Appendix A
Brooks' Fundamental Paper

A.1 On Colouring the Nodes of a Network

R. L. Brooks

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The purpose of this note is to prove the following theorem.

Let N be a network (or linear graph) such that at each node not more than n lines meet (where $n > 2$), and no line has both ends at the same node. Suppose also that no connected component of N is an n-simplex. Then it is possible to colour the nodes of N with n colours so that no two nodes of the same colour are joined.

An n-simplex is a network with $n+1$ nodes, every pair of which are joined by one line.

N may be infinite, and need not lie in a plane.

A network in which not more than n lines meet at any node is said to be *of degree not greater than n*. The colouring of its nodes with n colours so that no two nodes of the same colour are joined is called an "n-coloring".

Without loss of generality we may suppose that N is connected, for otherwise the theorem can be proved for each connected component; and that it is not a simplex. With these suppositions, N is finite or enumerable, both as regards nodes and lines.

Now we can $(n+1)$-colour N with colours $c_0, c_1, \dots, c_n$, by giving to each node in turn a colour different from all those already assigned to nodes to which it is

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directly joined. We can then apply the following operations, in which the colours of directly joined nodes remain distinct.

(1) A node directly joined to not more than $n-1$ colours can be recoloured not-c_0. (In the term "recolouring" we include for convenience the case in which no colour is altered.) In particular, a node directly joined to two nodes of the same colour may be recoloured not-c_0.

(2) If P and Q are directly joined they can be recoloured without altering any other nodes, so that P is not-c_0. For neglecting the join PQ, we may recolour P not-c_0, by (1); and Q can then be recoloured (possibly c_0).

(3) Let $P, P', P'', \dots, Q$ be a path, i.e. suppose every consecutive pair of nodes directly joined. Then we can recolour $P, P', \dots, Q$ successively, without altering any other nodes, so that at most Q has finally the colour c_0.

Corollary 1. If N is finite, choose Q arbitrarily in N. Since there is a path joining Q to every node P in N, we can recolour N with at most the node Q coloured c_0.

Corollary 2. If N is infinite, let F be any connected finite part, and Q a node directly joined to, but not in, F. Then we can recolour F and Q so that no node of F is coloured c_0.

PROOF OF THE MAIN THEOREM FOR A FINITE NETWORK

Case 1. If any node X meets fewer than n lines, we can n-colour N. For let N be $(n+1)$-coloured, with at most the node X coloured c_0. Then by (1), X also can be recoloured not-c_0.

Case 2. Suppose that if P, Q, A, B are any four distinct nodes, there is a path from P to Q not including A or B

Since N is not a simplex, we can find nodes P, Q not directly joined. Let N be $(n+1)$-coloured so that only Q is coloured c_0. Then P and all nodes directly joined to it are not-c_0. Either P meets fewer than n lines, when N may be n-coloured by case 1, or there are two nodes A, B directly joined to P, which have the same colour. But there is a path joining P to Q, not including A or B. Hence, by (3), N can be recoloured, without altering A or B, so that at most P is coloured c_0. Since A and B have the same colour, P can be recoloured not-c_0, by (1). Thus N is n-coloured.

Case 3. Suppose there exist distinct nodes P, Q, A, B, such that every path from P to Q passes through A or B.

Then consider the networks, contained in N, with the following specifications:

N_1. *Nodes:* P, and all nodes joined to P by some path not passing through A or B as an intermediate point.
Lines: all lines connecting the above nodes in N.

N_2. *Nodes:* A, B, and all nodes not in N_1.
Lines: all lines connecting the above nodes in N.

Thus N_1 and N_2 are connected non-null networks, together making up the whole network N, and having in common at least one of the nodes A, B, and at most A, B, and any lines AB. Therefore, if m_i is the number of lines in N_i, and m_0 is the number of lines in AB,

$$m_1 + m_2 \leq m_0 + n. \tag{A.1}$$

Clearly, N_1 and N_2 have degree not exceeding n.

There are three subcases, 3.1, 3.2 and 3.3.

Case 3.1. Suppose N_1 and N_2 have only one node, say A, in common. Then in each of them, the node A meets fewer than n lines. Thus by case 1, N_1 and N_2 may be n-coloured; and if we permit the colours of N_2 so that the colours of A in N_1 and N_2 become the same, the whole network N is n-coloured.

Case 3.2. One of N_1 and N_2 (say N_1), is such that when the line AB is added it becomes an n-simplex.

N_1 can be n-coloured by assigning arbitrary colours to the $n-1$ nodes other than A and B, and the remaining colour to A and B. By (A.1) there is just one line in N_2 meeting A and just just one meeting B or case 3.1 holds. Thus if A and B were identified in N_2, there would still (since $n > 2$) be fewer than n lines meeting A $(= B)$. Hence by case 1 the resulting network can be n-coloured.

Case 3.3. Neither N_1 nor N_2 becomes an n-simplex on adding a join AB.

Suppose they become M_1 and M_2 respectively by this addition. Then M_1 and M_2 each contain fewer nodes than N, and by (A.1) they are of degree not greater than n. (If not we should have case 3.1.)

If M_1 and M_2 are n colourable so is N. For since both contain a line AB, in any n-colouring A must have a different colour from B in each network. We can permute the colours of M_1 so that the colours of A and B are the same as in M_2, and then by combination obtain an n-colouring of N.

Thus if N is a finite network of degree not exceeding n, and is not an n-simplex, either it is n-colourable, or it is n-colourable if two networks which satisfy the same conditions, but have fewer nodes, are n-colourable. Now it is obvious that the theorem is true for a network with less than four nodes. Therefore, by induction over the number of nodes, N is always n-colourable.

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If F is a network or a set of nodes in N, we denote by $N - F$ the network composed of the nodes of N not in F, and the lines of N neither end of which is in F.

LEMMA. For each positive r we can find a connected finite network F_r such that

(i) the connected component of $N - F_1 - F_2 - \cdots - F_r$ are infinite for all r.
(ii) every node of N lies in one and only one F_r.

Let the nodes of N be enumerated as $P_1, P_2, P_3, \ldots$, and let F_r be defined inductively as follows.

When F_s has been chosen, for $s < r$ (or without any choice when $r = 1$), let R_r be the first P_m not in any F_s. Suppose further that the connected components of $N - F_1 - F_2 - \cdots - F_{r-1}$ are infinite. Then all the finite connected components of $N - F_1 - F_2 - \cdots - F_{r-1} - R_r$ must contain a node directly joined to R_r in N. Therefore the number of these components cannot exceed n, and at least one infinite component of $N - F_1 - F_2 - \cdots - F_{r-1} - R_r$ contains a node joined by a line to R_r.

Take F_r to be the "logical sum" of R_r, all finite connected components of $N - F_1 - F_2 - \cdots - F_{r-1} - R_r$, and all lines joining them in N. Thus F_r is a finite connected network, and has no node in common with F_s, for $s < r$. Further, the connected components of $N - F_1 - F_2 - \cdots - F_r$ are simply the infinite components of $N - F_1 - F_2 - \cdots - F_{r-1} - R_r$. The inductive construction is therefore complete.

By the method of choosing R_r, P_m must lie in some F_s $(s \leq m)$.

A node Q_r can be chosen in an infinite connected component of

$$N - F_1 - F_2 - \cdots - F_{r-1} - R_r,$$

so that Q_r is directly joined in N to R_r, which lies in F_r. Thus Q_r does not lie in F_s, if $s \leq r$.

Now N can be $(n+1)$-coloured; and by (3), corollary 2, we can recolour F_r in n colours, altering only F_r and Q_r, i.e. not altering F_s for $s < r$. Thus we can recolour $F_1, F_2, \ldots,$ in turn in n colours, each recolouring not affecting the nodes already recoloured: that is, we can n-colour N.

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A.2 Rowland Leonard Brooks 1916–1993

R. L. Brooks was born on 6 February 1916 in Lincolnshire, England, and he died 18 June 1993 in London. He studied mathematics at the University of Cambridge from 1935. In Cambridge he met Cedric A. B. Smith, Arthur H. Stone and William T. Tutte, who all became life-long friends. Also, the four were close friends of Blanche Descartes. In November 1940 Tutte communicated Brooks' paper [173], containing what later became famous as Brooks' Theorem, to the Cambridge Philosophical Society. After finishing his studies at Cambridge, Brooks worked as an income-tax inspector in London. He kept a strong interest in mathematics throughout his life. A joint paper [177] by the Trinity 4, Brooks, Smith, Stone and Tutte, on determinants and current flows in electrical networks was published in the Julius Petersen volume 100 of Discrete Mathematics in 1992. In 1940 the four had published their paper [175] on the dissection of rectangles into squares in volume 7 of Duke Mathematical Journal, see also [176]. The 1992 paper [177] may be seen as a sequel.

Cedric A. B. Smith sent each year a Christmas letter to his friends, containing thoughts, history, happenings and interesting quotations from the year that had passed. We shall take the liberty to quote from these letters.

From Cedric A. B. Smith's Christmas letter 1992:

Some 56 years ago there were three mathematics students, Arthur, Leonard and Cedric, who were friends. They were joined by a chemist, Bill, when he was not doing experiments. Arthur said, "I've heard about a problem. Show that you can not divide a (geometric) square into squares of all different sizes." Sounds easy. Doesn't it? They quickly discovered that if you take 9 squares, with sides 1, 4, 7, 8, 9, 10, 14, 15, 18, you can fit them together, like a jigsaw, to make what looks exactly like a square. Try it for yourself. But if you add up the sides, alas, they are just slightly different – it is a rectangle, pretending to be a square. They did not know it, but it had been found many years before by a Pole, named Moron. Just as many of the most famous musicians are German or Italian, so, for no obvious reason, many distinguished mathematicians are Polish or Hungarian. But to return to the point. For 3 years Arthur, Leonard, Cedric and Bill worked hard, producing many rectangles filled with unequal squares, but never one which looked remotely like a square, and never any hint as to why a square was impossible. Leonard tried a jigsaw on his mother, who put it together unexpectedly differently. Surprising, but not solving our problem. But one day Arthur suggested to Cedric, "if only we make it sufficiently complicated (following a certain plan) we might succeed." There were no nice pocket calculaters then, but after 3 days hard calculations on paper, there really and truly was before them a square divided into unequal squares. Arthur and Cedric invited Leonard and Bill to coffee, saying "we have something to show you". But Leonard insisted we should have coffee with him. So Arthur and Cedric carefully drew the divided square on paper, went over to Leonard's rooms, walked in and said "look at this". But Leonard came towards them, holding a piece of paper, and said "look at this". And on it was another square divided into unequal squares in a different way. Cheers!

From Cedric A. B. Smith's Christmas letter 1993:

Cedric's great ambition was to go to Cambridge to study mathematics. He was thrilled to go to his very first lecture, on geometry. Perhaps you think that geometry is about squares and triangles, circles and spheres, cones, and all that sort of thing. Cedric thought so too. But the lecturer talked about adding points together, and multiplying points by numbers. At the end of the lecture, Cedric turned to the young man next to him, and said "That was very confusing". "No", replied his companion, "I thought it was a very good lecture. When is the next lecture?" – "at 11" – "No, it's at 10". – They went together into the lecture room. But by a misprint in the timetable, they had walked into an advanced lecture. But, happily, they were blissfully unaware of that, because the lecturer had so thick a Russian accent that for half an hour they literally did not understand one word he said. Later on in the term, when the class roared with laughter at some awful distortion of some word, the lecturer sternly explained, "Yew may laff. Baat feefty meelion people speak yore kind of Eengleesh. Fife haandred meelion people speak my kind of Eengleesh". Anyway, Cedric found *that his new friend was called Leonard Brooks. And that was the start of a friendship to last over 57 years. Recently Leonard had been developing on his computer some ideas about the representation of numbers which could be made to produce some dazzling colored patterns. On June 15, this summer, he and Cedric were to have lunched together. But at the last minute he found that it was not possible, and it was put off a week, to June 22. Then it suddenly occurred to Cedric how one could produce a lot of unexpected Brooks patterns, and he worked hard to get them ready to show Leonard on the 22nd. But at the Saturday before that, Leonard's wife Gladys rang up to say that he had quite unexpectedly had a bad heart attack. And he had not survived to see the new pattern.*

In a private letter Cedric A. B. Smith, July 2001, said:

Leonard was rather a shy person, and he did not want his photograph published, which is why it is difficult to find a photo of him.

A.3 Brooks' Theorem and Beyond

It took several years for Brooks' theorem to become widely known. In the late 1940s Gabriel Andrew Dirac, influenced by Peter Ungar and supervised by Richard Rado, took up graph coloring theory in his doctoral studies at King's College, University of London. The title of his thesis, submitted in 1951, is *On the Colouring of Graphs. Combinatorial Topology of Linear Complexes*. In the thesis Dirac defined critical graphs and rediscovered Brooks' theorem. The external examiner Cedric A. B. Smith made Dirac aware of Brooks' 1941 paper, which led to some small changes in the final version (Senate House Library, University of London). In particular, Dirac's proof of Brooks' theorem was deleted from the thesis. Thereafter, through Dirac's papers in the 1950s, Brooks' theorem became well known and the cornerstone of abstract graph coloring. To this day, Brooks' 1941 paper [173] is stimulating for many mathematicians; see the two surveys of Brooks' theorem, one by Stiebitz and Toft [976] and another by Cranston and Rabern [260].

Over time, several authors have published alternative proofs of Brooks' theorem that have some advantages compared to the original proof and to those published by other authors. The first new proof of Brooks' theorem was published by Gerencsér [413] in 1965, but only in Hungarian. Gerencsér's proof is by contradiction; so let G be a counterexample to Brooks' theorem whose order is minimum. Then G is Δ-regular, $K_{\Delta+1} \not\subseteq G$, and every induced subgraph of G except G itself has a Δ-coloring, where $\Delta \geq 3$. First, note that G does not contain a $K^-_{\Delta+1}$ (a $K_{\Delta+1}$ minus an edge) as a subgraph. For otherwise, if $D = K_{\Delta+1} - vw \subseteq G$, then $G' = G - V(D)$ has a Δ-coloring. Since both v and w have only one neighbor in G', we can extend this Δ-coloring by first coloring v and w the same color and then coloring the remaining $\Delta - 1$ vertices of D. This results in a Δ-coloring of G, giving a contradiction. Let I be a maximum independent set of G. Then $G - I$ has maximum degree at most $\Delta - 1$. If no subgraph of $G - I$ is a K_Δ if $\Delta \geq 4$ and an odd cycle if $\Delta = 3$, then the minimality of G implies that $G - I$ has a $(\Delta - 1)$-coloring and so G has a Δ-coloring, a contradiction. It remains to consider the case that G contains an induced subgraph K, where $K = K_\Delta$ if $\Delta \geq 4$ and K is an odd cycle if $\Delta = 3$. Since $K^-_{\Delta+1} \nsubseteq G$ and G is Δ-regular, we obtain that each vertex of K has exactly one neighbor in $G' = G - V(K)$ and there are at least two different such neighbors, say u and w. Then $G' + uw$ has a Δ-coloring φ (since G is a minimum counterexample). By using a simple list coloring argument, it is easy to show that φ can be extended to a Δ-coloring of G, a contradiction. Gerencsér's proof was rediscovered by Rabern [842]; the case $\Delta = 3$ of Gerencér's proof was also obtained by Dirac [298, Theorem 2]. Rabern [839] gave a refinement of Gerencér's proof presented in Section 1.2.

The list of papers giving alternative proofs of Brooks' theorem keeps growing, see the journal papers [75], [141], [183], [195], [367], [377], [413], [500], [694], [731], [746], [817], [839], [842], [888], [945], [979], [1043], and [1088]. A second group of papers provide proofs of results that directly imply Brooks' theorem, see e.g. [148], [355], [397], [636], [653], [948], [1012], and [1051]. For instance, Gallai's result in [397], saying that the low vertex subgraph of a critical graph is a Gallai forest, immediately implies Brooks' theorem. Borodin [148, 149] as well as Erdős, Rubin, and Taylor [355] proved a choosability version of Brooks' theorem.

The algorithmic aspects of Brooks' theorem have also been studied extensively, for classical algorithms, but also for PRAM algorithms and LOCAL algorithms; see the following papers for corresponding results and further references: Panconesi and Srinivasan [796], Assadi, Kumar, and Mittal [72], and Fischer, Halldórsson, and Maus [373].

Brooks also proved his theorem for locally finite graphs. Schmerl [900] gave a different proof, by constructing a recursive set hitting all maximum cliques; a shorter proof of Schmerl's result was given by Tverberg [1044]. Conley, Marks, and Tucker-Drop [254] examines measurable generalizations of Brooks' theoem.

To date, there are probably several hundred papers related to Brooks' theorem; most of these papers deal with generalizations of Brooks' theorem to other coloring parameters or with strengthening of Brooks' theorem. Already in the 1950s, Dirac noticed that the inequality $2\text{ext}(k,n) \geq (k-1)n+1$ for $n > k \geq 4$ is equivalent to Brooks' theorem, but far from optimal. However, more than half a century passed before Kostochka and Yancey asymptotically determined the function $\text{ext}(k,n)$ for any fixed $k \geq 4$, see Chapter 5. What remains is a complete proof of Ore's conjecture that $\text{ext}(k,n) = \text{ore}(k,n)$.

Let us briefly discuss a precoloring extension version of Brooks' theorem. For a graph G and a vertex set $P \subseteq V(G)$, let

$$\text{mpd}(P : G) = \min\{\text{dist}_G(u,v) \mid u,v \in P,\ u \neq v\}.$$

Note that $\text{mpd}(P : G) \geq 2$ if and only if P is an independent vertex set in G. Let S be a set of colors. Following Brooks, a coloring $\varphi : V(G) \to S$ of the vertices of G with colors from S so that no two vertices of the same color are joined by an edge is called an S**-coloring** of G. A mapping $L : V(G) \to 2^S$ is called a k-assignment for

G (with color set S) if $|L(v)| \geq k$ for every vertex v of G. An **L-coloring** of G is an S-coloring of G such that $\varphi(v) \in L(v)$ for every vertex $v \in V(G)$.

The following result was obtained by Albertson, Kostochka, and West [39] and, independently, by Axenovich [73], answering a question proposed by Albertson at the Southeastern Conference on Graph Theory, Combinatorics and Computing, Boca Raton, March 2002 (see [73, Reference 1]).

Theorem A.1 (Axenovich / Albertson, Kostochka, West) *Let G be a graph with maximum degree $\Delta \geq 3$ not containing $K_{\Delta+1}$ as a subgraph, let P be a vertex set in G such that* $\text{mpd}(P : G) \geq 8$*, and let L be a Δ-assignment for G. Then every L-coloring of $G[P]$ can be extended to an L-coloring of G.*

As pointed out in [39] the distance threshold 8 in Theorem A.1 is sharp even for the constant list assignments. Let $k \geq 3$, and let H be a graph obtained from $K_{k+1} - uv$ by adding a vertex w and joining w to v; note that w is the only vertex having degree 1 in H and u is the only vertex having degree $k-1$ in H. Let G be the graph obtained from k (disjoint) copies of H by adding all possible edges between the vertices of degree 1, let P be the set of of vertices having degree $k-1$, and let $S = \{1, 2, \ldots, k\}$ be a set of k colors. Then it is easy to check that G has maximum degree $\Delta = k$, $\text{mpd}(P : G) = 7$, and there is no coloring φ of G with color set S such that $\varphi(u) = 1$ for all vertices $u \in P$.

The following strengthening of Theorem A.1 was obtained by Rackham [843]; again the new distance threshold is sharp.

Theorem A.2 (Rackham) *Let G be a connected graph with maximum degree $\Delta \geq 3$ not containing $K^-_{\Delta+1}$ as a subgraph, let P be a vertex set in G such that* $\text{mpd}(P : G) \geq 4$*, and let L be a Δ-assignment for G. Then every L-coloring of $G[P]$ can be extended to an L-coloring of G.*

Appendix B
Tutte's Lecture from 1992

B.1 Fifty Years of Graph Colouring

W. T. Tutte

Friday, June 26, 1992

I learned the Four Colour Problem from school, when I borrowed Rouse Ball's "Mathematical Recreations and Essays" from the school library. At the University, in the late Thirties I sought more information. I learned that workers on the problem were studying irreducible configurations. Every so often a paper would appear giving new reducible configurations and raising the minimum number of faces for a 5-chromatic planar cubic map.

I learned also that Hassler Whitney had posed colouring problems for graphs in general. He sought colourings of the vertices of a graph in n colours so that no two of the same colour were adjacent [here Tutte refers to the paper [1068]].

I played with colouring problems myself, with no significant result. But my fellow-student, Leonard Brooks, asked me to check the proof of his memorable theorem, and I had the honour of transmitting his paper to the Cambridge Philosophical Society. It was published in their mathematical Proceedings in 1941. Let me remind you of what this theorem says. If a [connected] graph G, with no loops or multiple joins, cannot be coloured in n colours, where $n > 2$, then either G is a complete graph of $n+1$ vertices or it has a vertex whose valency exceeds n. All graphs of this lecture are to be without loops or multiple joins.

Some time before I began my Ph. D. work in 1945 I learned of two conjectures that would, if valid, generalize Brooks' Theorem. There was the conjecture of Hajós, that every graph not colourable in n colours must contain a subdivision of a $K(n+1)$, that is of a complete graph on $n+1$ vertices. But as we know this conjecture was disproved by P. A. Catlin in a paper of 1979 (J. Combinatorial Theory). Then there was a conjecture of H. Hadwiger, that every graph not colourable in n colours could

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be changed, by deleting some edges and contracting others, into a $K(n+1)$. Nowadays we would express this condition by saying that the graph must have $K(n+1)$ as a "minor". Another way of putting it is to say that the graph must have $n+1$ disjoint connected subgraphs, each being joined to each of the others to an edge. As far as I know Hadwiger's Conjecture still stands.

The Conjectures of Hajós and Hadwiger were of interest because each implied the Four Colour Conjecture.

Some time in the Fifties I learned of Hajós' Theorem. Then I had it explained to me by Hajós himself. That would have been at a Hungarian Conference in 1959. Hajós Theorem was a disappointment to its discover, it seemed to promise so much and yet achieved so little. It had little publicity, and yet I want to expand upon it in this lecture. I will give you a version of the Theorem that I believe to be a little stronger than the one first presented. Let n be a fixed integer greater than 2. Let us define an "achrome" as a graph that cannot be coloured in n colours., and a "minimal achrome" as an achrome that does not contain another as a proper subgraph. The complete graph $K(n+1)$ is the simplest achrome, and it is clearly minimal. It is easy to show that a minimal achrome must be connected and non-separable. Hajós was interested in operations that changed an achrome, or a pair of achromes , into a new achrome.

For example let G be an achrome in which there are two non-adjacent vertices p and q. Let H be the graph got from G by identifying p and q to make a new vertex r. Any digons introduced by this operation are to be replaced by single edges. Then H is another achrome, for any n-colouring of H would determine one of G in which p and q had the same colour. We say that H is formed from G by the Hajós operation of "vertex–identification".

Now consider two disjoint achromes J and L. Let E be an edge of J with ends p and q. Deleting E from J we obtain a graph $J(1)$. Of this we can assert that in any n-colouring p and q must have the same colour. Now let us construct a graph $L(1)$ by splitting a vertex v of L into two new vertices x and y. Each edge originally incident with v is to be incident with one of the new vertices x and y, but not with both. Incidences of edges with other vertices are remain unchanged. We can assert that in any n-colouring of $L(1)$ the vertices x and y must have different colours. We now unite $J(1)$ and $L(1)$ to form a single graph M by identifying p with x and q with y, no other vertex–identifications being made. Evidently M is an achrome. We refer to the construction of M from J and L as the Hajós operation of "achrome–junction". Hajós was interested in the achromes that could be constructed from copies of $K(n+1)$ by repeated application of his two operations. In his Theorem he asserts that every minimal achroma could be constructed in this way.

Actually that is a weakened form of the Theorem. Hajós required, for the operation of achrome–junction, that neither x nor y should be isolated in L. But now I will state a version that I believe to be stronger than Hajós' original.

Let us say that an achrome G is "H–decomposable" if it is the union of two subgraphs $J(1)$ and $L(1)$ having two common vertices x and y such that the following conditions are fulfilled.

(i) J(1) becomes an achrome J when its vertices x and y are joined by a new edge E.
(ii) L(1) becomes a minimal achrome L when its vertices x and y are identified.

We then say that $\{x, y\}$ is the ”junctive pair” of the H–decomposition, that $J(1)$ is the first ”part” and $L(1)$ its second, and that J is its first ”constituent” and L its second.

If G satisfies these conditions it is obtainable by Hajós’ operations from J and L, first an achrome–junction with joining vertices x and y and then appropriate vertex–identifications, each of a vertex of $J(1)$ with a vertex of $L(1)$.

Next we define recursively a class of ”H–constructible” graphs. The rules are as follows.

(iii) The achrome $K(n+1)$ is H–constructible.
(iv) An achrome M is H-constructible if it is non-separable and has a H–decomposition whose constituent achromes are H–constructible.

This recursively definition makes it clear that every H–constructible achrome is non-separable, but not that every such achrome is minimal.

So a graph is H–constructible if and only if it can be reduced to a set of copies of $K(n+1)$ by a number of H–decompositions into constituent achromes. Let us now attempt a proof of the following version of Hajós’ Theorem.

Theorem I. *Every achrome G has as a subgraph an H-constructible achrome Q with the following property. Either Q is a* $K(n+1)$ *or it has an H–decomposition whose second part does not include all the edges of G.*

Proof. Assume the Theorem is false. Then there is an achrome G for which the Theorem fails, which has the least number of vertices consistent with this and, subject to these two conditions has the greatest possible number of edges.

We note first that any minimal achrome M with fewer vertices than G must be H–constructible. For it contains an H–constructible achrome R by the choice of G, and it is identical with R by minimality. We note also that G must be non-separable since otherwise it would contain such an M.

Suppose that G is a complete graph $K(m)$. Then $m > n$, for otherwise G would be n-colourable. So G has an H–constructible achroma $K(n+1)$ as a subgraph. From this contradiction we deduce that G is not a $K(m)$, that is it has two non-adjacent vertices p and q. Since G is connected we can choose these so that they are joined by an arc of length 2, that is so that each is joined to some third vertex r.

Let us join p and q by a new edge E to get a new achrome U. By the choice of G it contains an H–constructible achrome J, and J canot be a subgraph of G. Hence J contains the edge E joining p and q. Form $J(1)$ by deleting E from J.

In another construction we identify p and q to get a new achrome V. This contains a minimal achrome L. But L has fewer vertices than G and is therefore H–constructible. By an appropriate splitting of the new vertex in L, into the vertices p and q, we get a subgraph $L(1)$ of G. Note that in the construction of V one of the edges pr and qr is erased, by the rule for eliminating digons. Hence there is an edge of G that does not belong to $L(1)$.

Let M be the union of $J(1)$ and $L(1)$, a subgraph of G. Then Conditions (i) and (ii) are satisfied. Evidently J and L are the constituent achromes of an H–decomposition of M. So if M is non-separable it must be H–constructible, by the H–constructibility of J and L and the minimality of L. But this contradicts the choice of G. On the other hand if M is separable it contains a minimal achrome R with fewer vertices than G. By the choice of G the theorem is true for R and therefore it is true for G. This contradiction establishes the Theorem. □

Corollary. *Let G be a minimal achrome. Then G is H–constructible. Moreover if G is not a $K(n+1)$ it has an H–decomposition whose constituent achromes are H–constructible, and whose second part is a proper subgraph of G.*

Theorem II. *Let G be a minimal achrome and let x and y be two non-adjacent vertices of G. Let G be the union of $k > 1$ subgraphs $G(1), G(2), \ldots, G(k)$ such that the intersection of any two of them consists solely of the vertices x and y. Then $k = 2$. Moreover we can adjust the notation so that $\{G(1), G(2)\}$ is an H–decomposition of G whose constituent achromes are both minimal.*

Proof. Form a new achroma P from G by adjoining a new edge E with ends x and y. Let the adjunction of E to $G(i)$ change it into $P(i)$, where i ranges from 1 to k. Then one of the $P(i)$, let us say $P(1)$, is an achrome; otherwise n-colourings of $P(i)$ could be combined to give an n-colouring of P. Let J be a minimal achrome contained in $P(1)$. Since it is not a subgraph of G it includes E, with its ends x and y. Define $J(1)$ as before, the result of deleting E from J.

In an alternative construction we identify x and y in G, as a new vertex z, to get a graph Q. Let this identification change $G(i)$ into $Q(i)$ for each i. Clearly there is a $Q(j)$ that is achrome. It contains a minimal achrome L. Since no isomorph of L can be a subgraph of G the graph L must include z, and give rise under a vertex-splitting to a subgraph $L(1)$ of G that includes x and y.

The union of $J(1)$ and $L(1)$ is an achroma, identical with G since G is minimal. Hence $k = 2$, $J(1) = G(1)$ and $L(1) = G(2)$. We observe that J and L are the constituent achromes of an H–decomposition of G. They are both H–constructible by the Corollary to Theorem I. □

I noticed a few weeks ago that the strengthened version of Hajós' Theorem can actually be used to prove something. We can prove Brooks' Theorem from the above result.

Theorem III. (Brooks' Theorem). *If a minimal achrome G is not a $K(n+1)$ it has a vertex whose valency exceeds n.*

Proof. We note first that the valency of any vertex of a minimal achrome must be at least n. Otherwise we could n-color the rest of the graph, by minimality, and then we could find a color for v.

If G is not a $K(n+1)$ it has a H–decomposition whose first constituent achrome J is H–constructible, whose second constituent achrome L is minimal, and whose secons part $L(1)$ is a proper subgraph of G. We note that $J(1)$ has an edge E not belonging to $L(1)$.

Now E does not join x and y, by the definition of an H–decomposition. So it has an end z that is neither x nor y. If z is a vertex of $L(1)$ its valency in G exceeds its valency in $L(1)$ and is therefore at least $n+1$, by the minimality of L.

If z is not a vertex of $L(1)$ we can represent G as the union of two proper subgraphs P and Q having only vertices of L in common, where P contains L and z is a vertex of Q. The number k of common vertices is at least 2, since G is non-separable. If one of the common vertices is not x or y we can argue as before. So we have only to consider the case in which x and y are the only common vertices of P and Q.

In that case we use Theorem II. By minimality of constituent achromes we infer that in one of P and Q each of x and y has valency at least $n-1$, and in the other their combined valency is at least n. Since n is at least 3 it follows that one of x and y has valency exceeding n in G. □

We still do not see how to use Hajós' Theorem, even in a strengthened version, to prove Hadwiger's Conjecture. But at least we have made it prove one well-known theorem.

Progress on the planar Four Colour Problem in the last fifty years can be summed up quite briefly. People went on collecting reducible configurations until they believed they had enough. But now the wish they did not need quite so many.

There is also a Three Colour Problem for the plane, the problem of finding necessary and sufficient conditions for a planar graph G to be vertex-colourable in 3 colours. Some progress has been made with it; H. Grötzsch proved that G can be 3-coloured if it has no triangle. An extension by B. Grünbaum asserts that G can be 3-coloured if it has fewer than 4 triangles.

I wondered if it would be possible to settle the Three Colour Problem by applying a planar version of Hajós' Theorem. Now Hajós' Theorem for n colours reduces any given achrome, through a sequence of H–decompositions, to a set of copies of $K(n+1)$. But $K(n+1)$ is non-planar when $n > 3$, so then no planar version of the Theorem is to be expected. So let us now discuss the possibility of a planar version for $n = 3$, the coloring problem for $n = 1$ and $n = 2$ being adequately solved already.

I have found no difficulty in putting forward a conjectural planar form of Hajós' Theorem, but I have no proof of it. It is based on the notion that each of the Hajós operations must be carried out in such a way as to preserve planarity, as follows.

The splitting of a vertex v of L into x and y must be carried out planely. That is the edges left incident with x must form a consecutive block in the cyclic sequence of edges round v, and those left incident with y must make up the complementary consecutive block. And in each of these blocks the original cyclic order of the edges must be preserved.

The operation of achrome-junction, in which achromes J and L are combined to form M, does make M planar when J and L are planar. Note that $L(1)$ is confined to the face of $J(1)$ that contained the deleted edge, and $J(1)$ to the face of $L(1)$ that is formed from the two faces of L in the vertex splitting.

After such an achrome-junction some operations of vertex identification are permitted, each is to identify a vertex of $J(1)$ with a vertex of $L(1)$, neither being x or y, and no vertex of one part is to be identified with more than one vertex of the other. Moreover each identification is to be between two vertices with a common incident face, to preserve planarity, and to be made across that face. There are only two possible faces for this operation, each having one bounding arc in $J(1)$ and the complementary bounding arc in $L(1)$.

Of course two vertex-identifications across the same face must not interfere with one another. One pair of vertices must not separate the other in the boundary of the face.

The student is invited to construct some planar achromes for $n = 3$ out of tetrahedra, by Hajós operations restricted as above. We can conjecture that every such achrome that is minimal can be constructed in this way. If true this proposition is a planar version of Hajós' Theorem. But then it can equally well be called a solution of the Three Colour Problem. There is, I think, one encouraging thing about the conjecture. You can prove that every constructible achrome has at least four triangles, a result that Grünbaum proved for all achromes.

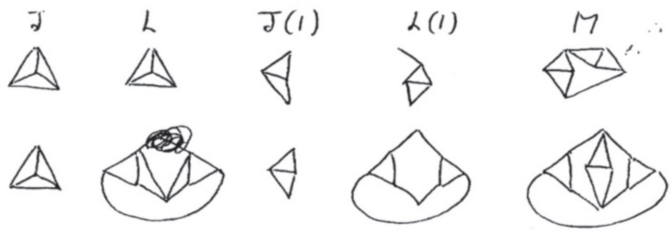


Fig. B.1 Some drawings by Tutte.

B.2 William Thomas Tutte 1917–2002

Bill Tutte was a leading graph and matroid theorist in the second half of the 1900s, but his greatest intellectual feat was probably achieved during World War Two, breaking the German Army High Command's extremely complex Lorenz code (that the British called FISH), uncovering the structure of the machine that generated it, without ever seeing it. In an interview [669] Donald Michie, founder of the Center for Machine Intelligence at the University of Edinburgh, said: *Well, the giant on whose shoulders that everybody was standing, including Allan Turing, was Bill Tutte – he got the original break.*

After studies in chemistry in the second half of the 1930s at Trinity College Cambridge Tutte was called up for national service, and he joined Bletchley Park in 1941 and worked there for the rest of the war. He regarded signing the Official Secrets Act as a lifelong obligation, and only very late in life he felt able to talk about Bletchley Park, when at least some of the secrets were no longer official.

In 1948 Tutte took up a post at the University of Toronto, and later in 1962 he moved to the University of Waterloo. He retired in 1985, but continued to play an important role as Professor Emeritus until his death in 2002. During the many years in Canada he often travelled to England and met there with colleagues and friends, in particular C.A.B. Smith and R.L. Brooks. He was a fellow of the Royal Society of Canada and also of the Royal Society of London. In 2001 he was awarded the Order of Canada.

Tutte's contributions to graph and matroid theory are immense. It is difficult in a short summary to describe all the branches of research pursued by Tutte and the results obtained. He himself gave such summaries in [1036] and [1037]. Also the obituaries [117], [1081] and [1082] give fine descriptions.

B.3 Tutte's Flow Conjectures

One of Tutte's accomplishments was to establish a relationship between colorings and flows of plane graphs. This led him to formulate three flow conjectures for graphs in general; these conjectures are among the most interesting graph problems. Although the conjectures are still unsolved, they have led to a variety of interesting graph-theoretical investigations and results. To formulate the conjectures we need some notation.

Let G be a graph, let Γ be an abelian group, and let $k \in \mathbb{N}$. A **Γ-flow** of G is an assignment to each edge of G of a direction and a value from Γ such that for each vertex v the sum (with respect to the group operation in Γ) of the values of the edges directed out of v is equal to the sum of the values of the edges directed into v. More formally, a Γ-flow is a pair (D, ϕ) such that D is an orientation of G (see Section 3.1) and $\phi : E(D) \to \Gamma$ is a mapping such that for every vertex $v \in V(D)$, we have

$$\sum_{e \in E_D^+(v)} \phi(e) = \sum_{e \in E_D^-(v)} \phi(e),$$

where the sum stands for the addition in the group Γ. A **nowhere zero Γ-flow** of G is a Γ-flow (D, ϕ) of G such that $\phi(e) \neq 0$ for every $e \in E(D)$. A **k-flow** of G is a $\mathbb{Z}$-flow (D, ϕ) of G such that $|\phi(e)| \leq k - 1$ for all $e \in E(D)$. A **nowhere zero k-flow** of G is a k-flow (D, ϕ) of G such that $\phi(e) \neq 0$ for all $e \in E(D)$. Note that if D is an orientation of G, then $V(D) = V(G)$ and $E(D) = E(G)$. Suppose that (D, ϕ) is a

Γ-flow of G and D' is an orientation of G. Then there is a mapping $\phi' : E(D') \to \Gamma$ such that (D', ϕ') is a Γ-flow of G (replace $\phi(e)$ by $-\phi(e)$ if the direction of e is changed). Furthermore, (D, ϕ) is nowhere zero if and only if (D', ϕ') is nowhere zero. Hence, the problem whether G has a nowhere zero Γ-flow is a problem about graphs and not digraphs; see e.g. Jaeger [514], [516], Jensen and Toft [530, Chapter 13], and Seymour [930]. A fine general exposition of flows was given by Younger [1080]. Tutte [1032] made two important observations.

Theorem B.1 (TUTTE) *Let G be a graph, let Γ be an abelian group of order k. Then G has a nowhere zero Γ-flow if and only if G has a nowhere zero k-flow.*

Theorem B.2 (TUTTE) *Let G be a 2-edge-connected plane graph, and let $k \in \mathbb{N}$. Then G has a face coloring with a set of k colors if and only if G has a nowhere zero k-flow.*

The proof of Theorem B.1 is by a nonconstructive counting argument using the principle of inclusion and exclusion. Combining Theorem B.2 with the four color theorem, it follows that every 2-edge connected planar graph has a nowhere zero 4-flow. The Petersen graph shows that not every 2-edge-connected graph has a nowhere zero 4-flow, but Tutte proposed the following two conjectures; the first was published in [1032] and the second in [1035].

Conjecture B.3 (TUTTE'S 5-FLOW CONJECTURE) *Every 2-edge-connected graph admits a nowhere zero 5-flow.*

Conjecture B.4 (TUTTE'S 4-FLOW CONJECTURE) *Every 2-edge-connected graph containing no Petersen graph as a minor admits a nowhere zero 4-flow.*

In 1981, Seymour [929] proved that every 2-edge-connected graph admits a nowhere zero 6-flow using Theorem B.1 with the abelian group $\Gamma = \mathbb{Z}_3 \times \mathbb{Z}_2$; an alternative short proof of this result was given by DeVos and Nurse [283]. In 2005, Lai, Li, and Poon [663] proved that every 2-edge-connected graph containing no K_5 as a minor admits a nowhere zero 4-flow, thus solving a conjecture proposed by Jensen and Toft [530, Problem 13.2]. For the history of the next flow conjecture, we refer the reader to the paper by Steinberg [961]. That the conjecture holds for the class of planar graphs follows from Theorem B.2 and the dual version of Grötzsch's Theorem (Theorem 5.13).

Conjecture B.5 (TUTTE'S 3-FLOW CONJECTURE) *Every 4-edge-connected graph admits a nowhere zero 3-flow.*

By Theorem B.1, a graph G has a nowhere zero 3-flow if and only if G has a nowhere zero $\mathbb{Z}_3$-flow. By reversing the direction of the edges of value 2 in a $\mathbb{Z}_3$-flow, it follows that a graph G has a nowhere zero 3-flow if and only if G has an orientation D such that $d_D^+(v) \equiv d_D^-(v) \pmod 3$ for all $v \in V(D)$. For further information about Tutte's 3-flow conjecture, we refer the reader to the paper by Li, Luo, and Wang [673] and the paper by Thomassen, Wu, and Zhang [1017].

Appendix C
Basic Graph Theory Concepts

This appendix presents basic definitions, terminology and notation of graph theory.

