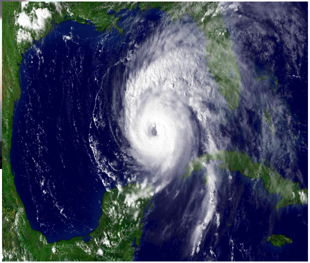
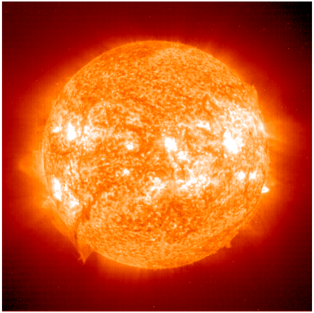
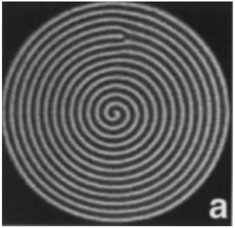
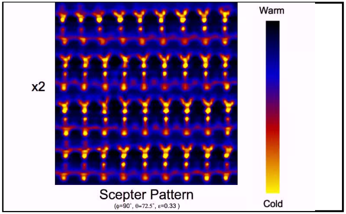
Thermal Convection

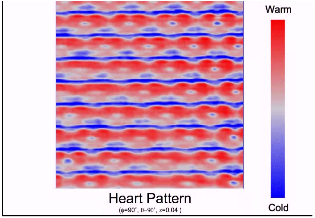
= flow driven by a thermal gradient











Rayleigh - Bénard Convection

horizontal fluid layer heated from below and cooled from above

Henri Bénard

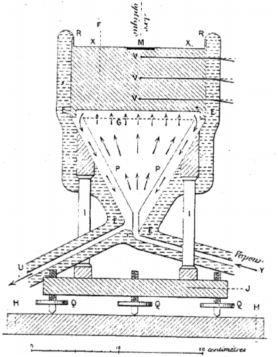
Bénard, H. (1900a). Etude expérimentale du mouvement des liquides propageant de la chaleur par convection. Régime permanent: tourbillons cellulaires. *Compte-rendus de l'Académie des Sciences, 130*, 1004–1007.

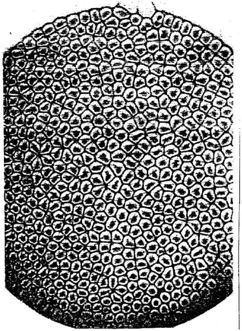
Bénard, H. (1900b). Mouvements tourbillonnaires à structure cellulaire: Etude optique de la surface. *Compte-rendus de l'Académie des Sciences, 130*, 1065–1068.

Bénard, H. (1900c). Les tourbillons cellulaires dans une nappe liquide, I. Description générale des phénomènes. *Revue Générale des Sciences Pures et Appliquées, 11*, 1261–1271.

Bénard, H. (1900d). Les tourbillons cellulaires dans une nappe liquide, II. Procédés mécaniques et optiques d'examens, lois numériques des phénomènes. *Revue Générale des Sciences Pures et Appliquées, 11*, 1309–1328.

Bénard, H. (1901). Les tourbillons cellulaires dans une nappe liquide transportant de la chaleur par convection en régime permanent. *Annales de Chimie et de Physique, 23*, 62–144.



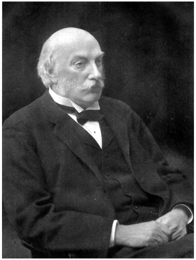


"The shape of the currents I was able to observe in liquids offering no other heterogeneity than temperature differences are, if I am not mistaken, especially interesting and novel in that they are examples of remarkably simple physical phenomena able to create from scratch a cellular structure that seemed, up until now, to be particular to living beings and characteristic of the organic world."

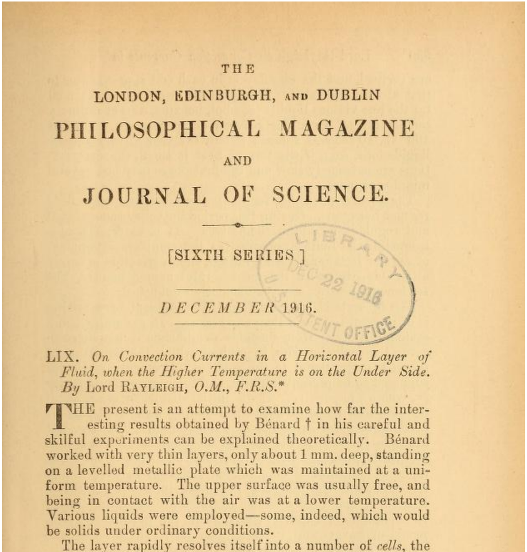
from: Aubin, *Stud. Hist. Phil. Sci.* **29**, 359 (2008)

Rayleigh - Bénard Convection

horizontal fluid layer heated from below and cooled from above



John William Strutt
(Lord Rayleigh)



THE
LONDON, EDINBURGH, AND DUBLIN
PHILOSOPHICAL MAGAZINE
AND
JOURNAL OF SCIENCE.

[SIXTH SERIES.]

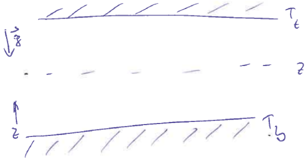
DECEMBER 1916.

