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Extreme vorticity growth in Navier–Stokes turbulence

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ABSTRACT

According to statistical turbulence theory, the ensemble averaged squared vorticity ρ_E is expected to grow not faster than $d\rho_E/dt \sim \rho_E^{3/2}$. Solving a variational problem for maximal bulk enstrophy (E) growth, velocity fields were found for which the growth rate is as large as $dE/dt \sim E^3$. Using numerical simulations with well resolved small scales and a quasi-Lagrangian advection to track fluid subvolumes with rapidly growing vorticity, we study spatially resolved statistics of vorticity growth. We find that the volume ensemble averaged growth bound is satisfied locally to a remarkable degree of accuracy. Elements with $dE/dt \sim E^3$ can also be identified, but their growth tends to be replaced by the ensemble-averaged law when the intensities become too large.

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1. Introduction

Fluid turbulence may be characterized as a tangle of intermittent vortices embedded in regions of straining motion [1,2]. Theoretical and computational studies of the evolution of isolated intense vortices [3,4], pairs of vortices [5–8], highly-symmetric vortex tangles or ensembles of randomly distributed vortices [9,10] have contributed considerably to our understanding of the statistical properties of homogeneous isotropic Navier–Stokes turbulence, but numerical computation of high-vorticity events in a turbulent flow remains elusive because of the high demands on spatial and temporal resolution. The properties of such events, their frequency and maximal intensities, are important for small scale mixing, the efficiency of combustion processes, and for modeling turbulence. The situation for the inviscid Euler equation is not much different. Recently, Bustamente and Kerr [11] discussed in detail the sensitivity of vorticity growth on grid resolution and de-aliasing techniques for interacting highly-symmetric anti-parallel vortices. They came to different conclusions from Hou and Li [12] who observed a depletion of the vortex stretching in their Fourier smoothing method. The question of finite-time blow-up in solutions of the Euler equations remains an open area of investigation.

For viscous fluids there is a bound on the relation between the growth rate of the volume integrated squared vorticity E , namely $dE/dt \leq aE^3$ with known prefactor a , and there is also a calcula-

tion of optimal fields for which $dE/dt \sim E^3$. On the other hand, the ensemble averaged squared vorticity ρ_E is not expected to grow faster than $d\rho_E/dt \sim \rho_E^{3/2}$. In an effort to understand the relation between these two results and how they translate to turbulence, we conducted high-resolution direct numerical simulations with a special focus on the dynamics on small scales and the evolution of strong vorticity amplification elements. This is the objective of the present work.

The strong temporal variations near “almost singular” events in turbulence can only be resolved with a sufficiently small time step, which together with a high spatial resolution requires the storage of huge data files. We use a cubic box with periodic boundaries in all directions and solve the Navier–Stokes equations numerically,

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla) \mathbf{u} = -\nabla p + \nu \Delta \mathbf{u} + \mathbf{f}, \quad (1)$$

$$\nabla \cdot \mathbf{u} = 0, \quad (2)$$

where \mathbf{f} is a large-scaling forcing. We apply the pseudospectral method with 2/3 de-aliasing and obtain a homogeneous isotropic and statistically stationary three-dimensional flow at a Taylor microscale Reynolds number of $R_\lambda = 107$ [13]. The grid size is 2048^3 , so that when expressed in terms of the Kolmogorov length $\eta_K = \nu^{3/4}/\langle \epsilon \rangle^{1/4}$ (with the mean energy dissipation rate $\langle \epsilon \rangle$) the sides of the box are $683\eta_K$ long and there are 3 grid points per η_K . Since the crossover from viscous to inertial range occurs near a scale of $8\eta_K$, we can resolve singular events into the viscous range. The time step $\Delta t = 0.003\tau_\eta$, where $\tau_\eta = \sqrt{\nu/\langle \epsilon \rangle}$ is the Kolmogorov time, is well within the limits discussed by Donzis and Sreenivasan [14]. The events we would like to study are followed for a time interval of about $55\tau_\eta$ units or 1830 output steps. This

