Some basics

I) Physical quantities and their SI units

Quantity Φ	SI unit $[\Phi]$
Temperature T	K
Velocity $\vec{u}$	$m s^{-1}$
Mass density ρ_0	$kg\, m^{-3}$
Heat Q	J (= Ws = Nm)
Specific heat c_p	$J K^{-1} kg^{-1}$
Heat conductivity λ	$W m^{-1} K^{-1}$
Local heat flux density $\vec{q}$	$W m^{-2}$
Thermal diffusivity κ	$m^2 s^{-1}$

II) Fourier law of heat conduction

$$\vec{q} = -\lambda \nabla T$$

Remarks : i) $\int_A \vec{q} \cdot d\vec{A} = \frac{dQ}{dt}$

ii) $\kappa := \frac{\lambda}{\rho_0 c_p}$

κ obeys the dimension of a diffusion constant

Other examples :

$\nu \rightarrow$ momentum diffusion

$D \rightarrow$ mass diffusion

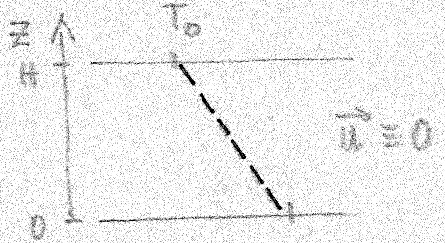
iii) We define $\vec{j}_d := \frac{1}{\rho_0 c_p} \vec{q} = -\kappa \nabla T$

$\vec{j}_d$ is called the diffusive heat current

$$[\vec{j}_d] = m s^{-1} K = [\vec{u} T]$$

III) Nusselt number Nu

(S1)



$T_0 + \Delta T, \ \Delta T > 0$

fluid layer below onset of convection

$$Nu := \frac{\langle q_{z,\text{total}} \rangle}{q_z} = \frac{\langle j_{z,\text{total}} \rangle}{j_{z,d}} \qquad (*)$$

$$q_z = -\lambda \frac{\partial T}{\partial z} = -\lambda \left(-\frac{\Delta T}{H} \right) = \lambda \frac{\Delta T}{H}$$

$$j_{z,d} = -\kappa \frac{\partial T}{\partial z} = \kappa \frac{\Delta T}{H}$$

Remarks:

i) $\langle \cdot \rangle$ in (*) is an average that will be specified later

ii) $Nu \geq 1$
For $\vec{u} \equiv 0$: $Nu = 1$ (see S1)

iii) Definition (*) holds for $\vec{u} \equiv 0$ and $\vec{u} \neq 0$

Purely diffusive heat transfer with the fluid being at rest corresponds to $Nu = 1$.
Above the onset of convection (with $\vec{u} \neq 0$) we have $Nu > 1$.

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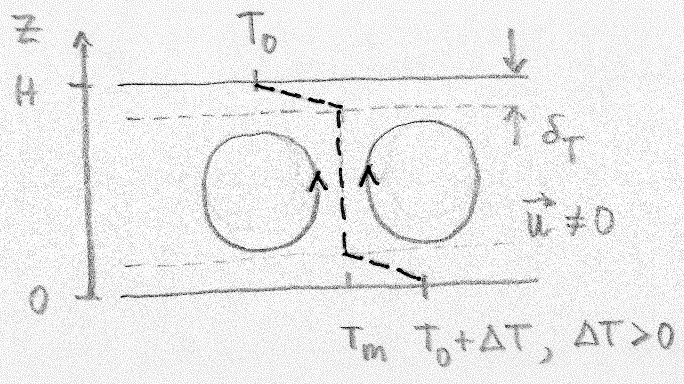


fluid layer above onset of convection

Remarks: i) δ_T is the thickness of the thermal boundary layer, same at top/bottom

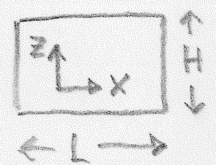
ii) $\vec{u} \equiv 0$ at $z = 0, H$ no-slip boundary condition

iii) $u_z = 0$

$\frac{\partial u_x}{\partial z} = \frac{\partial u_y}{\partial z} = 0$ at $z = 0, H$ free-slip boundary condition

iv) $T_m = \Delta T / 2$

Take temperature equation; 2d case in box:



$$\frac{\partial T}{\partial t} + \vec{u} \cdot \nabla T = \kappa \nabla^2 T$$

with $\nabla \cdot \vec{u} = 0$ $\rightsquigarrow$ $\frac{\partial T}{\partial t} + \nabla \cdot \overbrace{\left(\vec{u} T - \kappa \nabla T \right)}^{=: \vec{j}} = 0$

Convection flow is in statistically steady state

Take time average $\langle \cdot \rangle_t$

$\rightsquigarrow$ $\nabla \cdot \left\langle \vec{u} T - \kappa \nabla T \right\rangle_t = 0$

$\rightsquigarrow$ $\frac{\partial}{\partial x} \left\langle u_x T - \kappa \frac{\partial T}{\partial x} \right\rangle_t + \frac{\partial}{\partial z} \left\langle u_z T - \kappa \frac{\partial T}{\partial z} \right\rangle = 0$

Take additional average $\langle \cdot \rangle_x = \frac{1}{L} \int_0^L \cdot \, dx$

$\rightsquigarrow$ First term $= 0$, check boundary conditions

$$\rightsquigarrow \quad \frac{\partial}{\partial z}\left\langle u_z T - \kappa \frac{\partial T}{\partial z}\right\rangle_{x,t} = 0$$

$$\rightsquigarrow \quad \langle u_z T\rangle_{x,t} - \kappa \frac{\partial}{\partial z}\langle T\rangle_{x,t} = \text{const.} \quad (**)$$

<u>Remarks</u>: i) $(**)$ holds $\forall z \in [0,H]$

ii) 1st term of $(**)$:

mean convective heat current

$$\langle j_{z,c}(z)\rangle_{x,t} = \langle u_z T\rangle_{x,t}$$

2nd term of $(**)$:

$$\langle j_{z,d}(z)\rangle_{x,t} = -\kappa \frac{\partial \langle T\rangle_{x,t}}{\partial z}$$ mean diffusive heat current

iii) Nusselt number definition can be specified now to:

$$\boxed{Nu = Nu(z) = \frac{\langle u_z T\rangle_{x,t} - \kappa \frac{\partial \langle T\rangle_{x,t}}{\partial z}}{\kappa \frac{\Delta T}{H}}}$$

iv) $Nu(z=0) = -\frac{H}{\Delta T}\left.\frac{\partial \langle T\rangle_{x,t}}{\partial z}\right|_{z=0}$

since $u_z\big|_{z=0} \equiv 0$

S2 suggests that mean temperature profile can be approximated by piecewise linear function

$$\rightsquigarrow \left.\frac{\partial \langle T\rangle_{x,t}}{\partial z}\right|_{z=0} = -\frac{\Delta T}{2\delta_T} \quad \rightsquigarrow \quad \boxed{Nu = \frac{H}{2\delta_T}}$$