LIX. *On Convection Currents in a Horizontal Layer of Fluid, when the Higher Temperature is on the Under Side. By* Lord RAYLEIGH, *O.M., F.R.S.**

THE present is an attempt to examine how far the interesting results obtained by Bénard † in his careful and skilful experiments can be explained theoretically. Bénard worked with very thin layers, only about 1 mm. deep, standing on a levelled metallic plate which was maintained at a uniform temperature. The upper surface was usually free, and being in contact with the air was at a lower temperature. Various liquids were employed—some, indeed, which would be solids under ordinary conditions.

The layer rapidly resolves itself into a number of *cells*, the

Rayleigh - Bénard - Convection



$\Delta T = T_b - T_t$

Governing equations:

(1) $\frac{\partial \rho}{\partial t} + \nabla(\rho \cdot \vec{v}) = 0$ (continuity)

(2) $\rho \left[\frac{\partial \vec{v}}{\partial t} + (\vec{v}\nabla)\vec{v} \right] = -\nabla p + \rho \nu \Delta \vec{v} - \rho g \hat{z}$ (momentum)

(3) $\frac{\partial T}{\partial t} + (\vec{v}\nabla) T = \kappa \Delta T$ + internal heating + frictional heating (energy equation)

$\kappa = \frac{\lambda}{c_p \cdot \rho}$ thermal conductivity (thermal diffusivity) density heat capa

Boundary conditions:

@ $z = \frac{L}{2}$: $T = T_t$, $\vec{v} = 0$

$z = -\frac{L}{2}$: $T = T_b$, $\vec{v} = 0$

Boussinesq approximation:

Fluid properties constant, except for density in the buoyancy term:

$\Rightarrow$

$$\boxed{\rho = \rho_0 \left(1 - \alpha (T - T_0)\right)} \quad ; \quad \alpha \dots \text{thermal expansion}$$

$$\boxed{P = -\rho_0 g z + P_0 + P_s}$$

(1) $\rightarrow$ $\nabla \vec{v} = 0$

(2) $\rightarrow$ $$\frac{\nabla P}{\rho} \approx -g \hat{z} + g \alpha (T - T_0) \hat{z} + \frac{\nabla P_s}{\rho_0}$$

$\hookrightarrow$ NS: $$\frac{\partial \vec{v}}{\partial t} + (\vec{v} \nabla) \vec{v} = -\frac{\nabla P_s}{\rho_0} + g \alpha (T - T_0) \hat{z} + \nu \nabla^2 \vec{v}$$

(3) $$\frac{\partial T}{\partial t} + (\vec{v} \nabla) T = \kappa \nabla^2 T$$

Trivial solution:

(conduction)

$$\vec{v} = 0$$

$$T_{cond} = \underbrace{\frac{T_b + T_t}{2}}_{T_m} + \Delta T \cdot \frac{z}{L}$$

$$P_{s,cond} = \rho_0 g \alpha \left(T_m \cdot z + \frac{1}{2} \Delta T \frac{z^2}{L}\right)$$

Stability of the conduction solution:

$$T = T_{cond} + \Theta$$

$$P_s = P_{cond} + p'$$

NS:) $$\frac{\partial \vec{v}}{\partial t} + (\vec{v}\nabla)\vec{v} = -\frac{\nabla p'}{\rho_0} + g\alpha\Theta\hat{z} + \nu\nabla^2\vec{v}$$

Heat : $$\frac{\partial \Theta}{\partial t} + (\vec{v}\nabla)\Theta = \kappa\nabla^2\Theta + \frac{\Delta T}{L}\vec{v}\cdot\hat{z}$$

$$\vec{v} \cdot 0$$

Dimensionless units:

times: $t = \tilde{t} \cdot \frac{L^2}{\kappa}$, length: $\vec{r} = \tilde{\vec{r}} \cdot L$

Pressure: $p = \tilde{p} \cdot \frac{\rho_0 \kappa \nu}{L^2}$, temperature: $T = \tilde{T} \frac{\nu \kappa}{\alpha g L^3}$

$$(4) \quad \frac{1}{Pr}\left(\frac{\partial \tilde{\vec{v}}}{\partial \tilde{t}} + (\tilde{\vec{v}}\tilde{\nabla})\tilde{\vec{v}}\right) = -\tilde{\nabla}\tilde{p} + \tilde{\nabla}^2 \tilde{\vec{v}} + \tilde{\Theta} \cdot \hat{z}$$

$$(5) \quad \frac{\partial \tilde{\Theta}}{\partial \tilde{t}} + (\tilde{\vec{v}}\tilde{\nabla})\tilde{\Theta} = \tilde{\nabla}^2 \tilde{\Theta} + Ra\, \tilde{\vec{v}} \cdot \hat{z}$$

$$(6) \quad \tilde{\nabla} \tilde{\vec{v}} = 0$$

Boussinesq equation

$$Ra = \frac{g \alpha \Delta T \cdot L^3}{\nu \cdot \kappa}$$ ... Rayleigh Number

$$Pr = \frac{\nu}{\kappa}$$ ... Prandtl Number

Typical strategy for solving:

$\nabla \times (4)$: $\vec{\omega} = \nabla \times \vec{v}$ ... vorticity

$\nabla \times (\nabla \times (4))$

$\to$ (1) Eqn for ω_z

$\to$ (2) Eqn for v_z

(3) Eqn for Θ

$\Big\}$ => linearizing: Θ, v, ω are small

To check stability:

Ansatz:
$$v_z = V(z) \cdot e^{\sigma t} \cdot e^{i(q_x x + q_y y)}$$
$$\omega_z = W(z) \cdot e^{\sigma t} \cdot e^{i(q_x x + q_y y)}$$
$$\Theta = \Theta(z) \cdot e^{\sigma t} \cdot e^{i(q_x x + q_y y)}$$

$$\begin{matrix} V(z) \\ W(z) \\ \Theta(z) \end{matrix} = \begin{pmatrix} V_0 \\ W_0 \\ \Theta_0 \end{pmatrix} \cdot \cos\left(\frac{\pi z}{L}\right)$$

=> for example but other functions that fulfill BC are also possible

~~→ to~~

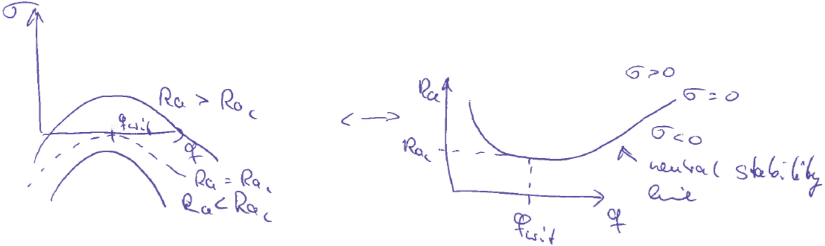
Set of algebraic equations, that can be solved numerically:

growth rate: $\sigma = \sigma(Ra, q)$; $q = \sqrt{q_x^2 + q_y^2}$

σ monotonic in Ra

growth rate: $\sigma = \sigma(Ra, q)$; $q = \sqrt{q_x^2 + q_y^2}$

σ monotonic in Ra

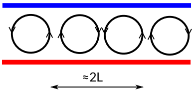


$q_{crit} = 3.117$
$Ra_c = 1708$

Independent of Pr

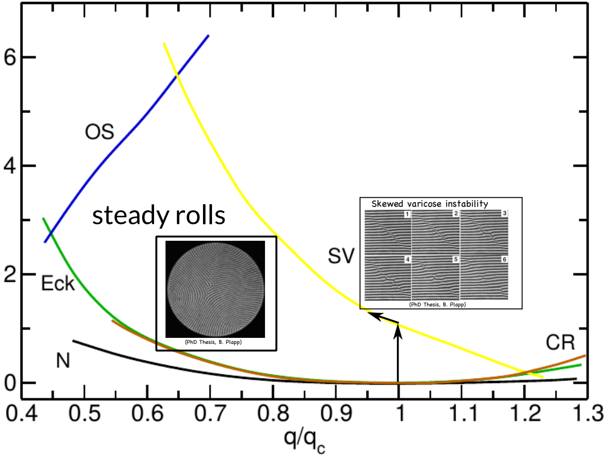
at Ra_c ; rolls of $\lambda = \frac{2\pi}{3.117} d \approx 2d$

Rolls periodicity of $\lambda \approx 2L$



Busse Balllon

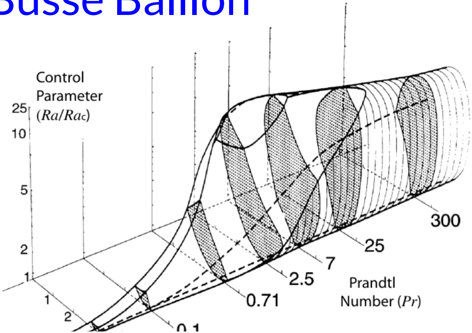
$$\frac{Ra \quad Ra_c}{Ra_c}$$

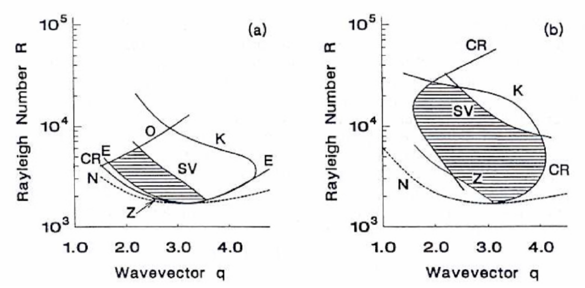


Stability region for stationary rolls @ Pr=1.27 (compressed gas)

(calculated using code by W. Pesch)

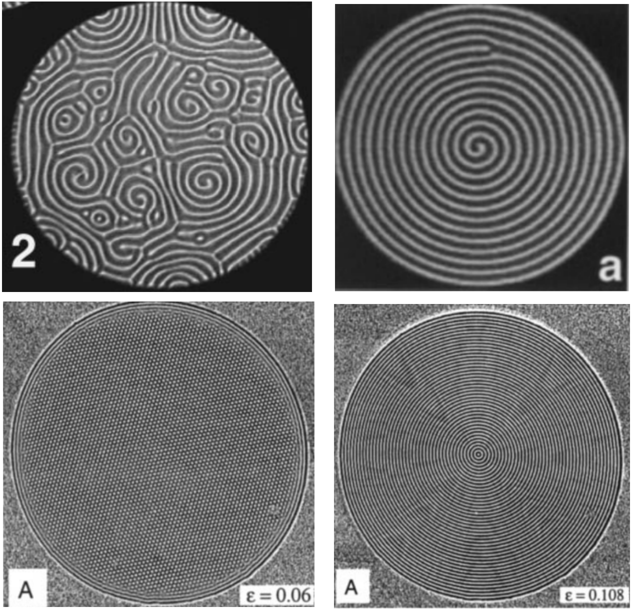
Busse Balllon





From: M. C. Cross and P. C. Hohenberg, Rev. of Mod. Phys. 65, 851 (1993).