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would add up to about 5×10^{13} velocity field values that have to be stored. In order to avoid this, we turn to the so-called quasi-Lagrangian method [15] which eliminates the large scale sweeping motion superimposed on the localized singular events we want to study. Specifically, we follow 100 Cartesian cubes V_L simultaneously through different regions of the evolving Navier–Stokes flow. The motion of the subvolumes V_L is fixed by the advection of a Lagrangian tracer in their center, and their sides are kept aligned with the outer coordinates. The boxes have a side length of $L = 17\eta_K$, corresponding to 51^3 grid points. This reduces the number of velocity field values that have to be stored to 1.4×10^{-3} of the original estimate.

2. Analytic predictions on the growth rate of enstrophy

The vorticity is the curl of the velocity field, $\boldsymbol{\omega} = \nabla \times \mathbf{u}$, and the enstrophy is the volume integral of its intensity,

$$E(t) = \int_V \boldsymbol{\omega}^2 dV. \quad (3)$$

It follows from the Navier–Stokes equation that in any incompressible Newtonian fluid the growth rate of the enstrophy, dE/dt , obeys,

$$\frac{dE(t)}{dt} = 2 \int_V (\boldsymbol{\omega} \cdot \nabla \mathbf{u}) \cdot \boldsymbol{\omega} dV - 2\nu \int_V (\nabla \boldsymbol{\omega})^2 dV. \quad (4)$$

From this it can be shown that $E(t)$ cannot grow faster than

$$\frac{dE(t)}{dt} \leq \frac{27c^3}{16\nu^3} E(t)^3, \quad (5)$$

with ν the kinematic viscosity of the fluid and $c = \sqrt{2/\pi}$ (for details see [16,17]). This holds for the volume integrated quantity and does not make any assumptions on the flow. Incompressible flow fields that maximize the enstrophy production, and thus the growth rate of enstrophy, were recently found by solving an optimization problem [16,17]. At high Reynolds number the maximum enstrophy growth rate (5) is realized by colliding, axially symmetric vortex rings. At lower Reynolds numbers, maximum enstrophy generation is realized by interacting Burgers vortices with $dE/dt \sim E^{7/4}$.

There is a second analytical result that pertains to the growth rate of the ensemble averaged squared vorticity (or enstrophy density)

$$\rho_E = \langle \boldsymbol{\omega}^2 \rangle \quad (6)$$

for the particular case of homogeneous and isotropic (box) turbulence. A direct consequence of the von Kármán–Howarth (KH) equation [18] for the velocity correlations, when the volume average $\langle \cdot \rangle$ in (7) agrees with the ensemble average that appears in the KH equation, is derived in [19,20] and states that

$$\frac{d}{dt} \rho_E = -\frac{7S}{3\sqrt{15}} \rho_E^{3/2} - 70\nu \langle (\partial_x^2 u_x)^2 \rangle, \quad (7)$$

where S is the skewness of the longitudinal velocity derivative, and u_x is the x -component of the turbulent velocity field. It is an empirical fact that $S < 0$. It has been observed that the skewness S is basically constant for Taylor microscale Reynolds numbers $R_\lambda \lesssim 200$ and it grows weakly as $|S| \sim R_\lambda^{0.11}$ for $R_\lambda > 200$ [21,22]. Thus we will assume for purposes of discussion and data analysis that $d\rho_E/dt \sim \rho_E^{3/2}$ holds approximately. This exponent is much smaller than the one in the upper bound (5).

