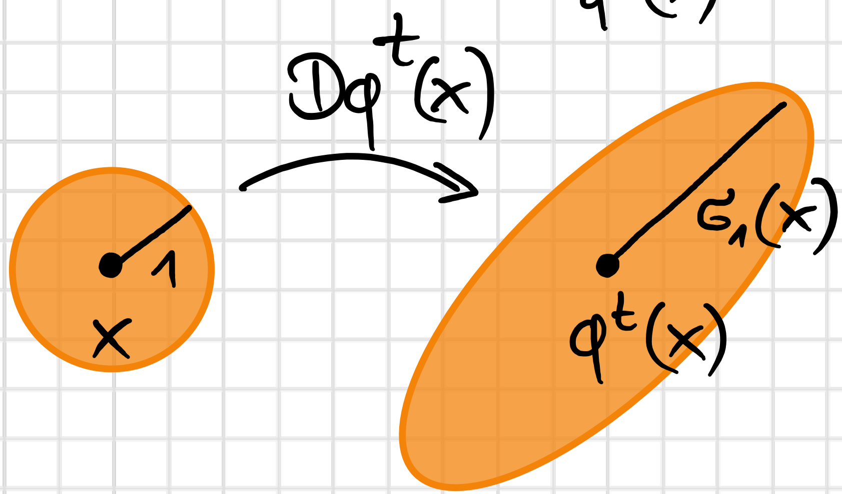
idea: $W^s = \{$ locations where φ^t stretches the most $\}$