C.1 Sets, Mappings, and Permutations

As usual, we denote by $\mathbb{R}$, $\mathbb{Z}$ and $\mathbb{N}$ the set of real numbers, the set of integers and the set of positive integers, respectively. Furthermore, let $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$ and $\mathbb{R}_{\geq 0} = \{x \in \mathbb{R} \mid x \geq 0\}$. For $a, b \in \mathbb{R}$, let $\mathbb{R}[a, b] = \{x \in \mathbb{R} \mid a \leq x \leq b\}$. For $p, q \in \mathbb{Z}$, let $[p, q] = \{h \in \mathbb{Z} \mid p \leq h \leq q\}$. For $x \in \mathbb{R}$, let $\lfloor x \rfloor$ denote the **lower integer part** of x, and $\lceil x \rceil$ the **upper integer part** of x.

If A and B are two sets, we write $A \subseteq B$ if A is a **subset** of B; and we write $A \subset B$ if A is a **proper subset** of B, i.e., $A \subseteq B$ and $A \neq B$. The **set difference** of A and B is denoted by $A \setminus B$, so $A \setminus B = \{x \mid x \in A \text{ and } x \notin B\}$. For a set A and an integer $p \geq 0$, we denote by $[A]^p$ the set of all subsets X of A of **cardinality** (or **size**) $|X| = p$; these sets are called the p**-element subsets** of A. The **power set** of A, written as 2^A, is the set of all subsets of A. A **partition** or k**-partition** of a nonempty set A is a set $\mathcal{A}$ of k pairwise disjoint nonempty subsets of A whose union $\bigcup \mathcal{A}$ is A.

Let A and B be two sets. If f is **function** from A to B, we write $f : A \to B$. A function is also called a **mapping**, or briefly a **map**. Two functions $f, g : A \to B$ are **equal**, written $f = g$, if $f(a) = g(a)$ for all $a \in A$.

Given a function $f : A \to B$, we use the following standard notation. If $A' \subseteq A$, then $f(A') = \{f(a) \mid a \in A'\}$ is the **image** of A' by f; and $\text{im}(f) = f(A)$ is the **image** or **range** of f. The function $f : A \to B$ is **surjective** if $\text{im}(f) = B$. If f is surjective we also say that f is a function from A **onto** B. The **restriction** of f to A' (or f **restricted** to A') is the function $f' : A' \to B$ with $f'(a) = f(a)$ for all $a \in A'$, and we write $f' = f|_{A'}$. If $B' \subseteq B$, then $f^{-1}(B') = \{a \in A \mid f(a) \in B'\}$ is the **preimage** or **inverse image** of B' by f. If $B' = \{b_1, b_2, \ldots, b_n\}$, we also write $f^{-1}(b_1, b_2, \ldots, b_n)$ instead of $f^{-1}(B')$. In particular, $f^{-1}(b) = \{a \in A \mid f(a) = b\}$ for $b \in B$. The function f is **injective** if, for every $a, a' \in A$, $f(a) = f(a')$ implies $a = a'$. So f is injective if and only if $|f^{-1}(b)| \leq 1$ for all $b \in B$. The function f

M. Stiebitz et al., *Brooks' Theorem*, Springer Monographs in Mathematics,
https://doi.org/10.1007/978-3-031-50065-7

is **bijective** or **one-to-one** if f is surjective and injective. A **bijection** from A to B is just a bijective function from A onto B. For a set X, let $\mathrm{id}_X : X \to X$ denote the **identity function** of X, defined by $\mathrm{id}_X(x) = x$ for all $x \in X$; clearly id_X is bijective.

Given two functions $f : A \to B$ and $g : B \to C$, then the function $h : A \to C$ with $h(a) = g(f(a))$ for all $a \in A$ is called the **composition** of g and f, written as $h = g \circ f$. A function $f' : B \to A$ is called the **inverse function** of $f : A \to B$ if $f \circ f' = \mathrm{id}_B$ and $f' \circ f = \mathrm{id}_A$. If f has an inverse function, this function is unique and denoted by f^{-1}. Furthermore, f has an inverse function if and only if f is bijective.

For a set A, a bijection from A to A is called a **permutation** of A. The set of all permutations of A forms a group, under composition of functions, called the **symmetric group** on A, written as $\mathrm{Sym}(A)$. So

$$\mathrm{Sym}(A) = \{\pi \mid \pi : A \to A \text{ is bijective}\}.$$

When $A = [1,n]$ we write S_n instead of $\mathrm{Sym}(A)$. A **permutation group** is just a subgroup of a symmetric group. A permutation group $\Gamma \subseteq \mathrm{Sym}(A)$ is called **transitive** if for every pair $(a,a') \in A \times A$ there is a permutation $\gamma \in \Gamma$ such that $\gamma(a) = a'$; in this case we also say that Γ **acts transitively** on A.

C.2 Graphs and Subgraphs

A **graph** G is a pair of sets, $V(G)$ and $E(G)$, where $V(G)$ is finite and $E(G)$ is a set of 2-element subsets of $V(G)$. The set $V(G)$ is the **vertex set** of G and its elements are the **vertices** of G. The set $E(G)$ is the **edge set** of G and its elements are the **edges** of G. For convenience, we allow a graph G to be **empty**, i.e., $V(G) = E(G) = \emptyset$. In this case we write briefly $G = \emptyset$. The empty graph is sometimes called the **null graph**.

Let G be a graph. The number of vertices of G is its **order**, denoted by $|G|$. A vertex v is **incident** with an edge e if $v \in e$. The two vertices incident with an edge e are the **ends** of e; and e is said to **join** its ends. An edge $\{u,v\}$ is also written as uv or vu. Two distinct vertices u,v of G with $uv \in E(G)$ are called **adjacent vertices** and **neighbors**. The set of all neighbors of v in G is denoted by $N_G(v)$, i.e., $N_G(v) = \{u \in V(G) \mid uv \in E(G)\}$. If $X \subseteq V(G)$, then $N_G(X) = \bigcup_{v \in X} N_G(v)$ is the **neighborhood** of X in G. For a vertex $v \in V(G)$, let $E_G(v) = \{e \in E(G) \mid v \in e\}$. Furthermore, for $X,Y \subseteq V(G)$, let $E_G(X,Y)$ denote the set of all edges of G joining a vertex of X with a vertex of Y. When $Y = V(G) \setminus X$, then $E_G(X,Y)$ is called the **coboundary** of X in G and is denoted by $\partial_G(X)$. The **degree** of a vertex $v \in V(G)$ is $d_G(v) = |E_G(v)|$. If $G \neq \emptyset$, then let

$$\delta(G) = \min_{v \in V(G)} d_G(v) \text{ and } \Delta(G) = \max_{v \in V(G)} d_G(v)$$

denote the **minimum degree** and the **maximum degree** of G, respectively. For $G = \emptyset$, we define $\delta(G) = \Delta(G) = 0$. A vertex of degree 0 is an **isolated vertex**, and a vertex of degree 1 is an **endvertex**. For an integer $p \geq 0$, let $V^{(p)}(G)$ denote the set of all vertices v in G with $d_G(v) = p$. So $V^{(0)}(G)$ is the set of isolated vertices and $V^{(1)}(G)$ is the set of endvertices of G. A nonempty graph G is called **regular** and r**-regular** if all vertices of G have the same degree r. A 3-regular graph is also referred to as a **cubic graph**.

Since the ends of each edge are distinct vertices, it follows by a simple counting argument that every nonempty graph G with n vertices and m edges satisfies

$$\sum_{v \in V(G)} d_G(v) = \sum_{v \in V(G)} |E_G(v)| = 2m;$$

so $\frac{2m}{n}$ is the **average degree** of G and we obtain that

$$\delta(G) \leq \frac{2m}{n} \leq \Delta(G).$$

Let H and G be two graphs. If $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$, we write briefly $H \subseteq G$ and call H a **subgraph** of G. Obviously, $H = G$ if and only if $H \subseteq G$ and $G \subseteq H$. A subgraph H of G with $H \neq G$ is said to be a **proper subgraph** of G, written as $H \subset G$. A **spanning subgraph** or **factor** of G is a subgraph of G that contains every vertex of G. An r**-factor** of G is a factor of G that is r-regular.

If there exists a bijection $\phi : V(G) \to V(H)$ such that $vw \in E(G)$ if and only if $\phi(v)\phi(w) \in E(H)$ for all $u, v \in V(G)$, then we write briefly $H \cong G$ and say that H is **isomorphic** to G. A graph isomorphic to a graph G is also said to be a **copy** of G.

Let G be a graph and $X \subseteq V(G)$. Then $G[X]$ is the subgraph of G with $V(G[X]) = X$ and $E(G[X]) = E_G(X, X)$. We call $G[X]$ the subgraph of G **induced** by X; and $H \subseteq G$ is an **induced subgraph** of G if $H = G[V(H)]$. If H is an induced subgraph of G, we write $H \leqslant G$, and $H < G$ if H is a proper induced subgraph of G. So $H < G$ if and only if $H \leqslant G$ and $H \neq G$. Clearly, $H \leqslant G$ implies $H \subseteq G$. Furthermore, let $G - X = G[V(G) \setminus X]$. We write $G - v$ instead of $G - \{v\}$. If $H \subseteq G$, $G - H$ means $G - V(H)$. For an edge set $F \subseteq E(G)$, let $G - F = (V(G), E(G) \setminus F)$. If $F = \{e\}$ is a singleton, we write $G - e$ rather than $G - \{e\}$. We call $G - v$ a **vertex-deleted subgraph** of G, and $G - e$ an **edge-deleted subgraph** of G; the two operations are called **vertex deletion** and **edge deletion**, respectively. If $u, v \in V(G)$ are nonadjacent vertices of G, then we denote by $G + uv$ the graph obtained from G by adding the edge joining u and v.

C.3 Path, Cycles and Complete Graphs

A **path** is a nonempty graph whose vertices can be arranged in a sequence with no repetitions such that two vertices are adjacent if and only if they are consecutive in the sequence. It is common to refer to a path by the sequence of its vertices. So we say that $P = (v_1, v_2, \ldots, v_n)$ is a path if

$$V(P) = \{v_1, v_2, \ldots, v_n\} \text{ and } E(P) = \{v_1v_2, v_2v_3, \ldots, v_{n-1}v_n\},$$

where the vertices $v_1, v_2, \ldots, v_n$ are distinct. In this case we say that v_1 and v_n are **joined** by P, and that P is a path **between** v_1 and v_n (respectively, **from** v_1 **to** v_n), and that P is a v_1-v_n **path**. The vertices v_1 and v_n are the **ends** of P; and the remaining vertices $v_2, \ldots, v_{n-1}$ are the **inner vertices** of P. Note that if $n \geq 2$, the ends of P are the endvertices of P.

If $P = (v_1, v_2, \ldots, v_n)$ is a path and $n \geq 3$, then $C = P + v_nv_1$ is a **cycle**. As for graphs, we refer to a cycle by the cyclic sequence of its vertices, so for C we also write $C = (v_1, v_2, \ldots, v_n, v_1)$.

The number of edges of a path (or cycle) G is its **length**, written $\ell(G)$. So $\ell(P) = |P| - 1$ if P is a path; and $\ell(C) = |C|$ if C is a cycle. A path or cycle is **odd** or **even**, depending on whether its length is odd or even. A cycle of length n is also called an n**-cycle**. If a path (or cycle) H is a subgraph of a graph G, then we also say that H is a path (or cycle) in G (respectively, of G). A **Hamilton path** of a graph G is a path in G that contains every vertex of G; and a **Hamilton cycle** of G is a cycle in G that contains every vertex of G.

A graph in which any two distinct vertices are adjacent is **complete**. If G is a complete graph on n vertices, then it is common to write $G = K_n$ (and neither $G \cong K_n$ nor $G \in K_n$). Note that up to isomorphism, there is a unique complete graph on n vertices; and K_n represents the isomorphism class and not a graph with a specific vertex set. In the same sense, we use P_n for a path on n vertices and C_n for a cycle on n vertices. A K_3 (isomorphic to C_3) is often called a **triangle**.

A **vertex order** or **vertex enumeration** of a graph G of order $n \geq 1$ is a sequence $\ell = (v_1, v_2, \ldots, v_n)$ of all its vertices, that is, $V(G) = \{v_1, v_2, \ldots, v_n\}$ and all the v_i are distinct. Each vertex order ℓ of G induces a **linear order** $\leq$ on its vertex set, where $u \leq v$ if $u = v$ or u is before v in the sequence ℓ.

C.4 Connectivity and Blocks

A nonempty graph G is **connected** if there is a path in G between any two of its vertices, and otherwise G is **disconnected**. A (connected) **component** of a nonempty graph G is a maximal connected subgraph of G. Alternatively, the vertex sets of the connected components of G are the equivalence classes of $V(G)$ under the relation *to be equal or to be joined by a path in G*. Let $\text{com}(G)$ denote the number of components of G. By convention, we put $\text{com}(\varnothing) = 0$. A nonempty graph G is disconnected if and only if there exists a 2-partition $\{X, Y\}$ of $V(G)$ such that $E_G(X, Y) = \varnothing$. A set $S \subseteq V(G) \cup E(G)$ is defined to be a **separating set** of G if $\text{com}(G - S) \geq 2$. If u and w are vertices belonging to different components of $G - S$, then we also say that S **separates** u and w in G. A vertex v of G such that $S = \{v\}$ separates two other vertices of the same component of G is a **separating vertex** of G. So v is a separating vertex of G if and only if $\text{com}(G - v) > \text{com}(G)$. An edge e of G such that $S = \{e\}$ separates the two ends of e in G is a **bridge** of G. So the bridges in a graph G are precisely those edges that do not belong to any cycle of G. A separating set of k vertices of G is called a **separator** or k**-separator** of G. A separator S of G is **minimal** if no proper subset of S is a separator of G. By a **cut** or **edge cut** of a graph G we mean a triple (X,Y,F) such that X is a nonempty proper subset of $V(G)$, $Y = V(G) \setminus X$, and $F = \partial_G(X) = \partial_G(Y)$. Clearly, any edge set of a cut is a separating edge set. The converse is not true. However, if F is a minimal separating edge set of a connected graph G, then $G - F$ has precisely two components with vertex sets X and Y, implying that $F = E_G(X,Y)$, and thus (X,Y,F) is a cut.

A graph G is called k**-connected** ($k \in \mathbb{N}_0$) if $|G| \ge k+1$ and $G - S$ is connected for every set $S \subseteq V(G)$ with $|S| \le k-1$. So every nonempty graph is 0-connected, and every connected graph is 1-connected except K_1. Note that if G is k-connected, then G is ℓ-connected for $0 \le \ell \le k$. For a nonempty graph G, the **connectivity** $\kappa(G)$ is defined to be the largest integer k such that G is k-connected. By convention, $\kappa(\varnothing) = 0$. Note that $\kappa(K_n) = n-1$ for all $n \ge 1$. If G is not complete, then

$$\kappa(G) = \min\{|X| \mid X \subseteq V(G), \mathrm{com}(G - X) \ge 2\}.$$

A graph G is called k**-edge-connected** ($k \in \mathbb{N}_0$) if $|G| \ge 2$ and $G - F$ is connected for every set $F \subseteq E(G)$ with $|F| \le k-1$. For a graph G with $|G| \ge 2$, the **edge-connectivity** $\kappa'(G)$ is defined to be the largest integer k such that G is k-edge-connected. By convention, $\kappa(\varnothing) = \kappa(K_1) = 0$. If $|G| \ge 2$, then

$$\kappa'(G) = \min\{|\partial_G(X)| \mid \varnothing \neq X \subset V(G)\}.$$

Proposition C.1 *Every graph G satisfies $\kappa(G) \le \kappa'(G) \le \delta(G)$.*

Proof If G is a complete graph K_n with $n \ge 0$, all three numbers are equal. So assume that G is not complete. Then $\kappa'(G) \le \min\{|\partial_G(v)| \mid v \in V(G)\} = \delta(G)$. To prove the first inequality, choose a cut $F = \partial_G(X)$ with $\varnothing \neq X \subset V(G)$ such that $|F| = \kappa'(G)$. Let X_F denote the set of vertices in X incident with some edge in F; and let Y_F denote the set of vertices in $Y = V(G) \setminus X$ incident with some edge in F. If $X_F \neq X$, then X_F is a separating vertex set in G and hence $\kappa(G) \le |X_F| \le |F| = \kappa'(G)$, so we are done. By symmetry, we are also done if $Y_F \neq Y$. Hence it remains to consider the case that $X_F = X$ and $Y_F = Y$. Since G is not complete, there is a vertex v of G such that $N_G(v)$ is a separating vertex set of G. By symmetry, we may assume that $v \in X_F$. If $w \in N_G(v)$, then either $vw \in F$ or $w \in X_F$ and hence incident with an edge of F. This implies that $\kappa(G) \le |U| = d_G(v) \le |F| = \kappa'(G)$. Thus the proof is complete. □

The following result, known as Menger's theorem [732], is a basic tool in graph theory. We state the global version of Menger's theorem. A proof may be found, for example, in the book by Diestel [286, Theorem 3.3.6]. A set $\mathcal{X}$ of paths in a graph is said to be **internally disjoint** if no inner vertex of a path of $\mathcal{X}$ lies on another path of $\mathcal{X}$.

Theorem C.2 (Menger's Theorem) *For any graph G with $|G| \ge 2$ and any integer $k \ge 0$ the following statements hold:*

(a) *G is k-connected if and only if any two distinct vertices of G are joined by k internally disjoint paths.*
(b) *G is k-edge-connected if and only if any two distinct vertices of G are joined by k edge-disjoint paths.*

Let G be an arbitrary graph. A **block** of G is a maximal connected subgraph of G that has no separating vertex. Alternatively, the edges of the blocks are the equivalence classes under the relation *to be equal or to lie in a common cycle*. The set of blocks of G is denoted by $\mathfrak{B}(G)$, and for $v \in V(G)$, let $\mathfrak{B}_v(G) = \{B \in \mathfrak{B}(G) \mid v \in V(B)\}$. Note that each block of G is a connected induced subgraph of G, and so $\mathfrak{B}(\varnothing) = \varnothing$. If G is disconnected, then $\mathfrak{B}(G)$ is the union of $\mathfrak{B}(H)$ taken over all components H of G. If G is connected and has no separating vertex, then $\mathfrak{B}(G) = \{G\}$ and we say that G is a **block**; in particular, we have $\mathfrak{B}(K_1) = \{K_1\}$. If G is connected and has a separating vertex v, then G is the union of two proper induced subgraphs, say G_1 and G_2, having only v in common. Clearly, both subgraphs G_1 and G_2 are connected, and we have $\mathfrak{B}(G) = \mathfrak{B}(G_1) \cup \mathfrak{B}(G_2)$ and $\mathfrak{B}(G_1) \cap \mathfrak{B}(G_2) = \varnothing$. Any two distinct blocks of G have at most one vertex in common; and a vertex v is a separating vertex of G if and only if it belongs to more than one block, i.e. $|\mathfrak{B}_v(G)| \geq 2$. A block of G which contains at most one separating vertex of G is called an **end-block** of G. If G has a separating vertex, then G has at least two end-blocks. So either G itself is a block or G has at least two end-blocks.

The following proposition characterizes 2-connected graphs; the proof is left to the reader as an exercise.

Proposition C.3 *Let G be a connected graph with $|G| \geq 3$. Then the following statements are equivalent:*

(a) *G is 2-connected.*
(b) *Any two vertices are on a common cycle.*
(c) *Any two edges are on a common cycle.*
(d) *G has no separating vertex.*
(e) *For every vertex $v \in V(G)$, $G - v$ is connected.*
(f) $\mathfrak{B}(G) = \{G\}$.

C.5 Distance, Girth and Odd Girth

Let G be a graph and let $v, w \in V(G)$. If there exist a path in G between v and w, define the **distance** $\mathrm{dist}_G(v, w)$ of v nd w in G to be the length of a shortest v-w path contained in G, that is,

$$\mathrm{dist}_G(v, w) = \min\{\ell(P) \mid P \subseteq G \text{ is an } v\text{-}w \text{ path}\}.$$

If no such path exists, define $\mathrm{dist}_G(v, w) = \infty$. If G is a connected graph, then the distance is a metric, and we can use the classical concepts for metric spaces, as ball, sphere, diameter, and radius. For instance, the **radius** of a connected graph G is defined by

$$\mathrm{rad}(G) = \min_{v \in V(G)} \max_{w \in V(G)} \mathrm{dist}_G(v, w).$$

If a graph G contains a cycle, define the **girth** $g(G)$ of G to be the length of a shortest cycle contained in G, that is,

$$g(G) = \min\{\ell(C) \mid C \subseteq G \text{ is a cycle}\}.$$

If G contains no cycle, then define $g(G) = \infty$. If G contains an odd cycle, define the **odd girth** $g_o(G)$ of G to be the length of a shortest odd cycle in G, that is,

$$g_o(G) = \min\{\ell(C) \mid C \subseteq G \text{ is an odd cycle}\}.$$

If G contains no odd cycle, then define $g_o(G) = \infty$.

Let G be a graph and let $W = (v_0, v_1, \ldots, v_n)$ be a nonempty sequence of vertices of G. We call W a **walk** of G if $v_i v_{i+1} \in E(G)$ for $0 \le i \le n-1$. We say that W **connects** v_0 and v_n, and that W **starts** in v_0, and **terminates** in v_n, and that W is a v_0-v_n walk. The vertices v_0 and v_n are called the **ends** of the walk W, v_0 being its **initial vertex** and v_n its **terminal vertex**; the remaining vertices $v_1, v_2, \ldots, v_{n-1}$ are its **internal vertices**. The integer n is called the **length** of W, written as $\ell(W) = n$. The reverse sequence $W^{-1} = (v_n, v_{n-1}, \ldots, v_0)$ is also a walk in G. A walk is **odd** or **even**, depending on whether its length is odd or even. A walk is **closed** if its initial and terminal vertices are identical; otherwise it is **open**. If the edges $v_i v_{i+1}$ of a walk are all different it is called a **tour**. If the vertices of a walk are distinct, it defines a path in G; two vertices of G are connected by a walk if and only if they are joined by a path in G. A closed walk $W = (v_0, v_1, \ldots, v_n)$ with $n \ge 3$ whose vertices $v_0, v_1, \ldots, v_{n-1}$ are distinct defines a cycle; then the closed walk $W' = (v_i, v_{i+1}, \ldots, v_n, v_1, v_2, \ldots, v_{i-1}, v_i)$ (with $1 \le i \le n-1$) is different from W, but it defines the same cycle as W. So a path of length $n \ge 1$ gives rise to two different open walks; and a cycle of length $n \ge 3$ gives rise to $2n$ different closed walks.

Proposition C.4 *Every closed odd walk W in a graph G satisfies $g_o(G) \le \ell(W)$.*

Proof The proof is by induction on the length of W. Since W is a closed walk whose length is odd, $\ell(W) \ge 3$. If $\ell(W) = 3$, then W defines a cycle and $g_o(G) = 3$. So assume that $\ell(W) = n > 3$, say $W = (v_0, v_1, \ldots, v_n)$. If W defines a cycle, we have $g_o(G) = n = \ell(W)$. Otherwise, $v_i = v_j$ for $0 \le i < j \le n-1$. Then we can split W into two closed walks, namely $W_1 = (v_0, v_1, \ldots, v_i, v_{j+1}, \ldots, v_n)$ and $W_2 = (v_i, v_{i+1}, \ldots, v_j)$. Since $\ell(W) = \ell(W_1) + \ell(W_2)$ and $\ell(W) = n$ is odd, either W_1 or W_2 is odd, say W_1. Then $\ell(W_1) < \ell(W)$ and the induction hypothesis implies that $g_o(G) \le \ell(W_1)$, which gives $g_o(G) \le \ell(W)$. □

Proposition C.5 *Let G be a graph and suppose that $g_o(G) \ge 2p+1$ with $p \ge 1$. Then for two vertices $v \ne w$ of G the following conditions are equivalent:*

(a) *There exists an odd v-w walk W in G with $\ell(W) \le 2p+1$.*

(b) *There exists an odd v-w path P in G with $\ell(P) \leq 2p+1$.*

Proof Since each path defines an obvious walk of the same length, (b) implies (a). To show the converse implication, let $W = (v_0, v_1, \dots, v_{2p}, v_{2q+1})$ be an odd v-w walk, whose length is minimum. By (a), such a walk exists and $q \leq p$. We claim that the vertices of W are distinct and W therefore defines a path, which proves (b). For otherwise, $v_i = v_j$ for $0 \leq i < j \leq 2q$, since $v_0 = v \neq w = v_{2q+1}$. Then $W_1 = (v_i, v_{i+1}, \dots, v_j)$ is a closed walk with $2 \leq \ell(W_1) < 2q+1 \leq 2p+1$. Since $g_o(G) \geq 2p+1$, it then follows from Proposition C.4 that W_1 is even. But then $W_2 = (v_0, v_1, \dots, v_i, v_{j+1}, \dots, v_{2q+1})$ is an odd v-w walk with $\ell(W_2) \leq \ell(W) - 2$, contradicting the choice of W. □

C.6 Trees and Bipartite Graphs

An **acyclic graph**, that is, a graph not containing any cycle, is a **forest**, and it is a **tree** when it is connected. So any component of a forest is a tree. A graph is a forrest if and only if each of its edges is a bridge. An endvertex of a tree is also called a **leaf**. If T is a tree of order $n \geq 2$, then T has at least two leaves. This follows from the simple fact that the two ends of a longest path in T are leaves of T. If T is a tree and v is an leaf, then $T - v$ is a tree. It follows, by induction on its order, that every tree of order n has $n-1$ edges. If T is a tree and v, w are two vertices of T, then there is a unique v-w path in T; this path is denoted by vTw or wTv. Note that $\text{dist}_T(v,w) = \ell(vTw)$. Clearly, every connected graph G contains a **spanning tree**, that is, a tree T such that $V(T) = V(G)$ (take a minimal connected subgraph of G with vertex set $V(G)$, or take a maximal acyclic subgraph of G with vertex set $V(G)$).

Depth-first search (**DFS**) is a fundamental graph searching technique developed by Hopcroft and Tarjan [505] and by Tarjan [992]. The structure of DFS enables efficient graph algorithm for many graph problems. A **DFS tree** T of a given graph G of order $n \geq 1$ can be constructed as follows. First we choose a vertex r of G and a vertex order $\ell = (v_1, v_2, \dots, v_n)$ of G with $v_1 = r$. Next we construct recursively vertices u_i and trees T_i, where we start with the vertex $u_1 = r$ and the tree T_1 consisting of the single vertex r. If in step $i \geq 1$ the vertex u_i and the tree T_i has been obtained, we proceed as follows. If no vertex in $U = \{u_1, u_2, \dots, u_i\}$ has a neighbor in G belonging to $U' = V(G) \setminus U$, then we stop and put $T = T_i$. Otherwise, we choose the largest integer $j \in [1,i]$ such that u_j has a neighbor in U', and then we choose the neighbor $u \in U'$ of u_j which comes first in the vertex order ℓ. Then $u_{i+1} = u$ and T_{i+1} is the tree obtained from T_i by adding the vertex u and the edge $u_j u$. Since $V(G)$ is a finite set, the procedure terminates after a finite number of steps with a tree T. The vertex $u_1 = r$ is called the **root** of the DFS tree T. If G is disconnected, then the procedure stops with $T = T_i$ for some $i < n$ and $V(T) = \{u_1, u_2, \dots, u_i\}$ is the vertex set of a component of G. If G is connected, the procedure stops with $T = T_n$ and T is a spanning tree of G with $V(G) = V(T) = \{u_1, u_2, \dots, u_n\}$.

Proposition C.6 *Let G be a connected graph with order n. Then the following statements hold:*

(a) *Let T be a DFS tree of G with root r, and let vw be an edge of G. Then either v belongs to the path rTw, or w belongs to the path rTv.*
(b) *Let T be a DFS tree of G with root r, and let $V^{(1)}(T)$ be the set of leaves of T. Then $V^{(1)}(T) \setminus \{r\}$ is an independent set of G.*
(c) *Let $P \subseteq G$ be a path between r and v. Then there exists a DFS tree T of G with root r such that $P = rTv$.*

Proof To prove (a), let u_i be the vertex of G added in step i to T, so $V(G) = V(T) = \{u_1, u_2, \ldots, u_n\}$ and $u_1 = r$. If $vw \in E(G)$, then $v = u_i$ and $u = u_j$, where we may assume that $i < j$. Let U be the set of vertices $u \in V(G)$ for which v belongs to the path rTu. An inductive argument shows that $u_i, u_{i+1}, \ldots, u_j \in U$. Hence $w \in U$, which proves (a). Statement (b) is an immediate consequence of (a). To prove (c), suppose that $P = (v_1, v_2, \ldots, v_p)$ is a path with $v_1 = r$ and $v_p = v$. Then $(v_1, v_2, \ldots, v_p)$ can be extended to a vertex order $\ell = (v_1, v_2, \ldots, v_n)$ of G. If T is the DFS tree corresponding to the vertex order ℓ, then $u_i = v_i$ for $i \in [1, p]$. Hence T has root r and $rTv = v_1 T v_p = P$. □

A **rooted tree** (T, v_T) is a tree T with a designated vertex $v_T \in V(T)$ called its **root**; we then also say that T is a tree **rooted** at v_T. Designating a root imposes a hierarchy on the vertices of a rooted tree, according to their distance from that root. For a rooted tree (T, v_T) we use the following terminology. The **depth** or **level** of a vertex v of T is its distance $d_T(v_T, v)$ from the root in T; thus the root has depth 0. The **height** of (T, v_T) is the length of a longest path in T from the root, or equivalently the greatest depth in (T, v_T). If w is a vertex of T and v is the neighbor of w on the path $v_T T w$, then v is a **parent** of w and w is a **child** of v. If $e = vw$ is an edge of T such that v is a parent of w, then e is called an **outgoing edge** of v (or a **descending edge** of v) and an **incoming edge** of w. A vertex v is called an **ancestor** of w if v belongs to the path $v_T T w$; if, in addition, $v \neq w$, then v is a **proper ancestor** of w. A **leaf** of (T, v_T) is any vertex having no children. An **internal vertex** of (T, v_T) is any vertex that has at least one children. Note that the root v_T is an internal vertex of (T, v_T), unless it is the only vertex of T. If the root v_T has exactly one child, then it is a leaf of T, but not of (T, v_T); otherwise, the leaves of (T, v_T) are the same as the leaves of T.

Breadth-first search (**BFS**) is another natural way of searching a graph; this leads to a **BFS tree** of a given graph G. It seems that BFS and its application in finding components of a graph G were invented in 1945 by Konrad Zuse and Michael Burke. The method and its application in finding shortest path of a graph or digraph were reinvented in the 1950s by various researchers, including Bellman, Dijkstra, Dantzig, Ford, Moore, and possibly others. Those who are interested in the history of the shortest path problem, should read the article by Schrijver [904]. A BFS tree of a connected graph G is a rooted tree (T, v_T) such that T is a spanning subgraph of G and every vertex v of G satisfies $\mathrm{dist}_G(v_T, v) = \mathrm{dist}_T(v_T, v) = \ell(v_T T v)$.

A graph G is called **bipartite** if there are two disjoint sets X, Y such that $V(G) = X \cup Y$ and $E(G) = E_G(X,Y)$; the sets X and Y are called **partite sets** or **parts** of G, and (X,Y) is called a **bipartition** of G. A **complete bipartite** graph is a bipartite graph G with parts X and Y such that $E(G) = \{xy \mid x \in X, y \in Y\}$; if $|X| = p$ and $|Y| = q$ we briefly write $G = K_{p,q}$ $(p,q \geq 0)$. A graph isomorphic to $K_{1,n}$ is called a **star**; if $n = 3$, such a star is also called a **claw**. Clearly, any subgraph of a bipartite graph is bipartite, too. Furthermore, it is easy to see that every tree T is bipartite and has a bipartition (X,Y) such that the path vTw has even length whenever v and w belong to the same part X or Y (up to permuting the parts X and Y, this bipartition of T is unique). The sets X and Y may be described as follows. Fix a vertex r of T as a root of T and put $X = \{v \in V(T) \mid \mathrm{dist}_T(r,v) \text{ is even}\}$ and $Y = \{v \in V(T) \mid \mathrm{dist}_T(r,v) \text{ is odd}\}$. In 1916 Dénes König [609] provided a characterization of bipartite graphs by forbidden subgraphs.

Theorem C.7 (König) *A graph is bipartite if and only if it contains no odd cycle.*

Proof Since an odd cycle has no bipartition, no bipartite graph contains an odd cycle. Conversely, we shall show that if a graph G contains no odd cycle, then G has a bipartition. To do this, we may assume that G is connected and $|G| \geq 2$. Then G has a spanning tree T, and T has a bipartition $\{X,Y\}$. If this is a bipartition of G, we are done. Otherwise, there is an edge vw with both ends in one of the two partition classes, say in X. Then the path vTw has even length and hence $vTw + vw$ is an odd cycle, giving a contradiction. □

C.7 Graph Operations and Graph Modifications

Let H and G be two graphs. The **union** $H \cup G$ is the graph with $V(H \cup G) = V(H) \cup V(G)$ and $E(H \cup G) = E(H) \cup E(G)$; and the **intersection** $H \cap G$ is the graph with $V(H \cap G) = V(H) \cap V(G)$ and $E(H \cap G) = E(H) \cap E(G)$. We say that H and G are **disjoint** if $H \cap G = \varnothing$. The **complement** $\overline{G}$ of G is the graph with $V(\overline{G}) = V(G)$ and $E(\overline{G}) = [V(G)]^2 \setminus E(G)$. So if G has order $n \geq 0$, then $G \cup \overline{G} = K_n$ and $G \cap \overline{G} = \overline{K_n}$.

Let G be a graph and let X be a set of vertices of G. We call X an **independent set** or p**-independent set** of G if $G[X] = \overline{K_p}$, that is, $|X| = p$ and no two vertices of X are adjacent in G. The **independence number** of G, written $\alpha(G)$, is the largest integer p for which G contains a p-independent set. We call X a **clique** or p**-clique** of G if $G[X] = K_p$, that is, $|X| = p$ and any two vertices of X are adjacent in G. The **clique number** of G, written $\omega(G)$, is the largest integer p for which G contains a p-clique. Clearly, $\alpha(\overline{G}) = \omega(G)$ and $\omega(\overline{G}) = \alpha(G)$

Join Operation. If H and G are disjoint graphs, we define the **join** $H \boxplus G$ of these two graphs to be the graph obtained from $H \cup G$ by adding all edges uv with $u \in V(H)$ and $w \in V(G)$. Assuming that all graphs P_n, C_n and K_n are disjoint, we have $K_{p+q} = K_p \boxplus K_q$ and $K_{p,q} = \overline{K_p} \boxplus \overline{K_q}$. A **wheel** is a graph isomorphic to

$W_n = K_1 \boxplus C_n$. The wheel W_n is said to be **odd** or **even** depending on whether n is odd or even.

Vertex Splitting. Let H and G be disjoint graphs, let $X \subseteq V(H)$ and let $v \in V(G)$. Let G' denote a graph obtained from $H \cup (G - v)$ by joining each vertex of $N_G(v)$ to exactly one vertex of X such that each vertex in X has a at least one neighbor in $N_G(v)$. This is possible if $d_G(v) \geq |X|$. We say that G' is obtained from G and H by (proper) **splitting** v into X.

Vertex Identification. Let G be a graph and $X \subseteq V(G)$. We say that graph H is obtained from G by **identifying** X to a new vertex v, if $V(H) = V(G - X) \cup \{v\}$ and $E(H) = E(G - X) \cup \{vu \mid u \in V(G) \setminus X,\ N_G(u) \cap X \neq \varnothing\}$.

Vertex Duplication. Duplicating a vertex v of a graph G produces a new graph G' by adding a new vertex v' and joining v' to all neighbors of v in G (but not to v).

Edge Contraction. Let G be a graph and let e be an edge of G. By G/e we denote the graph obtained from G by identifying the ends of e; we then say that G/e is obtained from G by **contracting** the edge e.

Edge Subdivision. Let G be a graph and let e be an edge of G. We say that H is obtained from G by **subdividing** the edge e, if H is obtained from $G - e$ by adding a new vertex v_e and by joining v_e to the ends of e.

Hajós Construction. Suppose H and G are disjoint graphs, $x_1y_1 \in E(H)$ and $x_2y_2 \in E(G)$. The **Hajós join** of G and H with respect to (x_1, y_1) and (x_2, y_2) is the graph G' obtained from $H \cup G$ by deleting the edges x_1y_2 and x_2y_2, identifying x_1, x_2 to a new vertex x and adding the edge y_1y_2.

Graph Products. Let G and H be graphs. The **Cartesian product** $G \square H$ has vertex set $V(G) \times V(H)$, and (v, w) is adjacent to (v', w') if and only if either (1) v is adjacent to v' in G and $w = w'$, or (2) $v = v'$ and w is adjacent to w' in H. The **lexicographic product** (or **composition**) $G[H]$ is the graph with vertex set $V(G) \times V(H)$, in which (v, w) is adjacent to (v', w') if and only if either (1) $vv' \in E(G)$, or (2) $v = v'$ and $ww' \in E(H)$. The **direct product** (or **categorial product**) $G \times H$ has vertex set $V(G) \times V(H)$, and (v, w) is adjacent to (v', w') if and only if v is adjacent to v' in G and w is adjacent to w' in H.

C.8 Minors and Subdivisions

We have already discussed some simple ways of modifying a graph by means of local graph operations, such as vertex deletion, edge deletion, edge contraction and edge subdivision.

Let G be a graph. Any subgraph of G can be obtained from G by repeated application of the operations vertex deletion and edge deletion; any induced subgraph of G can be obtained from G by repeated application of the operation vertex deletion; and any spanning subgraph of G can be obtained from G by repeated application of the operation edge deletion.

A graph H is a **minor** of G if an isomorphic copy of H can be obtained from G by repeated application of the operations of vertex deletion, edge deletion and edge contraction, where these operation can be performed in any order. If H is a minor of G, we write $H \preceq G$. It follows from the definition that if H and H' are isomorphic graphs, then $H \preceq G$ if and only if $H' \preceq G$. Furthermore, every subgraph of G is a minor of G; and every minor of G can be obtained from a subgraph of G by a sequence of edge contractions. It is easy to prove that $H \preceq G$ if and only if there is a set $\{X_v \mid v \in V(H)\}$ of pairwise disjoint subsets of $V(G)$ such that $G[X_v]$ is connected for all $v \in V(H)$ and $E_G(X_v, X_w) \neq \emptyset$ whenever $vw \in E(H)$. The minor relation $\preceq$ is an partial order on the class of graphs, that is, the relation is reflexive, transitive, and antisymmetric in the sense that $H \preceq G$ and $G \preceq H$ implies $H \cong G$.

A graph H is a **subdivision** of G if an isomorphic copy of H can be obtained from G by successive edge subdivision; or equivalently, by replacing edges with pairwise internally disjoint paths (so an edge vw is replaced by an v-w path). A graph H is a **topological minor** of G if a subgraph of G is (isomorphic to) a subdivision of H; in this case we write $H \preceq_t G$. So $H \preceq_t G$ if and only if there is an injective mapping $\varphi : V(H) \to V(G)$ such that for every edge vw of H there is a $\varphi(v)$-$\varphi(w)$ path in G and all these paths are internally vertex disjoint. Clearly, $H \preceq_t G$ implies $H \preceq G$, but not conversely. However, if $\Delta(H) \leq 3$, then $H \preceq G$ and $H \preceq_t G$ are equivalent (see [286, Proposition 1.7.2])

C.9 Multipartite Graphs and Turán Graphs

A p**-partite graph** is a graph G whose vertex set is the disjoint union of p sets (called **partite sets** or **parts**), in such a way that no edge of G has both ends in the same part. If any two vertices in different partite sets are adjacent, then G is called a **complete multipartite graph**, respectively a **complete** p**-partite graph**completeppartitegraph. Thus a graph G is complete p-partite if and only if $\overline{G}$ is the disjoint union of p complete graphs (where one or more of the complete graphs may be empty).

Let n and p be positive integers. The **Turán graph** $TG(n,p)$ is the unique complete p-partite graph with n vertices whose partite sets differ in size by at most 1. Note that if n has the form $n = hp + r$ with $0 \leq r < p$, then

$$TG(n,p) \cong \overline{K}_{n_1} \boxplus \overline{K}_{n_2} \boxplus \cdots \boxplus \overline{K}_{n_p},$$

where $n_i = h+1$ for $i \in [1,r]$ and $n_i = h$ for $i \in [r+1,p]$ (see Figure C.1). In particular, $TG(n,p) = K_n$ if $n \leq p$. The Turán graph $G = TG(n,p)$ is an edge-maximal K_{p+1}-

free graph, that is, $\omega(G) \leq p$ but $\omega(G+vw) = p+1$ for every edge $vw \in E(\overline{G})$; where $G+vw$ is the graph obtained from G by adding the edge vw. The number of edges of $TG(n,p)$, denoted by $t(n,p)$, is called the **Turán number**. The next result, which was proved in 1941 by Turán [1031], opened a new branch in discrete mathematics, known as extremal graph theory.