In order to make the relation between the two relations more explicit, we rewrite the first one assuming that $E = L^3 \rho_E = L^3 \langle \boldsymbol{\omega}^2 \rangle$

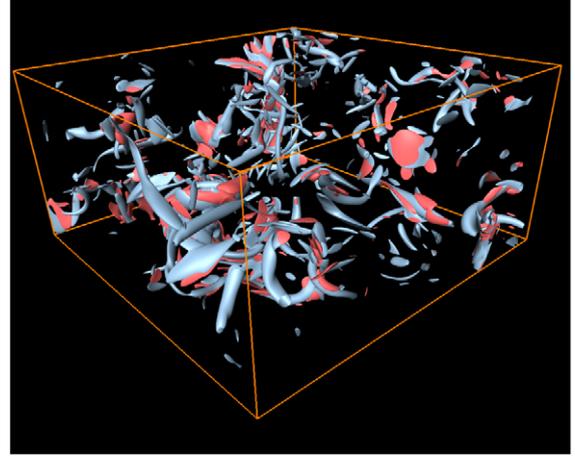


Fig. 1. (Color online.) Vortex tubes and dissipation sheets in homogeneous isotropic turbulence. Isosurfaces of the vorticity magnitude square (or local enstrophy) $\Omega = \boldsymbol{\omega}^2$ (cyan) and the energy dissipation rate $\epsilon = 2\nu S_{ij} S_{ij}$ (red) with $S_{ij} = (\partial u_i/\partial x_j + \partial u_j/\partial x_i)/2$ the rate of strain tensor. Both surfaces are shown at the level of ten times their mean. The displayed volume is 1/16 of the full simulation box.

for a box of length L and $\langle \epsilon \rangle = \nu \rho_E E$ to bring in the Kolmogorov length. Then the bound suggests that

$$\frac{d}{dt} \rho_E \leq \frac{27c^3}{16} \left(\frac{L}{\eta_K} \right)^6 \rho_E^{3/2}. \quad (8)$$

The key difference then is a ratio of lengths: if the sidelength L of the volume V is of the order of the Kolmogorov scale, both results imply the same exponent, despite the different boundary conditions and derivations. But if the volume is larger than η_K , a stronger variation is possible: in a situation where the vorticity content of the volume is below the mean that enters the definition of the Kolmogorov length, the local dissipation length is larger than the statistical average, and a built up of enstrophy will reduce the local value, thereby increasing the contribution from the factor (L/η_K) .

3. Results

3.1. Local quasi-Lagrangian analysis

Configurations as highly symmetric as the colliding vortex rings that realize the maximum instantaneous enstrophy generation cannot be generically expected in a turbulent flow. In particular, the high-amplitude vorticity events in turbulence arise preferentially in the form of tubes [1,2,23] which are rapidly stretched and deformed by background and self-induced straining motions [24,25]. Strong strain (or shear) results in high-amplitude energy dissipation rates. The spatial distribution of high-amplitude events of the local enstrophy and energy dissipation rate, illustrated in Fig. 1, underlines this behavior. Isosurfaces of the *local* enstrophy, $\Omega(\mathbf{x}, t) = \boldsymbol{\omega}^2(\mathbf{x}, t)$ (Fig. 1, cyan) at ten times the mean value show primarily elongated structures. The isosurfaces of energy dissipation rate, $\epsilon(\mathbf{x}, t) = (\nu/2)(\partial u_i/\partial x_j + \partial u_j/\partial x_i)^2$ (Fig. 1, red) also at ten times the mean value, reveal sheet-like structures between the high-vorticity events. This illustrates that while the ensemble averaged values of energy dissipation and vorticity are related by $\langle \epsilon \rangle = \nu \rho_E E$ this does not apply to their instantaneous and local values. It does show, however, that extreme events occur at neighboring locations.