$W^u = \{$ —— " —— φ^{-t} —— " —— $\}$

FINITE TIME LYAPUNOV EXPONENTS



$$\varphi^t(x+\Delta x)-\varphi^t(x) \approx D\varphi^t(x)\,\Delta x$$

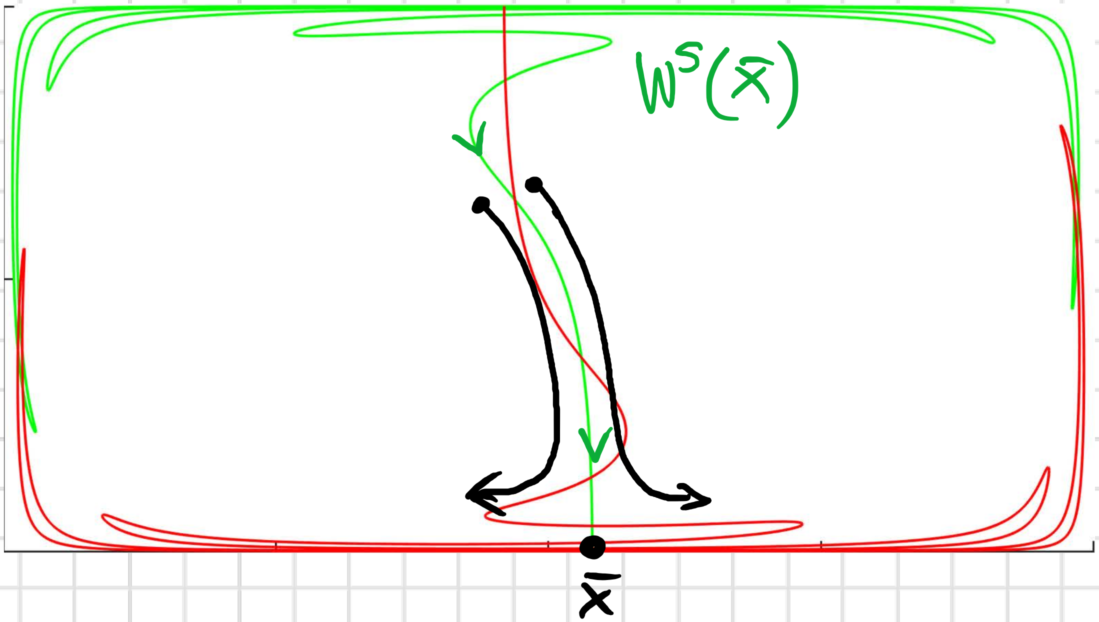


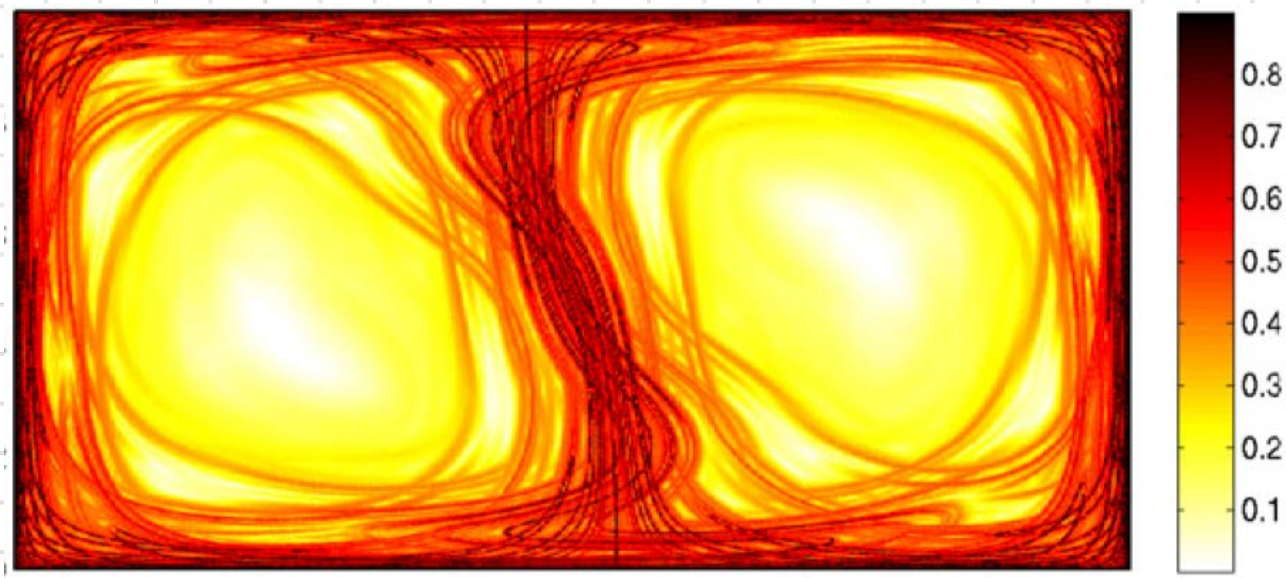
= largest singular value of $D\varphi^t(x)$

expect $\sigma_1(x) = e^{t\Lambda_1(x)} \longrightarrow \boxed{\Lambda_1(x) = \frac{1}{t}\log \sigma_1(x)}$