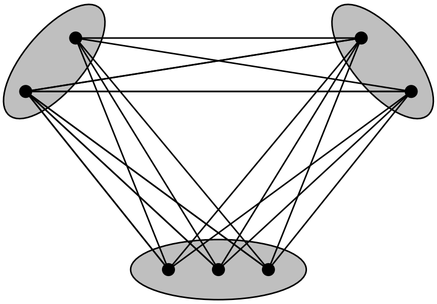


Fig. C.1 The Turán graph $TG(7,3)$.

Theorem C.8 (Turán's Theorem) *Let n and p be positive integers. If G is an edge-maximal K_{p+1}-free graph of order n, then $G \cong TG(n,p)$.*

Proof First we claim that the degrees of any two nonadjacent vertices of G are equal, and the degrees of any two adjacent vertices of G differ by at most one. For otherwise, deleting the vertex whose degree is smaller and duplicating the other vertex results in K_{p+1}-free graph of order n having more edges than G, giving a contradiction. Next we claim that G is a complete multipartite graph. For otherwise, G contains three vertices v, v_1, v_2 such that $vv_1, vv_2 \notin E(G)$ but $v_1v_2 \in E(G)$. By the first claim, all three vertices have the same degree. But then deleting both v_1 and v_2 and duplicating v twice results in a K_{p+1}-free graph of order n having more edges than G, giving a contradiction. Thus G is a complete p-partite graph and the sizes of any two nonempty parts differ by at most one. Consequently, $G \cong TG(n,p)$ as required. □

The above proof of Turán's theorem is due to Zykov [1105]. Turán's theorem implies, in particular, that

$$t(n,p) = \max\{|E(G)| \mid \omega(G) \leq p \wedge |G| = n\}. \tag{C.1}$$

for all positive integers n and p.

C.10 Automorphism Group of Graphs

Given two graphs G and H, an **isomorphism** from G to H is a bijective mapping $\phi : V(G) \to V(H)$ such that, for all vertices $v, w \in V(G)$,

$$vw \in E(G) \Longleftrightarrow \phi(v)\phi(w) \in E(H).$$

An isomorphism of a graph G to itself is called an **automorphism** of G. Note that any automorphism of G is a permutation on its vertex set V. The set of all automorphisms of G forms a group under composition of mappings; this group is called the **automorphism group** of G written as $\mathrm{Aut}(G)$. So $\mathrm{Aut}(G)$ is a subgroup of the symmetric group $\mathrm{Sym}(V)$. The graph G is said to be **transitive** if $\mathrm{Aut}(G)$ acts transitively on its vertex set V, that is, for all vertices $v, w \in V$ there exists an automorphism $\gamma \in \mathrm{Aut}(G)$ such that $\gamma(v) = w$.

Proposition C.9 *Let G be a graph, and let $\gamma \in \mathrm{Aut}(G)$ be an automorphism of G Then the following statements hold:*

(a) $d_G(\gamma(v)) = d_G(v)$ *for all* $v \in V(G)$.
(b) $\mathrm{dist}_G(\gamma(v), \gamma(w)) = \mathrm{dist}_G(v, w)$ *for all* $v, w \in V(G)$.

Proof Note that γ is bijective and, for $v, w \in V(G)$, we have $vw \in E(G)$ if and only if $\gamma(v)\gamma(w) \in E(G)$. Consequently, $N_G(\gamma(v)) = \gamma(N_G(v))$ and so $d_G(v) = |N_G(v)| = |N_G(\gamma(v))| = d_G(\gamma(v))$, which proves (a). If $P = (v_1, v_2, \ldots, v_n)$ is a v-w path in G, then $P' = (\gamma(v_1), \gamma(v_2), \ldots, \gamma(v_n))$ is a $\gamma(v)$-$\gamma(w)$ path in G (and vice versa) with $\ell(P') = \ell(P)$. Since $\gamma^{-1} \in \mathrm{Aut}(G)$, this implies (b). □

Let Γ be a permutation group acting on a finite nonempty set V. Then Γ is a finite set and $|\Gamma|$ is its **order**. Given a subgroup Γ' of Γ and a permutation $\gamma \in \Gamma$, the set

$$\gamma \circ \Gamma' = \{\gamma \circ \gamma' \mid \gamma' \in \Gamma'\}$$

is a **left coset** of Γ' in Γ, and Γ/Γ' denotes the set of all left cosets. It is well-known, that Γ/Γ' forms a partition of Γ and Lagrange's theorem says that

$$|\Gamma| = |\Gamma/\Gamma'||\Gamma'|. \tag{C.2}$$

For elements $v, w \in V$, define

$$v \sim_\Gamma w \Longleftrightarrow \exists \gamma \in \Gamma : w = \gamma(v).$$

Then $\sim_\Gamma$ is an equivalence relation on V, and an equivalence class is called an **orbit**. So $O \subseteq V$ is an orbit if and only if there is an $v \in V$ and $O = \{\gamma(v) \mid \gamma \in \Gamma\}$. Given an element $v \in V$, the set

$$\Gamma_v = \{\gamma \in \Gamma \mid \gamma(v) = v\}$$

is called the **stabilizer** of v in Γ; the stabilizer is an subgroup of Γ. There is a simple relation between the number m of orbits of Γ and the order of its stabilizers, namely

$$m|\Gamma| = \sum_{v\in V} |\Gamma_v|.$$

This result is due to Cauchy and Frobenius, but it is often referred to as the Burnside lemma. If $\emptyset \neq U \subseteq V$, then define

$$\Gamma_U = \bigcap_{v\in U} \Gamma_v \text{ and } \Gamma|_U = \{\gamma|_U \mid \gamma \in \Gamma\}.$$

Proposition C.10 *Let Γ be a permutation group on a finite nonempty set V, and let $\emptyset \neq U \subseteq V$. Then $|\Gamma| = |\Gamma|_U| \cdot |\Gamma_U|$.*

Proof Clearly, Γ_U is a subgroup of Γ. For $\gamma, \gamma' \in \Gamma$ we then deduce that

$$\begin{aligned} \gamma \circ \Gamma_U = \gamma' \circ \Gamma_U &\iff \gamma^{-1} \circ \gamma' \in \Gamma_U \\ &\iff (\gamma^{-1} \circ \gamma')(v) = v \; \forall v \in U \\ &\iff \gamma(v) = \gamma'(v) \; \forall v \in U \\ &\iff \gamma|_U = \gamma'|_U. \end{aligned}$$

Thus $|\Gamma|_U| = |\Gamma/\Gamma_U|$ and the result follows from equation (C.2). □

C.11 Graph Properties and Graph Invariants

Let $\mathcal{G}$ denote the class of all graphs. A **graph property** $\mathcal{P}$ is a subclass of $\mathcal{G}$ that is closed under isomorphism, that is, if a graph G belongs to $\mathcal{P}$, then so does every graph isomorphic to G. A graph property $\mathcal{P}$ is **trivial** if $\mathcal{P} = \mathcal{G}$, or $\mathcal{P} = \emptyset$, or $\mathcal{P}$ consists only of the empty graph. Thus a graph property is nontrivial if and only if it contains a nonempty graph, but not all graphs. A graph property is **monotone** if it is closed under taking subgraphs, and it is **hereditary** if it is closed under taking induced subgraphs. Obviously, every monotone graph property is a hereditary graph property, but not vice versa.

Every hereditary graph property can be defined in terms of forbidden induced subgraphs. Given a class or set $\mathcal{X}$ of graphs, let Forb($\mathcal{X}$) be the class of graphs $G \in \mathcal{G}$ such that no induced subgraph of G is isomorphic to a graph of $\mathcal{X}$. Clearly, Forb($\mathcal{X}$) is a hereditary graph property, and Forb($\mathcal{X}$) $= \mathcal{G}$ if and only if $\mathcal{X} = \emptyset$. If $\mathcal{X} = \{H_1, H_2, \dots, H_n\}$, then we abbreviate Forb($\mathcal{X}$) by Forb($H_1, H_2, \dots, H_n$). A graph G is said to be **$\mathcal{X}$-free** if $G \in$ Forb($\mathcal{X}$). If $\mathcal{X}$ consists of a single graph H, then the term becomes **H-free**; a K_3-free graph is also called **triangle-free**. The class Forb(H) is empty if and only if $H = \emptyset$; and the class Forb(H) consists only of the empty graph if and only if $H = K_1$. The class Forb(K_2) of K_2-free graphs consists of all edgeless graphs. Figure C.2 below shows a list of popular forbidden subgraphs; these small graphs have special names. For example, the class Forb($K_{1,3}$) of claw-free graphs has attracted particular attention, see the 2005 survey paper by Chudnowsky and Seymour [240].

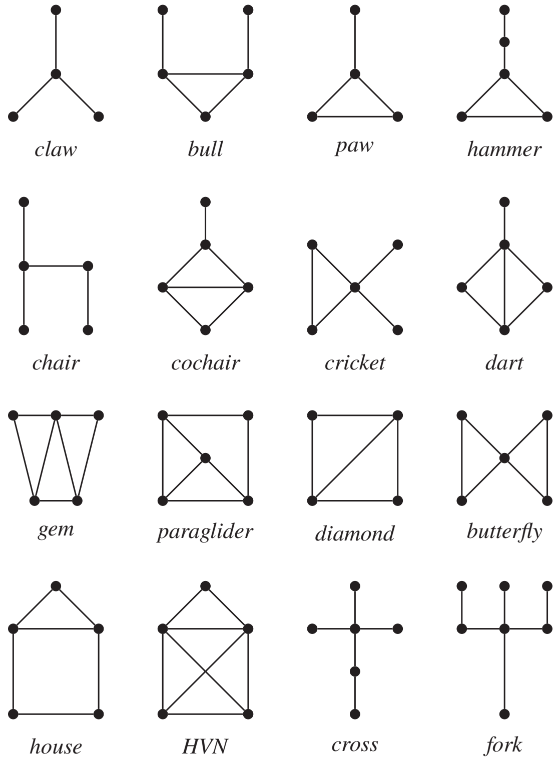


Fig. C.2 Small graphs with names.

In the class of graphs $\mathcal{G}$ we have the subgraph relation $\subseteq$, the induced subgraph relation $\leqslant$, the minor relation $\preceq$, and the topological minor relation $\preceq_t$. In what follows let $\trianglelefteq$ be one of these relations. If $H \trianglelefteq G$ and $H \not\cong G$, then we write $H \triangleleft G$. The relation $\trianglelefteq$ is reflexive, transitive, and antisymmetric in the sense that $H \trianglelefteq G$ and $G \trianglelefteq H$ implies $H \cong G$. Furthermore, every strictly descending chain in $(\mathcal{G}, \trianglelefteq)$ is finite. A graph property $\mathcal{P}$ is **$\trianglelefteq$-hereditary** if it is closed with respect to the relation $\trianglelefteq$, that is, if G belongs to $\mathcal{P}$, then so does every graph H with $H \trianglelefteq G$. Thus $\mathcal{P}$ is monotone if and only if it is $\subseteq$-hereditary; and $\mathcal{P}$ is hereditary if and only if it is $\leqslant$-hereditary. For a class $\mathcal{X}$ of graphs, let

$$\text{Forb}_{\trianglelefteq}(\mathcal{X}) = \{G \in \mathcal{G} \mid \forall H \in \mathcal{G}\, \forall H' \in \mathcal{X} : H \trianglelefteq G \Rightarrow H \not\cong H'\}$$

Clearly, $\text{Forb}_{\trianglelefteq}(\mathcal{X})$ is a $\trianglelefteq$-hereditary graph property and $\text{Forb}(\mathcal{X}) = \text{Forb}_{\leqslant}(\mathcal{X})$. For a graph property $\mathcal{P}$, let

$$\text{Crit}_{\trianglelefteq}(\mathcal{P}) = \{G \in \mathcal{G} \mid G \notin \mathcal{P} \wedge \forall H (H \triangleleft G \Rightarrow H \in \mathcal{P})\}.$$

Note that $\text{Crit}_{\trianglelefteq}(\mathcal{P})$ is a graph property, which is empty if and only if $\mathcal{P} = \mathcal{G}$. Since every strictly descending chain in $(\mathcal{G}, \trianglelefteq)$ is finite, it follows that for every graph

$G \notin \mathcal{P}$ there exists a graph $G' \in \mathrm{Crit}_{\trianglelefteq}(\mathcal{P})$ such that $G' \trianglelefteq G$. If $\mathcal{P}$ is a $\trianglelefteq$-hereditary graph property, then

$$\mathcal{P} = \mathrm{Forb}_{\trianglelefteq}(\mathrm{Crit}_{\trianglelefteq}(\mathcal{P})). \tag{C.3}$$

A **graph parameter** (or **graph invariant**) is a map ρ that assigns to each graph $G \in \mathcal{G}$ a real number $\rho(G)$ such that $\rho(H) = \rho(G)$ for any two isomorphic graphs $H, G \in \mathcal{G}$. A graph parameter ρ is called **$\trianglelefteq$-monotone** if $H \trianglelefteq G$ implies $\rho(H) \leq \rho(G)$, for any graphs $G, H \in \mathcal{G}$. A graph parameter is **monotone** if it is $\subseteq$-monotone. For a graph parameter ρ and a real number x, let

$$\mathcal{G}(\rho \leq x) = \{G \in \mathcal{G} \mid \rho(G) \leq x\}.$$

Clearly, $\mathcal{G}(\rho \leq x)$ is a graph property. If ρ is a $\trianglelefteq$-monotone graph parameter, then the graph property $\mathcal{G}(\rho \leq x)$ is $\trianglelefteq$-hereditary for all $x \in \mathbb{R}$, and vice versa. For $\mathrm{Forb}_{\trianglelefteq}(\mathcal{G}(\rho \leq x))$ we also write $\mathrm{Forb}_{\trianglelefteq}(\rho \leq x)$, and for $\mathrm{Crit}_{\trianglelefteq}(\mathcal{G}(\rho \leq x))$ we also write $\mathrm{Crit}_{\trianglelefteq}(\rho \leq x)$. Note that if ρ is a graph parameter, then

$$\mathrm{Crit}_{\trianglelefteq}(\rho \leq x) = \{G \in \mathcal{G} \mid \rho(G) > x \wedge \forall H (H \triangleleft G \Rightarrow \rho(H) \leq x)\}.$$

A graph property $\mathcal{P}$ is **additive** if $\mathcal{P}$ is closed under taking vertex disjoint unions. So if $\mathcal{P}$ is an additive graph property, then a nonempty graph belongs to $\mathcal{P}$ if and only if each of its components belong to $\mathcal{P}$. A graph parameter ρ is **additive** if $\rho(H \cup G) = \max\{\rho(G), \rho(H)\}$ whenever G and H are disjoint graphs. Clearly, if ρ is an additive graph parameter, then $\mathcal{G}(\rho \leq x)$ is an additive graph property for every $x \in \mathbb{R}$. Furthermore, if ρ is a monotone graph parameter, then ρ is additive if and only if $\rho(H \cup G) \leq \max\{\rho(G), \rho(H)\}$ whenever G and H are disjoint nonempty graphs. The maximum degree Δ, the clique number ω and the independence number α are examples of monotone graph parameters; while the minimum degree δ and the connectivity κ are nonmonotone graph parameters. For a nonempty graph G, let

$$\mathrm{ad}(G) = \frac{2|E(G)|}{|G|}$$

be the **average degree** of G, and let $\mathrm{mad}(G) = \max\{\mathrm{ad}(H) \mid \varnothing \neq H \subseteq G\}$ be the **maximum average degree**. Clearly, both the average degree and the maximum average degree are monotone graph parameters. To see that the average degree is an additive graph parameter, let G_1 and G_1 be disjoint nonempty graphs, and let $a_i = 2|E(G_i)|$ and $b_i = |G_i|$ $(i = 1,2)$. Then

$$\mathrm{ad}(G_1 \cup G_2) = \frac{a_1 + a_2}{b_1 + b_2} \leq \max\{\frac{a_1}{b_1}, \frac{a_2}{b_2}\} = \max\{\mathrm{ad}(G_1), \mathrm{ad}(G_2)\},$$

where the inequality holds, since if $B = b_1 + b_2$ and $M = \max\{\frac{a_1}{b_1}, \frac{a_2}{b_2}\}$, then we have

$$\frac{a_1 + a_2}{b_1 + b_2} = \frac{a_1}{b_1}\frac{b_1}{B} + \frac{a_2}{b_2}\frac{b_2}{B} \leq M\frac{b_1}{B} + M\frac{b_2}{B} = M.$$

As a consequence, we obtain that the maximum average degree is additive, too. Thus for every nonempty graph G, we have

$$\mathrm{mad}(G) = \max\{\mathrm{ad}(H) \mid H \subseteq G, H \text{ is connected}\}. \tag{C.4}$$

For the empty graph, we define $\mathrm{ad}(\varnothing) = 0$ and $\mathrm{mad}(\varnothing) = 0$.

C.12 Multigraphs and Directed Multigraphs

A **multigraph** is a triple $G = (V, E, i)$, where V and E are two finite disjoint sets, and $i : E \longrightarrow [V]^2$. As for graphs, $V = V(G)$ is the **vertex set** of G and $E = E(G)$ is the **edge set** of G; moreover, $i = i_G$ is called the **incidence function** of G. A vertex v is **incident** with an edge e if $v \in i_G(e)$. The two vertices incident with an edge e are the **ends** of e; and e is said to **join** its ends. Terminology introduced for graphs will be used correspondingly. For instance, for $v \in V(G)$, we put $E_G(v) = \{e \in E(G) \mid v \in i_G(e)\}$, and for $X, Y \subseteq V(G)$, we denote by $E_G(X, Y)$ the set of edges of G joining a vertex of X with an vertex of Y. The **degree** of a vertex v is $d_G(v) = |E_G(v)|$, and the **multiplicity** of two distinct vertices v and w of G is $\mu_G(v, w) = |E_G(v, w)|$; distinct edges in $E_G(v, w)$ are called **multiple edges** or **parallel edges**. The **maximum multiplicity** of G is denoted by $\mu(G)$. Thus, a graph is essentially the same as a multigraph without multiple edges, that is, with $\mu(G) \le 1$. Note that our multigraphs have no loops. If G is a multigraph without parallel edges, then for an edge e with $i_G(e) = \{v, w\}$ we also write $e = vw$ or $e = wv$, so we can identify e with the 2-subset $\{u, v\}$. Conversely, every graph G may be viewed as a multigraph whose incidence function is the identity function of its edge set, that is, $i_G(e) = e$ for every edge $e \in E(G)$.

Let G and H be two multigraphs. Then H is a **submultigraph** of G, written $H \subseteq G$, if $V(H) \subseteq V(G)$, $E(H) \subseteq E(G)$, and $i_G(e) = i_H(e)$ for all edges $e \in E(H)$. We say that G and H are **isomorphic** if there is a bijective map $\phi : V(G) \longrightarrow V(H)$ such $\mu_G(v, w) = \mu_H(\phi(v), \phi(w))$ whenever v and w are distinct vertices of G. If H_1 and H_2 are two submultigraphs of G, then the **union** $H = H_1 \cup H_2$ is the submultigraph of G with $V(H) = V(H_1) \cup V(H_2)$ and $E(H) = E(H_1) \cup E(H_2)$, and the **intersection** $H' = H_1 \cap H_2$ is the submultigraph of G with $V(H) = V(H_1) \cap V(H_2)$ and $E(H) = E(H_1) \cap E(H_2)$. A **decomposition** of a multigraph G is a list of submultigraphs of G such that each edge of G appears in exactly one subgraph in the list.

The **underlying graph** of a multigraph G is the graph H with $V(H) = V(G)$ and $E(H) = \{i_G(e) \mid e \in E(G)\}$; then for $X \subseteq V(G)$, we have $N_G(X) = N_H(X)$. Clearly, such G and H are isomorphic if and only if G has no parallel edges.

Let G be a multigraph. Two distinct edges of G which are incident with a common vertex are called **adjacent edges** of G. A **matching** of G is a set of pairwise nonadjacent edges, so no two edges of a matching have a common end. For a set $M \subseteq E(G)$, we denote by $V(M)$ the set of vertices which are incident to some edge of M; a vertex in $V(M)$ is said to be **covered** by M. In 1935 Hall [463] established a necessary and sufficient condition for the existence of a matching in a bipartite multigraph that covers every vertex in one of its part.

Theorem C.11 (Hall's Theorem) *Let G be a bipartite multigraph with parts X and Y. Then G has a matching M with $V(M) \supseteq X$ if and only if $|N_G(S)| \geq |S|$ for all $S \subseteq X$.*

Let G be a graph. The maximum number of edges in a matching of G is called the **matching number** of G, written $\alpha'(G)$. Furthermore, we denote by $\mathrm{odd}(G)$ the number of components of G having odd order. The following result is due to Tutte and Berge (see [144, Corollary16.12]).

Theorem C.12 (The Berge-Tutte Formula) *Every graph G satisfies $\alpha'(G) = \max\{\frac{1}{2}(|G - \mathrm{odd}(G-P) + |P||) \mid P \subset V(G)\}$.*

Note that our multigraphs may have parallel edges but no loops. A **pseudomultigraph** is a triple $G = (V, E, i_G)$, where V and E are two finite disjoint sets, and $i_G : E \longrightarrow [V]^1 \cup [V]^2$. As for graphs, the elements of V are the vertices and the elements of E are the edges. An edge e of a pseudomultigraph with $i_G(e) = \{v\}$ for a vertex $v \in V$ is called a **loop** at v. Similarly, a **pseudograph** is a pair $G = (V, E)$ with $E \subseteq [V]^1 \cup [V]^2$. Note that a vertex v of a pseudograph is **isolated** if there is no edge $\{v, w\}$ of the pseudograph with $v \neq w$.

A **multidigraph** (or **directed multigraph**) D is a pair of two finite disjoint sets, the **vertex set** $V(D)$ and the **edge set** $E(D)$, together with two maps

$$v_D^+ : E(D) \longrightarrow V(D) \text{ and } v_D^- : E(D) \longrightarrow V(D)$$

assigning to every edge e an **initial vertex** $v_D^+(v)$ and a **terminal vertex** $v_D^-(v)$ such that $v_D^+(e) \neq v_D^-(e)$. If it is clear that we refer to the digraph D we write $v^+(e)$ and $v^-(e)$ rather than $v_D^+(e)$ and $v_D^-(e)$, respectively. The edge e is said to be **directed from** $v^+(e)$ **to** $v^-(e)$. If there exists an edge in D directed from v to w, then w is called a **successor** or **out-neighbor** of v, and v is called a **predecessor** or **in-neighbor** of w. For a vertex v of a multidigraph D, let $N_D^+(v)$ be the set of all out-neighbors of v in D, and let $N_D^-(v)$ be the set of all in-neighbors of v in D. The number of vertices of a multidigraph D is its **order**, denoted by $|D|$.

A multidigraph may have several edges between the same two vertices. Such edges are called **multiple edges**; if they have the same direction, say from u to v, they are **parallel**. Note that our digraphs have no loops. A multidigraph without multiple edges is called a **simple digraph**. A **strict digraph** is a multidigraph without parallel edges. So if D is a strict digraph, then between any two distinct vertices v and w there is at most one edge directed from v to w and at most one edge directed from w to v. If e is an edge of a strict digraph D directed from u to v, we also write $e = (u, v)$, so the edge set of D may be regarded as a subset of $V(D)^2 = V(D) \times V(D)$.

Let D be a multidigraph. For a vertex $v \in V(D)$, define

$$E_D^+(v) = \{e \in E(D) \mid v_D^+(e) = v\} \text{ and } E_D^-(v) = \{e \in E(D) \mid v_D^-(e) = v\}.$$

Then $d_D^+(v) = |E_D^+(v)|$ is the **out-degree** of v in D, and $d_D^-(v) = |E_D^-(v)|$ is the **in-degree** of v in D. Furthermore, $\Delta^+(D) = \max_{v \in V(D)} d_D^+(v)$ is the **maximum out-degree** of D, and $\delta^+(D) = \min_{v \in V(D)} d_D^+(v)$ is the **minimum out-degree** of D. Similarly, we define the **maximum in-degree** $\Delta^-(D)$ and the **minimum in-degree** $\delta^-(D)$ of D. Note that if D is a strict digraph, then $d_D^+(v) = |N_D^+(v)|$ and $d_D^-(v) = |N_D^-(v)|$ for every vertex v of D.

Let D, D' be two multidigraphs. We call D' a **subdigraph** of D if $V(D') \subseteq V(D)$, $E(D') \subseteq E(D)$, and for all $e \in E(D')$ we have $v_{D'}^+(e) = v_D^+(e)$ as well as $v_{D'}^-(e) = v_D^-(e)$. If $X \subseteq V(G)$, then $D[X]$ is the **subdigraph induced by** X, that is, the subdigraph D' of D satisfying

$$V(D') = X \text{ and } E(D') = \{e \in E(D) \mid \{v^+(e), v^-(e)\} \subseteq X\}.$$

So D' is an **induced subdigraph** of G if $D' = D[V(D')]$. A **decomposition** of D is a list of subdigraphs of D such that each edge of D belongs to exactly one subdigraph in the list.

Let D be a multidigraph. The **underlying graph** of D, denoted by G_D, is the unique graph with vertex set $V(G_D) = V(D)$ and $E(G_D) = \{\{v_D^+(e), v_D^-(e)\} \mid e \in E(D)\}$. The multidigraph D is said to be **connected** if G_D is connected. A set $X \subseteq V(D)$ is called an **independent set** of D if X is an independent set of G_D, that is, $D[X]$ is a multidigraph without edges. A set $X \subseteq V(D)$ is called an **clique** of D if X is a clique of G_D. A **kernel** of D is an independent set S of D such that for every vertex $u \in V(D) \setminus S$ there is an edge in D directed from u to some vertex $v \in S$. We say that D is **kernel-perfect** if every induced subdigraph of D has a kernel.

A **directed path** P is a simple directed graph whose underlying graph is a path $(v_1, v_2, \ldots, v_n)$ such that $v_D^+(v_i v_{i+1}) = v_i$ for $i \in [1, n-1]$. In this case we also say that P is a **directed path from** v_1 **to** v_n. A **directed cycle** is a simple directed graph whose underlying graph is a cycle $(v_1, v_2, \ldots, v_n, v_1)$ such that $v_D^+(v_i v_{i+1}) = v_i$ for $i \in [1, n-1]$ and $v_D^+(v_n v_1) = v_n$. As for graphs, a directed cycle is called **odd** or **even** depending on whether its order is odd or even. A digraph is **acyclic** if it has no directed cycle. Note that if D is an acyclic digraph, then its underlying graph may have cycles. It is easy to show that every nonempty acyclic digraph D contains a vertex v with $d_D^+(v) = 0$ as well as a vertex w with $d_D^-(w) = 0$ (take a directed path in D whose order is maximum). As for graphs, a directed path or directed cycle of a digraph D is called a **directed Hamilton path** respectively a **directed Hamilton cycle** of D if it contains all vertices of D.

Let G be a graph or multigraph. An **orientation** of G is a multidigraph D with vertex set $V(D) = V(G)$ and edge set $E(D) = E(G)$ such that every edge $e \in E(D)$ satisfies $i_G(e) = \{v_D^+(e), v_D^-(e)\}$. So an orientation of G is a multidigraph obtained from G by assigning to each edge of G a direction from one of its ends to the other.

C.13 Graphs on Surfaces

Figure C.3 shows three different drawings of the Petersen graph in the plane, but none of the three drawings is crossing-free. The Petersen graph can be embedded on the torus, but not in the plane. In this section we review some basic facts for graphs embedded in the plane or in other (closed) surfaces. Here a **surface** is a connected compact 2-dimensional manifold without boundary. Such surfaces can be classified according to their Euler characteristic and orientability. The **orientable surfaces** are the surfaces $\mathbf{S}_g$, for $g \geq 0$, obtained from the (2-dimensional) sphere by attaching g handles. So $\mathbf{S}_0$ is the sphere and $\mathbf{S}_1$ is the torus. The **nonorientable surfaces** are the surfaces $\mathbf{N}_h$, for $h \geq 1$, obtained by taking the sphere with h holes and attaching h Möbius bands along their boundary to the boundary of their holes. So $\mathbf{N}_1$ is the projective plane and $\mathbf{N}_2$ is the Klein bottle. The **Euler characteristic** $\varepsilon(\mathbf{S})$ of the surface $\mathbf{S}$ is $2-2g$ if $\mathbf{S} = \mathbf{S}_g$ and $2-h$ if $\mathbf{S} = \mathbf{N}_h$. The derived invariant $\mathrm{eg}(\mathbf{S}) = 2 - \varepsilon(\mathbf{S})$ is the **Euler genus** of $\mathbf{S}$. The famous classification theorem due to Brahana [166] states that every (closed) surface is homeomorphic either to $\mathbf{S}_g$ for some $g \geq 0$, or to $\mathbf{N}_h$ for some $h \geq 1$. Each surface may be viewed as a topological subspace of either $\mathbb{R}^3$ or $\mathbb{R}^4$. By Brahana's theorem, two surfaces are topologically the same if and only if both are either orientable or nonorientable and both have the same Euler characteristic. The Euler characteristic of the sphere with g handles and h Möbius bands is $2-2g-h$, and it is nonorientable when $h \geq 1$.

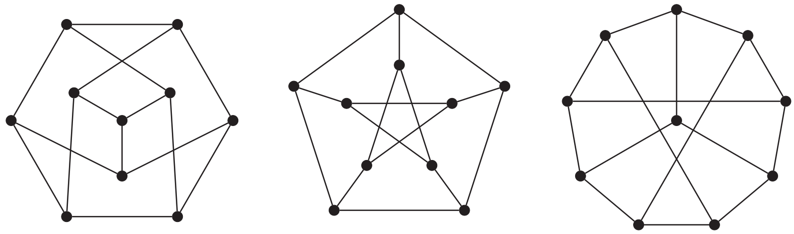


Fig. C.3 Three drawings of the Petersen graph.

Let $\mathbf{T} \subseteq \mathbb{R}^n$ be a topological subspace with the usual induced topology. If $\mathbf{X}$ is a subset of $\mathbf{T}$, then we denote its **boundary** by $\mathrm{bd}(\mathbf{X})$, its **interior** by $\mathrm{int}(\mathbf{X})$, and its **closure** by $\mathrm{cl}(\mathbf{X})$. An **arc**, a **circle**, a **closed disc**, and an **open disc** in $\mathbf{T}$ are subsets that are homeomorphic (in the subspace topology) to the real interval $\mathbb{R}[0,1]$, to the 1-sphere $\mathbf{S}^1 = \{\mathbf{x} \in \mathbb{R}^2 \mid \|\mathbf{x}\| = 1\}$, to the 2-ball $\mathbf{B}^2 = \{\mathbf{x} \in \mathbb{R}^2 \mid \|\mathbf{x}\| \leq 1\}$, and to the disc $\{\mathbf{x} \in \mathbb{R}^2 \mid \|x\| < 1\}$, respectively. If $\mathbf{a}$ is an arc in $\mathbf{T}$ and $g : \mathbb{R}[0,1] \to \mathbf{a}$ is the corresponding homomorphism, then the **endpoints** of $\mathbf{a}$ are $g(0)$ and $g(1)$; we also say that $\mathbf{a}$ is an **arc between** $g(0)$ and $g(1)$ or an $g(0)$**-**$g(1)$ **arc**; further we call the elements of $\mathbf{a} \setminus \{g(0), g(1)\}$ the **internal points** of $\mathbf{a}$. The **components** of a nonempty subset $\mathbf{X}$ of $\mathbf{T}$ are the equivalence classes of $\mathbf{X}$, where two points of $\mathbf{X}$ are **equivalent** if they are the same or if there exists an arc $\mathbf{a} \subseteq \mathbf{X}$ between these two points. A nonempty subset $\mathbf{X}$ of $\mathbf{T}$ is **connected** if it is its only component. Let

$\mathbf{C}$ be a circle in $\mathbf{T}$. Then $\mathbf{C}$ is a **nonseparating circle** if $\mathbf{T} \setminus \mathbf{C}$ is connected, and a **separating circle** otherwise. The circle $\mathbf{C}$ is **contractible** if it is the boundary of a closed disk in $\mathbf{T}$, and **noncontractible** otherwise. If $\mathbf{S}$ is a surface and $\mathbf{C}$ is a separating circle in $\mathbf{S}$, then $\mathbf{S} \setminus \mathbf{C}$ has exactly two components, both with boundary $\mathbf{C}$. If $\mathbf{S}$ is a surface different from the sphere and $\mathbf{C}$ is a contractible circle in $\mathbf{S}$, then a disk $\mathbf{D}$ such that $\mathbf{C} = \mathrm{bd}(\mathbf{D})$ is uniquely determined; in this case we call the open disk $\mathrm{int}(\mathbf{D})$ the **inside** of $\mathbf{C}$, denoted by $\mathrm{ins}(\mathbf{C})$, and we call $\mathbf{S} \setminus \mathbf{D}$ the **outside** of D. The Jordan curve theorem says that if $\mathbf{C}$ is a circle in the plane $\mathbb{R}^2$, then $\mathbb{R}^2 \setminus \mathbf{C}$ has exactly two components $\mathbf{X}_1$ and $\mathbf{X}_2$, and $\mathrm{bd}(\mathbf{X}_1) = \mathrm{bd}(\mathbf{X}_2) = \mathbf{C}$. Exactly one of the two components, say $\mathbf{X}_1$ is bounded and the other one is unbounded. Then the **inside** of $\mathbf{C}$ is $\mathrm{int}(\mathbf{X}_1)$ and the **outside** of $\mathbf{C}$ is $\mathrm{int}(\mathbf{X}_2)$. Furthermore, every surface different from the sphere contains a nonseparating circle.

Let $\mathbf{T} \subseteq \mathbb{R}^n$ be a topological space. An **embedded multigraph** in $\mathbf{T}$ (or a **$\mathbf{T}$-multigraph**) is a multigraph G such that every member of $V(G)$ is a point of $\mathbf{T}$, every member e of $E(G)$ is an arc of $\mathbf{T}$, such that both endpoints of an arc $e \in E(G)$, but no interior point of e, belong to $V(G)$, and such that any two different arcs of $E(G)$ have at most two endpoints and no interior point in common. Furthermore, for every arc $e \in E(G)$, $i_G(e)$ is the set consisting of the two endpoints of e. Then we denote by $\mathbf{G}$ the subspace of $\mathbf{T}$ induced by G, that is,

$$\mathbf{G} = V(G) \cup \bigcup_{e \in E(G)} e.$$

If $H \subseteq G$ is a subgraph of G, then $\mathbf{H}$ is a subspace of $\mathbf{G}$ and hence of $\mathbf{T}$. The **faces** of G are the components of $\mathbf{T} \setminus \mathbf{G}$; we denote by $F(G)$ the set of all faces of G. A vertex $v \in V(G)$ is **incident** to a face $f \in F(G)$ if $v \in \mathrm{bd}(f)$; an edge $e \in E(G)$ is **incident** to f if $\mathrm{bd}(f)$ contains an interior point of e. We say that G is **2-cell embedded** in $\mathbf{T}$ if all faces of G are open disks. Let G be an (abstract) graph or multigraph. We say that G is **embeddable** in $\mathbf{T}$ if G is isomorphic to a $\mathbf{T}$-multigraph G'; in this case we call G' an **embedding** of G in $\mathbf{T}$. The class of all graphs embeddable in $\mathbf{T}$ is denoted by $\mathcal{G}(\hookrightarrow \mathbf{T})$; clearly, this class is a graph property. A **2-cell embedding** of G is an embedding of G which is 2-cell embedded; a 2-cell embedding is also referred to as a **cellular embedding**. If $\mathbf{T}$ is the plane $\mathbb{R}^2$ or a surface, then a multigraph is embeddable in $\mathbf{T}$ if and only if its underlying graph is embeddable in $\mathbf{T}$. The same statement holds for pseudographs, that is, loops do not affect the embeddability.

Let $\mathbf{T}$ be a surface or the plane, and let G be a graph embedded in $\mathbf{T}$. Then every edge is incident either to exactly one face or to two different faces of G. Furthermore, an edge $e \in E(G)$ is incident to a face $f \in F(G)$ if and only if $e \subseteq \mathrm{bd}(f)$. Thus for every face $f \in F(G)$, there is a unique graph $H \subseteq G$ such that $\mathbf{H} = \mathrm{bd}(f)$. This graph H is denoted by $\mathrm{Bd}(f)$; it is called the **boundary subgraph** of f. If $f \in F(G)$ and H is a subgraph of G, we say that f is **bounded** by H if $H = \mathrm{Bd}(f)$. A subgraph of G is called a **facial subgraph** if it is the boundary subgraph of some face of G. A **facial cycle** is a facial subgraph that is a cycle. Every vertex of $\mathbf{T}$ has a (topological) neighborhood which is homeomorphic to an open disk. If $v \in V(G)$ is a vertex, then the edges incident with v are arranged in a cyclic order, which can be described by a cyclic permutation $\pi_v \in \mathrm{Sym}(E_G(v))$ (respectively by the inverse permutation π_v^{-1}). Then for all edges $e \in E(G)$ there is a face $f \in F(G)$ such that both e and $\pi_v(e)$ are incident with f. This face is not unique if $d_G(v) = 2$. Furthermore, if a vertex v is incident with an edge e and this edge is incident with a face f, then either $\pi_v(e)$ is incident with f or $\pi_v^{-1}(e)$ is incident with f. These observations can be used to define **facial walks** (see Thomassen [1011] or Mohar and Thomassen [752]). Let $f \in F(G)$ be a face, let $H = \mathrm{Bd}(f)$ the subgraph bounded by f, and let H' be a component of H. Then there is a closed walk $W = (v_1, v_2, \ldots, v_m, v_{m+1})$ with $v_{m+1} = v_1$ such that $V(H') = \{v_1, \ldots, v_m\}$, $E(H') = \{v_i v_{i+1} \mid i \in [1, m]\}$, every edge appears at most twice in this walk (if an edge appears twice in the walk, then it appears in both directions), and an edge appears twice if and only if f is the only face incident with that edge. If the boundary subgraph of f is connected, then modulo orientation and starting vertex, there is exactly one facial walk associated with f. The **degree** of a face $f \in F(G)$, denoted by $d_G(f)$, is the number of edges in the boundary subgraph of f plus the number of bridges of G belonging to the boundary subgraph of f. Note that if e is a bridge of G belonging to the boundary subgraph of f, then e does not belong to the boundary subgraph of any other face. Hence, each edge of G appears twice as an edge of the boundary subgraph of a face, implying that

$$\sum_{f \in F(G)} d_G(f) = 2|E(G)|. \tag{C.5}$$

If G is a connected graph and $|G| \geq 3$, then, for every face $f \in F(G)$, we have $d_G(f) \geq 3$ and equality hods if and only if f is bounded by a triangle K_3. If $C \subseteq G$ is a cycle, then $\mathbf{C}$ is a circle in $\mathbf{T}$. We say that C is a **contractible cycle**, if the circle $\mathbf{T}$ is contractible in $\mathbf{S}$; and a **noncontractible cycle** otherwise.