To capture this quantitatively, we study the probability density functions (pdf's) of local energy dissipation and vorticity in Fig. 2. The pdf's of $\epsilon(\mathbf{x}, t)$ and $\Omega(\mathbf{x}, t)$ show stretched exponential tails indicative of strong small-scale intermittency (see Fig. 2(a)). The

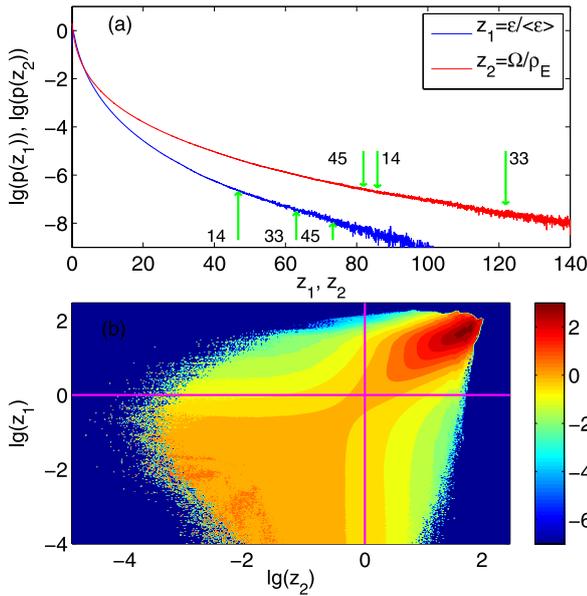


Fig. 2. (Color online.) Statistics of local enstrophy and energy dissipation rate. (a) Probability density functions of local enstrophy, Ω , and energy dissipation rate, ϵ , given in units of their means, respectively. The vertical arrows mark the global maxima of Ω and ϵ , respectively, in case of three Lagrangian tracers, No. 14, 33 and 45. (b) Joint probability density function of local enstrophy and energy dissipation rate. The distribution is normalized by both single quantity distributions, $p(z_1, z_2)/[p(z_1)p(z_2)]$, in order to highlight the statistical correlations between $z_1 = \epsilon/\langle\epsilon\rangle$ and $z_2 = \Omega/\rho_E$. Color coding is in decadic logarithm.

tail is more extended for Ω than for ϵ , in agreement with the observations in [26]. Fat tails imply that large amplitude events are significantly more probable than for a Gaussian distributed signal with the same second moment. In Fig. 2(a) we also mark the global maxima in ϵ and Ω identified within the advected volumes V_L for three particular Lagrangian trajectories. The locations far out in the tails document that our quasi-Lagrangian tracking is able to detect high-amplitude events and that extreme events in both quantities are spatially correlated and located within our advected volume. The local correlation between high-amplitude local enstrophy and energy dissipation events is further supported by the joint pdf in Fig. 2(b), where $p(\epsilon/\langle\epsilon\rangle, \Omega/\rho_E)/[p(\epsilon/\langle\epsilon\rangle)p(\Omega/\rho_E)]$ is shown. The maximum values appear in the upper right of the support where the largest amplitudes for both are present. High-amplitude fluctuations in energy dissipation and local enstrophy density are thus strongly statistically correlated and found very close together, in both, space and time.

We now turn to the study of the time evolution of extreme events within our subvolumes. Fig. 3(a) shows time traces of the local enstrophy

$$E_L(t) = \int_{V_L} \omega^2 dV. \quad (9)$$

Since we are interested in the relation between large values of dE_L/dt with E_L , we show in Fig. 3b the same data as a scatter plot in the plane spanned by E_L and dE_L/dt on a double logarithmic scale. The collection of the individual growth histories in the subvolumes (which can vary strongly from one to another) is bounded from above by the scaling $dE_L/dt \lesssim E_L^{3/2}$, indicated there by the dashed line. This shows that the local growth rate and enstrophy are related very much as are the volume averages, Eq. (4), so that the effects of sweeping are averaged out. It suggests that the estimate