Haller, 00

FTLEs IM DOUBLE GYRE

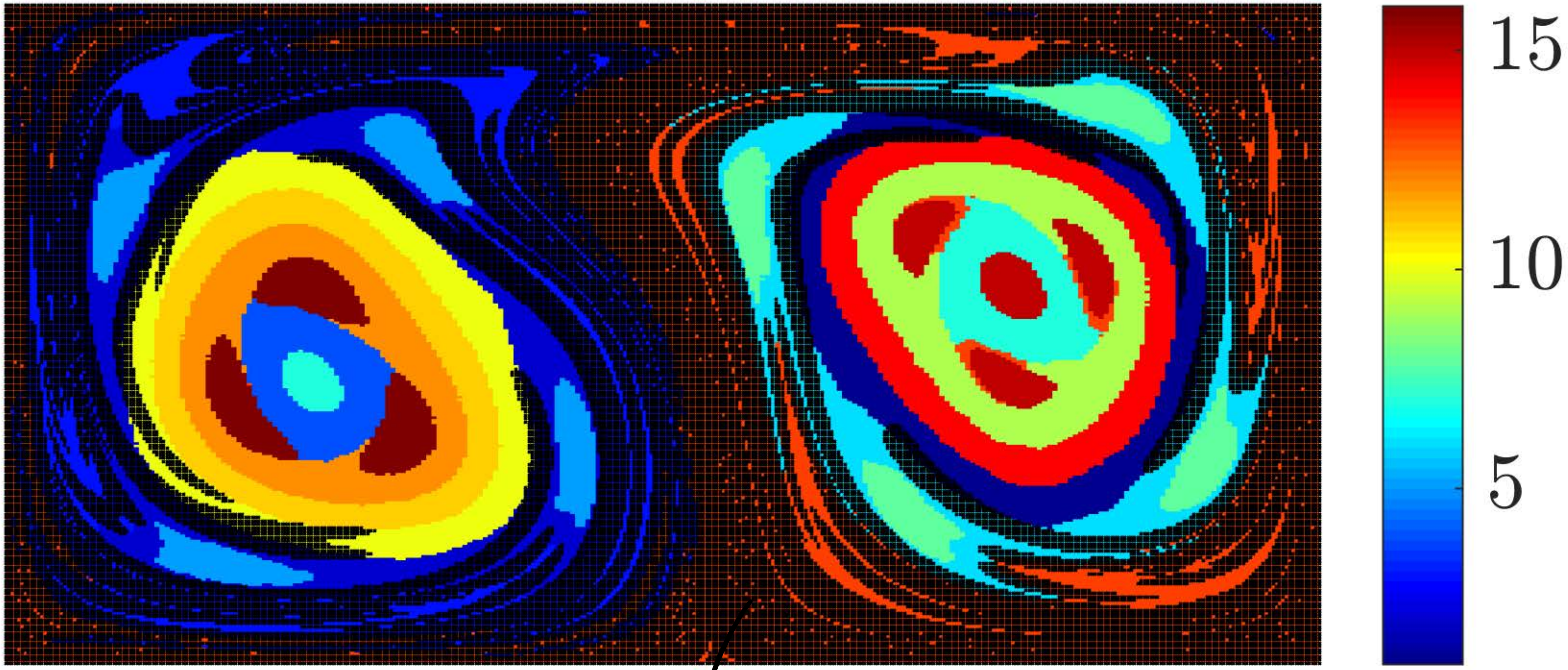




Λ_1 field

Fig: Froyland, Padberg, '09

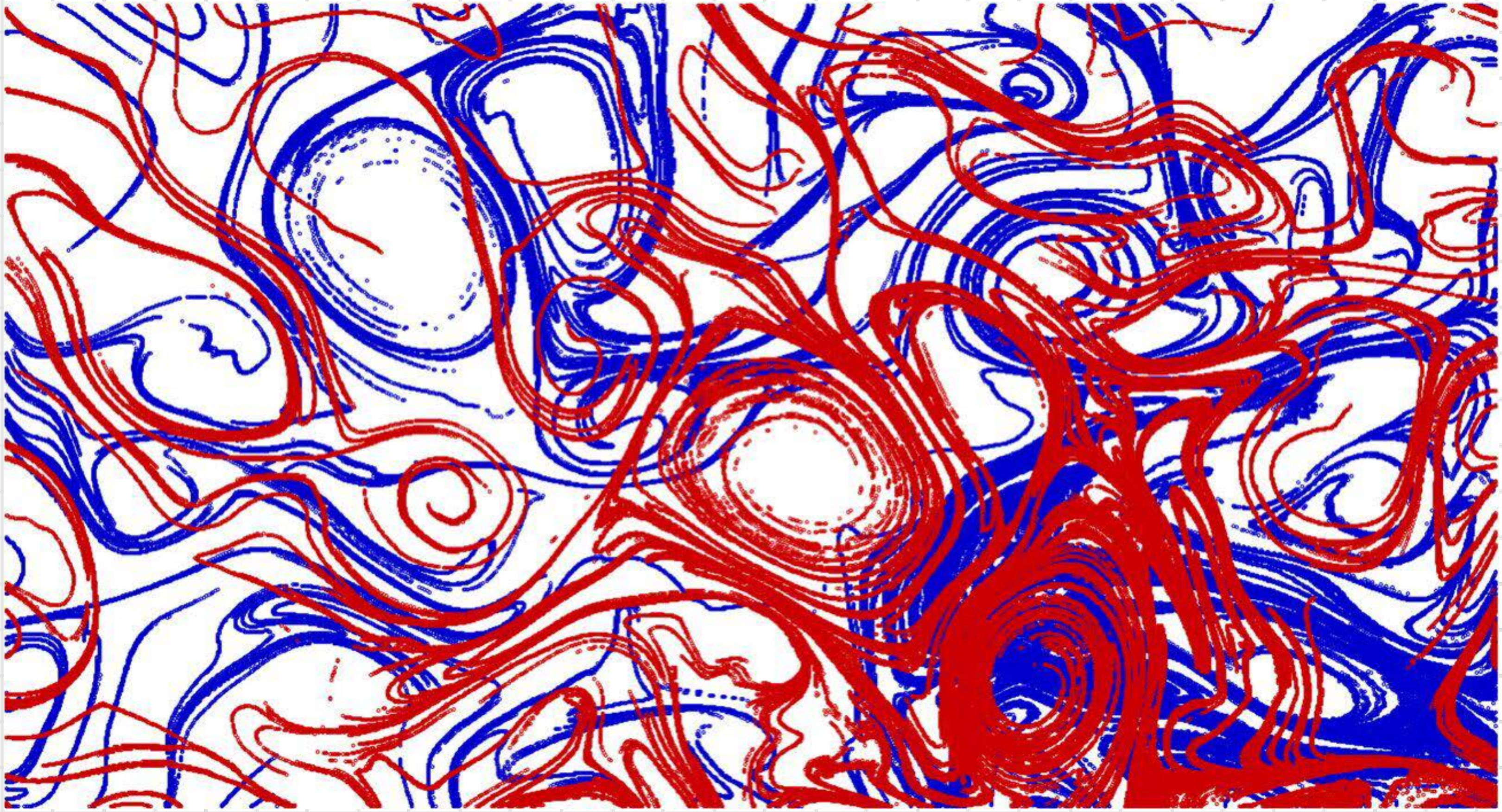