Proposition C.13 *Let* $\mathbf{T}$ *be a surface or the plane, let G be a graph embedded in* $\mathbf{T}$*, and let $C \subseteq G$ a contractible cycle, such that C is an induced subgraph of G and $G - V(C)$ is connected. Then C is a facial cycle of G.*

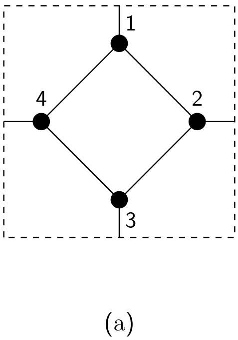
Proof As C is contractible, $\mathbf{T} \setminus \mathbf{C}$ has exactly two components $\mathbf{X}_1$ and $\mathbf{X}_2$ such that $\mathrm{bd}(\mathbf{X}_1) = \mathrm{bd}(\mathbf{X}_1) = \mathbf{C}$. Since $G - V(C)$ is a connected graph embedded in $\mathbf{S} \setminus \mathbf{C}$, we deduce that that either $V(G) \cap \mathbf{X}_1 = \varnothing$ or $V(G) \cap \mathbf{X}_2 = \varnothing$; say $V(G) \cap \mathbf{X}_2 = \varnothing$. This implies that $V(G) \subseteq \mathrm{cl}(\mathbf{X}_1)$. Let $e \in E(G)$. If both ends of e are in C, then $e \in E(C)$ (since C is an induced cycle of G) and so $e \subseteq \mathbf{C} \subseteq \mathrm{cl}(\mathbf{X}_1)$. If one endpoint of e lies in $\mathbf{X}_1$, then $e \subseteq \mathrm{cl}(\mathbf{X}_1)$. It follows that $\mathbf{G} \subseteq \mathrm{cl}(\mathbf{X}_1)$, which implies that $\mathbf{X}_2$ is a face of G whose boundary subgraph is C. □

A proof of the following well known results can be found in Diestel's book [286, Proposition 4.2.6 and Proposition 4.2.7]. Let G be an arbitrary graph, and let $C \subseteq G$ be a cycle. We call C a **separating cycle** of G if $G - V(C)$ is disconnected, and a **nonseparating cycle** otherwise.

Proposition C.14 *Let G be a 2-connected graph embedded in the plane or the sphere. Then for all faces $f \in F(G)$, the closure* $\mathrm{cl}(f)$ *is a closed disk and so* $\mathrm{Bd}(f)$ *is a cycle.*

Proposition C.15 *The facial subgraphs in a 3-connected graph embedded in the plane or the sphere are precisely its nonseparating induced cycles.*

In order to view an embedding of a graph or multigraph on a surface, we may use a polygonal representation of that surface. A torus can be obtained from a rectangle $ABCD$ by identifying the side AB with the side DC and the side AD with the side BC. A projective plane may be represented by the disk $\mathbf{B}^2$ in which every point on the boundary is identified with its antipodal point. Figure C.4(a) shows an 2-cell embedding of K_4 in the torus, it has two faces bounded by the cycle $(1,2,3,4,1)$ and the closed walk $(1,2,4,1,3,4,2,3,1)$, respectively. Figure C.4(b) shows a non-2-cell embedding of K_4 in the torus, it has three faces and one of its faces is homeomorphic to a cylinder and is bounded by the facial walk $(1,2,3,1,4,3,1)$. Figure C.5 shows a 2-cell embedding of the Petersen graph in the projective plane. Note that 2-cell embedded graphs are connected.



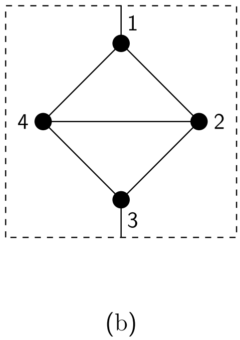


Fig. C.4 Two embeddings of K_4 in the torus.

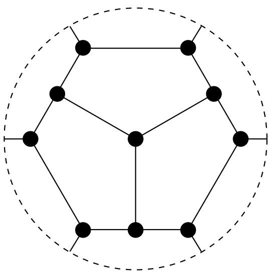


Fig. C.5 An embedding of the Petersen graph in the projective plane.

Many results about graphs embedded on surfaces are just consequences of the following result, which is known as Euler's Formula. A proof of this result may be found in [865] or in [752].

Theorem C.16 (Euler's Formula) *If a connected graph G is 2-cell embedded in a surface* $\mathbf{S}$*, then*

$$|V(G)| - |E(G)| + |F(G)| = \varepsilon(\mathbf{S}).$$

If a connected graph G has an embedding in a surface $\mathbf{S}$, but not a 2-cell embedding in $\mathbf{S}$, then we cannot apply Euler's formula directly. However in this case we can apply the following result obtained by Kagno [544] (see also [752] or [1086]).

Theorem C.17 *Let* $\mathbf{S}$ *be a surface, and let* $G \in \mathcal{G}(\hookrightarrow \mathbf{S})$ *be a connected graph embeddable in* $\mathbf{S}$*. Then there is a surface* $\mathbf{S}'$ *with* $\varepsilon(\mathbf{S}') \geq \varepsilon(\mathbf{S})$ *such that G has a 2-cell embedding in* $\mathbf{S}'$*.*

Theorem C.18 *Let* $\mathbf{S}$ *be a surface. If* $G \in \mathcal{G}(\hookrightarrow \mathbf{S})$ *is a connected graph with n vertices, m edges and girth at least g, where $n \geq g \geq 3$, then*

$$m \leq \frac{g}{g-2}(n - \varepsilon(S)).$$

Proof By Theorem C.17 there is a surface $\mathbf{S}'$ such that $\varepsilon(\mathbf{S}') \geq \varepsilon(\mathbf{S})$ and G has a 2-cell embedding G' in $\mathbf{S}$. Clearly, G' has n vertices and m edges. Let $\ell = |F(G')|$. Since G' is 2-cell embedded in $\mathbf{S}'$, for every face $f \in F(G')$ the facial subgraph $\mathrm{Bd}(f)$ is connected and gives rise to exactly one facial walk, which is a closed walk in G'. Since G' has girth at least g and $n \geq g$, we deduce that every facial walk in G' has at least g edges. Since each edge occurs exactly twice in facial walks, we conclude that $g\ell \leq 2m$. Then, according to Theorem C.16, we obtain that

$$\varepsilon(\mathbf{S}) \leq \varepsilon(\mathbf{S}') = n - m + \ell \leq n - m + \frac{2m}{g},$$

from which the result follows. □

Note that the condition $n \geq g$ in the above theorem is necessary, since we consider graphs with girth at least g and not graphs with girth exactly g, and any tree has girth at least g. Since any graph has girth at least three, Theorem C.18 immediately implies the following result.

Corollary C.19 *Let* $\mathbf{S}$ *be a surface. If* $G \in \mathcal{G}(\hookrightarrow \mathbf{S})$ *is a connected graph with n vertices and m edges, where $n \geq 3$, then $m \leq 3(n - \varepsilon(S))$.*

Corollary C.20 *Let* $\mathbf{S}$ *be a surface. Suppose that G is connected graph that is 2-cell embedded in* $\mathbf{S}$*, and G has n vertices and m edges, where $n \geq 3$. Then $m \leq 3(n - \varepsilon(\mathbf{S}))$ and equality holds if and only if $d_G(f) = 3$ for all faces $f \in F(G)$.*

Proof Let $\ell = |F(G)|$. Since G is connected and 2-cell embedded in $\mathbf{S}$, we have $m - n + \ell = \varepsilon(\mathbf{S})$ (by Theorem C.16). Since G is connected and $n \geq 3$, we have $d_G(f) \geq 3$ for all $f \in F(G)$. By (C.5), this leads to

$$2m = \sum_{f \in F(G)} d_G(f) \geq 3\ell = 3(m - n + \varepsilon(\mathbf{S})), \tag{C.6}$$

which is equivalent to

$$m \leq 3(n - \varepsilon(\mathbf{S})). \tag{C.7}$$

Equality holds in (C.7) if and only if it holds in (C.6), that is, if and only if $d_G(f) = 3$ for all $f \in F(G)$. □

Since the graph property $\mathcal{G}(\hookrightarrow \mathbf{S})$ is monotone, Corollary C.19 leads to an upper bound for the maximum average degree of graphs embeddable in $\mathbf{S}$ in terms of its Euler characteristic. Such a bound was established by Heawood [487]. For a surface $\mathbf{S}$, we call

$$H(\mathbf{S}) = \left\lfloor \frac{7 + \sqrt{49 - 24\,\varepsilon(\mathbf{S})}}{2} \right\rfloor \tag{C.8}$$

the **Heawood number** of $\mathbf{S}$. We may also express the Heawood number of $\mathbf{S}$ in terms of its Euler genus $\mathrm{eg}(\mathbf{S}) = 2 - \varepsilon(\mathbf{S})$, so

$$H(\mathbf{S}) = \left\lfloor \frac{7 + \sqrt{24\,\mathrm{eg}(\mathbf{S}) + 1}}{2} \right\rfloor .$$

For instance, the Heawood number of the sphere is $H(\mathbf{S}_0) = 4$. In his fundamental paper [487] published in 1890 Heawood proved the following result.

Theorem C.21 *Let* $\mathbf{S}$ *be a surface, and let* G *be a graph embeddable in* $\mathbf{S}$. *If* $\varepsilon(\mathbf{S}) \geq 1$, *then* $\mathrm{mad}(G) < 6$ *and so* $\delta(G) \leq 5$; *and if* $\varepsilon(\mathbf{S}) \leq 0$, *then*

$$\mathrm{mad}(G) \leq \frac{5 + \sqrt{49 - 24\varepsilon(\mathbf{S})}}{2}$$

and so $\delta(G) \leq H(\mathbf{S}) - 1$.

Proof Note that $\delta(G) \leq \lfloor \mathrm{mad}(G) \rfloor$. Thus, because of (C.4) and since $\mathcal{G}(\hookrightarrow \mathbf{S})$ is monotone, it suffices to show that every connected graph $G \in \mathcal{G}(\hookrightarrow \mathbf{S})$ satisfies $\mathrm{ad}(G) < 6$ if $\varepsilon(\mathbf{S}) \geq 1$, and

$$\mathrm{ad}(G) \leq h(\mathbf{S}) = \frac{5 + \sqrt{49 - 24\varepsilon(\mathbf{S})}}{2}$$

otherwise. So let $G \in \mathcal{G}(\hookrightarrow \mathbf{S})$ be a connected graph. Suppose that G has n vertices, m edges and average degree D. If $n \leq 2$, then $D \leq 1$ and there is nothing to prove. So assume that $n \geq 3$. Then it follows from Corollary C.19 that $m \leq 3(n - \varepsilon(\mathbf{S}))$, and since $D = \frac{2m}{n}$ this gives $nD \leq 6(n - \varepsilon(\mathbf{S}))$. If $\varepsilon(\mathbf{S}) \geq 1$, this implies $D < 6$ and we are done. So assume that $\varepsilon(\mathbf{S}) \leq 0$. Then $h(\mathbf{S}) \geq 6$ and there is nothing to prove if $D \leq 6$. So assume that $D > 6$. Since $m \leq 3(n - \varepsilon(\mathbf{S}))$ and $D = 2m/n$, we get again $Dn \leq 6(n - \varepsilon(\mathbf{S}))$, which is equivalent to $n(D - 6) + 6\varepsilon(\mathbf{S}) \leq 0$. Since $D > 6$ and $n \geq \Delta(G) + 1 \geq D + 1$, we get $(D + 1)(D - 6) + 6\varepsilon(\mathbf{S}) \leq 0$, from which we obtain that

$$D \leq \frac{5+\sqrt{49-24\varepsilon(\mathbf{S})}}{2}$$

and so $\delta(G) \leq \lfloor D \rfloor \leq H(\mathbf{S})-1$. Thus the proof is complete. □

For a surface **S** and a graph parameter ρ, let

$$\rho(\mathbf{S}) = \max\{\rho(G) \mid G \in \mathcal{G}(\hookrightarrow \mathbf{S})\}.$$

Theorem C.21 then says that every surface **S** different from the sphere satisfies $\delta(\mathbf{S}) \leq H(\mathbf{S})-1$ and so $\omega(\mathbf{S}) \leq H(\mathbf{S})$. For the sphere we have $\delta(\mathbf{S}_0) \leq 5$ and $\omega(\mathbf{S}_0) = H(\mathbf{S}_0) = 4$. For every surface **S** distinct from the Klein bottle, the Heawood number $H(\mathbf{S})$ is, in fact, the maximum order of a complete graph enbeddable in **S**, that is, $\omega(\mathbf{S}) = H(\mathbf{S})$. This landmark result, that was conjectured by Heawood [487], is due to Ringel [863] and Ringel and Youngs [866]. For the Klein bottle, we have $\omega(\mathbf{N}_2) = 6$, but $H(\mathbf{N}_2) = 7$.

The **(orientable) genus** $\mathrm{og}(G)$ and the **nonorientable genus** $\mathrm{ng}(G)$ of a graph G is the minimum g and the minimum h, respectively, for which G is embeddable in the orientable surface $\mathbf{S}_g$, respectively, in the nonorientable surface $\mathbf{N}_h$. So if $\mathrm{og}(G) = g$, then G is embeddable in an orientable surface **S** if and only if $\mathrm{eg}(G) \geq 2g$. It is known, see [752], that $\mathrm{ng}(G) \leq 2\mathrm{og}(G)+1$ for every graph G. If $\mathrm{ng}(G) = 2\mathrm{og}(G)+1$, then G is said to be **orientably simple**. For instance, every planar graph G is orientably simple, since $\mathrm{og}(G) = 0$ and $\mathrm{ng}(G) = 1$. Note that we consider trees as orientably simple graphs, although some authors exclude them. An embedding of G in the surface $\mathbf{S}_g$ with $g = \mathrm{og}(G)$ is called a **minimum genus embedding** of G. Similarly, a **nonorientable minimum genus embedding** of G is an embedding of G in the surface $\mathbf{N}_h$ with $h = \mathrm{ng}(G)$. For the next theorem the reader is referred to Mohar and Thomassen [752] and to Parson et al. [798].

Theorem C.22 *Let G be a connected graph. Then the following statements hold:*

(a) *Every minimum genus embedding of G is a 2-cell embedding.*
(b) *If* $\mathrm{ng}(G) \leq 2\mathrm{og}(G)$, *then every nonorientable minimum genus embedding of G is a 2-cell embedding.*

Remark C.23 Let G be a connected graph with n vertices and m edges, where $m \geq 3n+2$ and $n \geq 4$. If $G \in \mathcal{G}(\hookrightarrow \mathbf{S})$, then $3n+2 \leq m \leq 3(n-\varepsilon(\mathbf{S}))$ (by Corollary C.19), which leads to $\varepsilon(\mathbf{S}) \leq -1$. Hence $\mathrm{ng}(G) \geq 3$ and $\mathrm{og}(G) \geq 2$. Suppose that G can be embedded in $\mathbf{N}_3$. Then $\mathrm{ng}(G) = 3 \leq 2\mathrm{og}(G)$, and Theorem C.22(b) implies that every embedding of G in $\mathbf{N}_3$ is a 2-cell embedding. Now we fix such an embedding G in $\mathbf{N}_3$, which we may also denote by G. Then $m \leq 3(n+1)$ (by Corollary C.19), and so $3n+2 \leq m \leq 3n+3$. If $m = 3n+3$, then $d_G(f) = 3$ for all $f \in F(G)$ (by Corollary C.20). If $m = 3n+2$, then it follows from the proof of Corollary C.20 that there is a face $f' \in F(G)$ such that $d_G(f) = 3$ for all $f \in F(G) \setminus \{f'\}$ and $d_G(f') = 4$. Since $n \geq 4$ and G is connected, this implies that each face of the embedding G is bounded by a 3-cycle except for at most one face which is bounded by a 4-cycle.

The genus of the complete graphs was established by Ringel [863] and by Ringel and Youngs [866]; they used this result to give a complete answer to the Heawood problem concerning the clique number of surfaces.

Theorem C.24 *The following statements hold:*

(a) $\mathrm{og}(K_n) = \left\lceil \frac{(n-3)(n-4)}{12} \right\rceil$ *for* $n \geq 3$.

(b) $\mathrm{ng}(K_n) = \left\lceil \frac{(n-3)(n-4)}{6} \right\rceil$ *for* $n \geq 3$ *and* $n \neq 7$.

(c) *If* $\mathbf{S}$ *is a surface distinct from the Klein bottle, then* $\omega(\mathbf{S}) = H(\mathbf{S})$*; moreover,* $\omega(\mathbf{N}_2) = 6$ *and* $\mathrm{ng}(K_7) = 3$.

A **planar graph** is a graph embeddable in the plane; and a **plane graph** is a graph embedded in the plane. So each planar graph is isomorphic to a plane graph, and vice versa. If G is a plane graph, then exactly one of its faces is unbounded. This face is the **outer face** of G, the other faces are its **inner faces**.

A graph is embeddable in the plane $\mathbb{R}^2$ if and only if it is embeddable in the sphere $\mathbf{S}_0$, that is, the two classes $\mathcal{G}(\hookrightarrow \mathbb{R}^2)$ and $\mathcal{G}(\hookrightarrow \mathbf{S}_0)$ are the same. This follows from the fact that if x is any point of the sphere, then $\mathbf{S}_0 \setminus \{x\}$ is homeomorphic to $\mathbb{R}^2$; and the stereographic projection $\pi_x : \mathbf{S}_0 \setminus \{x\} \to \mathbb{R}^2$ is a homeomorphism between these two spaces. Let G be a planar graph, let G' be an embedding of G in the sphere, let $f \in F(G')$ be an arbitrary face, and let $x \in f$ be an arbitrary point. Then the image G^* of G' under the stereographic projection π_x is an embedding of G in the plane, and there is a one-to-one correspondence between the faces of G' and G^*, where the image of the face $f \setminus \{x\}$ with respect to π_x is the outer face of G^*. Conversely, if G^* is an embedding of G in the plane, then its preimage with respect to π_x is an embedding of G on the sphere. This can be used to prove the following result (see also Bondy and Murty [144, Proposition 10.5]).

Proposition C.25 *Let* G *be a planar graph, and let* f *be a face in some embedding of* G *in the plane. Then there is an embedding of* G *in the plane whose outer face has the same boundary subgraph as* f.

By Theorem C.22(a) it follows that every embedding of a planar connected graph in the sphere is a 2-cell embedding; and the same holds for every embedding of a planar connected graph in the plane. Furthermore, if G is a plane connected graph, then

$$|V(G)| - |E(G)| + |F(G)| = 2, \tag{C.9}$$

implying, by (C.5), that

$$\sum_{x \in F(G) \cup V(G)} (d_G(x) - 4) = -8. \tag{C.10}$$

Moreover, it follows from the proof of Theorem C.18 that every connected plane graph G with $n \geq 3$ vertices has at most $3n-6$ edges, where equality hods if and only if $d_G(f) = 3$ for all $f \in F(G)$, that is, each face is bounded by a triangle. A plane graph in which each face is bounded by a triangle is called a **plane triangulation**. A **maximal planar graph** is a planar graph G such that adding any new edge (from its complement $\overline{G}$) to G results in a nonplanar graph. Hence, each planar graph is a spanning subgraph of a maximal planar graph. If G is a planar graph of order $n \le 4$, then G is maximal planar if and only if G is a K_n. A proof of the following proposition can be found in [286, Proposition 4.2.8].

Proposition C.26 *If G is a maximal planar graph with $|G| \ge 3$, and G' is an embedding of G in the plane, then G' is a triangulation.*

Summarizing all results about planar and plane graphs mentioned in this section, we obtain the following result.

Corollary C.27 *Let G be a planar graph with n vertices and m edges, where $n \ge 3$. Then $m \le 3n-6$ and equality holds if and only if G is maximal planar.*

Proof There is a maximal planar graph $\tilde{G}$ containing G as a subgraph and with $|\tilde{G}| = |G|$. Let G' be an embedding of $\tilde{G}$ in the plane and let $\tilde{m}$ be the number of edges of $\tilde{G}$. Then G' is a triangulation (by Proposition C.26), and so $\tilde{G}$ is connected. Hence, $m \le \tilde{m} = 3n-6$ (by Corollary C.20). Furthermore, $m = 3n-6$ if and only if $G = \tilde{G}$, that is, if and only if G is maximal planar. □

Corollary C.28 *Let G be a planar graph with $|G| \ge 3$, and let G' be an embedding of G in the plane. Then G is maximal planar if and only if G' is a plane triangulation.*

A **triangulation** (respectively, **quadrangulation**) of a surface **S** is a graph G that is cellular embedded in **S** such that each facial subgraph of G is a C_4 (respectively, a C_3); the corresponding embedding of G in **S** is said to be **triangular** (respectively **quadrangular**). A graph G embedded in the sphere is a triangulation of the sphere if and only if G is a maximal planar graph. Let G be an embedding of K_3 in the torus, such that the triangle is a contractible cycle of the torus. Then G has two faces both of degree 3, but G is not 2-cell embedded in the torus and hence G is not a triangulation of the torus. On the other hand, every embedding of K_7 in the torus is a triangular embedding. There is an quadrangular embedding of K_4 in the projective plane $\mathbf{N}_1$ with three facial 4-cycles, namely (u,v,w,x,u), (u,v,x,w,u) and (u,x,v,w,u).

Suppose that G is a graph embedded in a surface or the plane and $\delta(G) \ge 2$. Then every face of G of degree at most five is bounded by a cycle.

Remark C.29 Let G be a nonbipartite graph that is embedded in the projective plane such that each facial subgraph of G is a C_4. Then G is a quadrangulation of $\mathbf{N}_1$. To prove this it suffices to show that the embedding of G in $\mathbf{N}_1$ is a cellular embedding. Let $n = |V(G)|$, $m = |E(G)|$ and $\ell = |F(G)|$. Let C be a shortest odd cycle of G, and let $p = |C|$. Since G is nonbipartite such a cycle exists. Since every facial subgraph is a C_4, we obtain that G is connected and C is a noncontractible cycle of $\mathbf{N}_1$. Clearly, C has no chords in G. Then, by cutting the projective plane $\mathbf{N}_1$ along C, we obtain a plane graph G_C with outer cycle O_C, where $|O_C| = 2p$ and G can be obtained from

G_C by identifying opposite vertices of O_C. In particular, G_C has order $n_C = n + p$ and $m_C = |E(G_C)| = m + p$. Since every facial subgraph of G is a C_4, we obtain that G_C is connected and $\ell_C = |F(G_C)| = \ell + 1$. Clearly, G_C is 2-cell embedded in the plane and so $n_C - m_C + \ell_C = 2$ which implies that $n - m + \ell = 1 = \varepsilon(\mathbf{N}_1)$. From this it follows that G is 2-cell embedded in $\mathbf{N}_1$.

C.14 Hypergraphs

A **hypergraph** H is a pair of sets, $V(H)$ and $E(H)$, where $V(H)$ is finite and $E(H)$ is a subset of $2^{V(H)}$ such that $|e| \geq 2$ for all $e \in E(H)$. The set $V(H)$ is the **vertex set** of H and its elements are the **vertices** of H. The set $E(H)$ is the **edge set** of H and its elements are the **edges** of H. Thus graphs are special hypergraphs and we adopt the notation for graphs. A hypergraph H is **empty** if $V(H) = E(H) = \varnothing$; in this case we write $H = \varnothing$.

Let H be a hypergraph. The number of vertices of H is its **order**, denoted by $|H|$. An edge e with $|e| \geq 3$ is called a **hyperedge**, and an edge e with $|e| = 2$ is an **ordinary edge**. If all edges of H have the same size p, then H is said to be **p-uniform**. So a graph is a 2-uniform hypergraph, that is, a hypergraph in which each edge is ordinary. A vertex v is **incident** with an edge e if $v \in e$. For a vertex v of H, let $E_H(v) = \{e \in E(H) \mid v \in e\}$. The **degree** of v in H is $d_H(v) = |E_H(v)|$. Let $\delta(H)$ and $\Delta(H)$ denote the **minimum degree** and the **maximum degree** of H, respectively. A nonempty hypergraph H is said to be **regular** and **r-regular** if all vertices of H have the same degree r, that is, if $\delta(G) = \Delta(G) = r$. By double counting, we deduce that

$$\sum_{v \in V(H)} d_H(v) = \sum_{e \in E(H)} |e|, \tag{C.11}$$

which yields $r|H| = p|E(H)|$ if the hypergraph H is p-uniform and r-regular.

Let H and H' be hypergraphs. If $V(H') \subseteq V(H)$ and $E(H') \subseteq E(H)$, then H' is called a **subhypergraph** of H and we write $H' \subseteq H$. Clearly, $H = H'$ if and only if $H' \subseteq H$ and $H \subseteq H'$. A subhypergraph H' of H with $H' \neq H$ is said to be a **proper subhypergraph** of H, written as $H' \subset H$. As for graphs, $H' \subseteq H$ is a **spanning subhypergraph** if $V(H') = V(H)$. Let $X \subseteq V(H)$. We define two hypergraphs $H[X]$ and $H\langle X\rangle$ by

$$V(H[X]) = X \text{ and } E(H[X]) = E(H) \cap 2^X,$$

and

$$V(H\langle X\rangle) = X \text{ and } E(H\langle X\rangle) = \{e \cap X \mid e \in E(H) \text{ and } |e \cap X| \geq 2\}.$$

While $H[X]$ is a subhypergraph of H (the subhypergraph of H **induced** by X), $H\langle X\rangle$ need not be a subhypergraph of H. Clearly, if H is a graph, then $H[X] = H\langle X\rangle$. Furthermore, let $H - X = H[V(H) \setminus X]$ and $H \div X = H\langle V(H) \setminus X\rangle$. As usual, we abbreviate $H - \{v\}$ by $H - v$ and $H \div \{v\}$ by $H \div v$. A subhypergraph H' of H is called an **induced subhypergraph** of H if $H' = H[V(H')]$. For a set $F \subseteq 2^{V(H)}$, we define $H - F = (V(H), E(H) \setminus F)$ and $H + F = (V(H), E(H) \cup F)$. When $F = \{e\}$, we abbreviate $H - F$ by $H - e$ and $H + F$ by $H + e$.

The hypergraphs H and H' are **isomorphic**, written $H \cong H'$, if there is a bijection $\phi : V(H) \to V(H')$ such that, for all sets $e \subseteq V(H)$, we have $e \in E(H)$ if and only if $\phi(e) \in E(H')$.

For hypergraphs H and H' let $H \cup H' = (V(H) \cup V(H'), E(H) \cup E(H'))$ be the **union** of H and H', and let $H \cap H' = (V(H) \cap V(H'), E(H) \cap E(H'))$ be the **intersection** of H and H'. We say that H and H' are **disjoint** if $H \cap H' = \varnothing$.

Let H be a hypergraph and let $X \subseteq V(H)$ be a vertex set. We call X an **independent set** or p**-independent set** of H if $|X| = p$ and $H[X]$ has no edges. The **independence number** of H, written $\alpha(H)$, is the largest integer p for which H contains a p-independent set. We call X a **clique** or p**-clique** of H if $|X| = p$ and $[X]^2 \subseteq E(H)$. The **clique number** of H, written $\omega(G)$, is the largest integer p for which H contains a p-clique. If X is a p-clique of H, then $H[X]$ contains K_p as a spanning subhypergraph, but not necessarily as an induced subhypergraph.

Let $q \geq 2$ be an integer. For a set X, let $K^q(X)$ denote the hypergraph with $V(K^q(X)) = X$ and $E(K^q(X)) = [X]^q$. We call $K^q(X)$ an **complete** q**-uniform hypergraph**. So $K^2(X)$ is a complete graph with vertex set X, and $K^q(X)$ is edgeless if $|X| < r$. If $|X| = q$, then we write $K(X)$ instead of $K^q(X)$. So $K(X)$ is the hypergraph with $V(K(X)) = X$ and $E(K(X)) = \{X\}$, where $|X| \geq 2$. If H is a complete q-uniform hypergraph of order n, then we also write $H = K_n^q$ and say that H is a K_n^q.

A **bipartite hypergraph** is a hypergraph H whose vertex set is the disjoint union of two sets (called **partite sets** or **parts**), each of which is an independent set of H. So if H is a bipartite hypergraph with parts X and Y, then both hypergraphs $H[X]$ and $H[Y]$ are edgeless, $V(H) = X \cup Y$ and $X \cap Y = \varnothing$.

Let H be a hypergraph and let $W = (v_0, e_1, v_1, \ldots, v_{n-1}, e_n, v_n)$ be a sequence with $n \geq 0$ such that $v_0, v_1, \ldots, v_n$ are vertices of H and $e_1, e_2, \ldots, e_n$ are edges of H. We call W a **hyperwalk** of H if $\{v_{i-1}, v_i\} \subseteq e_i$ for $1 \leq i \leq n$. We say that W **connects** or **joins** v_0 and v_n, and that W **starts** in v_0, and **terminates** in v_n, and that W is a v_0-v_n hyperwalk. The vertices v_0 and v_n are called the **ends** of the hyperwalk W, v_0 being its **initial vertex** and v_n its **terminal vertex**; the remaining vertices $v_1, v_2, \ldots, v_{n-1}$ are its **internal vertices**. The integer n is called the **length** of W, written as $\ell(W) = n$. The reverse sequence $W^{-1} = (v_n, e_n, v_{n-1}, \ldots, v_1, e_1, v_0)$ is also a hyperwalk in G. A walk is **odd** or **even**, depending on whether its length is odd or even. The hyperwalk W is called a **hyperpath** if all the vertices v_i and all the edges e_j are distinct. The hyperwalk W with $n \geq 2$ is called a **hypercycle** if the the vertices $v_0, v_1, \ldots, v_{n-1}$ are distinct, $v_0 = v_n$ and all the edges e_j are distinct. As for graphs, the **girth** $g(G)$ of a hypergraph is the length of a shortest hypercycle of G; if G has no hypercycle we define $g(G) = \infty$.

Proposition C.30 *Let H be a hypergraph and let v, w be vertices of H. Then the following statements are equivalent:*

(a) *There exists a v-w hyperwalk in H.*

(b) *There exists a v-w hyperpath in H.*

Proof Since every hyperpath is a hyperwalk, (b) implies (a). To see that the reverse implication is true let $W = (v_0, e_1, v_1, \dots, v_{n-1}, e_n, v_n)$ be a v-w hyperwalk, whose length is minimum. By (a) such a hyperwalk exists. If W is no hyperpath, then $n \geq 1$ and either two vertices of W or two edges of W are equal. If $v_i = v_j$ with $0 \leq i < j \leq n$, then $W' = (v_0, e_1, v_1, \dots, e_{i-1}, v_j, e_j, \dots, v_{n-1}, e_n, v_n)$ is a v-w hyperwalk of H with $\ell(W') < \ell(W)$. If $e_i = e_j$ with $0 \leq i < j \leq n-1$, then $W' = (v_0, e_1, v_1, \dots, e_{i-1}, v_i, e_j, v_{j+1} \dots, v_{n-1}, e_n, v_n)$ is a v-w hyperwalk of H with $\ell(W') < \ell(W)$. So in booth cases we obtain a contradiction. □

A nonempty hypergraph H is **connected** if there exists a hyperpath between any two of its vertices, and otherwise it is **disconnected**. A (connected) **component** of a nonempty hypergraph H is a maximal connected subhypergraph of H. As for graphs, let $\mathrm{com}(H)$ denote the number of components of H. By convention, we put $\mathrm{com}(\emptyset) = 0$.

Let H be a nonempty hypergraph. For vertices v and w of H, let $v \sim w$ if there exists a v-w hyperwalk in H. Then it is easy to check that $\sim$ is an equivalence relation on $V(H)$. By Proposition C.30 it follows that H' is a component of H if and only if H' is an induced subhypergraph of H and $V(H')$ is an equivalence class of $V(H)$ with respect to the relation $\sim$.

Proposition C.31 *Let H be a nonempty hypergraph, let U be the vertex set of a component of H, and let e be an edge of H. Then either $e \subseteq U$ or $e \subseteq V(G) \setminus U$.*

Proof Suppose, to the contrary, that there is an edge $e \in E(H)$ such that e contains a vertex u of U as well as a vertex w of $V(G) \setminus U$. The set U is an equivalence class with respect to the above defined relation $\sim$. Since $F = (u, e, w)$ is a hyperwalk, we obtain $u \sim w$ implying that $w \in U$, a contradiction. □

Let H be a hypergraph, and let $e \in E(H)$. Then it is easy to see that $\mathrm{com}(H) \leq \mathrm{com}(H - e) \leq \mathrm{com}(H) + |e| - 1$. We call e a **bridge** of H if $\mathrm{com}(H - e) > \mathrm{com}(H)$; and we call e a **strong bridge** of H if $\mathrm{com}(H - e) = \mathrm{com}(H) + |e| - 1$. Obviously, e is a bridge of H if and only if two vertices of e belong to different components of $H - e$; and e is a strong bridge of H if and only if no two vertices of e belong to the same component of $H - e$. The proof of the following proposition is straight forward.

Proposition C.32 *Let H be a connected hypergraph and let e be an edge of H. Then the following statements are equivalent:*

(a) *e is a strong bridge of H.*
(b) *e contains exactly one vertex from each component of $H - e$.*
(c) *e lies in no hypercycle of H.*

Let H be a hypergraph. We call $F \subseteq E(H)$ a **separating edge set** of H if we have $\mathrm{com}(H-F) \geq 2$. If $X \subseteq V(H)$, then we denote by $\partial_H(X)$ the set of edges of H containing a vertex of X as well as a vertex of $V(H) \setminus X$. By a **cut** or **edge cut** of H we mean a triple (X,Y,F) such that X is a nonempty proper subset of $V(H)$, $Y = V(H) \setminus X$, and $F = \partial_H(X) = \partial_H(Y)$. Clearly, if H is connected and $\varnothing \neq X \subset V(H)$, then $F = \partial_H(X)$ is a separating edge set of H, but not necessarily a minimal one.

Proposition C.33 *Let H be a connected hypergraph and let $F \subseteq E(H)$ be a minimal separating edge set. Then $F = \partial_H(X)$ for a set X with $\varnothing \neq X \subset V(H)$, and so $(X, V(H) \setminus X, F)$ is a cut.*

Proof Since H is connected and F is a separating edge set of H, $H - F$ is the union of two nonempty disjoint hypergraphs, say H_1 and H_2. Then $F_1 = \partial_H(V(H_1))$ is a separating edge set of H and $F_1 \subseteq F$. Since F is a minimal separating edge set, $F = F_1$ and we are done. □

A hypergraph H is called k**-edge-connected** ($k \in \mathbb{N}_0$) if $|H| \geq 2$ and $H - F$ is connected for every set $F \subseteq E(H)$ with $|F| \leq k-1$. For a hypergraph H with $|H| \geq 2$, the **edge-connectivity** $\kappa'(H)$ is defined to be the largest integer k such that H is k-edge-connected. By Proposition C.33 it follows that if $|H| \geq 2$, then

$$\kappa'(H) = \min\{|\partial_H(X)| \mid \varnothing \neq X \subset V(H)\}. \tag{C.12}$$

Let H be a hypergraph. A **separation** of H is a triple (S, H_1, H_2) consisting of a set $S \subseteq V(H)$ and two induced subhypergraphs H_1 and H_2 of H such that $H = H_1 \cup H_2$, $|H_i| > |S|$ $(i = 1,2)$ and $V(H_1) \cap V(H_2) = S$. If (S, H_1, H_2) is a separation of H, then S is called a **separation set** of H. A separation set S of H is **minimal** if no proper subset of S is a separation set of H. Clearly, the empty hypergraph and the hypergraph consisting of one vertex have no separation set and the only minimal separation set of a disconnected hypergraph is the empty set.

Proposition C.34 *Let H be a connected hypergraph and let $S \subseteq V(H)$. Then S is a separation set of H if and only if $H \div S$ is disconnected.*

Proof If S is a separation set of H, then there is a separation (S, H_1, H_2) of H. Then there is a vertex $v \in V(H_1) \setminus S$ as well as a vertex $w \in V(H_2) \setminus S$. We claim that v and w belong to different components of $H' = H \div S$. If not, then there is a v-w hyperwalk W of H'. Then W contains a subhyperwalk $W' = (v', e', w')$ with $v' \in V(H_1) \setminus S$, $w' \in V(H_2) \setminus S$ and $e' \in E(H')$. But then there is an edge $e \in E(H)$ such that $\{v', w'\} \subseteq e \subseteq S \cup \{v', w'\}$ and so $e \notin E(H_1) \cup E(H_2)$, a contradiction. Hence v and w belong to different components of H' as claimed and so H' is disconnected.

If $H' = H \div S$ is disconnected, then $\mathrm{com}(H') = p \geq 2$. Let $H'_1, H'_2, \ldots, H'_p$ be the components of H'. Let $H_1 = H[V(H'_1) \cup V(H'_2) \cup \cdots \cup V(H'_{p-1}) \cup S]$ and let $H_2 = H[V(H'_p) \cup S]$. We claim that (S, H_1, H_2) is a separation of H and so S is a separation set of H. If not then one edge e of H does not belong to $H_1 \cup H_2$. So e contains a vertex of $V(H'_p)$ as well as a vertex of $V(H'_1) \cup V(H'_2) \cup \cdots \cup V(H'_{p-1})$. But then H' contains also such an edge, which is impossible. □

A hypergraph H is k**-connected** ($k \in \mathbb{N}_0$) if $|H| \geq k+1$ and every separation set S of H satisfies $|S| \geq k$. For a nonempty hypergraph H the **connectivity** $\kappa(H)$ is the largest integer k such that H is k-connected. Note that Proposition C.1 is not true for hypergraphs. It is not difficult to show that a hypergraph H is k-connected if $|H| \geq k+1$ and $H \div S$ is connected for every set $S \subseteq V(H)$ with $|S| \leq k-1$. Note that if H is a graph, then $H \div S = H - S$.

Let H be a hypergraph. A set $S \subseteq V(H)$ is called a **separating vertex set** of H if $\text{com}(H \div S) \geq 2$. Then Proposition C.34 says that if H is a connected hypergraph and $S \subseteq V(H)$, then S is a separating vertex set of H if and only if S is a separation set of H. A **separating vertex** of H is a vertex v of H such that $\text{com}(H \div v) > \text{com}(H)$. This obviously implies that the separating vertices of a disconnected hypergraph are those of its components. If H is a connected hypergraph, then v is a separating vertex of H if and only if $S = \{v\}$ is a separation set of H (by Proposition C.34), that is, H is the union of two (induced) subhypergraphs having only vertex v in common and having both at least two vertices.

Let H be a hypergraph. A **block** of H is a maximal connected subhypergraph of H that has no separating vertex. Let $\mathfrak{B}(H)$ denote the set of all blocks of H. For a vertex $v \in V(H)$, let $\mathfrak{B}_v(H) = \{B \in \mathfrak{B}(H) \mid v \in V(B)\}$. Note that $\mathfrak{B}(\varnothing) = \varnothing$ and that every block of H is a connected induced subhypergraph of H. If H is disconnected, then $\mathfrak{B}(H)$ is the union of $\mathfrak{B}(H')$ taken over all components H' of H. If H is connected and has no separating vertex, then $\mathfrak{B}(H) = \{H\}$ and we say that H is a **block**. If H is connected and $(\{v\}, H_1, H_2)$ is a separation of H, $\mathfrak{B}(H) = \mathfrak{B}(H_1) \cup \mathfrak{B}(H_2)$ and $\mathfrak{B}(H_1) \cap \mathfrak{B}(H_2) = \varnothing$. As for graphs it is not difficult to show that any two distinct blocks of H have at most one vertex in common, and a vertex of H is a separating vertex of H if and only if it belongs to more that one block. A block of H, which contains at most one separating vertex of H is called an **end-block** of H. If H contains a separating vertex then H has at least two end-blocks. Note that for a hypergraph H the relation on $E(H)$ *to be equal or to lie on a common cycle* is not necessarily an equivalence relation.

C.15 Lovász Local Lemma

A (finite) **probability space** is a pair (Ω, Pr), where Ω is a finite set, called the **sample space**, and $\text{Pr} : \Omega \to \mathbb{R}$ is a **probability function**, that is, $0 \leq \text{Pr}(\omega) \leq 1$ for all $\omega \in \Omega$ and $\sum_{\omega \in \Omega} \text{Pr}(\omega) = 1$.

Let (Ω, Pr) be a probability space. Any subset A of Ω is called an **event** and the **probability** of A is defined by:

$$\text{Pr}(A) = \sum_{\omega \in A} \text{Pr}(\omega).$$

Two events A and B are **independent** if $\text{Pr}(A \cap B) = \text{Pr}(A)\text{Pr}(B)$; otherwise, they are **dependent**. Clearly, the independence relation is symmetric and if $\text{Pr}(B) > 0$, then A and B are independent if and only if $\Pr(A|B) = \Pr(A)$, where $\Pr(A|B)$ is the **conditional probability** $\Pr(A \cap B)/\Pr(B)$ that the event A occurs given that event B occurs. If A and B are independent events, then so are A and the negation $\overline{B}$ of B. If $A_1, A_2, \ldots, A_n$ are events and $S \subseteq [1,n]$, we put $A_S = \bigcup_{i \in S} A_i$. Events $A_1, A_2, \ldots, A_n$ are **mutually independent** if

$$\Pr(A_S) = \prod_{i \in S} \Pr(A_i)$$

for all $S \subseteq [1,n]$. If $A_1, A_2, \ldots, A_n$ are mutually independent events, then so are $\overline{A}_1, \overline{A}_2, \ldots, \overline{A}_n$. An event A is mutually independent of the events $A_1, A_2, \ldots, A_n$ if

$$\Pr(A \cap A_S) = \Pr(A)\Pr(A_S)$$

for all $S \subseteq [1,n]$. A **dependency digraph** for events $A_1, A_2, \ldots, A_n$ is a strict digraph D with vertex set $V(D) = [1,n]$ such that for each $i \in I$, the event A_i is mutually independent of the events A_j with $(i,j) \notin E(D)$. Note that the dependency digraph is not unique. The following result, first stated, proved and used in Erdős and Lovász [354], is known as the Local Lemma (see the book of Alon and Spencer [57]); it is a powerful tool in combinatorics and graph theory.