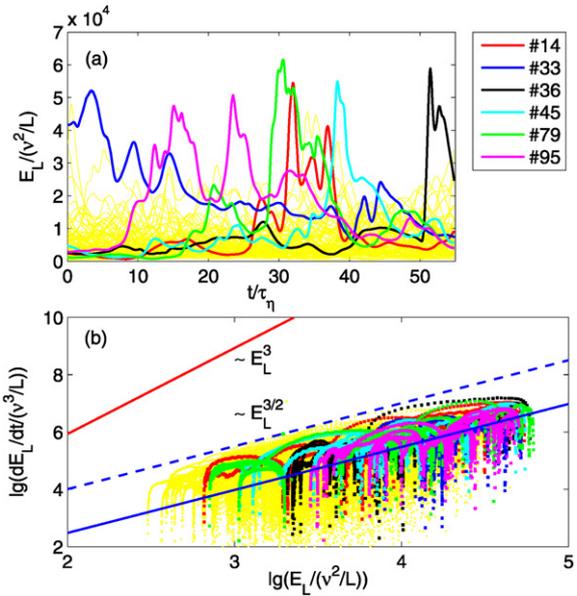


Fig. 3. (Color online.) Quasi-Lagrangian analysis of enstrophy. (a) Time series of $E_L(t)$ for all 100 subvolumes V_L are plotted. The time traces that reach the largest local maxima for E_L are colored differently and their labels are indicated in the legend. Enstrophy is given in units of v^2/L with $L = 16\eta\kappa$. (b) Enstrophy growth rate versus enstrophy. The enstrophy growth rate, dE_L/dt , is given in units of v^3/L^3 . The *a priori* upper bound $dE_L/dt = 27\sqrt{2}/(8v^3\sqrt{\pi^3})E_L^3$ is indicated as a red line. The growth that follows from the von Kármán–Howarth equation [18], $dE_L/dt \approx -7S/3\sqrt{15}E_L^{3/2}$ with a derivative skewness of $S = -0.5$, is indicated as a solid blue line. The dashed blue line has the same slope and serves as a guide to the eye. Color coding is as in panel (a).

$$\frac{dE_L}{dt} \sim \int_{V_L} (\boldsymbol{\omega} \cdot \nabla \mathbf{u}) \cdot \boldsymbol{\omega} dV \sim \langle \omega^2 \rangle^{3/2} V_L \sim \sqrt{E_L^3/V_L} \quad (10)$$

also holds locally. We have also verified this scaling in a volume $V_L/8$, i.e. in a cube with half the sidelength. The striking observation is that the envelope of the local enstrophy growth follows the scaling of (7). However, we wish to stress that (generically) there are nonvanishing enstrophy fluxes, $\int_{\partial V_L} \omega^2 \mathbf{u} \cdot d\mathbf{A} \neq 0$, across the bounds of V_L . Therefore (10) is only a heuristic estimate.

3.2. Local analysis of extreme events

The previous analysis shows that when the average volume has a diameter of about the Kolmogorov length or above, the bulk estimate for the enstrophy growth is recovered. Stronger growths, therefore, may only occur on smaller scales. To probe for this, we turn to the study of the time evolution of local extrema within the cell. We select the grid point \mathbf{x}^* with the fastest local growth rate within the subvolume, $d\Omega/dt|_{\max} = \max_{\mathbf{x} \in V_L(t)} [d\Omega/dt]$. As an indication of the numerical uncertainty, we also show the 26 growth rates for points on a 3^3 cube surrounding \mathbf{x}^* . Figs. 4(a) and (b) demonstrate that the maximum position always gives the outer envelope of the curve. The curve is continuous by construction, but its slope is discontinuous when the position \mathbf{x}^* of the point of maximal growth rate jumps discontinuously within the cell. Panel (a) shows also that long periods of low variability and small growth rates are interrupted by short violent outbursts of the local enstrophy growth rate. The two subsequent intervals I and II for tracer No. 45 mark exactly such a rapid growth event.