FTLE = TRANSPORT BARRIERS ?



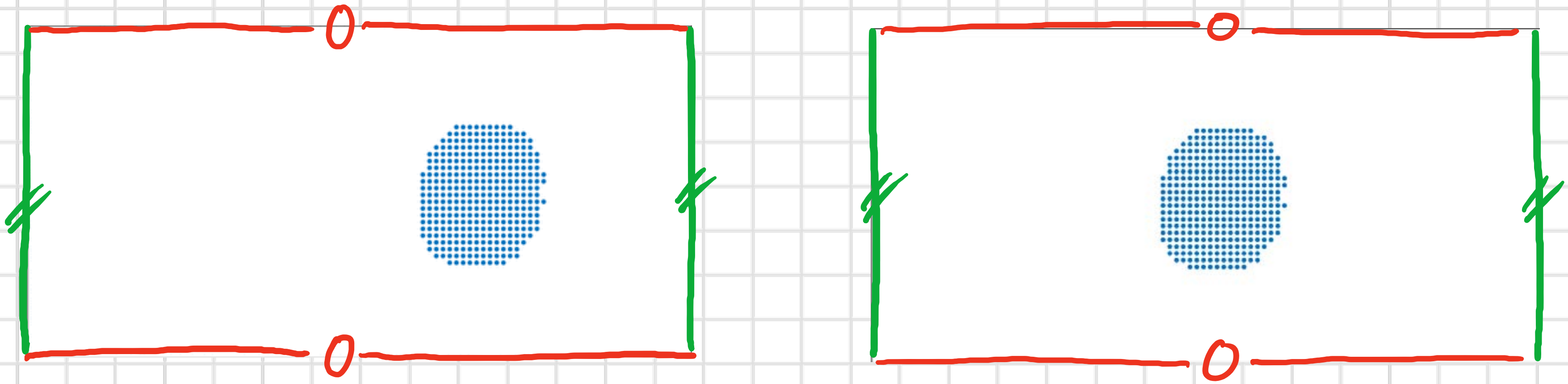
$\Lambda_1 > 0.1$

Lagrangian Coherent Structures

→ George Haller (talk on Wednesday)