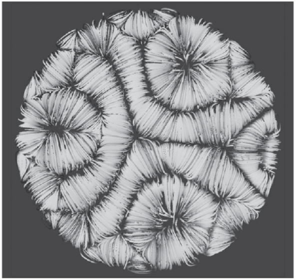
A model system for pattern formation



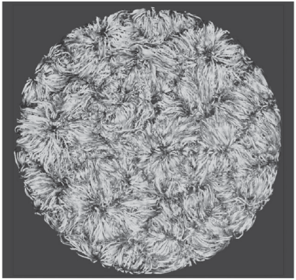
from: Bodenschatz et al, ***Annu. Rev. Fluid Mech.***, **32,** 709 (2000)

Transition to turbulence

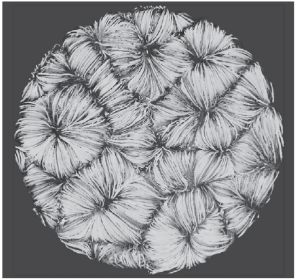
(Numerics)



Ra=6000



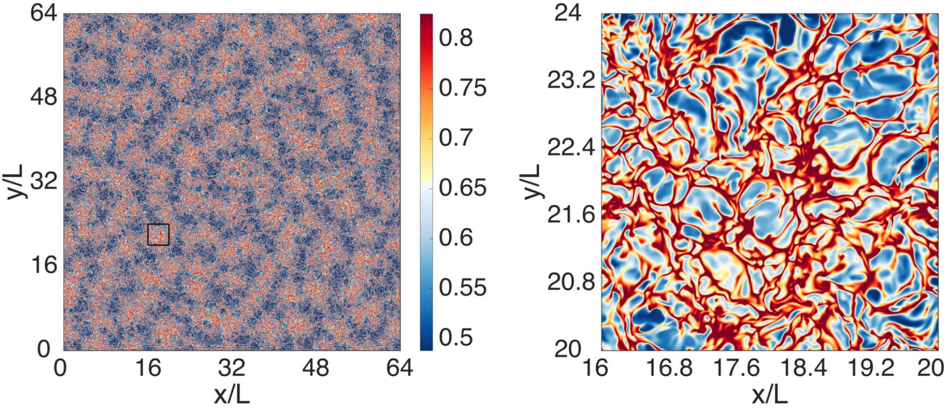
Ra=10^7

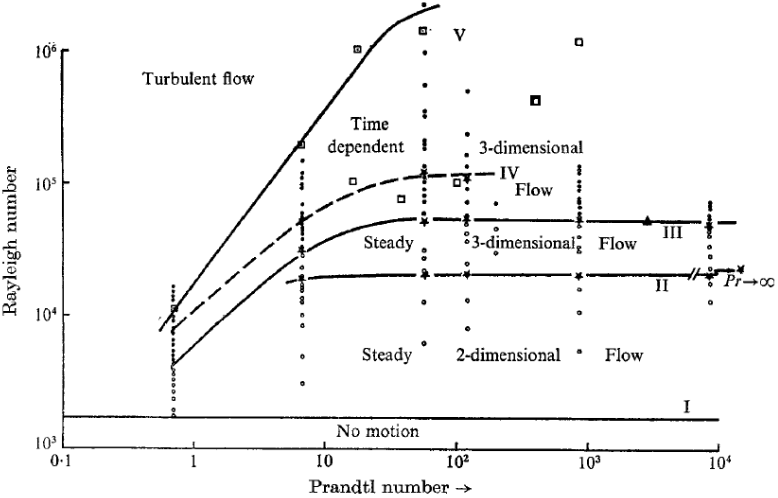


Ra=10^7
(time averaged)

from: Bailon-Cuba et al., JFM, (2010)

Transition to turbulence
(Numerics)

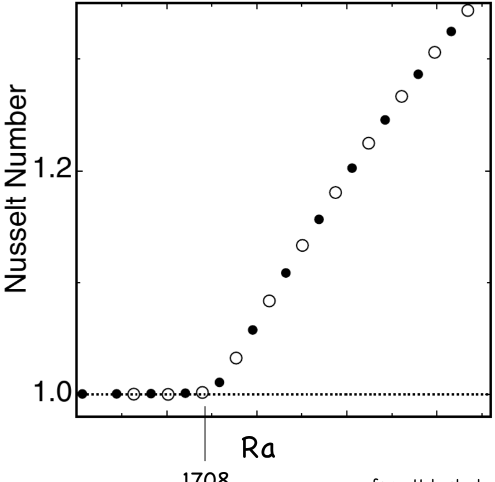




from: Krishnamurti, JFM (1970)