Fig. 4(b) shows the same data as (a) in the plane spanned by the local enstrophy and its growth rate, $d\Omega/dt$ and Ω . The first part follows a scaling that resembles the global enstrophy growth bound, $d\Omega/dt \sim \Omega^3$. It is connected with a rapid stretching by two

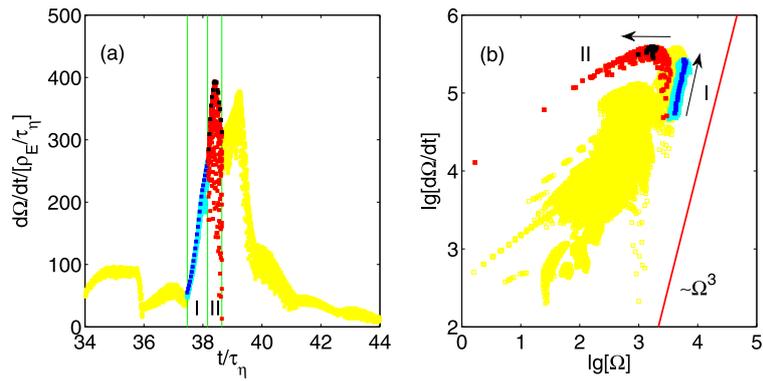


Fig. 4. (Color online.) Local analysis of very rapid enstrophy growth events. (a) Time evolution of the maximum of $d\Omega/dt$ in V_L for tracer No. 45. Growth rates at the grid point of the maximum (blue for I and black for II) and the 27 neighboring points (cyan for I and red for II) are shown. The curve is piecewise continuous since one local maximum in V_L takes over a former at a different grid point. (b) Replot of the data from (a) in the $d\Omega/dt-\Omega$ plane. The arrows in panel (b) indicate the time evolution.