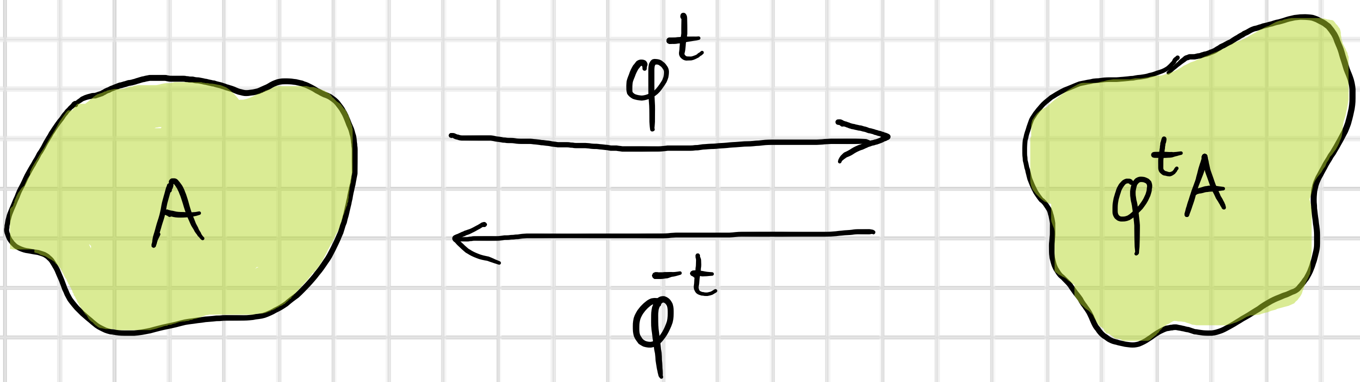


FINITE TIME COHERENT SETS



problem: „coherent" set moves

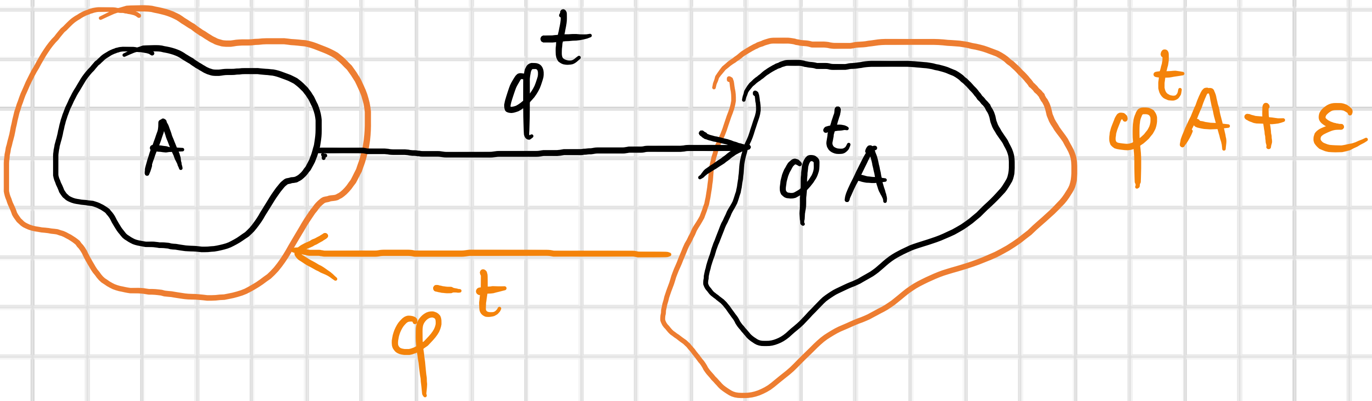
idea: consider $\varphi_*^{-t}\,\varphi_*^{t}$