Heat Transport

Nusselt Number: $$Nu = \frac{q}{q_{cond}} = \frac{qd}{\lambda \Delta T}$$

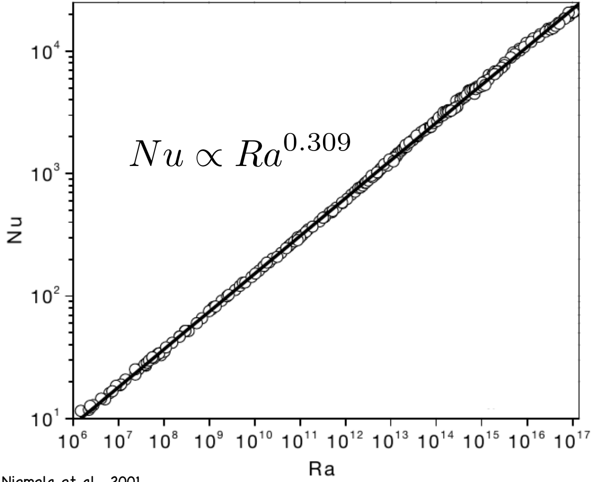


from: Hebert et al., PRE 2010

Heat Transport

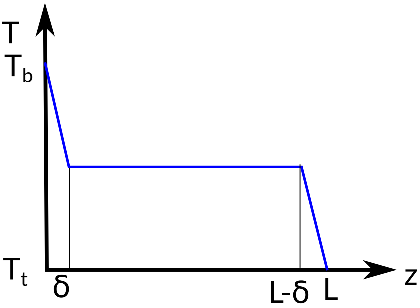
Nusselt Number:

$$Nu = \frac{q}{q_{cond}} = \frac{qd}{\lambda \Delta T}$$



from: Niemela et al., 2001

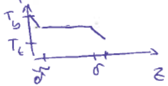
Malkus (1954)



1) Boundary layer at the top and bottom of size δ
2) Heat transported in BL by conduction
3) δ such that BL are marginally stable

A simple Model:

Malkus 1954:



from 1): $Nu = \frac{q \cdot \cancel{H} L}{\lambda \cdot \Delta T}$ with $q = \frac{\lambda \cdot \Delta T}{2\delta}$ $\Rightarrow$ $Nu = \frac{L}{2\delta}$

from 3): $Ra_c = \frac{\alpha g \frac{\Delta T}{2} \delta^3}{\nu \cdot \kappa}$

$\hookrightarrow$ $Ra = 2 \cdot Ra_c \cdot \left(\frac{\cancel{H} L}{\delta}\right)^3$ $\Rightarrow$ $\delta = L \cdot \left(\frac{Ra}{2 Ra_c}\right)^{-1/3}$

$\hookrightarrow$ $Nu = \frac{1}{2} Ra_c^{1/3} \cdot Ra^{1/3}$ $\Rightarrow$ $\underline{\underline{Nu \propto Ra^{1/3}}}$

Pretty good, but independent of Pr

$$Nu \propto Ra^{\gamma}$$

Reference	fluid	Pr	Ra range	γ
Ashkenazi & Steinberg (1999)	SF_6	1–93	10^9–10^{14}	0.30 ± 0.03
Garon & Goldstein (1973)	H_2O	5.5	10^7–3×10^9	0.293
Tanaka & Miyata (1980)	H_2O	6.8	3×10^7–4×10^9	0.290
Goldstein & Tokuda (1980)	H_2O	6.5	10^9–2×10^{11}	$\frac{1}{3}$
Qiu & Xia (1998)	H_2O	≈ 7	2×10^8–2×10^{10}	0.28
Lui & Xia (1998)	H_2O	≈ 7	2×10^8–2×10^{10}	0.28 ± 0.06
Shen *et al.* (1996)	H_2O	≈ 7	8×10^7–7×10^9	0.281 ± 0.015
Threlfall (1975)	He	0.8	4×10^5–2×10^9	0.280
Castaing *et al.* (1989)	He	0.7–1	$\lesssim 10^{11}$	0.282 ± 0.006
Wu & Libchaber (1991)	He	0.6–1.2	4×10^7–10^{12}	0.285
Chavanne *et al.* (1997)	He	0.6–0.73	3×10^7–10^{11}	$\frac{2}{7}$
Davis (1922)	air	≈ 1	$\lesssim 10^8$	0.25
Rossby (1969)	Hg	0.025	2×10^4–5×10^5	0.247
Takeshita *et al.* (1996)	Hg	0.025	10^6–10^8	0.27
Cioni *et al.* (1997)	Hg	0.025	5×10^6–5×10^8	0.26 ± 0.02
Cioni *et al.* (1997)	Hg	0.025	4×10^8–2×10^9	0.20
Glazier *et al.* (1999)	Hg	0.025	2×10^5–8×10^{10}	0.29 ± 0.01
Horanyi *et al.* (1998)	Na	0.005	$\lesssim 10^6$	0.25

TABLE 1. Power-law exponents γ of the power law $Nu \sim Ra^{\gamma}$ for various experiments. The experiments were done with different aspect ratios; however, no strong dependence of the scaling exponent γ on the aspect ratio is expected (in contrast to the prefactors, which do have an aspect ratio dependence as found by Wu & Libchaber 1992).