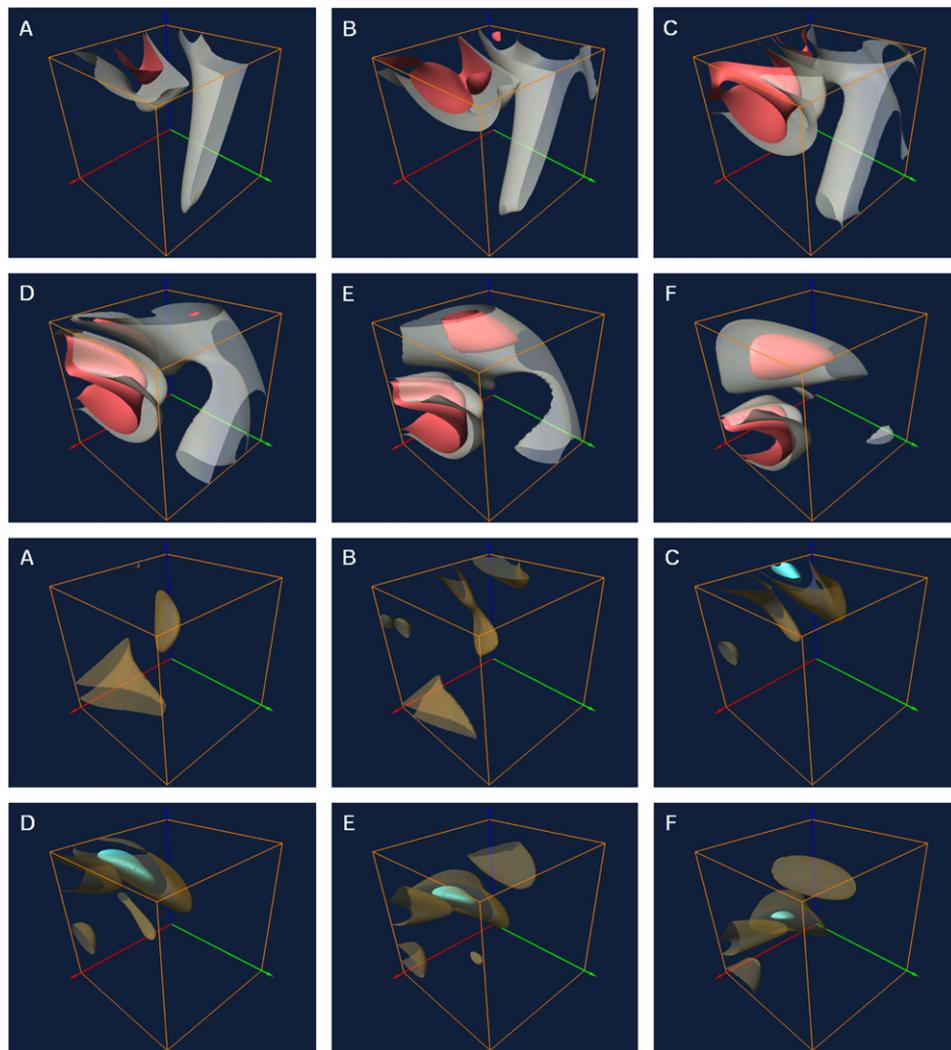


Fig. 5. (Color online.) Structures in an extreme enstrophy growth event. Upper six panels: Isosurfaces of Ω (red: 20 (Ω), gray: 10 (Ω)). Lower six panels: Isosurfaces plots of ϵ (yellow: 2 (ϵ), cyan: 20 (ϵ)). Panels (A) for $t/\tau_\eta = 37.44$, (B) for $t/\tau_\eta = 37.8$, (C) for $t/\tau_\eta = 38.16$ correspond with time interval I in Fig. 4; (D) for $t/\tau_\eta = 38.52$, (E) for $t/\tau_\eta = 38.88$ and (F) for $t/\tau_\eta = 39.24$ with time interval II.

approaching and interacting tubular vortex segments and thereby is similar to the vortex ring collision seen for the optimal growth events. However, when dissipation kicks in and the high-amplitude dissipation region between the tubes forms, the growth is weaker as can be seen Fig. 5 where we plot a sequence of isosurface plots

of $\Omega(\mathbf{x}, t)$ and $\epsilon(\mathbf{x}, t)$.¹ We might interpret it as a precursor of an

¹ Although we tracked 100 subvolumes simultaneously within the evolving turbulent flow, this specific event was the only one observed that displayed such a rapid growth in connection with a large local enstrophy amplitude.

“attempted” finite-time blowup. Further unbounded growth is prohibited by the imperfect collision of the tube segments, since it is highly unlikely that fully developed turbulence generates highly symmetric flow configurations that pin the local vorticity maximum fixed in space [9].

4. Conclusions

Our results show that the local enstrophy growth $dE_L/dt \sim E_L^{3/2}$ expected from the volume and ensemble averaged Navier–Stokes equations can also be observed locally in volumes of a diameter equal to 17 Kolmogorov lengths that are advected by the flow. The typical enstrophy growth time scales in these local volumes are of the order of the Kolmogorov time scale τ_η . They show that there is negligible influence from the sweeping motion, so that enstrophy growth is dominated by local events. Within the boxes larger exponents can be observed when the total vorticity inside the volume is weak, since then the instantaneous, local vorticity gives a larger (local) Kolmogorov length, and the increase in enstrophy and corresponding decrease in the (local) Kolmogorov length contributes to an increasing ratio (L/η_K). During such events the local vorticity can be amplified with a rate $dE/dt \sim E^3$, but this growth is cut off after some τ_η . A further growth of enstrophy is then limited by the shear layer and thus the enhanced dissipation that is formed between the interacting vortex filaments. This conclusion is supported both by displaying the evolution of vortex filaments (as seen in Fig. 5) and by monitoring the energy dissipation rate and enstrophy accumulated in the comoving volumes (not shown). Moreover, they seem to be rare and overwhelmed by the $E^{3/2}$ behavior. Nevertheless, already for the present moderate Reynolds number we found such rapid growth events and expect their more frequent appearance for larger ones.

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