Lemma C.35 (The Local Lemma; General Case) *Let $A_1, A_2, \ldots, A_n$ be events in an arbitrary probability space, and let D be a dependency digraph for these events. Suppose that there exist real numbers $x_1, x_2, \ldots, x_n$ such that $0 \leq x_i < 1$ and*

$$\Pr(A_i) \leq x_i \cdot \prod_{(i,j) \in E(D)} (1 - x_j) \tag{C.13}$$

for all $i \in [1,n]$. Then

$$\Pr\left(\bigcap_{i=1}^{n} \overline{A}_i\right) \geq \prod_{(i,j) \in E(D)} (1 - x_j).$$

In particular, with positive probability no event A_i $(i \in [1,n])$ holds.

Corollary C.36 (The Local Lemma; Symmetric Case) *Let $A_1, A_2, \ldots, A_n$ be events in an arbitrary probability space. Suppose that each event A_i is mutually independent of all the other events A_j except at most d, and that $\Pr(A_i) \leq p$ for all $i \in [1,n]$. If*

$$ep(d+1) \leq 1,$$

then $\Pr\left(\bigcap_{i=1}^{n} \overline{A}_i\right) > 0$ (the constant e is Euler's number 2.718..).

Proof If $d = 0$, then the events $A_1, A_2, \ldots, A_n$ are mutually independent and the result is trivial. So assume that $d \geq 1$. By assumption, there is a dependency digraph D for the events $A_1, A_2, \ldots, A_n$ with $\Delta^+(D) \leq d$. We now apply Lemma C.35 with

$x_i = \frac{1}{d+1}$ ($i \in [1,n]$). Clearly, $0 \le x_i < 1$ and the proof is completed by showing that (C.13) holds. Since $ep(d+1) \le 1$ and $\Delta^+(D) \le d$ we deduce that

$$\Pr(A_i) \le p \le \frac{1}{d+1} \cdot \frac{1}{e} \le \frac{1}{d+1} \cdot \left(1 - \frac{1}{1+d}\right)^d \le x_i \prod_{(i,j)\in E(D)} (1 - x_j).$$

Thus Corollary C.36 is proved. □

Meanwhile, an incredible number of applications of the Lovás Local Lemma (LLL) are known, some interesting applications were discovered in 1994 by Alon [46]. For later use we want to discuss one of his results here in detail, see [46, Proposition 5.3].

Lemma C.37 (ALON) *Let H be a graph, and let $\mathcal{X}$ be a partition of the vertex set of H. Suppose there is a positive integer k such that $|X| \ge 2ek$ for all $X \in \mathcal{X}$ and $\Delta(H) \le k$. Then H has an independent set I such that $|I \cap X| = 1$ for all $X \in \mathcal{X}$.*

Proof For the proof it suffices to consider the case that each set of $\mathcal{X}$ has cardinality $\ell = \lceil 2ek \rceil$ and is an independent set of H. For each set $X \in \mathcal{X}$, choose independently and uniformly a single vertex of X. Let I denote the resulting random subset of $V(H)$. Clearly, $|I \cap X| = 1$ for all $X \in \mathcal{X}$. To complete the proof it suffices to show that with positive probability I is an independent set. To do this, we shall use Corollary C.36. For an edge uv of H, let A_{uv} denote the event that $\{u, v\} \subseteq I$. Since each set in $\mathcal{X}$ has size ℓ, we have $\Pr(A_{uv}) = \ell^{-2}$. Now let uv be an edge of H. Then there are two sets in $\mathcal{X}$, say U and V, such that $u \in U$ and $v \in V$. Let $F(uv)$ be the set of edges of H such that one of its end belongs to $U \cup V$. Note that $uv \in F(uv)$ and

$$|F(uv)| \le \sum_{u' \in U} d_H(u') + \sum_{v' \in V} d_H(v') \le 2\ell k.$$

An event A_{uv} is mutually independent of all the events $A_{u'v'}$, provided that $u'v' \notin F(uv)$. So an event A_{uv} is mutually independent of all other events $A_{u'v'}$ except at most $d = 2\ell k - 1$. Since

$$e\, \ell^{-2}(d+1) = \frac{2ek}{\ell} = \frac{2ek}{\lceil 2ek \rceil} \le 1,$$

Corollary C.36 implies that I is an independent set of H with positive probability, as desired. □

In 1975, Bollobás, Erdős, and Spencer [130] conjectured that the quantity $2ek$ in the above lemma can be replaced by $2k$. This is best possible in the sense that $2k$ could not be replaced by ck for any constant $c < 2$. This was first proved in [130]; further constructions were later given by Yuster [1087]. Alon's proof of his lemma indicates that $2ek$ might be best value that can be obtained by applying the LLL. A proof that the conjectured value $2k$ is the right one was obtained in 2001 by Aharoni and Haxell as a special case of a more general result (unpublished); the theorem was first stated in a short paper by Haxell [482], but without a proof. Meanwhile several extensions of the Aharoni–Haxell result have been obtained by applying methods from topology (see [482]), but a direct proof would still be of interest.

Theorem C.38 (Aharoni and Haxell) *Let H be a graph, and let $\mathcal{X}$ be a partition of the vertex set of H. Suppose there is a positive integer k such that $|X| \geq 2k$ for all $X \in \mathcal{X}$ and $\Delta(H) \leq k$. Then H has an independent set I such that $|I \cap X| = 1$ for all $X \in \mathcal{X}$.*

Alon's decomposition lemma and its proof is a typical example how the LLL is used in order to prove the existence of combinatorial objects satisfying certain constraints. In 2010 Moser and Tardos [761] obtained a fully constructive proof of the LLL, thereby extending an earlier approach of Beck [88]. They also developed a general framework, the so-called *entropy compression method*, that can be used to provide an efficient algorithms for finding the combinatorial objects whose existence are guaranteed by the LLL. It tur ned out that one could obtain better combinatorial results by a direct application of the entropy compression method instead of applying the LLL in its original form. The LLL requires most events to be independent of each other. Erdős and Spencer [357] presented another version of the LLL, the lopsided LLL, which relaxes the requirement of independence to negative dependence, which is more general but also more difficult to identify. We state here the symmetric version, see, for example, [57, Theorem x.x].

Lemma C.39 (The Lopsided Local Lemma) *Let $A_1, A_2, \dots, A_n$ be events in an arbitrary probability space. Suppose that for every $i \in [1,n]$ there is a set $N_i \subseteq [1,n]$ such that $|N_i| \leq d$ and for all $M \subseteq [1,n] \setminus N_i$,*

$$\Pr\left(A_i \mid \bigcap_{j\in M} \overline{A}_j\right) \leq p$$

If

$$4pd \leq 1,$$

then $\Pr\left(\bigcap_{i=1}^n \overline{A}_i\right) > 0$

A collection $X_1, X_2, \dots, X_n$ of $\{0,1\}$-valued random variables is called **negatively correlated** if for all $I \subseteq [1,n]$,

$$\Pr(X_i = 1 \text{ for all } i \in I) \leq \prod_{i\in I} \Pr(X_i = 1). \tag{C.14}$$

Panconesi and Srinivasan [797] noted that several Chernoff-type bounds for mutually independent variables also hold on negatively correlated variables. For the following result see also [754, Lemma 3]. Here $\mathbb{E}(X)$ denotes the **expectation** of X.

Lemma C.40 (Chernoff bounds) *Let $X_1, X_2, \dots, X_n$ be $\{0,1\}$-valued random variables, let $Y_i = 1 - X_i$ for $i \in [1,n]$, let $X = \sum_{i=1}^n X_i$, and let $0 < t < \mathbb{E}(X)$. Then the following statements hold:*

(a) *If the variables* $X_1, X_2, \ldots, X_n$ *are negatively correlated, then*

$$\Pr(X > \mathbb{E}(X) + t) < \exp(-t^2/3\mathbb{E}(X))$$

(b) *If the variables* $Y_1, Y_2, \ldots, Y_n$ *are negatively correlated, then*

$$\Pr(X < \mathbb{E}(X) - t) < \exp(-t^2/2\mathbb{E}(X)).$$

If a random variable is the sum of mutually independent $\{0,1\}$-valued random variables, the following standard concentration bound can be used, see for instance [729, Theorem 2.3] or [306, Theorem 1.10.1].

Lemma C.41 *Let* X *be the sum of* n *mutually independent* $\{0,1\}$*-valued random variables, and let* $\vartheta > 0$. *Then*

$$\Pr(X \geq (1+\vartheta)\mathbb{E}(X)) \leq \exp\left(-\frac{\vartheta^2 \mathbb{E}(X)}{2 + \frac{2}{3}\vartheta}\right) \leq \exp\left(-\frac{\min\{\vartheta^2, \vartheta\}\mathbb{E}(X)}{3}\right)$$

Let $((\Omega_i, \Pr_i))_{i=1}^n$ be a family of finite probability spaces, and let $(\Omega, \Pr)$ be their product space. Let $X : \Omega \to \mathbb{R}$ be a nonnegative random variable, and let $\ell, r > 0$. We say that X is (r,ℓ)**-certifiable** if for every $s > 0$ and every $\omega = (\omega_1, \omega_2, \ldots, \omega_n)$ such that $\omega \in \Omega$ and $X(\omega) \geq s$, there exists an index set $I \subseteq [1,n]$ with $|I| \leq rs$ such that, for all $k \geq 0$, we have $X(\omega') \geq s - k\ell$, for all $\omega' = (\omega'_1, \omega'_2, \ldots, \omega'_n) \in \Omega$ such that $\omega_i \neq \omega'_i$ for at most k indices $i \in I$. For the following combinatorial version of Talagrand's inequality see Kelly and Postle [566, Theorem 6.3 with $\Omega^* = \emptyset$].

Theorem C.42 (Talagrand's Inequality) *Let* $((\Omega_i, \Pr_i))_{i=1}^n$ *be a family of finite probability spaces, and let* $(\Omega, \Pr)$ *be their product space. Let* $X : \Omega \to \mathbb{R}$ *be a nonnegative random variable, not identical* 0*, which is* (r,ℓ)*-certifiable, where* $\ell, r > 0$. *Then*

$$\Pr\left(|X - \mathbb{E}(X)| > t)\ \right) \leq 4\exp\left(-\tfrac{t^2}{8r\ell^2(4\mathbb{E}(X)+t)}\right)$$

for any real number t *with* $t > 96\ell\sqrt{r\mathbb{E}(X)} + 128r\ell^2$.

C.16 Fekete's Lemma

The following result is due to Pólya and Szegő [816]; in the literature it is also known as Fekete's lemma [369].

Lemma C.43 (Fekete's lemma) *Let* $f : \mathbb{N} \to \mathbb{R}$ *be a function such that*

$$f(m+n) \leq f(m) + f(n)$$

for all integers $m, n \geq 1$. *Then the limit* $\lim_{n\to\infty} \frac{f(n)}{n}$ *exists and*

$$\lim_{n\to\infty} \frac{f(n)}{n} = \inf_{n\in\mathbb{N}} \frac{f(n)}{n}.$$

Proof Let c be an arbitrary real number satisfying $c > \inf_n \frac{f(n)}{n}$. Then there exists an integer $p \in \mathbb{N}$ with $\frac{f(p)}{p} < c$. Now let $n \in \mathbb{N}$. Then $n = gp + r$ with $0 \le r < p$. From the hypothesis of the lemma we deduce that

$$f(n) = f(gp+r) \le f(gp) + f(r) \le gf(p) + rf(1),$$

which gives

$$\frac{f(n)}{n} \le \frac{gf(p)+rf(1)}{gp+r} = \frac{f(p)+\frac{r}{g}f(1)}{p+\frac{r}{g}}.$$

If n tends to infinity, then $\frac{r}{g}$ tends to zero. Hence the above inequality implies that

$$\limsup_{n\to\infty} \frac{f(n)}{n} \le \frac{f(p)}{p} < c.$$

By the choice of c, we then we deduce that

$$\limsup_{n\to\infty} \frac{f(n)}{n} \le \inf_n \frac{f(n)}{n},$$

which yields $\lim_{n\to\infty} \frac{f(n)}{n} = \inf_n \frac{f(n)}{n}$. □

A function $f : \mathbb{N} \to \mathbb{R}$ satisfying the hypothesis of Fekete's lemma is called **subadditive**. We need the following strengthening of Fekete's lemma.

Lemma C.44 *Let $f : \mathbb{N} \to \mathbb{R}_{\ge 0}$ be a function and let $k, h \in \mathbb{N}$. Suppose that*

$$f(m+n) \le f(m) + f(n+h) \tag{C.15}$$

for all integers $m, n \ge k$. Then the limit $\lim_{n\to\infty} \frac{f(n)}{n}$ exists and is finite.

Proof First we claim that $f(n) \le cn$ for all $n \in \mathbb{N}$, where

$$c = \max\{\frac{f(k)}{k}, \frac{f(k+h)}{k}, \frac{f(k+1+h)}{k+1}, \ldots, \frac{f(2k-1+h)}{2k-1}\}.$$

To see this, let $n \in \mathbb{N}$. Then $n = gk + r$ with $g \ge 0$ and $0 \le r \le k-1$. By repeated application of (C.15), we deduce that

$$\begin{aligned} f(n) &= f(gk+r) \\ &\le f(k) + (g-2)f(k+h) + f(k+r+h) \\ &\le ck + (g-2)ck + c(k+r) = c(gk+r) = cn. \end{aligned}$$

Since $0 \le f(n) \le cn$ for all $n \in \mathbb{N}$, there are real numbers a and b such that

$$a = \liminf_{n\to\infty} \frac{f(n)}{n} \le \limsup_{n\to\infty} \frac{f(n)}{n} = b$$

To show that $a = b$, let $\varepsilon > 0$ be given. Then there exists an integer $p \ge \max\{\frac{c}{\varepsilon}, k\}$ for which

$$\frac{f(p)}{p} \le a + \varepsilon.$$

Now let n be some suitable large integer. Then $n = gp + r$ for some integers g, r satisfying

$$\max\{k, \frac{c(h+k+p)}{\varepsilon p}\} \le g \text{ and } k \le r < k + p \tag{C.16}$$

Then by repeated application of (C.15) we obtain

$$f(n) = f(gp+r) \le gf(p) + f(gh+r) \le gf(p) + f(gh) + f(h+r).$$

Using (C.16) and the fact that $f(n) \le cn$ for all $n \in \mathbb{N}$, we deduce from the above inequality that

$$\begin{aligned}
\frac{f(n)}{n} &\le \frac{gf(p) + f(gh) + f(h+r)}{gp+r} \\
&\le \frac{gf(p) + cgh + c(h+r)}{gp} \\
&\le \frac{f(p)}{p} + \frac{c}{p} + \frac{c(h+k+p)}{gp} \\
&\le (a+\varepsilon) + \varepsilon + \varepsilon = a + 3\varepsilon.
\end{aligned}$$

Since this holds for all $\varepsilon > 0$ and all sufficiently large n, we deduce that $b \le a$, so $a = b$, and the desired limit exists and is finite. □

References

1. Abbott, H.L. and Hanson, D. (1969). On a combinatorial problem of Erdős. *Can. Math. Bull.,* 12:823–829.
2. Abbott, H.L. and Hare, D.R. (1989). Sparse color-critical hypergraphs. *Combinatorica,* 9:233–243.
3. Abbott, H.L., Hare, D.R. and Zhou, B. (1994). Sparse color-critical graphs and hypergraphs with no short cycles. *J. Graph Theory,* 18:373–388.
4. Abbott, H.L., Hare, D.R. and Zhou, B. (1998). Color-critical graphs and hypergraphs with few edges and no short cycles. *Discrete Math.,* 182:3–11.
5. Abbott, H.L. and Liu, A. (1978). The existence problem for color critical linear hypergraphs. *Acta Math. Hungar.,* 32:273–282.
6. Abbott, H.L. and Liu, A. (1980). On property B of families of sets. *Can. Math. Bull.,* 23:429–435.
7. Abbott, H.L., Liu, A. and Toft, B. (1980). The enumeration problem for color critical linear hypergraphs. *J. Combin. Theory Ser. B,* 29:106–115.
8. Abbott, H.L. and Moser, L. (1964). On a combinatorial Problem of Erdős and Hajnal. *Can. Math. Bull.,* 7:177–181.
9. Abbott, H.L. and Zhou, B. (1991). The edge density of 4-critical planar graphs. *Combinatorica,* 11:185–189.
10. Abbott, H.L. and Zhou, B. (1991). On small faces in 4-critical planar graphs. *Ars Combin.,* 32:203–207.
11. Abbott, H. L. and Zhou, B. (1992). On a conjecture of Gallai concerning complete subgraphs of k-critical graphs. *Discrete Math.,* 100:223–228.
12. Abbott, H.L. and Zhou, B. (1995). Some remarks on $(k-1)$-critical subgraphs of k-critical graphs. *Combinatorica,* 15:469–474.
13. Aboulker, P., Aubian, G. and Charbit, P. (2023). Digraph colouring and arc-connectivity. Manuscript arXive:2304.04690
14. Aboulker, P., Bellitto, T., Havet, F. and Rambaud, C. (2022). On the minimum number of arcs in k-critical oriented graphs. Manuscript arXive:2207.01051
15. Aboulker, P., Brettell, N., Havet, F., Marx, D. and Trotignon, N. (2017). Colouring graphs with constraints on connectivity. *J. Graph Theory,* 85:814–838.
16. Aboulker, P., Havet, F., Knauer, K. and Rambaud, C. (2022). On the dichromatic number of surfaces. *Electron. J. Combin.,* 29:#P1.30.
17. Aboulker, P. and Vermande, Q. (2022). Various bounds on the minimum number of arcs in a k-dicritical digraph. Manuscript arXive:2208.02112
18. Abu-Khzam, F.N., Feghali, C. and Heggernes, P. (2020). Partitioning a graph into degenerate subgraphs. *European J. Combin.,* 83:103015.
19. Addario-Berry, L., Havet, F. and Thomassé, S. (2007). Paths with two blocks in n-chromatic digraphs. *J. Combin. Theory Ser. B,* 97:620–626.

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https://doi.org/10.1007/978-3-031-50065-7

20. Aglave, S., Amarnath, V.A., Shannigrahi, S. and Singh, S. (2020). Improved bounds for uniform hypergraphs without property B. *Australas. J. Combin.,* 76:73–86.
21. Aharoni, R., Ben-Arroyo Hartman, I., and Hofmann, A.J. (1985). Path partitions and packs of acyclic digraphs. *Pacific J. Math.,* 118:249–259.
22. Aharoni, R., Berger, E. and Kfir, O. (2008). Acyclic systems of representatives and acyclic colorings of digraphs. *J. Graph Theory,* 59:177–189.
23. Aharoni, R. and Holzman, R. (1998). Fractional kernels in digraphs. *J. Combin. Theory Ser. B,* 73:1–6.
24. Ajtai, M., Komlós, J. and Szemerédi, E. (1980). A note on Ramsey numbers. *J. Combin. Theory Ser. A,* 3:354–360.
25. Akbari, S., Mirrokni, V. S. and Sadjad, B.S. (2006). A relation between choosability and uniquely list colorability. *J. Combin. Theory Ser. B,* 96:577–583.
26. Akolzin, I. and Shabanov, D. (2016). Colorings of hypergraphs with large number of colors. *Discrete Math.,* 339:3020–3031.
27. Aksionov, V.A. (1974). The extension of a 3-colouring on planar graphs (in Russian). *Diskret. Analiz,* 26:3–19.
28. Aksionov, V.A. (1977). Chromatic connected vertices in planar graphs (in Russian). *Diskret. Analiz,* 31:5–16.
29. Aksionov, V.A. and Mel'nikov, L.S. (1980). Some counterexamples associated with the three-color problem. *J. Combin. Theory Ser. B,* 28:1–9.
30. Albar, B. and Gonçalves, D. (2018). On triangles in K_r-minor free graphs. *J. Graph Theory,* 88:154–173.
31. Albertson, M.O. (1998). You can't paint yourself into a corner. *J. Combin. Theory Ser. B,* 73:189–194.
32. Albertson, M.O., Bollobás, B. and Tucker, S. (1976). The independence ratio and maximum degree of a graph. *Congr. Numer.,* 17:43–50.
33. Albertson, M.O. and Collins, K.L. (1985). Homomorphisms of 3-chromatic graphs. *Discrete Math.,* 54:127–132.
34. Albertson, M.O. and Hutchinson, J.P. (1979). The three excluded cases of Dirac's map-color theorem. (1979). *Ann. New York Acad. Sci.,* 319:7–17.
35. Albertson, M.O. and Hutchinson, J.P. (1980). On six-chromatic toroidal graphs. *Proc. London Math. Soc.,* 3:533–556.
36. Albertson, M.O. and Hutchinson, J.P. (1980). Hadwiger's conjecture for graphs on the Klein bottle. *Discrete Math.,* 29:1–11.
37. Albertson, M.O. and Hutchinson, J.P. (2004). Extending precoloring of subgraphs of locally planar graphs. *European J. Combin.,* 25:863–871.
38. Albertson, M.O., Jamison, R.E., Hedetniemi, S.T. and Locke, S.C. (1989). The subchromatic number of a graph. *Discrete Math.,* 74:33–49.
39. Albertson, M.O., Kostochka, A.V. and West, D. (2005). Precoloring extensions of Brooks' theorem. *SIAM J. Discrete Math.,* 18:542–553.
40. Albertson, M.O. and Stromquist, W.R. (1982). Locally planar toroidal graphs are 5-colorable. *Proc. Amer. Math. Soc.,* 84:449–457.
41. Alishahi, M., Hajiabolhassan, H. and Meunier, F. (2017). Strengthening topological colorful results for graphs. *European J. Combin.,* 64:27–44.
42. Alon, N. (1985). Hypergraphs with high chromatic number. *Graphs Combin.,* 1:387–389.
43. Alon, N. (1992). The strong chromatic number of a graph. *Random Struct. Alg.,* 3:1–7.
44. Alon, N. (1992). Choice numbers of graphs: a probabilistic approach. *Combin. Probab. Comput.,* 1:107–114.
45. Alon, N. (1993). Restricted colorings of graphs. In: *Surveys in Combinatorics, 1993*, pages 1–33. London Math. Soc. Lecture Notes Series, Vol. 187, Cambridge University Press.
46. Alon, N. (1994). Probabilistic methods in coloring and decomposition problems. *Discrete Math.,* 127:31–46.
47. Alon, N. (1996). Independence numbers of locally sparse graphs and a Ramsey type problem. *Random Struct. Alg.,* 9:271–278.

48. Alon, N. (1999). Combinatiorial Nullstellensatz. *Combin. Probab. Comput.,* 8:7–29.
49. Alon, N. (2000). Degrees and choice numbers. *Random Struct. Alg.,* 16:364–368.
50. Alon, N. and Bregman, Z. (1988). Every 8-uniform 8-regular hypergraph is 2-colorable. *Graphs Combin.,* 4:303–306.
51. Alon, N., Frankl, P. and Lovász, L. (1986). The chromatic number of Kneser hypergraphs. *Trans. Amer. Math. Soc.,* 298:359–370.
52. Alon, N., Kostochka, A.V., Reiniger, B., West, D.B. and Zhu, X. (2016). Coloring, sparseness and girth. *Israel J. Math.,* 214:315–331.
53. Alon, N., Krivelevich, M. and Seymour, P. (2000). Long cycles in critical graphs. *J. Graph Theory,* 35:193–196.
54. Alon, N., Krivelevich, M. and Sudakov, B. (1999). Coloring graphs with sparse neighborhoods. *J. Combin. Theory Ser. B,* 77:73–82.
55. Alon, N., Krivelevich, M. and Sudakov, B. (2023). Complete minors and average degree: A short proof. *J. Graph Theory,* 103:599–602.
56. Alon, N., Pach, J. and Solymosi, J. (2001). Ramsey-type theorems with forbidden subgraphs. *Combinatorica,* 21:155–170.
57. Alon, N. and Spencer, J. (1992). *The Probabilistic Method.* Wiley-Interscience Series in Discrete Mathematics and Optimization, John Wiley & Sons.
58. Alon, N. and Tarsi, M. (1992). Colorings and orientations of graphs. *Combinatorica,* 12:125–134.
59. Alon, N. and Tarsi, M. (1997). A note on graph colorings and graph polynomials. *J. Combin. Theory Ser. B,* 70:197–201.
60. Alon, N., Voigt, M. and Tuza, Zs. (1997). Choosability and fractional chromatic number. *Discrete Math.,* 165:31–38.
61. Andersen, L.D. (1977). On edge-colourings of graphs. *Math. Scand.,* 40:161–175.
62. Anderson, J., Bernshteyn, A., and Dhawan, A. (2023). Coloring graphs with forbidden almost bipartite subgraphs. *Combin. Probab. Comput.,* 32:45–67.
63. Andrásfai, B. (1964). Graphentheoretische Extremalprobleme. *Acta Math. Hungar.,* 15:413–438.
64. Andrásfai, B., Erdős, P. and Sós, V. (1974). On the connection between chromatic number, maximal clique and minimal degree of a graph. *Discrete Math.,* 8:205–218.
65. Andres, S.D. and Hochstättler, W. (2015). Perfect digraphs. *J. Graph Theory,* 79:21–29.
66. Andrews, J.A. and Jacobsen, M.S. (1985). On a generalization of chromatic number. *Congr. Numer.,* 47:33-48.
67. Aouchiche, M. and Hansen, P. (2013). A survey of Nordhaus-Gaddum type relations. *Discrete Math.,* 161:466-546.
68. Appel, K. and Haken, W. (1977). Every planar map is 4-colorable. Part I: Discharging. *Illinois J. Math,* 21:429–490.
69. Appel, K., Haken, W. and Koch, J. (1977). Every planar map is 4-colorable. Part II: Reducibility. *Illinois J. Math,* 21:491–567.
70. Aravind, N.R., Karthick, T., and Subramanian, C.R. (2011). Bounding χ in terms of ω and Δ for some classes of graphs. *Discrete Math.,* 311:911–920.
71. Asplund, E. and Grünbaum, B. (1960). On a coloring problem. *Math. Scand.,* 8:181–188.
72. Assadi, S., Kumar, P. and Mittal, P. (2022). Brooks' theorem in graph streams: a single-pass semi-streaming algorithm for Δ-coloring. In: *Prooceedings of the 54th Annual ACM SIGACT Symposium on Theory of Computing, STOC*, pages 234–247.
73. Axenovich, M. (2003). A note on graph coloring extensions and list-colorings. *Electron. J. Combin.,* 10:#N1.
74. Babson, E. and Kozlov, D.N. (2006). Complexes of graph homomorphisms. *Israel J. Math.,* 152:285–312.
75. Baetz, B. and Wood, D.R. (2014). Brooks' vertex-colouring theorem in linear time. Manuscript arXive:1401.8023
76. Balogh, J., Kostochka, A., Prince, N. and Stiebitz, M. (2009). The Erdős–Lovász Tihany conjecture for quasi-line graphs. *Discrete Math.,* 309:3985–3991.

77. Bang-Jensen, J., Bellitto, T., Schweser, T. and Stiebitz, M. (2020). On DP-coloring of digraphs. *J. Graph Theory,* 95:76–98.
78. Bang-Jensen, J., Bellitto, T., Schweser, T. and Stiebitz, M. (2020). Hajós and Ore construction for digraphs. *Electron. J. Combin.,* 27:#P1.63.
79. Bang-Jensen, J., Reed, B., Schacht, M., Šámal, R., Toft, B. and Wagner, U. (2006). On six problems posed by Jarik Nešetřil. In: *Topics in Discrete Mathematics,* pages 613–627. Algorithms and Combinatorics, Vol. 26, Springer.
80. Bang-Jensen, J., Schweser, T. and Stiebitz, M. (2022). Digraphs and variable degeneracy. *SIAM J. Discrete Math.,* 36:578–595.
81. Bárány, I. (1978). A short proof of Kneser's conjecture. *J. Combin. Theory Ser. A,* 25:325–326.
82. Barát, J., Joret, G. and Wood, D.R. (2011). Disproof of the list Hadwiger conjecture. *Electron. J. Combin.,* 18:#P232.
83. Bartnicki, T., Grytczuk, J., Kierstead, H.A. and Zhu, X. (2007). The map-coloring game. *Amer. Math. Monthly,* 114:793–803.
84. Baum, S. (2005). *Färbungen und Homomorphismen von Graphen.* Diploma thesis, Technical University Ilmenau.
85. Baum, S. and Stiebitz, M. (2005). Coloring of graphs without short odd paths between vertices of the same color. Preprint University of Southern Denmark, Department of Mathematics and Computer Science.
86. Beaudou, L., Naserasr, R. and Tardif, C. (2015). Homomorphisms of binary Cayley graphs. *Discrete Math.,* 338:2539–2544.
87. Beck, J. (1978). On 3-chromatic hypergraphs. *Discrete Math.,* 24:127–137.
88. Beck, J. (1991). An algorithmic approach to the Lovász local lemma. *Random Struct. Alg.,* 2:343–365.
89. Behzad, M. (1965). *Graphs and Their Chromatic Numbers*. Ph.D. Thesis, Michigan State University.
90. Beineke, L.W. (1968). On derived graphs and digraphs. In: *Beiträge zur Graphentheorie* (Manebach, Germany, 1967), pages 17–23. B. G. Teubner-Verlagsgesellschaft.
91. Beineke, L.W. (1970). Characterizations of derived graphs. *J. Combin. Theory,* 9:129–135.
92. Beineke, L.W., Wilson, R. and Toft, B. (2023). *Milestones in Graph Theory.* Manuscript submitted for publication
93. Bellitto, T., Bousquet, N., Kabela, A, and Pierron, T. (2023). The smallest 5-chromatic tournament. Manuscript arXive:2210.09936
94. Benedict, J.M. and Chinn, P.Z. (1978). On graphs having prescribed clique number, chromatic number, and maximum degree. In: *Theory and Applications of Graphs* (Michigan, 1976), pages 132–140. Lecture Notes in Math., Vol. 642, Springer.
95. Bensmail, J., Harutyunyan, A. and Le, N. (2018). List coloring digraphs. *J. Graph Theory,* 87:492–508.
96. Benzer, S. (1959). On the topology of the genetic fine structure. *Proc. Nat. Acad. Sci. U.S.A.,* 45:1607–1620.
97. Berge, C. (1961). Färbung von Graphen, deren sämtliche bzw. deren ungerade Kreise starr sind. *Wiss. Z. Martin-Luther-Univ. Halle-Wittenberg Math.-Natur. Reihe,* 10:114.
98. Berge, C. (1963). Some classes of perfect graphs. In: *Six Papers on Graph Theory,* pages 1–21. Indian Statistical Institute, Calcutta.
99. Berge, C. (1978). *Fractional Graph Theory*. Lecture Notes, Vol. 1, Indian Statistical Institute, Mc Millan, Calcutta.
100. Berge, C. (1982). k-optimal partitions of a directed graph. *European J. Combin.,* 3:97–101.
101. Berge, C. (1994). About vertex-critical non-bicolorable hypergraphs. *Australas. J. Combin.,* 9:211–220.
102. Berge, C. (1997). Motivations and history of some of my conjectures. *Discrete Math.,* 165/166:61–70.
103. Berge, C. and Duchet, P. (1983). *Problème*, Séminaire MSH, Paris.
104. Bernshteyn, A. (2016). The asymptotic behavior of the correspondence chromatic number. *Discrete Math.,* 339:2680–2692.

105. Bernshteyn, A. (2017). The local cut lemma. *European J. Combin.,* 63:95–114.
106. Bernshteyn, A. (2019). The Johannson-Molloy theorem for DP-coloring. *Random Struct. Alg.,* 54:653–664.
107. Bernshteyn, A., Brazelton, T., Cao, R. and Kang, A. (2023). Counting colorings of triangle-free graphs. *J. Combin. Theory Ser. B,* 161:86–108.
108. Bernshteyn, A. and Kostochka, A.V. (2018). Sharp Dirac's theorem for DP-critical graphs. *J. Graph Theory,* 88:521–546.
109. Bernshteyn, A. and Kostochka, A.V. (2019). DP-colorings of hypergraphs. *European J. Combin.,* 78:134–146.
110. Bernshteyn, A., Kostochka, A.V., and Pron, S. (2017). On DP-coloring of graphs and multigraphs. *Sib. Math. J.,* 58:28–36.
111. Bernshteyn, A., Kostochka, A.V., and Zhu, X. (2020). Fractional DP-colorings of sparse graphs. *J. Graph Theory,* 93:203–221.
112. Bernshteyn, A. and Lee, E. (2023). Weak degeneracy of graphs. *J. Graph Theory,* 103:607–634.
113. Bernstein, F. (1908). Zur Theorie der trigonometrischen Reihen. *Leipz. Ber.,* 60:325–338.
114. Beutelspacher, A. and Hering, P.-R. (1984). Minimal graphs for which the chromatic number equals the maximal degree. *Ars Combin.,* 18:201–216.
115. Bielecki, A. (1948). Problem 56. *Colloq. Math.,* 1:333.
116. Bigalke, H.-G. (1988). *Heinrich Heesch: Kristallgeometrie, Parkettierungen, Vierfarbenforschung.* Vita Mathematica Band 3, Birkhäuser Verlag.
117. Biggs, N. (2003). W.T. Tutte 1917-2002. In: *Surveys in Combinatorics, 2003*, pages 1–6. London Math. Soc. Lecture Notes Series, Vol. 307, Cambridge University Press.
118. Birkhoff, G.D. (1913). The reducibility of maps. *Amer. J. Math.,* 35:114–128.
119. Biró, M., Hujter, M. and Tuza, Zs. (1992). Precoloring extension. I. Interval graphs. *Discrete Math.,* 100:267–279.
120. Björner, A. (1995). Topological metheods. In: *Handbook of Combinatorics,* Vol. 2, pages 1819–1872. Elsevier.
121. Björner, A. and de Longueville, M. (2003). Neighborhood complexes of stable Kneser graphs. *Combinatorica,* 23:23–34.
122. Blasiak, J. (2007). A special case of Hadwiger's conjecture. *J. Combin. Theory Ser. B,* 97:1056–1073.
123. Bodlaender, H.L. (1991). On the complexity of some coloring games. *Intern. J. Found. Comput. Sci.,* 2:133–147.
124. Böhme, T., Broersma, H.J., Göbel, F., Kostochka, A.V. and Stiebitz, M. (1997). Spanning trees with pairwise nonadjacent endvertices. *Discrete Math.,* 170:219–222.
125. Böhme, T., Mohar, B. and Stiebitz, M. (1999). Dirac's map-color theorem for choosability. *J. Graph Theory,* 32:327–339.
126. Bokal, D., Fijavž, G., Juvan, M., Kayll, P.M. and Mohar, B. (2004). The circular chromatic number of a digraph. *J. Graph Theory,* 46:227–240.
127. Bollobás, B. (1978). Chromatic number, girth and maximal degree. *Discrete Math.,* 24:311-314.
128. Bollobás, B. (1984). The evolution of sparse graphs. In: *Graph Theory and Combinatorics* (Conference in honor of Paul Erdős, Cambridge, 1983), pages 36–58. Academic Press.
129. Bollobás, B., Catlin, P.A. and Erdős, P. (1980). Hadwiger's conjecture is true for almost every graph. *European J. Combin.,* 1:195–199.
130. Bollobás, B., Erdős, P. and Szemerédi, E. (1975). On complete subgraphs of r-chromatic graphs. *Discrete Math.,* 13:97-107.
131. Bollobás, B. and Harary, F. (1975). Point arboricity critical graphs exist. *J. London Math. Soc.,* 2:97–102.
132. Bollobás, B. and Harris, A.J. (1985). List-colourings of graphs. *Graphs Combin.,* 1:115–127.
133. Bollobás, B. and Hind, H.R. (1989). A new upper bound for the list chromatic number. *Discrete Math.,* 74:65–75.

134. Bollobás, B. and Manvel, B. (1979). Optimal vertex partitions. *Bull. London Math. Soc.,* 11:113–116.
135. Bollobás, B. and Thomason, A. (1979/2006). Set colourings of graphs. *Discrete Math.,* 25:21–26 and 306:948-952.
136. Bonamy, M., Charbit, P., Defrain, O., Joret, G., Lagoutte, V., Limouzy, V., Pastor, L. and Sereni, J.S. (2020). Revisiting a theorem by Folkman on graph colouring. *Electron. J. Combin.,* 27:#P1.56.
137. Bonamy, M., Dabrowski, K.K., Feghali, C., Johnson, M. and Paulusma, D. (2021). Recognizing graphs close to bipartite graphs with an application to colouring reconfiguration. *J. Graph Theory,* 98:81–109.
138. Bonamy, M., Kelly, T., Nelson, P. and Postle, L. (2022). Bounding χ by a fraction of Δ for graphs without large cliques. *J. Combin. Theory Ser. B,* 157:263–282.
139. Bonamy, M., Perrett, T. and Postle, L. (2022). Coloring graphs with sparse neighborhoods: bounds and applications. *J. Combin. Theory Ser. B,* 155:278–317.
140. Bondy, J.A. (1976). Disconnected orientations and a conjecture of Las Vergnas. *J. London Math. Soc.,* 2:277–282.
141. Bondy, J.A. (2003). Short proofs of classical theorems. *J. Graph Theory,* 44:159–165.
142. Bondy, J.A. and Hell, P. (1990). A note on the star chromatic number. *J. Graph Theory,* 14:479–482.
143. Bondy, J.A. and Locke, S.C. (1981). Relative lengths of paths and cycles in 3-connected graphs. *Discrete Math.,* 33:111–122.
144. Bondy, J.A. and Murty, U.S.R. (2008). *Graph Theory*. Graduate Texts in Mathematics, Vol. 244, Springer.
145. Bonnet, É., Bourneuf, R., Duron, J., Geniet, C., Thomassé, S. and Trotignon, N. (2023). A tamed family of triangle-free graphs with unbounded chromatic number. Manuscript arXive:2304.04296
146. Bonomo, F., Chudnovsky, M., Maceli, P., Schaudt, O., Stein, M. and Zhong, M. (2018). Three-coloring and list three-coloring of graphs without induced paths on seven vertices. *Combinatorica,* 38:779–801.
147. Borodin, O.V. (1976). On decomposition of graphs into degenerate subgraphs (in Russian). *Diskret. Analiz,* 28:3–12.
148. Borodin, O.V. (1977). A criterion of choosability of graphs with degree size list assignments (in Russian). In: Abstracts of IV. All-Union Conference on Theoretical Cybernetics, Novosibirsk, 1977, pages 127–128.
149. Borodin, O.V. (1979). *Problems of Colouring and of Covering the Vertex Set of a Graph by Induced Subgraphs (in Russian).* Ph.D. Thesis, Novosibirsk State University, Novosibirsk.
150. Borodin, O.V. (1996). Structural properties of plane graphs without adjacent triangles and an application to 3-colorings. *J. Graph Theory,* 21:183–186.
151. Borodin, O.V. (1997). A new proof of Grünbaum's 3-color theorem. *Discrete Math.,* 169:177–183.
152. Borodin, O.V. (2013). Colorings of plane graphs: A survey. *Discrete Math.,* 313:517–539.
153. Borodin, O.V., Dvořák, Z., Kostochka, A. V., Lidický, B. and Yancey, M. (2014). Planar 4-critical graphs with four triangles. *European J. Combin.,* 41:138–151.
154. Borodin, O.V., Glebov, A.N., Montassier, M. and Raspaud, A. (2009). Planar graphs without 5- and 7-cycles and without adjacent triangle are 3-colorable. *J. Combin. Theory Ser. B,* 99:668–673.
155. Borodin, O.V., Glebov, A.N., Raspaud, A. and Salavatipour, M.R. (2005). Planar graphs without cycles of length from 4 to 7 are 3-colorable. *J. Combin. Theory Ser. B,* 93:303–311.
156. Borodin, O., Kim, S-J., Kostochka, A.V. and West, D.B. (2004). Homomorphisms from sparse graphs with large girth. *J. Combin. Theory Ser. B,* 90:147–159.
157. Borodin, O.V. and Kostochka, A.V. (1977). On an upper bound of a graph's chromatic number, depending on the graph's degree and density. *J. Combin. Theory Ser. B,* 23:247–250.
158. Borodin, O.V., Kostochka, A.V., Lidický, B. and Yancey, M. (2014). Short proofs of coloring theorems on planar graphs. *European J. Combin.,* 36:314–321.