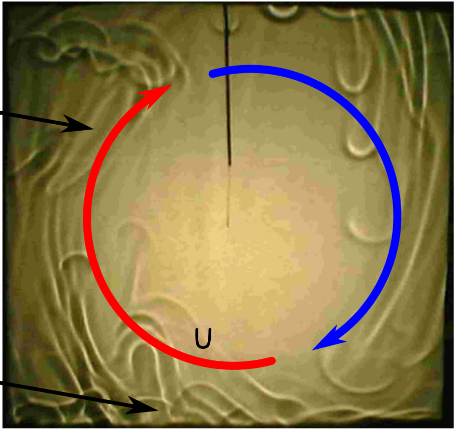
from Grossmann & Lohse, JFM (2000)

Shadowgraphy from the side

Aspect ratio = 1

Pr = 600 Ra=7×10^8

Large Scale Circulation (LSC)

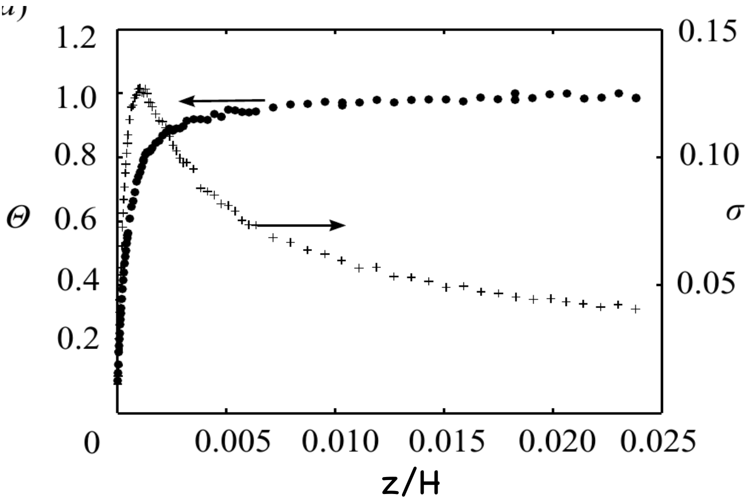


Plumes

Courtesy: Shang et al, PRL 2003

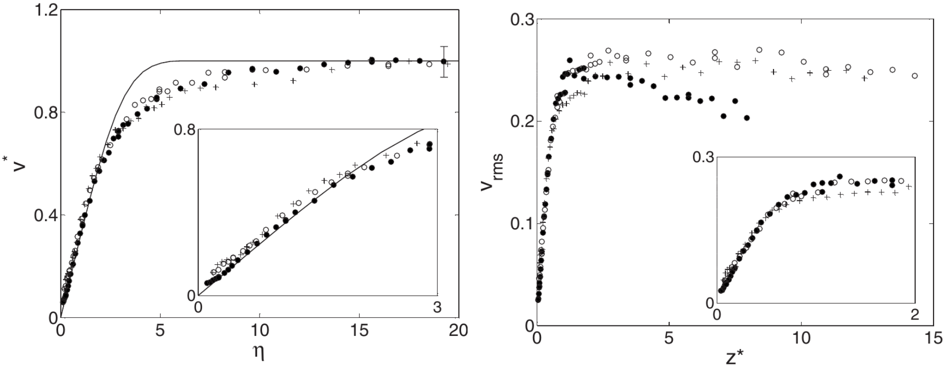
temperature profile (at distance from the top plate)

Ra=8x10^{11} Pr=0.7



from: duPuits et all, JFM 2007

Ra= 1x10^11 -> 1x10^12   Pr = 0.7



from: duPuits et al, PRE, 2009

Grossmann & Lohse (2000, 2001)

Control parameter:

$$Ra = \frac{\alpha g \Delta T L^3}{\nu \kappa} \quad ; \quad Pr = \nu/\kappa$$

Response:

$$Re = \frac{U \cdot L}{\nu} \quad ; \quad Nu = \frac{\langle u_z \Theta \rangle_{x,y,t} - \kappa \partial_z \langle \Theta \rangle_{x,y,z}}{\kappa \Delta T / L}$$

U ... maximal velocity of the mean wind

The Grossmann - Lohse model

Governing equations

$$\partial \vec{u}/\partial t + \vec{u}\cdot\nabla\vec{u} = -\nabla p + \nu\nabla^2\vec{u} + \alpha g T\hat{z}$$ momentum

$$\partial T/\partial t + \vec{u}\cdot\nabla T = \kappa\nabla^2 T$$ energy

Kinetic and thermal dissipation

$$\epsilon_u = \nu\left(\frac{\partial u_i}{\partial x_j}\right)^2$$

$$\epsilon_\theta = \kappa\left(\frac{\partial T}{\partial x_j}\right)^2$$

Volumen averaged kinetic and thermal dissipation

$$\langle\epsilon_u\rangle_V = \frac{\nu^3}{L^4}(Nu-1)RaPr^{-2}$$

$$\langle\epsilon_\theta\rangle_V = \kappa\frac{\Delta T^2}{L^2}Nu$$

Heights of kinetic and thermal boundary layers

$$\delta_u \propto \frac{L}{Re^{1/2}}$$

$$\delta_\theta = \frac{1}{2}\frac{L}{Nu}$$

Decompose dissipation in bulk and boundary layer contributions

$$\langle\epsilon_u\rangle_V = \epsilon_{u,BL} + \epsilon_{u,bulk}$$

$$\langle\epsilon_\theta\rangle_V = \epsilon_{\theta,BL} + \epsilon_{\theta,bulk}$$

This suggests 4 different regimes

1)	$\epsilon_{u,BL} \gg \epsilon_{u,bulk}$ $\epsilon_{\theta,BL} \gg \epsilon_{\theta,bulk}$	small Ra (thick BL)
2)	$\epsilon_{u,bulk} \gg \epsilon_{u,BL}$ $\epsilon_{\theta,BL} \gg \epsilon_{\theta,bulk}$	small Pr -> $\delta_u \ll \delta_\theta$
3)	$\epsilon_{u,BL} \gg \epsilon_{u,bulk}$ $\epsilon_{\theta,bulk} \gg \epsilon_{\theta,BL}$	large Pr -> $\delta_u \gg \delta_\theta$
4)	$\epsilon_{u,bulk} \gg \epsilon_{u,BL}$ $\epsilon_{\theta,bulk} \gg \epsilon_{\theta,BL}$	large Ra (thin BL)