FINITE TIME COHERENT SETS

but $A = \varphi^{-t}\varphi^{t} A$ for any set

→ apply perturbation



eigenproblem: $\varphi^{-t}_{*,\varepsilon}\,\varphi^{t}_{*,\varepsilon}\,u = \lambda u$

i.e. we look for singular values/vectors of $\varphi^{t}_{*,\varepsilon}$

Froyland, Lloyd, Santitissadeekorn, '10

Recent developments

DIRECT SOLUTION OF FOKKER–PLANCK

recall : $\varphi_*^t u_0$ solves the Fokker–Planck equation

$$u_t = \varepsilon^2 \Delta u - \mathrm{div}(uv) \quad , \quad u(\cdot,0) = u_0$$

- spectral discretization $\rightsquigarrow$ Δ = diagonal matrix

enables

- stiffness : linear implicit time-stepping

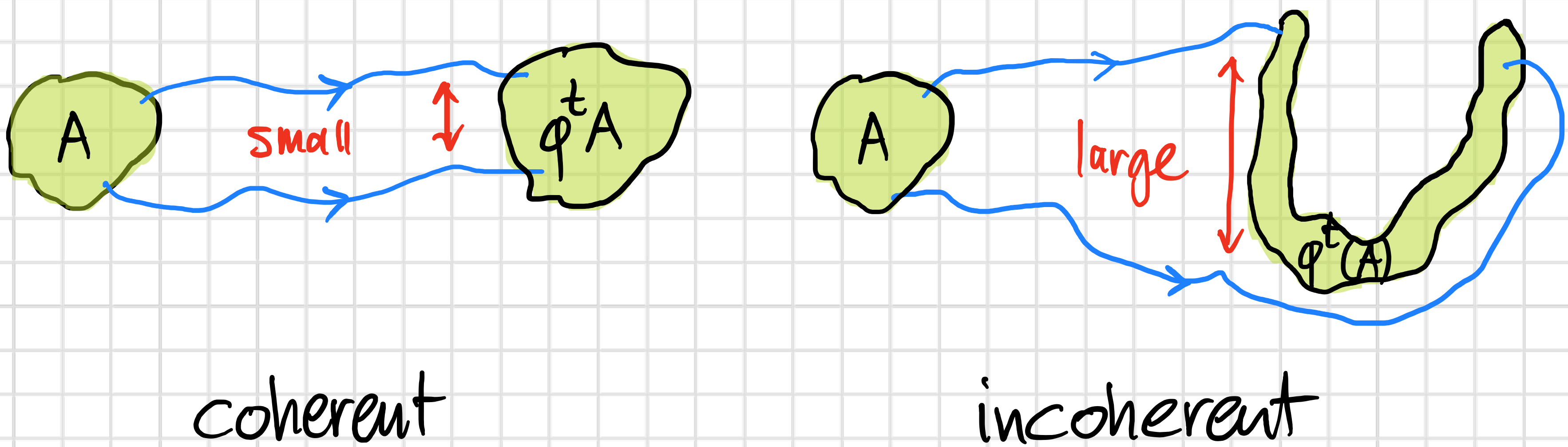
$\rightsquigarrow$ coherent sets

without integrating Lagrangian trajectories

(cf. Péter Koltai's project)

Denner, Junge, Matthes, 16

LAGRANGIAN TRAJECTORY CLUSTERING

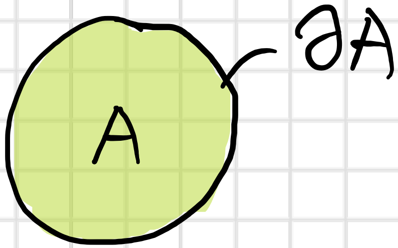


→ clustering directly on Lagrangian trajectories

applies to very sparse and incomplete data

Froyland, Padberg, 15; Karrasch, Haller, et al., 16; Banisch, Koltai, 16.