159. Borodin, O.V., Kostochka, A.V. and Toft, B. (2000). Variable degeneracy: extensions of Brooks' and Gallai's theorems. *Discrete Math.,* 214:101–112.
160. Borodin, O.V., Kostochka, A.V. and Woodall, D.R. (1997). List edge and list total colourings of multigraphs. *J. Combin. Theory Ser. B,* 71:184–204.
161. Boros, E. and Gurvich, V. (1996). Perfect graphs are kernel-solvable. *Discrete Math.,* 159:35–55.
162. Boros, E. and Gurvich, V. (2006). Perfect graphs, kernels, and cores of cooperative games. *Discrete Math.,* 306:2336–2354.
163. Borowiecki, M., Drgas-Burchardt, E. and Mihók, P. (1995). Generalized list colourings of graphs. *Discuss. Math., Graph Theory,* 15:185–193.
164. Borowiecki, M. and Mihók, P. (1991). Hereditary properties of graphs. In: *Advances in Graph Theory*, pages 41–68. Vishwa International Publisher.
165. Bosse, C. (2019). A note on Hadwiger's conjecture for W_5-free graphs with independence number two. *Discrete Math.,* 342:111595.
166. Brahana, H.R. (1921). Systems of circuits of two-dimensional manifolds. *Ann. of Math.,* 23:144-168.
167. Brandt, S. and Thomassé, S. (2005). Dense triangle-free graphs are four-colorable: A solution to the Erdős–Simonovits problem. Manuscript unpublished
168. Briański, M., Davies, J. and Walczak, B. (2023). Separating polynomial χ-boundedness from χ-boundedness. *Combinatorica,* in press.
169. Broere, I. and Mynhardt, C.M. (1985). Generalized colorings of outerplanar graphs and planar graphs. In: *Graph Theory with Applications to Algorithms and Computer Science* (Kalamazoo. Michigan, 1984), pages 151–161. John Wiley & Sons.
170. Broersma, H., Fomin, F.V., Golovach, P.A. and Paulusma, D. (2013). Three complexity results on coloring P_k-free graphs. *European J. Combin.,* 34:609–619.
171. Broersma, H., Golovach, P.A., Paulusma, D. and Song, J. (2012). Updating the complexity status of coloring graphs without a fixed induced linear forest. *Theoret. Comput. Sci.,* 414:9–19.
172. Broersma, H., Golovach, P.A., Paulusma, D. and Song, J. (2012). Determining the chromatic number of triangle-free $2P_3$-free graphs in polynomial time. *Theoret. Comput. Sci.,* 423:1–10.
173. Brooks, R.L. (1941). On colouring the nodes of a network. *Proc. Cambridge Philos. Soc.,* 37:194–197.
174. Brooks, R.L. (1970). A procedure for dissecting a rectangle into squares, and an example for the rectangle whose sides are in the ratio 2:1. *J. Combin. Theory,* 8:232–243.
175. Brooks, R.L., Smith, C.A.B., Stone, A.H. and Tutte, W.T. (1940). The dissection of rectangles into squares. *Duke Math. J.,* 7:312–340.
176. Brooks, R.L., Smith, C.A.B., Stone, A.H. and Tutte, W.T. (1947). A simple perfect square. *Nederl. Akad. Wetensch. Proc.,* 50:1300–1301.
177. Brooks, R.L., Smith, C.A.B., Stone, A.H. and Tutte, W.T. (1992). Determinants and current flows in electric networks. *Discrete Math.,* 100:291–301.
178. Brown, J.I. (1992). A vertex critical graph without critical edges. *Discrete Math.,* 102:99–101.
179. Brown, J.I. and Corneil, D.G. (1987). On generalized graph colorings. *J. Graph Theory,* 11:87–99.
180. Brown, J.I. and Corneil, D.G. (1991). Graph properties and hypergraph colourings. *Discrete Math.,* 98:81–93.
181. Brown, W.G. and Jung, H.A. (1969). On odd circuits in chromatic graphs. *Acta Math. Hungar.,* 20:129–134.
182. Brown, W.G. and Moon, J.W. (1969). Sur les ensembles de sommets indépendants dans les graphes chromatiques minimaux. *Canad. J. Math.,* 21:274–278.
183. Bryant, V. (1996). A characterisation of some 2-connected graphs and a comment on an algorithmic proof of Brooks' theorem. *Discrete Math.,* 158:279–281.
184. Burling, J.P. (1965). *On coloring problems of families of prototypes.* Ph.D. Thesis, University of Colorado.

185. Burr, S.A., Erdős, P. and Lovász, L. (1976). On graphs of Ramsey type. *Ars Combin.,* 1:167–190.
186. Burstein, M.I. (1976). Critical hypergraphs with minimal number of edges (in Russian). *Bull. Acad. Sci. Georgian SSR,* 83:285–288.
187. Cambie, S. (2021). Hadwiger's conjecture implies a conjecture of Füredi-Gyárfás-Simonyi. Manuscript arXive:2108.10303
188. Cameron, B., Hoàng, C.T. and Sawada, J. (2022). Dichotomizing k-vertex-critical H-free graphs for H of order four. *Discrete Appl. Math.,* 312:106–115.
189. Cameron, K., Goedgebeur, J., Huang, S. and Shi, Y. (2021). k-critical graphs in P_5-free graphs. *Theoret. Comput. Sci.,* 864:80–91.
190. Cameron, K. and Vušković, K. (2020). Hadwiger's conjecture for some hereditary classes of graphs: A survey. *Bulletin EATCS,* 131:pp. 19.
191. Carter, D. (2022). Hadwiger's conjecture with certain forbidden induced subgraphs. (2022). Manuscript arXive:2211.00259
192. Catlin, P.A. (1976). *Embedding Subgraphs and Coloring Graphs Under Extremal Degree Conditions.* Ph.D. Thesis, The Ohio State University.
193. Catlin, P.A. (1978). A bound on the chromatic number of a graph. *Discrete Math.,* 22:81–83.
194. Catlin, P.A. (1979). Hajós' graph-coloring conjecture: variations and counterexamples. *J. Combin. Theory Ser. B,* 26:268–274.
195. Catlin, P.A. (1979). Brooks' graph-coloring theorem and the independence number. *J. Combin. Theory Ser. B,* 27:42–48.
196. Catlin, P.A. and Lai, H.-J. (1995). Vertex arboricity and maximum degree. *Discrete Math.,* 141:37–46.
197. Chandran, L.S., Issac, D., and Zhou, S. (2019). Hadwiger's conjecture is true for squares of 2-trees. *European J. Combin.,* 76:159–174.
198. Chang, G.J., Liu, D.D.-F., Zhu, X. (2013). A short proof for Chen's alternative Kneser coloring lemma. *J. Combin. Theory Ser. A,* 120:159–163.
199. Chang, G.J., Tong, L.-D., Yan, J.-H. and Yeh, H.-G. (2002). A note on the Gallai-Roy-Vitaver theorem. *Discrete Math.,* 256:441–444.
200. Chang, M.S., Kloks, T. and Lee, C.M. (2001). Maximum clique transversal. In: *Graph-Theoretic Concepts in Computer Science* (Boltenhagen, 2001), pages 32–43. Lecture Notes in Comput. Sci., Vol. 2204, Springer.
201. Char, A. and Karthick, T. (2022). Coloring (P_5,4-wheel)-free graphs. *Discrete Math.,* 345:112795.
202. Char, A. and Karthick, T. (2023). On graphs with no induced P_5 and $K_5 - e$. Manuscript arXive:2308.08166
203. Chartrand, G., Geller, D. and Hedetniemi, S. (1968). A generalization of the chromatic number. *Proc. Cambridge Philos. Soc.,* 64:265–271.
204. Chartrand, G., Geller, D. and Hedetniemi, S. (1971). Graphs with forbidden subgraphs. *J. Combin. Theory,* 10:12–41.
205. Chartrand, G. and Kronk, H.V. (1968). Randomly traceable graphs. *SIAM J. Appl. Math.,* 16:696–700.
206. Chartrand, G., Kronk, H.V. and Wall, C.E. (1968). The point-arboricity of a graph. *Israel J. Math.,* 6:169–175.
207. Chen, B.-L., Lih, K.-W. and Wu, P.-L. (1994). Equitable coloring and the maximum degree. *European J. Combin.,* 15:443–447.
208. Chen, C.C., Jin, G. and Koh, K.M. (1997). Triangle-free graphs with large degree. *Combin. Probab. Comput.,* 6:381–396.
209. Chen, G., Jing, G. and Zang, W. (2019). Proof of the Goldberg-Seymour conjecture on edge-colorings of multigraphs. Manuscript arXive:1901.10316.
210. Chen, M. and Raspaud, A. (2010). Homomorphisms from sparse graphs to the Petersen graph. *Discrete Math.,* 310:2705–2713.
211. Chen, P.-A. (2011). A new coloring theorem of Kneser graphs. *J. Combin. Theory Ser. A,* 118:1062–1071.

212. Chen, R. (2022). Graphs with girth at least $2\ell+1$ and without longer odd holes are 3-colorable. Manuscript arXive:2301.00112
213. Chen, Z., Ma, J. and Zang, W. (2015). Coloring digraphs with forbidden cycles. *J. Combin. Theory Ser. B,* 115:210–223.
214. Chenette, N., Postle, L., Streib, N., Thomas, R. and Yerger, C. (2012). Five-coloring graphs on the Klein bottle. *J. Combin. Theory Ser. B,* 102:1067–1098.
215. Cherkashin, D. and Kozik, J. (2015). A note on random greedy coloring of uniform hypergraphs. *Random Struct. Alg.,* 47:407–413.
216. Cherkashin, D. and Petrov, F. (2019). Regular behaviour of the maximal hypergraph chromatic number. *SIAM J. Discrete Math.,* 34:1326–1333.
217. Chetwynd, A.G. and Häggkvist, R. (1989). A note on list-colorings. *J. Graph Theory,* 13:87–95.
218. Chiu, M.-K., Felsner, S., Scheucher, M., Schröder, F., Steiner, R. and Vogtenhuber, B. (2023). Coloring circle arrangements: New 4-chromatic planar graphs. *European J. Combin.,* 103839.
219. Choi, I., Liu, C.-H. and Oum, S.-I. (2018). Characterization of cycle obstruction sets for improper coloring planar graphs. *SIAM J. Discrete Math.,* 32:1209–1228.
220. Choudum, S.A. (1977). Chromatic bounds for a class of graphs. *Quart. J. Math. Oxford,* 28:257–270.
221. Chudnovsky, M. (2014). The Erdős–Hajnal conjecture– a survey. *J. Graph Theory,* 75:178–190.
222. Chudnovsky, M., Cook, L., Davies, J. and Oum, S. (2023). Reuniting χ-boundedness with polynomial χ-boundedness. Manuscript arXive:2310.11167
223. Chudnovsky, M., Cornuéjols, G., Liu, X., Seymour, P. and Vušković, K. (2005). Recognizing Berge graphs. *Combinatorica,* 25:143–186.
224. Chudnovsky, M. and Fradkin A.O. (2008). Hadwiger's conjecture for quasi-line graphs. *J. Graph Theory,* 59:17–33.
225. Chudnovsky, M. and Fradkin, A.O. (2010). An approximate version of Hadwiger's conjecture for claw-free graphs. *J. Graph Theory,* 63:259–278.
226. Chudnovsky, M., Fradkin, A. and Plumettaz, M. (2013). On the Erdős–Lovász Tihany conjecture for claw-free graphs. Manuscript arXive:1309.1020.
227. Chudnovsky, M., Goedgebeur, J., Schaudt, O. and Zhong, M. (2020). Obstructions for three-coloring and list three-coloring H-free graphs. *SIAM J. Discrete Math.,* 34:431–469.
228. Chudnovsky, M., Karthick, T., Maceli, P. and Maffray, F. (2020). Coloring graphs with no induced five-vertex path or gem. *J. Graph Theory,* 95:527–542.
229. Chudnovsky, M., King, A.D., Plumettaz, M. and Seymour, P. (2013). A local strengthening of Reed's ω, Δ, χ conjecture for quasi-line graphs. *SIAM J. Discrete Math.,* 27:95–108.
230. Chudnovsky, M., Robertson, N., Seymour, P. and Thomas, R. (2006). The strong perfect graph theorem. *Ann. of Math.,* 164:51–229.
231. Chudnovsky, M. and Safra, S. (2008). The Erdős–Hajnal conjecture for bull-free graphs. *J. Combin. Theory Ser. B,* 98:1301–1310.
232. Chudnovsky, M., Schaudt, O., Spirkl, S., Stein, M. and Zhong, M. (2019). Approximately coloring graphs without long induced paths. *Algorithmica,* 81:3186–3199.
233. Chudnovsky, M., Scott, A. and Seymour, P. (2016). Induced subgraphs of graphs with large chromatic number. II. Three steps towards Gyárfás' conjectures. *J. Combin. Theory Ser. B,* 118:109–128.
234. Chudnovsky, M., Scott, A. and Seymour, P. (2017). Induced subgraphs of graphs with large chromatic number. III. Long holes. *Combinatorica,* 37:1057–1072.
235. Chudnovsky, M., Scott, A. and Seymour, P. (2019). Induced subgraphs of graphs with large chromatic number. XI. Orientations. *European J. Combin.,* 76:53–61.
236. Chudnovsky, M., Scott, A. and Seymour, P. (2019). Induced subgraphs of graphs with large chromatic number. XII. Distant stars. *J. Graph Theory,* 92:237–254.
237. Chudnovsky, M., Scott, A. and Seymour, P. (2021). Induced subgraphs of graphs with large chromatic number. V. Chandeliers and strings. *J. Combin. Theory Ser. B,* 150:195–243.

238. Chudnovsky, M., Scott, A., Seymour, P. and Spirkl, S. (2020). Induced subgraphs of graphs with large chromatic number. VIII. Long odd holes. *J. Combin. Theory Ser. B,* 140:84–97.
239. Chudnovsky, M., Scott, A., Seymour, P. and Spirkl, S. (2023). Erdős–Hajnal for graphs with no 5-hole. *Proc. London Math. Soc.,* 126:997–1014.
240. Chudnovsky, M. and Seymour (2005). The structure of claw-free graphs. In: *Surveys in Combinatorics, 2005*, pages 153–171. London Math. Soc. Lecture Notes Series, Vol. 327, Cambridge University Press.
241. Chudnovsky, M. and Seymour (2012). Packing seagulls. *Combinatorica,* 32:251–282.
242. Chudnovsky, M. and Seymour (2023). Proof of a conjecture of Plummer and Zha. *J. Graph Theory,* 103:437–450.
243. Chudnovsky, M., Spirkl, S. and Zhong, M. (2018). Four-coloring P_6-free graphs. I. Extending an excellent precoloring. Manuscript arXive:1802.02282.
244. Chudnovsky, M., Spirkl, S. and Zhong, M (2018). Four-coloring P_6-free graphs. II. Finding an excellent precoloring. Manuscript arXive:1802.02283.
245. Chvátal, V. (1970). The smallest triangle-free 4-chromatic 4-regular graph. *J. Combin. Theory,* 9:93–94.
246. Chvátal, V. (1973). On the computational complexity of finding a kernel. Report No. CRM-300, Centre de Recherches Mathématiques, Université de Montréal.
247. Chvátal, V. (1974). The minimality of the Mycielski graph. In: *Graphs and Combinatorics*, pages 243–247. Lecture Notes in Math., Vol. 406, Springer.
248. Chvátal, V. (1975). On certain polytopes associated with graphs. *J. Combin. Theory Ser. B,* 18:138–154.
249. Chvátal, V., Garey, M.R. and Johnson, D.S. (1978). Two results concerning multicolorings. *Ann. Discrete Math.,* 2:151–154.
250. Clarke, F.H. and Jamison, R.E. (1976). Multicolorings, measures and games on graphs. *Discrete Math.,* 14:241–245.
251. Cockayne, E.J. (1972). Colour classes for r-graphs. *Canad. Math. Bull.,* 15:349–354.
252. Cohen-Addad, V., Hebdige, M., Kráľ, D., Li, Z. and Salgado, E. (2017). Steinberg's conjecture is false. *J. Combin. Theory Ser. B,* 122:452–456.
253. Collins, K.L., Heenehan, M.E. and McDonald, J. (2021). A note on the immersion number of generalized Mycielski graphs. Manuscript: arXive:2105.05724
254. Conley, C., Marks, A. and Tucker-Drop, R. (2016). Brooks' theorem for measurable colorings. *Forum Math. Sigma,* 4: E16.
255. Corsini, T., Deschamps, Q., Feghali, C., Goncalves, D. Langlois, H. and Talon, A. (2023). Partitioning into degenerate graphs in linear time. *European J. Combin.,* 114:103771.
256. Cranston, D.W., Lafayette, H. and Rabern, L. (2020). Coloring (P_5, gem)-free graphs with $\Delta - 1$ colors. *J. Graph Theory,* 101:633–642.
257. Cranston, D.W. and Rabern, L. (2012). Conjectures equivalent to the Borodin-Kostochka conjecture that are a priori weaker. *European J. Combin.,* 44:23–42.
258. Cranston, D.W. and Rabern, L. (2013). Coloring claw-free graphs with $\Delta - 1$ colors. *SIAM J. Discrete Math.,* 27:534–549.
259. Cranston, D.W. and Rabern, L. (2015). Graphs with $\chi = \Delta$ have big cliques. *SIAM J. Discrete Math.,* 29:1792–1814.
260. Cranston, D.W. and Rabern, L. (2015). Brooks' theorem and beyond. *J. Graph Theory,* 80:199–225.
261. Cranston, D.W. and Rabern, L. (2017). The fractional chromatic number of the plane. *Combinatorica,* 37:837–861.
262. Cranston, D.W. and Rabern, L. (2018). Edge lower bounds for list critical graphs, via discharging. *Combinatorica,* 38:1045–1065.
263. Cranston, D.W. and Rabern, L. (2018). Planar graphs are 9/2-colorable. *J. Combin. Theory Ser. B,* 133:32–45.
264. Cristofides, D., Edwards; K. and King, A.D. (2013). A note on hitting maximum and maximal cliques with a stable set. *J. Graph Theory,* 73:354–360.

265. Csiszár, I. and Körner, J. (1981). Graph decomposition: a new key for coding theorems. *IEEE Trans. Inform. Theory,* 27:5–12.
266. Csorba, P. (2005). *Non-tidy Spaces and Graph Colorings*. Doctoral Thesis, ETH Zürich.
267. Csorba, P. (2008). Fold and Mycielskian on homomorphism complexes. *Conributions Discrete Math.,* 3:1–8.
268. Csorba, P. and Osztényi, J. (2010). On the topological lower bound for the multichromatic number. *Discrete Math.,* 310:1334–1339.
269. Culver, E. and Hartke, S.G. (2023). Relation between the correspondence chromatic number and the Alon–Tarsi number. *Discrete Math.,* 346:113347.
270. Davies, E., de Joannis de Verclos, R., Kang, R.J. and Pirot, F. (2020). Coloring triangle-free graphs with local list sizes. *Random Struct. Alg.,* 57:730–744.
271. Davies, E., Kang, R.J. Pirot, F. and Sereni, J.-S. (2020). Graph structure via local occupancy. Manuscript arXive:2003.14361
272. De Bruijn, N.G. and Erdős, P. (1951). A colour problem for infinite graphs and a problem in the theory of infinite relations. *Indag. Math.,* 13:371–373.
273. De Grey, A.D.N.J. (2018). The chromatic number of the plane is at least 5. *Geombinatorics,* 28:18–31.
274. Delcourt, M. and Postle, L. (2022). Reducing linear Hadwiger's conjecture to coloring small graphs. Manuscript arXiv: 2108.01633
275. Deligkas, A., Eiben, E., Gutin, G. Neary, P.R., and Yeo, A. (2023). Complexity Dichotomies for the maximum weighted digraph partition problem. Manuscript arXive: 2307.01109
276. De Loera, J.A., Goaoc, X., Meunier, F., and Mustafa, N.H. (2019). The discrete yet ubiquitous theorems of Carathéodory, Helly, Sperner, Tucker, and Tverberg. *Bull. Am. Math. Soc.,* 56:415–511.
277. De Longueville, M. (2004). 25 years proof of the Kneser conjecture. The advent of topological combinatorics. *EMS Newsletter,* 53:16–19.
278. De Longueville, M. (2012). *A Course in Topological Combinatorics*. Springer Science & Business Media.
279. Demidovich, Y.A. (2022). On some generalizations of the property B-problem of an n-uniform hypergraph. *J. Math. Sci.,* 262:457–475.
280. Descartes, B. (1947). A three colour problem. *Eureka,* 9:24–25.
281. Deuber, W.A., Kostochka, A.V. and Sachs, H. (1996). A shorter proof of Dirac's theorem on the number of edges in chromatically critical graphs (in Russian). *Diskret. Anal. Issled. Oper.,* 3:28–34.
282. DeVos, M. (2007). Mapping planar graphs to odd cycles. Open Problem Garden, Graph Theory, Coloring, Homomorphisms.
283. DeVos, M. and Nurse, K. (2023). A short proof of Seymour's 6-flow theorem. Manuscript arXive:2302.08625
284. De Vries, H.L. (1977). On property B and on Steiner systems. *Math. Z.,* 153:155–159.
285. Dhurandhar, M (1982). Improvement on Brooks' chromatic bound for a class of graphs. *Discrete Math.,* 42:51–56.
286. Diestel, R. (2017). *Graph Theory*. Fifth Edition. Graduate Texts in Mathematics, Vol. 173, Springer.
287. Diks, K., Kowalik, L. and Kurowski, M. (2002). A new 3-color criterion for planar graphs. In: *Graph-Theoretic Concepts in Computer Science, WG 2002,* pages 138–149. Lecture Notes in Comput. Sci., Vol. 2573, Springer.
288. Dilworth, R.P. (1950). A decomposition theorem for partially ordered sets. *Ann. of Math.,* 51:161–166.
289. Dirac, G.A. (1951). Note on the colouring of graphs. *Math. Z.,* 54:347–353.
290. Dirac, G.A. (1952). Map-colour theorems. *Canad. J. Math.,* 4:480–490.
291. Dirac, G.A. (1952). A property of 4-chromatic graphs and some remarks on critical graphs. *J. London Math. Soc.,* 27:85–92.
292. Dirac, G.A. (1952). Some theorems on abstract graphs. *Proc. London Math. Soc.,* 3:69–81.

293. Dirac, G.A. (1953). The structure of k-chromatic graphs. *Fund. Math.,* 40:42–55.
294. Dirac, G.A. (1955). Circuits in critical graphs. *Monatsh. Math.,* 59:178–187.
295. Dirac, G.A. (1956). Map colour theorems related to the Heawood colour formula. *J. London Math. Soc.,* 31:460–471.
296. Dirac, G.A. (1957). Map colour theorems related to the Heawood colour formula (II). *J. London Math. Soc.,* 32:436–455.
297. Dirac, G.A. (1957). Short proof of a map-colour theorem. *Canad. J. Math.,* 9:225–226.
298. Dirac, G.A. (1957). A theorem of R. L. Brooks and a conjecture of H. Hadwiger. *Proc. London Math. Soc.,* 3:161–195.
299. Dirac, G.A. (1960). Trennende Knotenpunktmengen und Reduzibilität abstrakter Graphen mit Anwendung auf das Vierfarbenproblem. *J. Reine Angew. Math.,* 204:116–131.
300. Dirac, G.A. (1961). On rigid circuit graphs. *Abh. Math. Sem. Univ. Hamburg,* 25:71–76.
301. Dirac, G.A. (1963). Percy John Heawood. *J. London Math. Soc.,* 38:263–277.
302. Dirac, G.A. (1964). On the structure of 5- and 6-chromatic abstract graphs. *J. Reine Angew. Math.,* 214/215:43–52.
303. Dirac, G.A. (1974). The number of edges in critical graphs. *J. Reine Angew. Math.,* 268:150–164.
304. Diwan, A.A., Kenkre, S., and Vishwanathan, S. (2013). Circumference, chromatic number and online coloring. *Combinatorica,* 33:319–334.
305. Dobrynin, A.A. and Mel'nikov, L.S. (2006). Counterexamples to Grötzsch–Sachs–Köster's conjecture. *Discrete Math.,* 306:591–594.
306. Doerr, B. (2020). Probabilistic tools for the analysis of randomized optimization heuristics. In: *Theory of Evolutionary Computation: Recent Developments in Discrete Optimization*, pages 1–87. Springer. Also available arXive:1801.06733
307. Dol'nikov, V.L. (1988). A certain combinatorial inequality. *Sib. Math. J.,* 29:375–397.
308. Du, D.Z., Hsu, D.F. and Hwang, F.K. (1993). The Hamiltonian property of consecutive-d digraphs. *Math. Comput. Modelling,* 17:61–63.
309. Dvořák, Z. (2022). Coloring of triangle-free graphs and the Rosenfeld counting method. Lecture Notes, Computer Science Institute of Charles University, Prague.
310. Dvořák, Z. and Feghali, C. (2023). Solution to a problem of Grünbaum on the edge density of 4-critical planar graphs. Manuscript arXive:2311.03132
311. Dvořák, Z., Hu, X. and Sereni, J.-S. (2019). A 4-choosable graph that is not $(8:2)$-choosable. *Advances Combin.,* 5:9 pp..
312. Dvořák, Z., Král', D. and Thomas, R. (2020). Three-coloring triangle-free graphs on surfaces III. Graphs of girth five. *J. Combin. Theory Ser. B,* 145:376–432.
313. Dvořák, Z., Král', D. and Thomas, R. (2021). Three-coloring triangle-free graphs on surfaces V. Coloring planar graphs with distant anomalies. *J. Combin. Theory Ser. B,* 150:244–269.
314. Dvořák, Z., Král', D. and Thomas, R. (2022). Three-coloring triangle-free graphs on surfaces VII. A linear-time algorithm. *J. Combin. Theory Ser. B,* 152:483–504.
315. Dvořák, Z., Lidický, B., Mohar, B. and Postle, L (2017). 5-list-coloring planar graphs with distant precolored vertices. *J. Combin. Theory Ser. B,* 122:311–552.
316. Dvořák, Z. and Pekárek, J. (2022). Characterization of 4-critical triangle-free toroidal graphs. *J. Combin. Theory Ser. B,* 154:336–369.
317. Dvořák, Z. and Postle, L. (2017). Density of $5/2$-critical graphs. *Combinatorica,* 37:863–886.
318. Dvořák, Z. and Postle, L. (2018). Correspondence coloring and its application to list-coloring planar graphs without cycles of lengths 4 to 8. *J. Combin. Theory Ser. B,* 129:38–54.
319. Dvořák, Z., Sereni, J.-S. and Volec, J. (2015). Fractional coloring of triangle-free planar graphs. *Electron. J. Combin.,* 22:#P4.11.
320. Edmonds, J. (1960). A combinatorial representation for polyhedral surfaces. *Notices Amer. Math. Soc.,* 7:646.
321. Edmonds, J. (1965). Minimum partition of a matroid into independent subsets. *J. Res. Nat. Bur. Standards Sect. B,* 69:67–72.

322. Edwards, K. and King, A.D. (2014). A superlocal version of Reed's conjecture. *Electron. J. Combin.,* 21:#P4.48.
323. Ehrlich, G., Even, S. and Tarjan, R.E. (1976). Intersection graphs of curves in the plane. *J. Combin. Theory Ser. B,* 21:8–20.
324. Ellingham, M.N. and Goddyn, L. (1996). List edge colourings of some 1-factorable multigraphs. *Combinatorica,* 16:343–352.
325. El-Zahar, M. and Sauer, N.W. (1985). The chromatic number of the product of two 4-chromatic graphs is 4. *Combinatorica,* 5:121–126.
326. Emden-Weinert, T., Hougardy, S. and Kreuter, B. (1998). Uniquely colourable graphs and the hardness of colouring graphs of large girth. *Combin. Probab. Comput.,* 7:375–386.
327. Erdős, P. (1947). Some remarks on the theory of graphs. *Bull. Am. Math. Soc.,* 53:292–294.
328. Erdős, P. (1959). Graph theory and probability. *Canad. J. Math.,* 11:34–38.
329. Erdős, P. (1961). Graph theory and probability II. *Canad. J. Math.,* 13:346–352.
330. Erdős, P. (1962). On circuits and subgraphs of chromatic graphs. *Mathematika,* 9:170–175.
331. Erdős, P. (1963). On a combinatorial problem. *Nordisk Mat. Tidskr.,* 11:5–10.
332. Erdős, P. (1964). On a combinatorial problem II. *Acta Math. Hungar.,* 15:445–447.
333. Erdős, P. (1964). Problem 9. In: *Theory of Graphs and Its Application* (Smolenice, 1963), page 159. Academia Praha and Academic Press.
334. Erdős, P. (1968). Problem 2. In: *Theory of Graphs* (Tihany, Hungary, 1966), page 361. Academic Press.
335. Erdős, P. (1971). Some unsolved problems in graph theory and combinatorial analysis. In: *Combinatorial Mathematics and Its Applications* (Oxford, 1969), pages 97–109. Academic Press.
336. Erdős, P. (1979). Problems and result in graph theory and combinatorial analysis. In: *Graph Theory and Related Topics* (Waterloo, 1977), pages 153–163. Academic Press.
337. Erdős, P. (1980). Problems and results in number theory and graphs theory. *Congr. Numer.,* 27:3–21.
338. Erdős, P. (1985). Problems and results on chromatic numbers in finite and infinite graphs. In: *Graph Theory with Applications to Algorithms and Computer Science*, pages 201–213. John Wiley & Sons.
339. Erdős, P. (1989). On some aspects of my work with Gabriel Dirac. In: *Graph Theory in Memory of G. A. Dirac*, pages 111–116. Ann. Discrete Math., Vol. 41, North-Holland.
340. Erdős, P. (1990). Some of my favourite unsolved problems. In: *A Tribute to Paul Erdős*, pages 467–468. Cambridge University Press.
341. Erdős, P. (1992). Some of my favourite problems in various branches of combinatorics. *Le Matematiche,* 47:231–240.
342. Erdős, P. and Fajtlowicz, S. (1981). On the conjecture of Hajós. *Combinatorica,* 1:141–143.
343. Erdős, P., Faudree, R.J., Rousseau, C.C. and Schelp, R.H. (1999). The number of cycle lengths in graphs of given minimum degree and girth. *Discrete Math.,* 200:55–60.
344. Erdős, P., Füredi, Z., Hajnal. A., Komjáth, P., Rödl, V. and Seress, Á. (1986). Coloring graphs with locally few colors. *Discrete Math.,* 59:21–34.
345. Erdős, P. and Hajnal, A. (1961). On a property of families of sets. *Acta Math. Hungar.,* 12:87–123.
346. Erdős, P. and Hajnal, A. (1964). Some remarks on set theory. IX. Combinatorial problems in measure theory and set theory. *Michigan Math. J.,* 11:107–127.
347. Erdős, P. and Hajnal, A. (1966). On chromatic number of graphs and set-systems. *Acta Math. Hungar.,* 17:61–99.
348. Erdős, P. and Hajnal, A. (1967). On chromatic graphs (Hungarian). *Mat. Lapok.,* 18:1–4.
349. Erdős, P. and Hajnal, A. (1977). On spanned subgraphs of graphs. In: Beiträge zur Graphentheorie und deren Anwendungen (Int. Kolloquium in Oberhof, DDR, 1977), pages 80–96. Ilmenau: Techn. Hochschule.
350. Erdős, P. and Hajnal, A. (1985). On the chromatic number of finite and infinite graphs and hypergraphs. *Discrete Math.,* 53:281–285.

351. Erdős, P. and Hajnal, A. (1989). Ramsey-types theorems. *Discrete Appl. Math.,* 25:37–52.
352. Erdős, P. and Hanani, H. (1963). On a limit theorem in combinatorial analysis. *Publ. Math. Debrecen,* 10:10–13.
353. Erdős, P., Ko, C. and Rado, R. (1961). Intersection theorems for systems of finite sets. *Quart. J. Math. Oxford,* 12:313–320.
354. Erdős, P. and Lovász, L. (1973). Problems and results on 3-chromatic hypergraphs and some related questions. In: *Infinite and Finite Sets* (Keszthely, Hungary, 1973), pages 609–627. Coll. Math. Soc. János Bolyai , Vol. 10, North-Holland.
355. Erdős, P., Rubin, A.L. and Taylor, H. (1979). Choosability in graphs. *Congr. Numer.,* 26:125–157.
356. Erdős, P. and Simonovits, M. (1973). On a valence problem in extremal graph theory. *Discrete Math.,* 5:323–334.
357. Erdős, P. and Spencer, J. (1991). Lopsided Lovász Local Lemma and Latin transversals. *Discrete Appl. Math.,* 30:151–154.
358. Erdős, P. and Stone, A.H. (1946). On the structure of linear graphs. *Bull. Am. Math. Soc.,* 52:1087–1091.
359. Erdős, P. and Szekeres, G. (1935). A combinatorial problem in geometry. *Compos. Math.,* 2:463–470.
360. Esperet, L. (2017). *Graph Colorings, Flows and Perfect Matchings.* Habilitation thesis, Université Greneoble Alpes.
361. Exoo, G. and Goedgebeur, J. (2019). Bounds for the smallest k-chromatic graphs of given girth. *Discrete Math. Theor. Comput. Sci.,* 21:#9.
362. Fajtlowicz, S. (1978). On the size of independent sets in graphs. *Congr. Numer.,* 21:269–274.
363. Fan, K. (1952). A generalization of Tucker's combinatorial lemma with topological applications. *Ann. of Math.,* 56:431–437.
364. Farrugia, A. (2004). Vertex-partitioning into fixed additive induced-hereditary properties is NP-hard. *Electron. J. Combin.,* 11:#R46.
365. Farzad, B. and Molloy, M. (2009). On the edge-density of 4-critical graphs. *Combinatorica,* 29:665–689.
366. Farzad, B., Molloy, M. and Reed, B. (2005). $(\Delta - k)$-critical graphs. *J. Combin. Theory Ser. B,* 93:173–185.
367. Feghali, C. (2024). Dirac's theorem on chordal graphs implies Brooks' theorem. *Discrete Math.,* 347:113791.
368. Feige, U. and Kilian, J. (1998). Zero knowledge and the chromatic number. *J. Comput. System Sci.,* 57:187–199.
369. Fekete, M. (1923). Über die Verteilung der Wurzeln bei gewissen algebraischen Gleichungen mit ganzzahligen Koeffizienten. *Math. Z.,* 17:228–249.
370. Felsner, S., Hurtado, F., Noy, M. and Streinu, I. (2006). Hamiltonicity and colorings of arrangement graphs. *Discrete Appl. Math.,* 154:2470–2483.
371. Fernandez de la Vega, W. (1983). On the maximum density of graphs which have no subcontraction to K^s. *Discrete Math.,* 46:109–110.
372. Finck, H.J. and Sachs, H. (1969). Über eine von H. S. Wilf angegebene Schranke für die chromatische Zahl endlicher Graphen. *Math. Nachr.,* 39:373–386.
373. Fischer, M., Halldórsson, M.M. and Maus, Y. (2023). Fast distributed Brooks' theorem. In: Proceedings of the 2023 Annual ACM-SIAM Symposium on Discrete Algorithms (SODA), pages 2567–2588.
374. Fisher, D. and Ullman, D. (1992). The fractional chromatic number of the plane. *Geombinatorics,* 2:8–12.
375. Fisk, S. (1973). Combinatorial structure on triangulations. I. The structure of four colorings. *Advances Math.,* 11:326–338.
376. Fisk, S. (1978). The nonexistence of colorings. *J. Combin. Theory Ser. B,* 24:247–248.
377. Fleiner, T. (2014). Yet another proof of Brooks' theorem. EGRES, Quick Proof No. 2014-01
378. Fleiner, T. (2018). A note on restricted list edge-colourings. *Combinatorica,* 38:1265–1267.