Estimating bulk and BL contribution

with ν here

$$\partial_t \vec{u} + (\vec{u} \nabla)\vec{u} = -\nabla p + \nu \Delta \vec{u} + \alpha g \partial_z \Theta \qquad (1.5)$$

$$\partial_t \Theta + \vec{u} \nabla \Theta = \kappa \Delta \Theta \qquad (1.6.)$$

$$\varepsilon_{u,bulk} = \nu \langle (\partial_i u_j (x \in bulk(t)))^2 \rangle \approx u \cdot u^2 \frac{H-\delta_u}{LH} \sim \frac{U^3}{H} = \frac{\nu^3}{H^4} Re^3$$

this assumes that all the energy of the large scale circulation in the breaks down to smaller scale and gets dissipated ($E_{kin} \propto u^2$, its change: $\frac{u^2}{T}$ with $T = \frac{L}{u}$)

$\varepsilon_{\Theta,bulk}$ more difficult since it is not equivalent whether $\delta_u > \delta_\Theta$ or $\delta_u < \delta_\Theta$!

for $\delta_u < \delta_\Theta$ (small Pr)

analog:

$$\varepsilon_{\Theta,bulk} = \kappa \langle (\partial_i \Theta(x\in bulk,t))^2 \rangle \sim \frac{U \Delta T^2}{H} = \kappa \frac{\Delta T^2}{H^2} Pr\, Re$$

this is in couple to analogy to above

for $\delta_u > \delta_\Theta$

large Pr

the same, but the relevant velocity is reduced by $\frac{\delta_\Theta}{\delta_u}$

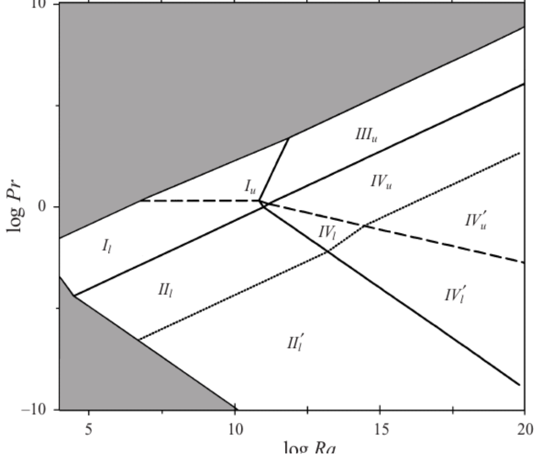
$$\varepsilon_{\Theta,bulk} \sim \frac{\delta_\Theta}{\delta_u} \frac{U \Delta T^2}{H} = \kappa \frac{\Delta T^2}{H^2} Pr\, Re^{3/2} Nu^{-1}$$

Boundary layer contributions:

$$\varepsilon_{u,BL} \sim \nu \frac{U^2}{\delta_u^2} \frac{\delta_u}{H} \sim \frac{\nu^3}{H^4} Re^{5/2}$$ (from Chavanne et al. 1997)

$$\varepsilon_{\Theta,BL} \sim \kappa \frac{\Delta T^2}{\delta_\Theta^2} \frac{\delta_\Theta}{H} \sim \kappa \frac{(\Delta T)^2}{H^2} Nu$$

(this [illegible] information, but coincides with 2.6 [illegible] constant is assumed)



Regime	Dominance of	BL	Nu	Re
I_l	$\epsilon_{u,BL}$, $\epsilon_{\theta,BL}$	$\lambda_u < \lambda_\theta$	$0.27Ra^{1/4}Pr^{1/8}$	$0.037Ra^{1/2}Pr^{-3/4}$
I_u		$\lambda_u > \lambda_\theta$	$0.33Ra^{1/4}Pr^{-1/12}$	$0.039Ra^{1/2}Pr^{-5/6}$
II_l	$\epsilon_{u,bulk}$, $\epsilon_{\theta,BL}$	$\lambda_u < \lambda_\theta$	$0.97Ra^{1/5}Pr^{1/5}$	$0.47Ra^{2/5}Pr^{-3/5}$
(II_u)		$\lambda_u > \lambda_\theta$	$(\sim Ra^{1/5})$	$(\sim Ra^{2/5}Pr^{-2/3})$
III_l	$\epsilon_{u,BL}$, $\epsilon_{\theta,bulk}$	$\lambda_u < \lambda_\theta$	$6.43 \times 10^{-6}Ra^{2/3}Pr^{1/3}$	$5.24 \times 10^{-4}Ra^{2/3}Pr^{-2/3}$
III_u		$\lambda_u > \lambda_\theta$	$3.43 \times 10^{-3}Ra^{3/7}Pr^{-1/7}$	$6.46 \times 10^{-3}Ra^{4/7}Pr^{-6/7}$
IV_l	$\epsilon_{u,bulk}$, $\epsilon_{\theta,bulk}$	$\lambda_u < \lambda_\theta$	$4.43 \times 10^{-4}Ra^{1/2}Pr^{1/2}$	$0.036Ra^{1/2}Pr^{-1/2}$
IV_u		$\lambda_u > \lambda_\theta$	$0.038Ra^{1/3}$	$0.16Ra^{4/9}Pr^{-2/3}$

TABLE 2. The power laws for Nu and Re of the theory presented, including the prefactors which are adopted from four pieces of experimental information in §4. The exact values of the prefactors depend also on how the Reynolds number is defined, see the first paragraph of §4. Regime II_u is in brackets as it turns out that it does not exist for this choice of prefactors.

from (

In GL-theory: laminar viscose boundary layers

What happens if viscose BL become turbulent?