DYNAMIC ISOPERIMETRY



$$\frac{|\partial A|}{|A|} \text{ small}$$

$$\frac{|\partial A|}{|A|} \text{ large}$$

coherent set : $\frac{1}{2}\left(\frac{|\partial A|}{|A|} + \frac{|\partial \varphi^t A|}{|\varphi^t A|}\right)$ small

$\rightsquigarrow$ eigenproblem : $\frac{1}{2}\left(\Delta + \varphi_*^{-t}\Delta\varphi_*^{t}\right)u = \lambda u$

Froyland 15

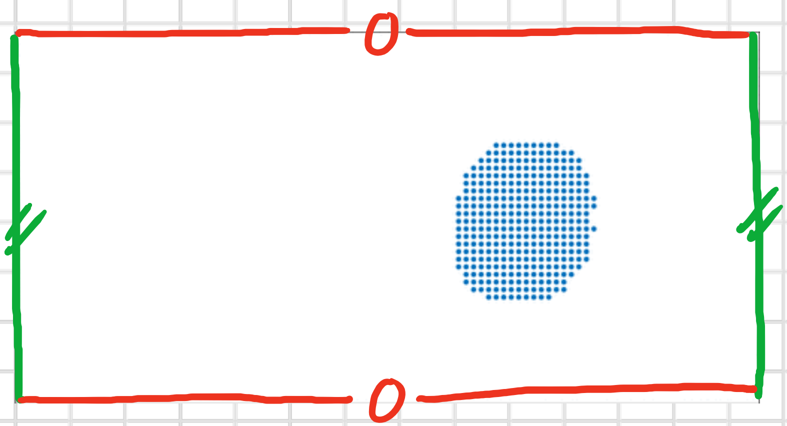
COMPUTATION

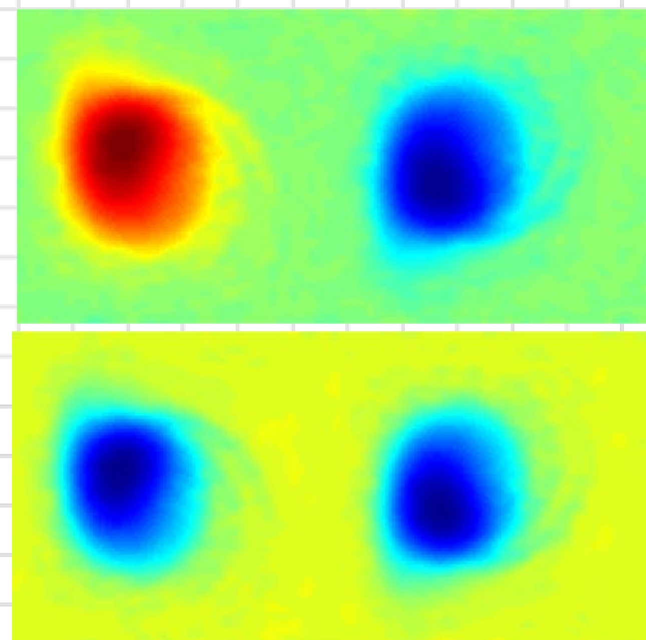
solve

$$\frac{1}{2}\left(\Delta + \varphi_*^{-t}\Delta\varphi_*^{t}\right)u = \lambda u \qquad + \text{R.B.}$$

by collocation with radial basis functions.

→ very high order, sparse data suffices





u_2

u_3

Froyland, Junge, 2015

GEOMETRIC HEAT FLOW

Fokker-Planck equation

Euclidean metric

$$u_t = \varepsilon^2 \Delta u - \mathrm{div}(uv) \qquad \text{on } (\varphi^t M, g)$$

can be written as

$$w_t = \varepsilon^2 \Delta_{g(t)} w \qquad \text{on } (M, \varphi^t g)$$

coherent set = almost invariant set on $(M, \bar{g})$
with

$$\Delta_{g(t)} \approx \Delta_{\bar{g}} \quad \leftarrow \quad \text{always self-adjoint}$$

Karrasch, Keller, 2016