379. Fleischner, H. and Stiebitz, M. (1992). A solution to a colouring problem of P. Erdős. *Discrete Math.,* 101:39–48.
380. Fleischner, H. and Stiebitz, M. (1997). Some remarks on the cycle plus triangles problem. In: *The Mathematics of Paul Erdős II,* pages 136–142. Algorithms and Combinatorics, Vol. 14, Springer.
381. Folkman, J.H. (1970). An upper bound on the chromatic number of a graph. In: *Combinatorial Theory and its Application* (Balatonfüred, Hungary, 1969), Vol II, pages 437–457. Coll. Math. Soc. János Bolyai, Vol. 4, North-Holland.
382. Foniok, J. and Tardif, C. (2017). Hedetniemi's conjecture and adjoint functors in thin categories. *Appl. Categor. Struct.,* 26:113–128.
383. Fouquet, J.L. and Vanherpe, J.M. (2012). Reed's conjecture on some special classes of graphs. Manuscript arXive:1205.0730
384. Fox, J. and Pach, J. (2012). Coloring K_k-free intersection graphs of geometric objects in the plane. *European J. Combin.,* 33:853–866.
385. Frank, A. (1995). Connectivity and network flows. In: *Handbook of Combinatorics*, Vol. I, pages 111–178. North-Holland.
386. Frank, A. (2011). *Connections in Combinatorial Optimization.* Oxford Lecture Series in Mathematics and its Application 38, Oxford University Press.
387. Frank, A. and Gyárfás, A. (1978). How to orient the edges of a graph? In: *Combinatorics* (Keszthely, Hungary, 1976), Vol 1, pages 353–364. Coll. Math. Soc. János Bolyai, Vol. 18, North-Holland.
388. Franklin, P. (1934). A six color problem. *J. Math. Phys.,* 13:363–369.
389. Franklin, P. (1939). The four color problem. *Scripta Mathematica,* 6:149–156, 197–210.
390. Freund, R.M. and Todd, M.J. (1981). A constructive proof of Tucker's combinatorial lemma. *J. Combin. Theory Ser. A,* 30:321–325.
391. Frieze, A. and Luczak, T. (1992). On the independence and chromatic numbers of random regular graphs. *J. Combin. Theory Ser. B,* 54:123-132.
392. Frieze, A. and Mubayi, D. (2013). Coloring simple hypergraphs. *J. Combin. Theory Ser. B,* 103:767-794.
393. Fritsch, R. and Fritsch, G. (1998). *The Four-Color Theorem: History, Topological Foundations, and Idea of Proof.* Springer, Translated from the 1994 German original by Julie Peschke.
394. Fulkerson, D.R. (1971). Blocking and anti-blocking pairs of polyhedra. *Math Program.,* 1:168-194.
395. Fulkerson, D.R. (1973). On the perfect graph theorem. In: *Mathematical Programming*, pages 69–76. Academic Press.
396. Gale, D. and Shapley, L.S. (1962). College admissions and the stability of marriage. *Amer. Math. Monthly,* 69:9–15.
397. Gallai, T. (1963). Kritische Graphen I. *Magyar Tud. Akad. Mat. Kutató Int. Közl.,* 8:165–192.
398. Gallai, T. (1963). Kritische Graphen II. *Magyar Tud. Akad. Mat. Kutató Int. Közl.,* 8:373–395.
399. Gallai, T. (1964). Critical graphs. In: *Theory of Graphs and its Applications* (Smolenice, 1963), pages 43–45. Academia Praha and Academic Press.
400. Gallai, T. (1968). On directed paths and circuits. In: *Theory of Graphs* (Tihany, Hungary, 1966), pages 115–118. Academic Press.
401. Gallai, T. and Milgram, A.N. (1960). Verallgemeinerung eines graphentheoretischen Satzes von Rédei. *Acta Sci. Math.,* 21:181–186.
402. Galvin, F. (1995). The list chromatic index of a bipartite multigraph. *J. Combin. Theory Ser. B,* 63:153–158.
403. Gao, J. and Ma, J. (2023). Tight bounds towards a conjecture of Gallai. *Combinatorica,* 43:447–453.
404. Gao, W. and Postle, L. (2018). On the minimal edge density of K_4-free 6-critical graphs. Manuscript arXiv:1811.02940

405. Gardner, M. (1960). Mathematical games. *Sci. Amer.,* 202:174–188.
406. Gardner, M. (1981). Mathematical games. *Sci. Amer.,* 244:18.
407. Garey, M.R. and Johnson, D.S. (1979). *Computers and Intractability: A Guide to the Theory of* NP*-completeness.* Freeman.
408. Garey, M.R., Johnson, D.S. and Stockmeyer, L. (1976). Some simplified NP-complete graph problems. *Theoret. Comput. Sci.,* 1:237–267.
409. Gasparian, G.S. (1996). Minimal imperfect graphs: a simple approach. *Combinatorica,* 16:209–212.
410. Gebauer, H. (2013). On the construction of 3-chromatic hypergraphs with few edges. *J. Combin. Theory Ser. A,* 120:1483–1490.
411. Geetha, J., Narayanan, N. and Somasundaram, K. (2023). Total colourings–a survey. *AKCE Int. J. Graphs Combin.,* in press.
412. Geller, D. (1976). r-tuple colorings of uniquely colorable graphs. *Discrete Math.,* 16:9–12.
413. Gerencsér, L. (1965). Szinezési problémákrol (On coloring problems). *Mat. Lapok.,* 16:274–277.
414. Gimbel, J., Kündgen, A., Li, B. and Thomassen, C. (2019). Fractional coloring methods with applications to degenerate graphs and graphs on surfaces. *SIAM J. Discrete Math.,* 33:1415-1430.
415. Gimbel, J. and Thomassen, C. (1997). Coloring graphs with fixed genus and girth. *Trans. Amer. Math. Soc.,* 349:4555–4564.
416. Gimbel, J. and Thomassen, C. (2000). Coloring triangle-free graphs with fixed size. *Discrete Math.,* 219:275–277.
417. Goddard, W. and Lyle, J. (2011). Dense graphs with small clique number. *J. Graph Theory,* 66:319–331.
418. Godsil, C. and Royle, G. (2001). *Algebraic Graph Theory*. Graduate Texts in Mathematics, Vol. 207, Springer.
419. Goedgebeur, J., Huang, S., Ju, Y. and Merkel, O. (2023). Colouring graphs with no induced six-vertex path or diamond. *Theoret. Comput. Sci.,* 941:278–299.
420. Goldberg, M.K. (1973). On multigraphs of almost maximal chromatic class (in Russian). *Diskret. Analiz,* 23:3–7.
421. Goldberg, M.K. (1984). Edge-coloring of multigraphs: Recoloring technique. *J. Graph Theory,* 8:123–137.
422. Golovach, P., Johnson, M., Paulusma, D. and Song, J. (2017). A survey on the computational complexity of coloring graphs with forbidden subgraphs. *J. Graph Theory,* 84:331–363.
423. Golowich, N. (2016). The m-degenerate chromatic number of a digraph. *Discrete Math.,* 339:1734–1743.
424. Golumbic, M.C. (2004). *Algorithmic Graph Theory and Perfect Graphs.* Elsevier.
425. Gould, R., Larsen, V. and Postle, L. (2022). Structure in sparse k-critical graphs. *J. Combin. Theory Ser. B,* 156:194–222.
426. Graham, R.L. and Nešetřil, J. (1997). Ramsey theory in the work of Paul Erdős. In: *The Mathematics of Paul Erdős II,* pages 193–209. Algorithms and Combinatorics, Vol. 14, Springer.
427. Gravier, S. (1996). A Hajós-like theorem for list coloring. *Discrete Math.,* 152:299–302.
428. Gravier, S., Hoàng, C.T. and Maffray, F. (2003). Coloring the hypergraph of maximal cliques of a graph with no long path. *Discrete Math.,* 272:285–290.
429. Gravin, N.V. and Karpov, D.V. (2012). On proper colorings of hypergraphs. *J. Math. Sci.,* 184:595–600.
430. Greene, J.E. (2002). A new short proof of Kneser's conjecture. *Amer. Math. Monthly,* 109:918–920.
431. Greenwell, D. and Lovász, L. (1974). Applications of product colouring. *Acta Math. Hungar.,* 25:335–340.
432. Grötschel, M., Lovász, L. and Schrijver, A. (1981). The ellipsoid method and its consequences in combinatorial optimization. *Combinatorica,* 1:169–197.
433. Grötschel, M., Lovász, L. and Schrijver, A. (1984). Polynomial algorithms for perfect graphs. *Ann. Discrete Math.,* 21:325–356.

434. Grötzsch, H. (1958/59). Ein Dreifarbensatz für dreikreisfreie Netze auf der Kugel. *Wiss. Z. Martin-Luther Univ. Halle-Wittenberg, Math.-Nat. Reihe,* 8:109–120.
435. Grünbaum, B. (1963). Grötzsch's theorem on 3-colorings. *Michigan Math. J.,* 10:303–310.
436. Grünbaum, B. (1988). The edge-density of 4-critical planar graphs. *Combinatorica,* 8:137–139.
437. Gupta, R.P. (1963). A note on a theorem on k-coloring the vertices of a graph. In: *Six Papers on Graph Theory,* pages 30–32. Indian Statistical Institute, Calcutta.
438. Gupta, R.P. (1967). *Studies in the Theory of Graphs*. Ph.D. Thesis, Tata Institute of Fundamental Research, Bombay.
439. Gupta, R.P. (1978). On the chromatic index and the cover index of a multigraph. In: *Theory and Applications of Graphs* (Michigan, 1976), pages 204–215. Lecture Notes in Math., Vol. 642, Springer.
440. Gupta, U.K. and Pradhan, D. (2021). Borodin–Kostochka's conjecture for (P_5, C_4)-free graphs. *J. Appl. Math. Comp.,* 65:877–884.
441. Guthrie, F. (1880). Note on the colouring of maps. *Proc. Roy. Soc. Edinburgh,* 10:727–728.
442. Gyárfás, A. (1975). On Ramsey covering-numbers. In: *Infinite and Finite Sets* (Keszthely, Hungary, 1973), pages 801–816. Coll. Math. Soc. János Bolyai Vol. 10, North-Holland.
443. Gyárfás, A. (1987). Problems from the world surrounding perfect graphs. *Zastos. Mat.,* 19:413–441.
444. Gyárfás, A. (1992). Graphs with k odd cycle lengths. *Discrete Math.,* 103:41–48.
445. Gyárfás, A., Jensen, T. and Stiebitz, M. (2004). On graphs with strongly independent color-classes. *J. Graph Theory,* 46:1–14.
446. Gyárfás, A. and Lehel, J. (1988). On-line and first fit colorings of graphs. *J. Graph Theory,* 12:217–227.
447. Gyárfás, A., Szemerédi, E. and Tuza, Zs. (1980). Induced subtrees in graphs of large chromatic number. *Discrete Math.,* 30:235–244.
448. Hadwiger, H. (1943). Über eine Klassifikation der Streckenkomplexe. *Vierteljschr. Naturforsch. Ges. Zürich,* 88:133–142.
449. Häggkvist, R. (1989). Towards a solution of the Dinitz problem? *Discrete Math.,* 75:247–251.
450. Häggkvist, R., Hell, P., Miller, D.J. and Neumann Lara, V. (1988). On multiplicative graphs and the product conjecture. *Combinatorica,* 8:63–74.
451. Häggkvist, R. and Janssen, J. (1997). New bounds on the list-chromatic index of the complete graph and other simple graphs. *Combin. Probab. Comput.,* 6:295–313.
452. Hahn, G. and Tardif, C. (1997). Graph homomorphisms: structure and symmetry. In: *Graph Symmetry,* pages 107–166. NATO ASI Series (Series C: Mathematical and Physical Sciences), Vol. 497, Springer.
453. Hajebi, S., Li, Y. and Spirkl, S. (2021). List-5-coloring graphs with forbidden induced subgraphs. Manuscript arXive:2105.01787
454. Hajiabolhassan, H. (2009). On colorings of graph powers. *Discrete Math.,* 309:4299–4305.
455. Hajiabolhassan, H. and Taherkhani, A. (2010). Graph powers and graph homomorphisms. *Electron. J. Combin.,* 17:#R17.
456. Hajnal, A. A theorem on k-saturated graphs. (1965). *Can J. Math.,* 10:720–724.
457. Hajnal, A. and Surányi, J. (1958). Über die Auflösung von Graphen in vollständige Teilgraphen. *Ann. Univ. Sci. Budapest Eötvös Sect. Math.,* 1:113–121.
458. Hajnal, A. and Szemerédi, E. (1970). Proof of a conjecture of P. Erdős. In: *Combinatorial Theory and its Application* (Balatonfüred, Hungary, 1969), Vol. II, pages 601–623. Coll. Math. Soc. János Bolyai, Vol. 4, North-Holland.
459. Hajnal, P. (1983). Partition of graphs with condition on the connectivity and minimum degree. *Combinatorica,* 3:95–99.
460. Hajós, G. (1956). Über eine Art von Graphen. *Intern. Math. Nachr.,* 47/48:65.
461. Hajós, G. (1961). Über eine Konstruktion nicht n-färbbarer Graphen. *Wiss. Z. Martin Luther Univ. Halle-Wittenberg, Math.-Natur. Reihe,* 10:116–117.
462. Hakimi, S.L. (1965). On the degrees of the vertices of a directed graph. *J. Franklin Inst.,* 279:290–308.

463. Hall, P. (1935). On representatives of subsets. *J. London Math. Soc.,* 10:26–30.
464. Halldórsson, M.M. and Lau, H.C. (1997). Low-degree graph partitioning via local search with application to constraint satisfaction, max cut, and coloring. *J. Graph Algorithms and Applications,* 1:1–13.
465. Hamaker, Z. and Vatter, V. (2022). Three coloring via triangle counting. Manuscript arXiv:2203.08136
466. Han, M., Li, J., Wu, Y. and Zhang, C. (2018). Counterexamples to Jaeger's circular flow conjecture. *J. Combin. Theory Ser. B,* 131:1–11.
467. Hanson, D., Robinson, G.C. and Toft, B. (1986). Remarks on the graph colour theorem of Hajós. *Congr. Numer.,* 55:69–76.
468. Harary, F. (1953–54). On the notion of balance of a signed graph. *Michigan Math. J,* 2:143–146. Addendum ibid., preceding p. 1.
469. Harary, F. (1985). Conditional colorability in graphs. In: *Graphs and Applications*, pages 127–136. John Wiley & Sons.
470. Harary, F. and Tutte, W.T. (1965). A dual form of Kuratowski's theorem. *Canad. Math. Bull.,* 8:17–20.
471. Hare, D.R. (2019). Tools for counting odd cycles in graphs. *J. Combin. Theory Ser. B,* 139:163–192.
472. Harris, D.G. (2019). Some results on chromatic number as a function of triangle count. *SIAM J. Discrete Math.,* 33:546–563.
473. Harutyunyan, A., Le, T.-N., Newmann, A. and Thomassé, S. (2019). Coloring dense digraphs. *Combinatorica,* 39:1021–1053.
474. Harutyunyan, A. and Mohar, B. (2011). Gallai's theorem for list coloring of digraphs. *SIAM J. Discrete Math.,* 25:170–180.
475. Harutyunyan, A. and Mohar, B. (2011). Strengthened Brooks' theorem for digraphs of girth at least three. *Electron. J. Combin.,* 18:#P195.
476. Harutyunyan, A. and Mohar, B. (2012). Two results on the digraph chromatic number. *Discrete Math.,* 312:1823–1826.
477. Harutyunyan, A. and Mohar, B. (2017). Planar digraphs of digirth five are 2-colorable. *J. Graph Theory,* 84:408–427.
478. Hasse, M. (1965). Zur algebraischen Begründung der Graphentheorie I. *Math. Nachr.,* 28:275–290.
479. Havel, I. (1969). On a conjecture of B. Grünbaum. *J. Combin. Theory Ser. B,* 7:184–186.
480. Havet, F., Picasarri-Arrieta, L. and Rambaud, C. (2023). On the minimum number of arcs in 4-dicritical oriented graphs. In: *Graph-Theoretic Concepts in Computer Science, WG 2023*, pages 376–387. Lecture Notes in Comput. Sci., Vol. 14093, Springer.
481. Havet, F. and Thomassé, S. (2000). Oriented Hamiltonian paths in tournaments: A proof of Rosenfeld's conjecture. *J. Combin. Theory Ser. B,* 78:243–273.
482. Haxell, P. (2001). A note on vertex list colouring. *Combin. Probab. Comput.,* 10:345–347.
483. Haxell, P. and MacDonald, C. (2023). Large cliques in graphs with high chromatic number. *Discrete Math.,* 346:113181.
484. Haxell, P. and McDonald, J. (2012). On characterizing Vizing's edge colouring bound. *J. Graph Theory,* 69:160–168.
485. Haxell, P. and Naserasr, R. (2023). A note on Δ-critical graphs. *Graphs Combin.,* 39:#101.
486. He, X. and Wigderson, Y. (2021). Hedetniemi's conjecture is asymptotically false. *J. Combin. Theory Ser. B,* 146:485–494.
487. Heawood, P.J. (1890). Map colour theorem. *Quart. J. Pure Appl. Math.,* 24:332–338.
488. Heawood, P.J. (1898). On the four-color map theorem. *Quart. J. Pure Appl. Math.,* 29:270–285.
489. Hedetniemi, S.T. (1966). Homomorphisms of graphs and automata., University of Michigan, Technical Report 03105-44-T.
490. Hedetniemi, S.T. (1968). On partitioning planar graphs. *Canad. Math. Bull.,* 11:203–211.
491. Heesch, H. (1969). *Untersuchungen zum Vierfarbenproblem,* BI Hochschulskripten Band 810.

492. Hefetz, D. (2011). On two generalizations of the Alon-Tarsi polynomial method. *J. Combin. Theory Ser. B,* 101:403–414.
493. Heffter, L. (1891). Über das Probem der Nachbargebiete. *Math. Ann.,* 38:477–508.
494. Hell, P. and Nešetřil, J. (1990). On the complexity of H-coloring. *J. Combin. Theory Ser. B,* 48:92–110.
495. Hell, P. and Nešetřil, J. (2004). *Graphs and Homomorphisms*. Oxford Lecture Series in Mathematics and its Application, Vol. 28, Oxford University Press.
496. Henning, M.A. and Yeo, A. (2015). On 2-colorings of hypergraphs. *J. Graph Theory,* 80:112–135.
497. Henning, M.A. and Yeo, A. (2018). Not-all-equal 3-SAT and 2-colorings of 4-regular 4-uniform hypergraphs. *Discrete Math.,* 341:2285–2292.
498. Hilton, A.J.W. and Johnson, P D. (1999). The Hall number, the Hall index, and the total Hall number of a graph. *Discrete Appl. Math.,* 94:227–245.
499. Hilton, A.J.W., Rado, R. and Scott, S.H. (1973). A (¡5)-colour theorem for planar graphs. *Bull. London Math. Soc.,* 5:302–306.
500. Hladký, J., Král', D. and Schauz, U. (2010). Brooks' theorem via the Alon-Tarsi theorem. *Discrete Math.,* 310:3426–3428.
501. Hoàng, C.T., Kamiński, M., Sawada, J. and Shu, X. (2010). Deciding k-colorability of P_5-free graphs in polynomial time. *Algorithmica,* 57:74–81.
502. Hochberg, R. and O'Donnell, P. (1993). A large independent set in the unit distance graph. *Geombinatorics,* 3:83–84.
503. Hochstättler, W., Schröder, F. and Steiner, R. (2019). On the complexity of digraph colourings and vertex arboricity. *Discrete Math. Theor. Comput. Sci.,* 22:#4.
504. Holyer, I. (1981). The NP-completeness of edge-coloring. *SIAM J. Comput.,* 10:718–720.
505. Hopcroft, J.E. and Tarjan, R.E. (1973). Efficient algorithms for graph manipulation. *Commun. ACM,* 16:372–378.
506. Hoshino, R. and Kawarabayashi, K. (2015). The edge density of critical digraphs. *Combinatorica,* 35:619–631.
507. Huang, S. (2016). Improved complexity results on k-coloring P_t-free graphs. *European J. Combin.,* 51:336–346.
508. Hungerford, T.W. (1974). *Algebra*. Graduate Texts in Mathematics, Vol. 73, Springer.
509. Hurley, E. and Pirot, F. (2020). Coloring locally sparse graphs with the first moment method. Manuscript arXive:2109.15215
510. Hutchinson, J.P. (1976). On coloring maps made from Eulerian graphs. *Congr. Numer.,* 15:343–354.
511. Hutchinson, J.P. (1995). Three-coloring graphs embedded on surfaces with all faces even-sided. *J. Combin. Theory Ser. B,* 65:139–155.
512. Imrich, W. and Klavžar, S. (2000). *Product Graphs: Structure and Recognition*. Wiley-Interscience Series in Discrete Mathematics and Optimization, John Wiley & Sons.
513. Jaeger, F. (1978). Sur les graphes couverts par leurs bicycles et la conjecture des quatre couleurs. In: *Problémes Combinatoires et Theorie des Graphes,* pages 243–247. Editions du Centre National de la Recherche Scientifique.
514. Jaeger, F. (1979). Flows and generalized coloring theorems in graphs. *J. Combin. Theory Ser. B,* 26:205–216.
515. Jaeger, F. (1981). A constructive approach to the critical problem for matroids. *European J. Combin.,* 2:137–144.
516. Jaeger, F. (1984). On circular flows in graphs. In: *Finite and Infinite Sets* (Eger, Hungary, 1981), pages 391–402. Coll. Math. Soc. János Bolyai, Vol. 37, North-Holland.
517. Jaeger, F., Linial, N., Payan, C. and Tarsi, M. (1992). Group connectivity of graphs – A non-homogenous analogue of nowhere-zero flow properties. *J. Combin. Theory Ser. B,* 56:165–182.
518. Jakobsen, I.T. (1971). *On Certain Homomorphism-Properties of Graphs With Applications to the Conjecture of Hadwiger.* Ph.D. Thesis, University of London. Various Publ. Ser. 15 (Aarhus University).

519. Jakobsen, I.T. (1971). A homomorphism theorem with an application to the conjecture of Hadwiger. *Studia Sci. Math. Hungar.,* 6:151–160.
520. Jakobsen, I.T. (1973). Some remarks on the chromatic index of a graph. *Arch. Math. (Basel),* 24:440–448.
521. Jakovac, M. and Štesl, D. (2023). On game chromatic vertex-critical graphs. *Bull. Malays. Math. Sci. Soc,* 46:#27.
522. Jamall, M.S. (2011). A Brooks' theorem for triangle-free graphs. Manuscript arXive:1106.1958
523. Jamall, M.S. (2011). A coloring algorithm for triangle-free graphs. Manuscript arXive:1101.5721
524. Javdekar, M. (1980). Note on Choudum's "chromatic bounds for a class of graphs". *J. Graph Theory,* 4:265–267.
525. Jensen, T.R. (2002). Dense critical and vertex-critical graphs. *Discrete Math.,* 258:63–84.
526. Jensen, T.R. (2017). Grassmann homomorphism and Hajós-type theorems. *Linear Algebra Appl.,* 522:140–152.
527. Jensen, T.R. and Royle, G.F. (1995). Small graphs with chromatic number 5: a computer search. *J. Graph Theory,* 19:107–116.
528. Jensen, T.R. and Royle, G.F. (1999). Hajós constructions of critical graphs. *J. Graph Theory,* 30:37–50.
529. Jensen, T.R. and Thomassen, C. (2000). The color space of a graph. *J. Graph Theory,* 34:234–245.
530. Jensen, T.R. and Toft, B. (1995). *Graph Coloring Problems*. Wiley-Interscience Series in Discrete Mathematics and Optimization, John Wiley & Sons.
531. Jensen, T.R, and Toft, B. (1995). Choosability versus chromaticity – the plane unit distance graph has a 2-chromatic subgraph of infinite list-chromatic number. *Geombinatorics,* 5:45–64.
532. Jiang, T. (2001). Small odd cycles in 4-chromatic graphs. *J. Graph Theory,* 37:115–117.
533. Jin, G. (1995). Triangle-free four-chromatic graphs. *Discrete Math.,* 145:151–170.
534. Johansson, A. (1996). Asymptotic choice number for triangle free graphs. Unpublished Manuscript.
535. Johansson, A. (1996). The choice number of sparse graphs. Unpublished Manuscript.
536. Johnson, A., Holroyd, F.C. and Stahl, S. (1997). Multichromatic numbers, star chromatic numbers and Kneser graphs. *J. Graph Theory,* 26:137–145.
537. Johnson, D.S. (1974). Worst case behavior of graph colouring algorithms. *Congr. Numer.,* 10:513–527.
538. Johnson, D.S. (1974). Approximation algorithms for combinatorial problems. *J. Comp. System Sci.,* 9:256–278.
539. Johnson, D.S. (1976). On property B_r. *J. Combin. Theory Ser. B,* 20:64–66.
540. Johnson, P.D. (1994). The choice number of the plane. *Geombinatorics,* 3:122–128.
541. Jones, R.P. (1975). Brooks' theorem for hypergraphs. *Congr. Numer.,* 15:379–384.
542. Juvan, M., Mohar, B. and Škrekovski, R. (1998). List-total colourings of graphs. *Combin. Probab. Comput.,* 7:181–188.
543. Juvan, M., Mohar, B. and Thomas, R. (1999). List edge-colorings of series-parallel graphs. *Electron. J. Combin.,* 6:#R42.
544. Kagno, I.N.. (1936). The triangulation of surfaces and the Heawood color formula. *J. Math. Phys.,* 15:179-186.
545. Kahle, M. and Martinez-Figueroa, F. (2020). The chromatic number of random Borsuk graphs. *Random Struct. Alg.,* 56:838–850.
546. Kahn, J. and Seymour, P.D. (1992). A fractional version of the Erdős-Faber-Lovász conjecture. *Combinatorica,* 12:155–160.
547. Kaiser, T. and Stehlík, M. (2015). Colouring quadrangulations of projective spaces. *J. Combin. Theory Ser. B,* 113:1–17.
548. Kaiser, T. and Stehlík, M. (2019). Generalised Mycielski graphs and the Borsuk–Ulam theorem. *Electron. J. Combin.,* 26:#P4.1.
549. Kaiser, T. and Stehlík, M. (2020). Edge-critical subgraphs of Schrijver graphs. *J. Combin. Theory Ser. B,* 144:191–196.

550. Kaiser, T. and Stehlík, M. (2020). Edge-critical subgraphs of Schrijver graphs II: The general case. *J. Combin. Theory Ser. B,* 152:453–482.
551. Kamiński, M. and Lozin, V.V. (2007). Coloring edges and vertices of graphs without short or long cycles. *Contrib. Discrete Math.,* 2:61–66.
552. Kang, D., Kelly, T., Kühn, D., Methuku, A. and Osthus, D. (2023). A proof of the Erdős–Faber–Lovász conjecture. *Ann. of Math.,* 189:537–618.
553. Kang, Y. (2018). *Coloring of signed graphs.* Dissertation, Universität Paderborn.
554. Kang, Y. (2018). Hajós-like theorem for signed graphs. *European J. Combin.,* 67:199–207.
555. Kang, Y., Jin, L. and Wang, Y. (2016). The 3-colorability of planar graphs without cycles of length $4,6$ and 9. *Discrete Math.,* 339:299–307.
556. Karp, R. M. (1972). Reducibility among combinatorial problems. In: *Complexity of Computer Computation*, pages 85–103. Plenum Press.
557. Karthick, T. and Maffray, F. (2018). Coloring (gem,co-gem)-free graphs. *J. Graph Theory,* 89:283–303.
558. Kaul, H. and Mudrock, J.A. (2019). A note on fractional DP-coloring of graphs. Manuscript arXive:1910:03416
559. Kawarabayashi, K. (2015). Hadwiger's conjecture. In: *Topics in Chromatic Graph Theory*, pages 73–93. Cambridge University Press.
560. Kawarabayashi, K., Král', D., Kynčl, J. and Lidický, B (2009). 6-critical graphs on the Klein bottle. *SIAM J. Discrete Math.,* 23:372–383.
561. Kawarabayashi, K, Pedersen, A.S., and Toft, B. (2010). Double-critical graphs and complete minors. *Electron. J. Combin.,* 17:#R87.
562. Kawarabayashi, K. and Toft, B. (2005). Any 7-chromatic graph has K_7 or $K_{4,4}$ as a minor. *Combinatorica,* 25:327–353.
563. Keevash, P., Li, Z., Mohar, B. and Reed, B. (2013). Digraph girth via chromatic number. *SIAM J. Discrete Math.,* 27:693–696.
564. Kelly, J.B. and Kelly, L.M. (1954). Path and circuits in critical graphs. *Amer. J. Math.,* 76:786–792.
565. Kelly, T. (2019). *Cliques, Degrees, and Coloring: Expanding the* ω,Δ,χ *Paradigm.* Ph.D. Thesis, University of Waterloo (Canada).
566. Kelly, T. and Postle, L. (2020). A local epsilon version of Reed's Conjecture. *J. Combin. Theory Ser. B,* 141:181–222.
567. Kelly, T. and Postle, L. (2020). Corrigendum to "A local epsilon version of Reed's Conjecture". *J. Combin. Theory Ser. B,* 151:509–512.
568. Kempe, A.B. (1879). On the geographical problem of four colours. *Amer. J. Math.,* 2:193–200.
569. Kenkre, S. and Vishwanathan, S. (2017). A bound on the chromatic number using the longest odd cycle length. *J. Graph Theory,* 54:267–276.
570. Kézdy, A.E. and Snevily, H.S. (1997). On extensions of a conjecture of Gallai. *J. Combin. Theory Ser. B,* 70:317–324.
571. Kierstead, H.A. (1984). On the chromatic index of multigraphs without large triangles. *J. Combin. Theory Ser. B,* 36:156-160.
572. Kierstead, H.A. (1989). Applications of edge coloring of multigraphs to vertex coloring of graphs. *Discrete Math.,* 74:117–124.
573. Kierstead, H.A. (2000). On the choosability of complete multipartite graphs with part size three. *Discrete Math.,* 211:255–259.
574. Kierstead, H.A. and Kostochka, A.V. (2008). A short proof of the Hajnal-Szemerédi theorem on equitable coloring. *Combin. Probab. Comput.,* 17:265–270.
575. Kierstead, H.A. and Kostochka, A.V. (2009). Ore-type version of Brooks' theorem. *J. Combin. Theory Ser. B,* 99:298–305.
576. Kierstead, H.A. and Kostochka, A.V. (2010). Equitable versus nearly equitable coloring and the Chen-Lih-Wu conjecture. *Combinatorica,* 30:201–216.
577. Kierstead, H.A., Kostochka, A.V., Mydlarz, M. and Szemerédi, E. (2010). A fast algorithm for equitable coloring. *Combinatorica,* 30:217–224.

578. Kierstead, H.A., Kostochka, A.V. and Yu, G. (2009). Extremal graph packing problems: Ore-type versus Dirac-type. In: *Surveys in Combinatorics, 2009,* pages 113–136. London Math. Soc. Lecture Notes Series, Vol. 365, Cambridge University Press.
579. Kierstead, H.A. and Penrice, S. (1994). Radius two trees specify χ-bounded classes. *J. Graph Theory,* 18:119–129.
580. Kierstead, H.A. and Rabern, L. (2017). Extracting list colorings from large independent sets. *J. Graph Theory,* 86:315–328.
581. Kierstead, H.A. and Rabern, L. (2020). Improved lower bounds on the number of edges in list critical and online list critical graphs. *J. Combin. Theory Ser. B,* 140:147–170.
582. Kierstead, H.A. and Rödl, V. (1996). Applications of hypergraph coloring to coloring graphs not inducing certain trees. *Discrete Math.,* 150:187–193.
583. Kierstead, H.A. and Schmerl, J.H. (1983). Some applications of Vizing's theorem to vertex colorings of graphs. *Discrete Math.,* 45:277–285.
584. Kierstead, H.A. and Schmerl, J.H. (1986). The chromatic number of graphs which induce neither $K_{1,3}$ nor $K_5 - e$. *Discrete Math.,* 58:253–262.
585. Kierstead, H.A., Szemerédi, E. and Trotter, W.T. (1984). On coloring graphs with locally small chromatic number. *Combinatorica,* 4:183–185.
586. Kierstead, H.A. and Trotter, W.T. (1994). Planar graph coloring with an uncooperative partner. *J. Graph Theory,* 18:569–584.
587. Kierstead, H.A. and Zhu, Y. (2004). Radius three trees in graphs with large chromatic number. *SIAM J. Discrete Math.,* 17:571–581.
588. Kilakos, K. and Reed, B. (1993). Fractionally colouring total graphs. *Combinatorica,* 13:435–440.
589. Kim, J.H. (1995). On Brooks' Theorem for sparse graphs. *Combin. Probab. Comput.,* 4:97–13.
590. Kim, J.H. (1995). The Ramsey number $R(3,t)$ has order of magnitude $t^2/\log t$. *Random Struct. Alg.,* 7:173–207.
591. Kim, S.-J. and Ozeki, K. (2019). A note on a Brooks' type theorem for DP-coloring. *J. Graph Theory,* 91:148–161.
592. Kim, S.-J. and Park, B. (2015). Counterexamples to the list square coloring conjecture. *J. Graph Theory,* 78:239–247.
593. King, A.D. (2009). *Claw-free Graphs and two Conjectures on omega, Delta and chi.* Ph.D. Thesis, McGill University (Canada).
594. King, A.D. (2011). Hitting all maximum cliques with a stable set using lopsided independent transversals. *J. Graph Theory,* 67:300–305.
595. King, A.D. and Reed, B. (2008). Bounding χ in terms of ω and Δ for quasi-line graphs. *J. Graph Theory,* 59:215–228.
596. King, A.D. and Reed, B. (2016). A short proof that χ can be bounded ε away from $\Delta+1$ towards ω. *J. Graph Theory,* 81:30–34.
597. King, A.D., Reed, B. and Vetta, A. (2007). An upper bound for the chromatic number of line graphs. *European J. Combin.,* 28:2182–2187.
598. Király, T. (2003). *Edge-connectivity of undirected and directed hypergraphs.* Dissertation, Eötvös Loránd University, Budapest.
599. Király, T. and Pap, J. (2009). A note on kernels and Sperner's lemma. *Discrete Appl. Math.,* 157:3327–3331.
600. Klein, R. (1992). *On $(m_1, m_2, \ldots, m_s)$-composed graphs.* Doctoral Thesis, Tel Aviv University.
601. Klein, R. and Schönheim, J. (1995). On the colorability of graphs decomposable into degenerate graphs with specified degeneracy. *Australas. J. Combin.,* 12:201–208.
602. Klimošová, T., Malík, J., Masařík, T., Novotná, J., Paulusma, D. and Slívová, V. (2020). Colouring $(P_r + P_s)$-free graphs. *Algorithmica,* 82:1833–1858.
603. Klostermeyer, W. and Zhang, C.Q. (2000). $(2+\epsilon)$-Coloring of planar graphs with large odd-girth. *J. Graph Theory,* 33:109–119.
604. Klostermeyer, W. and Zhang, C.Q. (2002). n-tuple coloring of planar graphs with large odd girth. *Graphs Combin.,* 18:119–131.

605. Kneser, M. (1955/56). Aufgabe 360. *Jahresber. Dtsch. Math.-Ver.,* 58 (2): 27.
606. Kohl, A. and Schiermeyer, I. (2010). Some results on Reed's conjecture about ω, Δ, and χ with respect to α. *Discrete Math.,* 310:1429–1438.
607. Komjáth, P. (2011). The list-chromatic number of infinite graphs defined on Euclidean spaces. *Discrete Comput. Geom.,* 45:497–502.
608. König, D. (1905). On map colouring (in Hungarian). *Mat. Fiz. Lapok,* 14:193–200.
609. König, D. (1916). Über Graphen und ihre Anwendung auf Determinantentheorie und Mengenlehre. *Math. Ann.,* 77:453–465.
610. König, D. (1936). *Theorie der Endlichen und Unendlichen Graphen*. Akademische Verlagsgesellschaft M.B.H., Leipzig. Reprinted by Chealsea 1950 and by B. G. Teubner Verlagsgesellschaft 1986. English translation published by Birkhäuser 1990.
611. Körner, J., Pilotto, C. and Simonyi, G. (2005). Local chromatic number and Sperner capacity. *J. Combin. Theory Ser. B,* 95:101–117.
612. Korzhik, V.P. (2022). A simpler proof of the Map Color Theorem for nonorientable surfaces. *J. Combin. Theory Ser. B,* 156:1–17.
613. Köster, G. (1985). Note to a problem of T. Gallai and G. A. Dirac. *Combinatorica,* 5:227–228.
614. Köster, G. (1990). 4-critical 4-valent planar graphs constructed with crowns. *Math. Scand.,* 67:15–22.
615. Köster, G. (1991). On 4-critical planar graphs with high edge density. *Discrete Math.,* 98:147–151.
616. Kostochka, A.V. (1978). Degree, girth and chromatic number. In: *Combinatorics* (Keszthely, Hungary, 1976), Vol II, pages 679–696., Coll. Math. Soc. János Bolyai, Vol. 18, North-Holland.
617. Kostochka, A.V. (1980). Degree, density, and chromatic number of graphs (in Russian). *Metody Diskret. Analiz.,* 35:45–70.
618. Kostochka, A.V. (1982). A modification of Catlin's algorithm (in Russian). *Methods and Programs of Solutions Optimization Problems on Graphs and Networks,* 2:75–79.
619. Kostochka, A.V. (1984). Lower bound of the Hadwiger number of graphs by their average degree. *Combinatorica,* 4:307–316.
620. Kostochka, A.V. (2004). Coloring uniform hypergraphs with few colors. *Random Struct. Alg.,* 24:1–10.
621. Kostochka, A.V. (2004). Coloring intersection graphs of geometric figures with a given clique number. In: *Towards a Theory of Geometric Graphs,* pages 127–138. Contemporary Mathematics, Vol. 342, American Mathematical Society.
622. Kostochka, A.V. (2006). Color-critical graphs and hypergraphs with few edges: A survey. In: *More Sets, Graphs and Numbers* (A Salute to Vera Sós and András Hajnal), pages 175–197. Bolyai Soc. Math. Stud., Vol. 15, Springer.
623. Kostochka, A.V. and Mazurova, N.P. (1977). An inequality in the theory of graph coloring (in Russian). *Metody Diskret. Analiz.,* 30:23–29.
624. Kostochka, A.V., Mubayi, D., Rödl, V. and Tetali, P. (2001). On the chromatic number of set systems. *Random Struct. Alg.,* 19:87–98.
625. Kostochka, A.V. and Nakprasit, K. (2005). On equitable Δ-coloring of graphs with low average degree. *Theoret. Comput. Sci.,* 349:82–91.
626. Kostochka, A.V. and Nešetřil, J (1999). Properties of Descartes' contruction of triangle-free graphs with high chromatic number. *Combin. Probab. Comput.,* 8:467–472.
627. Kostochka, A.V., Rabern, L. and Stiebitz, M. (2012). Graphs with chromatic number close to maximum degree. *Discrete Math.,* 312:1273–1281.
628. Kostochka, A.V., Schweser, T. and Stiebitz, M. (2023). Generalized DP-colorings of graphs. *Discrete Math.,* 346:113186.
629. Kostochka, A.V. and Stiebitz, M. (1998). Colour-critical graphs with few edges. *Discrete Math.,* 191:125–137.
630. Kostochka, A. . and Stiebitz, M. (1999). Excess in colour-critical graphs. In: *Graph Theory and Combinatorial Biology* (Balatonlelle, Hungary, 1996), pages 87–99. Bolyai Soc. Math. Stud., Vol. 7, Springer.

631. Kostochka, A.V. and Stiebitz, M. (2000). On the number of edges in colour-critical graphs and hypergraphs. *Combinatorica,* 20:521–530.
632. Kostochka, A.V. and Stiebitz, M. (2002). A list version of Dirac's theorem on the number of edges in colour-critical graphs. *J. Graph Theory,* 39:165–177.
633. Kostochka, A.V. and Stiebitz, M. (2003). A new lower bound on the number of edges in colour-critical graphs and hypergraphs. *J. Combin. Theory Ser. B,* 87:374–402.
634. Kostochka, A.V. and Stiebitz, M. (2008). Partitions and edge colourings of multigraphs. *Electron. J. Combin.,* 15:#N25.
635. Kostochka, A.V. and Stiebitz, M. (2020). The minimum number of edges in 4-critical digraphs of given order. *Graphs Combin.,* 36:703–718.
636. Kostochka, A.V., Stiebitz, M. and Wirth, B. (1996). The colour theorems of Brooks and Gallai extended. *Discrete Math.,* 162:299–303.
637. Kostochka, A.V., Sudakov, B. and Verstraëte, J. (2017). Cycles in triangle-free graphs of large chromatic number. *Combinatorica,* 37:481–494.
638. Kostochka, A.V. and Woodall, D.R. (2001). Choosability conjectures and multicircuits. *Discrete Math.,* 240:123–143.
639. Kostochka, A.V. and Yancey, M. (2014). Ore's conjecture for $k = 4$ and Grötzsch's theorem. *Combinatorica,* 34:323–329.
640. Kostochka, A.V. and Yancey, M. (2014). Ore's conjecture on color-critical graphs is almost true. *J. Combin. Theory Ser. B,* 109:73–101.
641. Kostochka, A.V. and Yancey, M. (2018). A Brooks-type result for sparse critical graphs. *Combinatorica,* 38:887–934.
642. Král', D. (2004). Hajós' theorem for list coloring. *Discrete Math.,* 287:161–163.
643. Král', D., Kratochvíl, J., Tuza, Zs. and Woeginger, G.J. (2001). Complexity of coloring graphs without forbidden induced subgraphs. In: *Graph-Theoretic Concepts in Computer Science, WG 2001,* pages 254–262. Lecture Notes in Comput. Sci., Vol. 2204, Springer.
644. Král', D., Mohar, B., Nakamoto, A., Pangrác, O. and Suzuki, Y. (2012). Coloring Eulerian triangulations of the Klein bottle. *Graphs Combin.,* 28:499–530.
645. Král', D., Pangrác, O. and Voss, H.-J. (2005). A note on group colorings. *J. Graph Theory,* 50:123–129.
646. Král', D. and Škrekovski, R. (2006). The last excluded case of Dirac's map-color theorem for choosability. *J. Graph Theory,* 51:319–354.
647. Kratochvíl, J. and Tuza, Zs. (1994). Algorithmic complexity of list colorings. *Discrete Appl. Math.,* 50:297–302.
648. Kratochvíl, J., Tuza, Zs. and Voigt, V. (1999). New trends in the theory of graph colorings: choosability and list colorings. In: *Contemporary Trends in Mathematics,* pages 183–198. DIMACS, Series in Discrete Mathematics and Theoretical Computer Science, Vol. 49.
649. Kriesell, M. (2010). On Seymour's strengthening of Hadwiger's conjecture for graphs with certain forbidden subgraphs. *Discrete Math.,* 310:2714–2724.
650. Kriesell, M. (2017). Unique colorability and clique minors. *J. Graph Theory,* 85:207–216.
651. Krivelevich, M. (1997). On the minimal number of edges in color-critical graphs. *Combinatorica,* 17:401–426.
652. Krivelevich, M. (1998). An improved bound on the minimal number of edges in color-critical graphs. *Electron. J. Combin.,* 5:#P4.
653. Krivelevich, M. (2022). The choosability version of Brooks' theorem - a short proof. Manuscript arXive:2205.08326
654. Kříž (1989). A hypergraph-free construction of highly chromatic graphs without short cycles. *Combinatorica,* 9:227–229.
655. Kronk, H.V. and Mitchem, J. (1972). On Dirac's generalization of Brooks' theorem. *Canad. J. Math.,* 24:805–807.
656. Kronk, H.V. and Mitchem, J. (1975). Critical point-arboritic graphs. *J. London Math. Soc.,* 2:459–466.
657. Kronk, H.V. and White, A.T. (1972). A 4-color theorem for toroidal graphs. *Proc. Amer. Math. Soc.,* 34:83–86.