Kraichnan 1962:

- for turbulent BL:

$$Nu \propto [Ra/(\ln Ra)^3]^{1/2}$$

The ultimate regime

Heat transport is bulk limited

Most relevant for geo/astro-physical applications

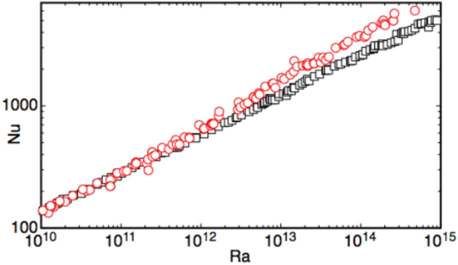
How to reach large Ra in the experiment?

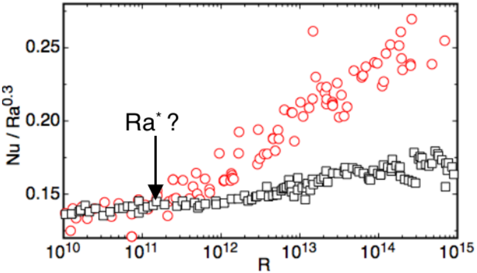
$$Ra = \frac{\alpha}{\nu\kappa} \cdot g \Delta T L^3$$

- using a gas near its critical point

Problem:

- Pr varies
- Non-Oberbeck- Boussinesq effects





- Chavanne, et al, Phys. Fluids 13, 1300 (2001)
  found a transition at Ra~10^{11}-10^{12}
- Niemela, et al, Nature 404, 837 (2000)
  found no transition

Similar experiments using Helium close to its critical point

$$Ra = \frac{\alpha}{\nu\kappa} \cdot g\Delta T L^3$$

Fluid: Sulfur hexafluoride (SF_6)

- non-toxic (but strong greenhouse gas)
- density: 6.63 kg/m^3 @ 1atm (6x air)

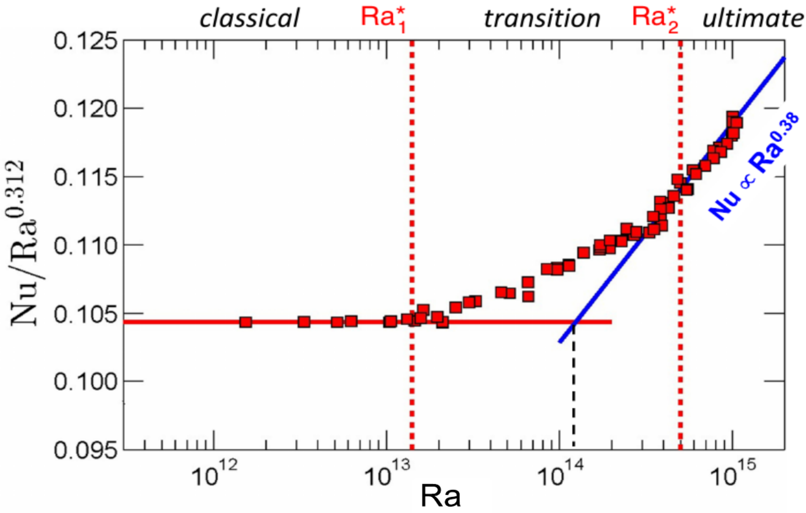
→ small kinematic viscosity ν

→ large Rayleigh numbers

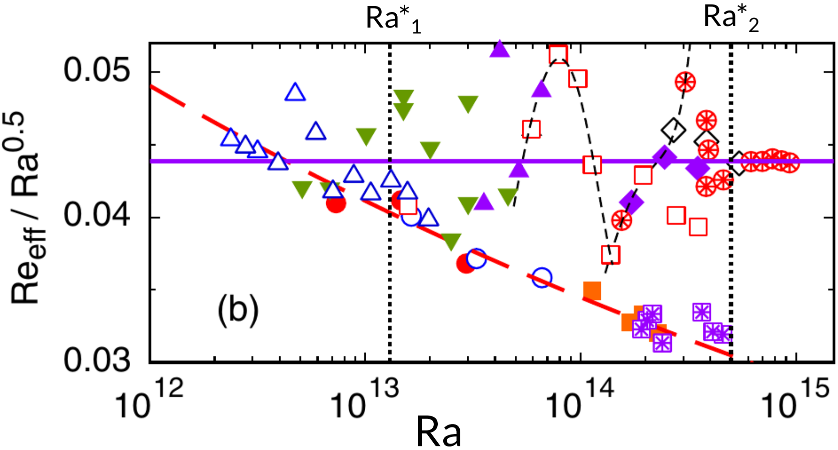
Pressure: 1×10^{-3} bar → 20 bar

→ variable density and kinematic viscosity

Γ = 0.5



He et al., PRL, 108, 024502, 2012



He et al., PRL, 108, 024502, 2012