658. Krul, M. and Thoma, L. (2019). 2-colorability of r-uniform hypergraphs. *Electron. J. Combin.,* 26:#P3.30.
659. Krusenstjerna-Hafstrøm, U. and Toft, B. (1980). Special subdivisions of K_4 and 4-chromatic graphs. *Monatsh. Math.,* 89:101–110.
660. Kühn, D. and Osthus, D. (2002). Topological minors in graphs of larger girth. *J. Combin. Theory Ser. B,* 86:364–380.
661. Kuratowski, C. (1930). Sur le problème des courbes gauches en Topologie. *Fund. Math.,* 15:271–283.
662. La, H., Lužar, B. and Štorgel, K. (2022). Further extensions of the Grötzsch theorem. *Discrete Math.,* 345:112849.
663. Lai, H.-J., Li, X. and Poon, H. (2005). Nowhere zero 4-flows in regular matroids. *J. Graph Theory,* 49:196–204.
664. Lai, H.-J. and Mazza, L. (2021). Group colorings and DP-colorings of multigraphs using edge-disjoint decompositions. *Graphs Combin.,* 37:2227–2243.
665. Lai, H.-J. and Zhang, X. (2002). Group chromatic number of graphs without K_5-minors. *Graphs Combin.,* 18:147–154.
666. Larsen, M., Propp, J. and Ullmann, D.H. (1995). The fractional chromatic number of Mycielski's graphs. *J. Graph Theory,* 19:411–416.
667. Larsen, V. (2015). *An Epsilon Improvement to the Asymptotic Density of k-Critical Graphs.* Ph.D. Thesis, Emory University.
668. Lawrence, J. (1978). Covering the vertex set of a graph with subgraphs of smaller degree. *Discrete Math.,* 21:61–68.
669. Lee, J.A.N. and Holtzman, G. (1995). 50 Years after breaking the codes: interviews with two of the Bletchley Park scientists. *IEEE Annals History Comp.,* 17:32–43.
670. Lekkerkerker, C. and Boland, J.Ch. (1962). Representation of a finite graph by a set of intervals on the real line. *Fund. Math.,* 51:45–64.
671. Leven, D. and Galil, Z. (1983). NP-completeness of finding the chromatic index of regular graphs. *J. Algorithms,* 4:35–44.
672. L'Huilier, S. (1812/1813). Géométrie. *Annales de Mathématiques, pures et appliquées,* 3:169–189.
673. Li, J., Luo, R. and Wang, Y. (2018). Nowhere-zero 3-flow of graphs with small independence number. *Discrete Math.,* 341:42–50.
674. Li, Z. and Mohar, B. (2017). Planar digraphs of digirth four are 2-colorable. *SIAM J. Discrete Math.,* 31:2201–2205.
675. Lick, D.R. and White, A. (1970). k-degenerate graphs. *Can. J. Math,* 22:1082–1096.
676. Lih, K.-W. (1998). The equitable coloring of graphs. In: *Handbook of Combinatorial Optimization*, Vol. 3, pages 543–566. Kluwer Academic Publisher.
677. Lih, K.-W. and Wu, P.-L. (1996). On equitable coloring of bipartite graphs. *Discrete Math.,* 151:155–160.
678. Lin, W., Wu, J., Lam, P.C.B. and Gu, G. (2006). Several parameters of generalized Mycielskians. *Discrete Appl. Math.,* 154:1173–1182.
679. Liu, A. (1976). *Some results on hypergraphs.* Ph.D. Thesis, University of Alberta (Canada).
680. Liu, C.-H. and Ma, J. (2018). Cycle lengths and minimum degree of graphs. *J. Combin. Theory Ser. B,* 128:66–95.
681. Liu, C.-H. and Postle, L. (2017). On the minimum edge-density of 4-critical graphs of girth five. *J. Graph Theory,* 86:387–405.
682. Liu, D.D.-F. and Zhu, X. (2016). A combinatorial proof for the circular chromatic number of Kneser graphs. *J. Comb. Optim.,* 32:765–774.
683. Liu, R., Loeb, S., Rolek, M., Yin, Y. and Yu, G. (2019). DP-3-coloring of planar graphs without 4,9-cycles and cycles of two length from $\{6,7,8\}$. *Graphs Combin.,* 35:695–705.
684. Liu, S. and Zhang, J. (2006). Using Hajós' construction to generate hard graph 3-colorability instances. In: *Artificial Intelligence and Symbolic Computation, AISC 2006,* pages 211–225. Lecture Notes in Comput. Sci., Vol. 4120, Springer.

685. Liu, W., Lawrencenko, S., Chen, B., Ellingham, M.N., Hartsfield, N. Yang, H., Ye, D. and Zha, X. (2019). Quadrangular embeddings of complete graphs and the even map color theorem. *J. Combin. Theory Ser. B,* 139:1–26.
686. Lovász, L. (1966). On decomposition of graphs. *Studia Sci. Math. Hungar.,* 1:237–238.
687. Lovász, L. (1967). Operations with structures. *Acta Math. Hungar.,* 18:321–328.
688. Lovász, L. (1968). On chromatic number of finite set-systems. *Acta Math. Hungar.,* 19:59–67.
689. Lovász, L. (1970). A generalization of König's theorem. *Acta Math. Hungar.,* 21:443–446.
690. Lovász, L. (1972). A characterization of perfect graphs. *J. Combin. Theory Ser. B,* 13:95–98.
691. Lovász, L. (1972). Normal hypergraphs and the perfect graph conjecture. *Discrete Math.,* 2:253–267.
692. Lovász, L. (1973). Coverings and colorings of hypergraphs. *Congr. Numer.,* 8:3–12.
693. Lovász, L. (1973). Independent sets in critical chromatic graphs. *Studia Sci. Math. Hungar.,* 8:165–168.
694. Lovász, L. (1975). Three short proofs in graph theory. *J. Combin. Theory Ser. B,* 19:269–271.
695. Lovász, L. (1975). On combinatorial min-max theorems (Hungarian). *Mat. Lapok,* 26:209–264.
696. Lovász, L. (1976). Chromatic number of hypergraphs and linear algebra. *Studia Sci. Math. Hungar.,* 11:113–114.
697. Lovász, L. (1978). Kneser's conjecture, chromatic number, and homotopy. *J. Combin. Theory Ser. A,* 25:319–324.
698. Lovász, L. (1979). *Combinatorial Problems and Exercises*. Second Edition 1993, North-Holland.
699. Lovász, L (1982). Bounding the independence number of a graph. In: *Bonn Workshop on Combinatorial Optimization,* pages 213–223. Ann. Discrete Math., Vol. 16, North-Holland.
700. Lovász, L (1983). Self-dual polytopes and the chromatic number of distance graphs on the sphere. *Acta Sci. Math.,* 45:317–323.
701. Lovász, L. (1994). Stable sets and polynomials. *Discrete Math.,* 124:137–153.
702. Lovász, L. (2012). *Large Networks and Graph Limits.* American Mathematical Society.
703. Lovász, L.M., Thomassen, C., Wu, Y. and Zhang, C.-Q. (2013). Nowhere-zero 3-flows and modulo k-orientations. *J. Combin. Theory Ser. B,* 103:587–598.
704. Lubotzky, A., Phillips, R. and Sarnak, P. (1988). Ramanujan graphs. *Combinatorica,* , 8:261–277.
705. Luczak, T. (2006). On the structure of triangle-free graphs of large minimum degree. *Combinatorica,* 26:489–493.
706. Lund, C. and Yannakakis, M. (1994). On the hardness of approximating minimization problems. *J. ACM,* 41:960–981.
707. Luo, C., Ma, J. and Yang, T. (2023). On the maximum number of edges in k-critical graphs. *Combin. Probab. Comput.,* 1–12.
708. Lyle, J. (2014). A not on independent sets in graphs with large minimum degree and small cliques. *Electron. J. Combin.,* 21:#P2.38.
709. Ma, J. and Yang, T. (2021). Counting critical subgraphs in k-critical graphs. *Combinatorica,* 41:669–694.
710. Mader, W. (1967). Über trennende Eckenmengen in homomorphiekritischen Graphen. *Math. Ann.,* 175:243–252.
711. Mader, W. (1968). Homomorphiesätze für Graphen. *Math. Ann.,* 178:154–168.
712. Mader, W. (1973). Grad und lokaler Zusammenhang in endlichen Graphen. *Math. Ann.,* 205:9–11.
713. Maffray, F. (1992). Kernels in perfect line-graphs. *J. Combin. Theory Ser. B,* 55:1–8.
714. Maffray, F. and Preissmann, M. (1996). On NP-completeness of k-colorability problem for triangle-free graphs. *Discrete Math.,* 162:313–317.
715. Mansfield, A.J. and Welsh, D.J.A. (1982). Some coloring problems and their complexity. *Ann. Discrete Math.,* 13:159–170.
716. Marshall, T., Naserasr, R. and Nešetřil, J. (2006). Homomorphism bounded classes of graphs. *European J. Combin.,* 27:592–600.

717. Martinsson, A. (2021). A simplified proof of the Johansson–Molly theorem using Rosenfeld counting method. Manuscript arXive:2111.06214
718. Martinsson, A. and Steiner, R. (2023). Vertex critical graphs far from edge-criticality. Manuscript arXive:2310.12891
719. Martinsson, A. and Steiner, R. (2024). Strengthening Hadwiger's conjecture for 4- and 5-chromatic graphs. *J. Combin. Theory Ser. B,* 164:1–16.
720. Matamala, M. (2007). Vertex partition and maximum degenerate subgraphs. *J. Graph Theory,* 55:227–232.
721. Mathews, J., Panda, M.K. and Shannigrahi, S. (2015). On the construction of non-2-colorable uniform hypergraphs. *Discrete Appl. Math.,* 180:181–187.
722. Matoušek, J. (2003). *Using the Borsuk–Ulam Theorem: Lectures on Topological Methods in Combinatorics and Geometry*, Springer.
723. Matoušek, J. (2004). A combinatorial proof of Kneser's conjecture. *Combinatorica,* 24:163–170.
724. Matoušek, J. and Ziegler, G. (2004). Topological lower bounds for the chromatic number: A hierarchy. *Jahresber. Dtsch. Math.-Ver.,* 106:71–90.
725. Matula, D.W. (1968). A min-max theorem for graphs with application to graph coloring. *SIAM Rev.,* 10:481–482.
726. Mayer, J. (1989). Conjecture de Hadwiger: Un graphe k-chromatique contraction-critique n'est pas k-régulier. *Ann. Discrete Math.,* 41:341–346.
727. McConnell, R.M., Mehlhorn, K., Näher, S. and Schweitzer, P. (2011). Certifying algorithms. *Comput. Sci. Rev.,* 5:119–161.
728. McDiarmid, C. (1997). Hypergraph colouring and the Lovász Local Lemma. *Discrete Math.,* 167/168:481–486.
729. McDiarmid, C. (1998). Concentration. In: *Probabilistic Methods for Algorithmic Discrete Mathematics*, pages 195–248. Springer.
730. McDonald, J. (2015). On Galvin orientations of line graphs and list-edge-colouring. Manuscript arXive:1508.01820.
731. Melnikov, L.S. and Vizing, V.G. (1969). New proof of Brooks' theorem. *J. Combin. Theory,* 7:289–290.
732. Menger, K. (1927). Zur allgemeinen Kurventheorie. *Fund. Math.,* 10:96–115.
733. Meyer, W. (1972). Five-coloring planar maps. *J. Combin. Theory Ser. B,* 13:72–82.
734. Meyer, W. (1973). Equitable coloring. *Amer. Math. Monthly,* 80:920–922.
735. Mihók, P. (1981). On the structure of the point arboricity critical graphs. *Math. Slovaca,* 31:101–106.
736. Mihók, P. (1992). An extension of Brooks' theorem. *Ann. Discrete Math.,* 51:235–236.
737. Mihók, P. and Schiermeyer, I. (2004). Cycle lengths and chromatic number of graphs. *Discrete Math.,* 286:147–149.
738. Mihók, P. and Škrekovsky, R. (2001). Gallai's inequality for critical graphs of reducible hereditary properties. *Discuss. Math., Graph Theory,* 21:167–177.
739. Miller, E.W. (1937). On a property of families of sets. *Comptes Rendus Varsovie,* 30:31–38.
740. Miller, Z. and Müller, H. (1981). Chromatic numbers of hypergraphs and coverings of graphs. *J. Graph Theory,* 5:299–305.
741. Minty, G.J. (1962). A theorem on n-colouring the points of a linear graph. *Amer. Math. Monthly,* 69:623–624.
742. Mirsky, L. (1971). A dual of Dilworth's decomposition theorem. *Amer. Math. Monthly,* 78:876–877.
743. Mirzakhani, M. (1996). A small non-4-choosable planar graph. *Bull. Inst. Combin. Appl.,* 17:15–18.
744. Mitchem, J. (1977). An extension of Brooks' theorem to n-degenerate graphs. *Discrete Math.,* 17:291–298.
745. Mitchem, J. (1978). A new proof of a theorem of Dirac on the number of edges in critical graphs. *J. Reine Angew. Math.,* 299/300:84–91.
746. Mitchem, J. (1978). A short proof of Catlin's extension of Brooks' theorem. *Discrete Math.,* 21:213–214.

747. Mohar, B. (1999). A linear time algorithm for embedding graphs in an arbitrary surface. *SIAM J. Discrete Math.,* 12:6–26.
748. Mohar, B. (2003). Circular colorings of edge-weighted graphs. *J. Graph Theory,* 43:107–116.
749. Mohar, B. (2004). Hajós theorem for colorings of edge-weighted graphs. *Combinatorica,* 25:65–76.
750. Mohar, B. (2010). Eigenvalues and colorings of digraphs. *Linear Algebra Appl.,* 432:2273–2277.
751. Mohar, B. (2015). Colouring graphs on surfaces. In: *Topics in Chromatic Graph Theory*, pages 13–35. Cambridge University Press.
752. Mohar, B. and Thomassen, C. (2001). *Graphs on Surfaces*. The John Hopkins University Press.
753. Molloy, M. (1999). Chromatic neighborhood sets. *J. Graph Theory,* 31:303–311.
754. Molloy, M. (2019). The list chromatic number of graphs with small clique number. *J. Combin. Theory Ser. B,* 134:264–284.
755. Molloy, M. and Reed, B. (1998). Coloring graphs whose chromatic number is almost their maximum degree. In: *LATIN'98: Theoretical Informatics* (Campinas, 1998), pages 216–225. Lecture Notes in Comput. Sci., Vol. 1380, Springer.
756. Molloy, M. and Reed, B. (2001). *Graph Colouring and the Probabilistic Method*. Algorithms and Combinatorics, Vol. 23, Springer.
757. Molloy, M. and Reed, B. (2014). Coloring graphs when the number of colours is almost the maximum degree. *J. Combin. Theory Ser. B,* 109:134–195.
758. Montassier, M., Ossona de Mendez, P., Raspaud, A. and Zhu, X. (2012). Decomposing a graph into forests. *J. Combin. Theory Ser. B,* 102:38–52.
759. Moore, B. and Smith-Roberge, E. (2023). A density bound for triangle-free 4-critical graphs. *J. Graph Theory,* 103:66–111.
760. Moser, L. and Moser, W. (1961). Solution to problem 10. *Can. Math. Bull.,* 4:187–189.
761. Moser, R.A. and Tardos, G. (2010). A constructive proof of the general Lovász local lemma. *J. ACM,* 57:1–15.
762. Mozhan, N.N. (1983). Chromatic number of graphs with a density that does not exceed two-thirds of the maximal degree (in Russian). *Metody Diskret. Analiz.,* 39:52-65.
763. Mozhan, N.N. (1987). On doubly critical graphs with chromatic number five. *Metody Diskret. Analiz.,* 46:50–59.
764. Müller, V. (1975). On colorable critical and uniquely colorable critical graphs. In: *Recent Advances in Graph Theory* (Prague, 1974), pages 385–386. Academia Praha.
765. Müller, V. (1979). On colorings of graphs without short cycles. *Discrete Math.,* 26:165–176.
766. Mycielski, J. (1955). Sur le coloriage des graphes. *Colloq. Math.,* 3:161–162.
767. Mynhardt, C.M. and Broere, I. (1985). Generalized colorings of graphs. In: *Graph Theory with Applications to Algorithms and Computer Science* (Kalamazoo, Michigan, 1984), pages 583–594. John Wiley & Sons.
768. Nash-Williams, C.St.J.A. (1961). Edge-disjoint spanning trees of finite graphs. *J. London Math. Soc.,* 36:445–450.
769. Nash-Williams, C.St.J.A. (1964). Decomposition of finite graphs into forests. *J. Combin. Theory Ser. B,* 39:12.
770. Năstase, E., Rödl, V. and Siggers, M. (2010). Note on robust critical graphs with large odd girth. *Discrete Math.,* 310:499–504.
771. Nešetřil, J. (1966). K-chromatic graphs without cycles of length ≤ 7 (in Russian). *Comment. Math. Univ. Carol.,* 7:373–376.
772. Nešetřil, J. (1999). The homomorphism structure of classes of graphs. *Combin. Probab. Comput.,* 8:177–184.
773. Nešetřil, J. (2013). A combinatorial classic - sparse graphs with high chromatic number. In: *Erdős Centennial,* pages 383–407. Bolyai Soc. Math. Stud., Vol 25, Springer.
774. Nešetřil, J. and Nigussie, Y. (2012). Finite dualities and map-critical graphs on a fixed surface. *J. Combin. Theory Ser. B,* 102:131–152.

775. Nešetřil, J. and Rödl, V. (1979). A short proof of the existence of highly chromatic hypergraphs without short cycles. *J. Combin. Theory Ser. B,* 27:225–227.
776. Nešetřil, J. and Rödl, V. (1985). Three remarks on dimensions of graphs. *Ann. Discrete Math.,* 28:199–207.
777. Nešetřil, J. and Rödl, V. (1989). Chromatically optimal rigid graphs. *J. Combin. Theory Ser. B,* 46:133–141.
778. Nešetřil, J. and Zhu, X. (1996). On bounded treewidth duality of graphs. *J. Graph Theory,* 23:151–162.
779. Neumann, J. von and Morgenstern, O. (1944). *Theory of Games and Economic Behavior*. Princeton University Press.
780. Neumann-Lara, V. (1982). The dichromatic number of a digraph. *J. Combin. Theory Ser. B,* 33:265–270.
781. Neumann-Lara, V. (1985). Vertex colourings in digraphs - some problems. Technical Report, University of Waterloo.
782. Neumann-Lara, V. (1994). The 3- and 4-chromatic tournaments of minimum order. *Discrete Math.,* 135:233–243.
783. Ngoc, N. Van and Tuza, Zs. (1995). 4-chromatic graphs with large odd girth. *Discrete Math.,* 138:387–392.
784. Nielsen, F. and Toft, B. (1975). On a class of planar 4-chromatic graphs due to T. Gallai. In: *Recent Advances in Graph Theory* (Prague, June 1974), pages 425–430. Academia Praha
785. Nigussie, Y. (2009). Extended Gallai's theorem. *Electron. Notes Discrete Math.,* 34:399–403.
786. Nikoforov, V. (2010). Chromatic number and minimum degree of K_r-free graphs. Manuscript arXive:1001.2070
787. Nilli, A. (1999). Short odd cycles in 4-chromatic graphs. *J. Graph Theory,* 31:145–147.
788. Nilli, A. (2000). Triangle-free graphs with large chromatic numbers. *Discrete Math.,* 211:261–262.
789. Noel, J.A., Reed, B.A. and Wu, H. (2014). A proof of a conjecture of Ohba. *J. Graph Theory,* 79:86–102.
790. Nordhaus, E.A. and Gaddum, J.W. (1956). On complementary graphs. *Amer. Math. Monthly,* 63:175–177.
791. Oberkampf, H. and Schacht, M. (2020). On the structure of dense graphs with bounded clique number. *Combin. Probab. Comput.,* 29:641–649.
792. Ohba, K. (2002). On chromatic-choosable graphs. *J. Graph Theory,* 40:130–135.
793. Ore, O. (1960). A note on hamiltonian circuits. *Amer. Math. Monthly,* 67:55.
794. Ore, O. (1967). *The Four-Color Problem.* Academic Press.
795. Östergård, P.R.J. (2014). On the minimum size of 4-uniform hypergraphs without property B. *Discrete Appl. Math.,* 163:199–204.
796. Panconesi, A. and Srinivasan, A. (1995). The local nature of Δ-coloring and its algorithmic applications. *Combinatorica,* 15:255–280.
797. Panconesi, A. and Srinivasan, A. (1997). Randomized distributed edge coloring via an extension of the Chernoff–Hoeffding bounds. *SIAM J. Comput.,* 26:350–368.
798. Parsons, T.D., Pica, G., Pisanski, T. and Ventre, A.G.S. (1987). Orientably simple graphs. *Math. Slovaca,* 37:391–394.
799. Pawlik, A., Kozik, J., Krawczyk, T., Lasoń, M., Micek, P., Trotter, W.T. and Walczak, B. (2014). Triangle-free intersection graphs of line segments with large chromatic number. *J. Combin. Theory Ser. B,* 105:6–10.
800. Payan, C. (1992). On the chromatic number of cube-like graphs. *Discrete Math.,* 103:271–277.
801. Pedersen, A.S., Plummer, M. and Toft, B. (2016). Inflations of anti-cycles and Hadwiger's conjecture. J. Combin.. 7:413–421.
802. Pedgen, W. (2017). Critical graphs without triangles: An optimum density construction. *Combinatorica,* 33:495–512.
803. Petersen, J. (1891). Die Theorie der regulären graphs. *Acta Math.,* 15:193–220.

804. Peterson, D. and Woodall, D.R. (1999). Edge-choosability in line perfect multigraphs. *Discrete Math.,* 202:191–199.
805. Pettie, S. and Su, H.-H. (2015). Distributed coloring algorithms for triangle-free graphs. *Inform. Comput.,* 243:263–280.
806. Picassari-Arrieta, L. and Stiebitz, M. (2023). Minimum number of arcs in k-critical digraphs with order at most $2k - 1$. Manuscript arXive:2310.03584
807. Pirnazar, A. and Ullman, D.H. (2002). Girth and fractional chromatic number of planar graphs. *J. Graph Theory,* 39:201–217.
808. Pitassi, T. and Urquhart, A. (1995). The complexity of the Hajós calculus. *SIAM J. Discrete Math.,* 8:464–483.
809. Plantholt, M.J. and Tipnis, S.K. (1999). On the list chromatic index of nearly bipartite multigraphs. *Australas. J. Combin.,* 19:157–170.
810. Plesnevič, G.S. and Vizing, V.G. (1965). On the problem of the minimal coloring of the vertices of a graph (in Russian). *Sibirsk. Mat. Zh.,* 6:234–236.
811. Pluhár, A. (2009). Greedy colorings of uniform hypergraphs. *Random Struct. Alg.,* 35:216–221.
812. Plummer, M. and Zha, X. (2014). On a conjecture concerning the Petersen graph: part II. *Electron. J. Combin.,* 21:#P1.34.
813. Plummer, M., Stiebitz, M. and Toft, B. (2003). On a special case of Hadwiger's conjecture. *Discuss. Math., Graph Theory,* 23:333-363.
814. Poljak, S. and Roberts, F.S. (2009). An application of Stahl's conjecture about the k-tuple chromatic numbers of Kneser graphs. In: *The Mathematics of Preference, Choice and Order,* pages 345–352. Studies in Choice and Welfare, Springer.
815. Poljak, S. and Rödl, V. (1981). On the arc-chromatic number of a digraph. *J. Combin. Theory Ser. B,* 31:190–198.
816. Pólya, G. and Szegő, G. (1925). *Aufgaben und Lehrsätze aus der Analysis*, Springer.
817. Ponstein, J. (1969). A new proof of Brooks's chromatic number theorem for graphs. *J. Combin. Theory,* 7:255–257.
818. Postel, J. von, Schweser, T. and Stiebitz, M. (2022). Point partition numbers: decomposable and indecomposable critical graphs. *Discrete Math.,* 345:112903.
819. Postle, L. (2012). *5-List-Coloring Graphs on Surfaces.* Ph.D. Thesis, Georgia Institute of Technology.
820. Postle, L. (2017). On the minimum number of edges in triangle-free 5-critical graphs. *European J. Combin.,* 66:264–280.
821. Postle, L. (2020). Further progress towards Hadwiger's conjecture. Manuscript arXive:2006.11798
822. Postle, L. (2021). 3-List-coloring graphs of girth at least five on surfaces. *J. Combin. Theory Ser. B,* 147:1–36.
823. Postle, L. and Smith-Roberge, E. (2022). On the density of C_7-critical graphs. *Combinatorica,* 42:253–300.
824. Postle, L. and Thomas, R. (2018). Hyperbolic families and coloring graphs on surfaces. *Trans. Amer. Math. Soc., Ser. B,* 5:167–221.
825. Potapov, V.N. (2022). DP-colorings of uniform hypergraphs and splitting boolean hypercubes into faces. *Electron. J. Combin.,* 29:#P3.37.
826. Pretzel, O.R.L. and Youngs, D. (1990). Cycle lengths and graph orientations. *SIAM J. Discrete Math.,* 3:544–553.
827. Pretzel, O.R.L. and Youngs, D. (1991). Balanced graphs and noncovering graphs. *Discrete Math.,* 88:279–287.
828. Prowse, A. and Woodall, D.R. (2003). Choosability of powers of circuits. *Graphs Combin.,* 19:137–144.
829. Przybylo, J. (2024). On triangle-free list assignments. *Discrete Math.,* 347:113779.
830. Puš, V. (1988). Chromatic number of products of graphs. *Comment. Math. Univ. Carolin.,* 29:457–463.
831. Rabern, L. (2006). On graph associations. *SIAM J. Discrete Math.,* 20:529–535.

832. Rabern, L. (2008). A note on Reed's conjecture. *SIAM J. Discrete Math.,* 22:820–827.
833. Rabern, L. (2011). The Borodin–Kostchka conjecture for graphs with a doubly critical edge. *Electron. J. Combin.,* 14:#N22.
834. Rabern, L. (2011). On hitting all maximum cliques with an independent set. *J. Graph Theory,* 66:32–37.
835. Rabern, L. (2011). A strengthening of Brooks' theorem for line graphs. *Electron. J. Combin.,* 18:#P145.
836. Rabern, L. (2012). Δ-Critical graphs with small high vertex cliques. *J. Combin. Theory Ser. B,* 102:126–130.
837. Rabern, L. (2013). Destroying noncomplete regular components in graph partitions. *J. Graph Theory,* 72:123–127.
838. Rabern, L. (2013). Partitioning and coloring graphs with degree constraints. *Discrete Math.,* 313:1028–1034.
839. Rabern, L. (2014). A different short proof of Brooks' theorem. *Discuss. Math., Graph Theory,* 34:633–634.
840. Rabern, L. (2016). A better lower bound on average degree of 4-list-critical graphs. *Electron. J. Combin.,* 23:#P3.37.
841. Rabern, L. (2016). A better lower bound on average degree of online k-list-critical graphs. *Electron. J. Combin.,* 25:#P1.51.
842. Rabern, L. (2023). Yet another proof of Brooks' theorem. *Discrete Math.,* 346:113261.
843. Rackham, T. (2009). A note on $K^-_{\Delta+1}$-free precolouring with Δ colours. *Electron. J. Combin.,* 16:#N28.
844. Radhakrishnan, J. and Srinivasan, A. (2000). Improved bounds and algorithms for hypergraph 2-coloring. *Random Struct. Alg.,* 16:4–32.
845. Raigorodskii, A.M. and Shabanov, D. (2011). The Erdős–Hajnal problem of hypergraph colouring, its generalization, and related problems (in Russian). *Russ. Math. Surv.,* 66:933–1002.
846. Ramirez-Alfonsin, J. and Reed, B. (eds.) (2001). *Perfect Graphs*. John Wiley & Sons.
847. Ramsey, F.P. (1930). On a problem of formal logic. *Proc. London Math. Soc.,* 30:264–286.
848. Randerath, B. (1998). *The Vizing Bound for the Chromatic Number Based on Forbidden Pairs*. Dissertation, RWTH Aachen, Shaker Verlag.
849. Randerath, B. and Schiermeyer, I. (2004). 3-Colorability $\in$ P for P_6-free graphs. *Discrete Appl. Math.,* 136:299–313.
850. Randerath, B. and Schiermeyer, I. (2004). Vertex colouring and forbidden subgraphs – a survey. *Graphs Combin.,* 20:1–40.
851. Rao, M. and Wang, T. (2021). DP-3-coloring of planar graphs without certain cycles. *Discrete Appl. Math.,* 297:35–45.
852. Read, R.C. (1957). Maximal circuits in critical graphs. *J. London Math. Soc.,* 32:456–466.
853. Rédei, L. (1934). Ein kombinatorischer Satz. *Acta. Litt. Sci. Szeged,* 7:39–43.
854. Reed, B. (1998). ω, Δ, and χ. *J. Graph Theory,* 27:177–212.
855. Reed, B. (1999). A strengthening of Brooks' theorem. *J. Combin. Theory Ser. B,* 76:136–149.
856. Reed, B. (2001). A gentle introduction to semi-definite programming. In *Perfect graphs*, Chapter 11, John Wiley & Sons
857. Reed, B. and Seymour, P. (1998). Fractional colouring and Hadwiger's conjecture. *J. Combin. Theory Ser. B,* 74:147–152.
858. Reed, B. and Seymour, P. (2004). Hadwiger's conjecture for line graphs. *European J. Combin.,* 25:873–876.
859. Reed, B. and Sudakov, B. (2002). Asymptotically the list coloring constants are 1. *J. Combin. Theory Ser. B,* 86:27–37.
860. Riasat, A. and Schauz, U. (2012). Critically paintable, choosable or colorable graphs. *Discrete Math.,* 312:3373-3383.
861. Richardson, M. (1953). Solutions of irreflexive relations. *Ann. of Math.,* 58:573–590.
862. Richter, R.B., Thomassen, C. and Younger, D.H. (2016). Group-colouring, group-connectivity, claw-decompositions, and orientations in 5-edge-connected planar graphs. *J. Combin,* 7:219–232.

863. Ringel, G. (1954). Bestimmung der Maximalzahl der Nachbargebiete auf nichtorientierbaren Flächen. *Math. Ann.,* 127:181–214.
864. Ringel, G. (1959). *Färbungsprobleme auf Flächen und Graphen*. VEB Deutscher Verlag der Wissenschaften.
865. Ringel, G. (1974). *Map Color Theorem*. Springer.
866. Ringel, G. and Youngs, J.W.T. (1968). Solution of the Heawood map-coloring problem. *Proc. Nat. Acad. Sci. U.S.A.,* 60:438–445.
867. Ritter, E. (1892). Die eindeutigen automorphen Formen vom Geschlecht Null, eine Revision und Erweiterung der Poincaréschen Sätze. *Math. Ann.,* 41:1-82.
868. Robertson, N., Sanders, D.P., Seymour, P. and Thomas, R. (1997). The four-colour theorem. *J. Combin. Theory Ser. B,* 70:2–44.
869. Robertson, N. and Seymour, P. (1983). Graph minors I. Excluding a forest. *J. Combin. Theory Ser. B,* 35:39–61.
870. Robertson, N. and Seymour, P. (2004). Graph minors XX. Wagner's conjecture. *J. Combin. Theory Ser. B,* 92:325–357.
871. Robertson, N., Seymour, P. and Thomas, R. (1993). Hadwiger's conjecture for K_6-free graphs. *Combinatorica,* 13:279–361.
872. Rödl, V. (1977). On the chromatic number of subgraphs of a given graph. *Proc. Amer. Math. Soc.,* 64:370–371.
873. Rödl, V. (1985). On a packing and covering problem. *European J. Combin.,* 6:69–78.
874. Rödl, V. and Siggers, M. (2006). Color critical hypergraphs with many edges. *J. Graph Theory,* 53:56–74.
875. Rödl, V. and Tuza, Zs. (1985). On colour critical graphs. *J. Combin. Theory Ser. B,* 38:204–213.
876. Rolek, M., Song, Z.-X. and Thomas, R. (2023). Properties of 8-contraction-critical graphs with no K_7 minor. *European J. Combin.,* 110:103711.
877. Rosenfeld, M. (2020). Another approach to non-repetitive colorings of graphs of bounded degree. *Electron. J. Combin.,* 27:#P3.43.
878. Roussel, F., Rusu, I. and Thuillier, H. (2009). The strong perfect graph conjecture: 40 years of attempts, and its resolution. *Discrete Math.,* 309:6092–6113.
879. Roy, B. (1967). Nombre chromatique et plus longs chemins d'un graphe. *ESAIM Math. Model. Numer. Anal.,* 1:129–132.
880. Saaty, T.L. and Kainen, P.C. (1977). *The Four-Color Problem*. McGraw-Hill.
881. Sabidussi, G. (1961). Graph derivatives. *Math. Z.,* 76:385–401.
882. Sachs, H. (1969). Finite graphs (investigations and generalizations concerning the construction of finite graphs having given chromatic number and no triangles). In: *Recent Progress in Combinatorics* (Proc. Third Waterloo Conf. on Combinatorics, 1968), pages 175–184. Academic Press.
883. Sachs, H. (1978). A three-colour conjecture of Grötzsch. In: *Problémes Combinatoires et Theorie des Graphes,* page 471. Editions du Centre National de la Recherche Scientifique.
884. Sachs, H. (1993). Elementary proof of the cycle-plus-triangles theorem. In: *Combinatorics, Paul Erdős is Eighty,* (Kesthely (Hungary), 1993), Vol. 1, pages 347–359. Bolyai Soc. Math. Stud., Vol. 2, János Bolyai Math. Soc.
885. Sachs, H. and Stiebitz, M. (1983). Construction of colour-critical graphs with given major-vertex subgraph. In: *Combinatorial Mathematics* (Marseille-Luminy, 1981), pages 581–598. Ann. Discrete Math., Vol. 17, North-Holland.
886. Sachs, H. and Stiebitz, M. (1988). Colour-critical graphs with vertices of low valency. In: *Graph Theory in Memory of G. A. Dirac*, pages 371–396. Ann. Discrete Math., Vol. 41, North-Holland.
887. Sachs, H. and Stiebitz, M. (1989). On constructive methods in the theory of colour-critical graphs. *Discrete Math.,* 74:201–226.
888. Sajith, G. and Saxena, S. (2022). On Brooks' theorem. Manuscript: arXive:2208.02186
889. Sanders, D.P. and Zhao, Y. (1995). A note on the three color problem. *Graphs Combin.,* 11:91–94.

890. Sauer, N. (1993). Problem 18 in "Open Problems"; International Colloquium on Combinatorics (Paul Erdős is Eighty), Keszthely (Hungary), July, 19–24, 1993. Unpublished.
891. Sauer, N. (2001). Hedetniemi's conjecture – a survey. *Discrete Math.,* 229:261–292.
892. Scarf, H.E. (1967). The core of an n person game. *Econometrica,* 35:50–69.
893. Schäuble, M. (1969). Bemerkungen zur Konstruktion dreikreisfreier k-chromatischer Graphen. *Wiss. Z. TH Ilmenau,* 15 (Heft 2) :59–63.
894. Schauz, U. (2009). Mr. Paint and Mrs. Correct. *Electron. J. Combin.,* 16:#R77.
895. Schauz, U. (2010). Flexible color lists in Alon and Tarsi's theorem, and time scheduling with unreliable participants. *Electron. J. Combin.,* 17:#R13.
896. Schauz, U. (2014). Proof of the list edge coloring conjecture for complete graphs of prime degree. *Electron. J. Combin.,* 21:#P3.43.
897. Scheide, D. (2010). Graph edge colouring: Tashkinov trees and Goldberg's conjecture. *J. Combin. Theory Ser. B,* 100:68–96.
898. Scheinerman, E.R. and Ullman, D.H. (1997). *Fractional Graph Theory, A Rational Approach to the Theory of Graphs*. Wiley-Interscience Series in Discrete Mathematics and Optimization, John Wiley & Sons.
899. Schiermeyer, I. (2016). Chromatic number of P_5-free graphs: Reed's conjecture. *Discrete Math.,* 339:1940–1943.
900. Schmerl, J.H. (1982). The efective version of Brooks' theorem. *Canad. J. Math.,* 34:1036–1046.
901. Schmerl, J.H. (1995). The list-chromatic number of Euclidean space. *Geombinatorics,* 5:65–68.
902. Schrijver, A. (1978). Vertex-critical subgraphs of Kneser-graphs. *Nieuw Arch. Wiskd.,* 26:454–461.
903. Schrijver, A. (2003). *Combinatorial Optimization: Polyhedra and Efficiency*. Algorithms and Combinatorics, Vol. 24, Springer.
904. Schrijver, A. (2012). On the history of the shortest path problem. *Documenta Math.,* 17:155–167.
905. Schweser, T. (2019). DP-degree colorable hypergraphs. *Theoret. Comput. Sci.,* 796:196–206.
906. Schweser, T. (2020). *Colorings of Graphs, Digraphs, and Hypergraphs.* Dissertation, TU Ilmenau (Germany).
907. Schweser, T. (2021). Generalized hypergraph coloring. *Discuss. Math., Graph Theory,* 41:103–121.
908. Schweser, T. and Stiebitz, M. (2017). Degree choosable signed graphs. *Discrete Math.,* 340:882–891.
909. Schweser, T. and Stiebitz, M. (2021). Partition of hypergraphs under variable degeneracy constraints. *J. Graph Theory,* 96:7–33.
910. Schweser, T. and Stiebitz, M. (2021). Vertex partition of hypergraphs and maximum degenerate subhypergraphs. *Electr. J. Graph Theory Appl.,* 9:1–9.
911. Schweser, T., Stiebitz, M., and Toft, B. (2022). Coloring hypergraphs of low connectivity. *J. Combin.,* 13:1–21.
912. Scott, A. (1997). Induced trees in graphs of large chromatic number. *J. Graph Theory,* 24:297–311.
913. Scott, A. and Seymour, P. (2016). Induced subgraphs of graphs with large chromatic number. I. Odd holes. *J. Combin. Theory Ser. B,* 121:68–84.
914. Scott, A. and Seymour, P. (2017). Induced subgraphs of graphs with large chromatic number. IX. Rainbow path. *Electron. J. Combin.,* 24:#P2.53.
915. Scott, A. and Seymour, P. (2018). Induced subgraphs of graphs with large chromatic number. IV. Consecutive holes. *J. Combin. Theory Ser. B,* 132:180–235.
916. Scott, A. and Seymour, P. (2019). Induced subgraphs of graphs with large chromatic number. X. Holes of specific residue. *Combinatorica,* 39:1105–1132.
917. Scott, A. and Seymour, P. (2020). Induced subgraphs of graphs with large chromatic number. VI. Banana trees. *J. Combin. Theory Ser. B,* 145:487–510.

918. Scott, A. and Seymour, P. (2020). Induced subgraphs of graphs with large chromatic number. VII. Gyárfás' complementation conjecture. *J. Combin. Theory Ser. B,* 142:43–55.
919. Scott, A. and Seymour, P. (2020). A survey of χ-boundedness. *J. Graph Theory,* 95:473–504.
920. Scott, A., Seymour, P. and Spirkl, S. (2022). Polynomial bounds for chromatic number. III. Excluding a double star. *J. Graph Theory,* 101:323–340.
921. Sebő, A. (2009). Path partitions, cycle covers and integer decomposition. In: *Graph Theory, Computational Intelligence and Thought,* pages 183–199. Lecture Notes in Comput. Sci., Vol. 5420, Springer.
922. Sebő, A. (2019). Color-critical graphs and hereditary hypergraphs. Manuscript arXive: 1910.11302
923. Seidman, S.B. (1983). Network structure and minimum degree. *Social Networks,* 5:269–287.
924. Seinsche, D. (1974). On a property of the class of n-colorable graphs. *J. Combin. Theory Ser. B,* 16:191–193.
925. Seymour, P. (1974). On the two-colouring of hypergraphs. *Quart. J. Math. Oxford,* 25:303–312.
926. Seymour, P. (1974). A note on a combinatorial problem of Erdős and Hajnal. *J. London Math. Soc.,* 8:681–682.
927. Seymour, P. (1979). Some unsolved problems on one-factorizations of graphs. In: *Graph Theory and Related Topics* (Waterloo, 1977), pages 367–368. Academic Press.
928. Seymour, P. (1979). On multi-colourings of cubic graphs, and conjectures of Fulkerson and Tutte. *Proc. London Math. Soc.,* 38:423–460.
929. Seymour, P. (1981). Nowhere zero 6-flows. *J. Combin. Theory Ser. B,* 30:130–135.
930. Seymour, P. (1995). Nowhere zero flows. In: *Handbook of Combinatorics,* Vol. 1, pages 290–299. Elsevier.
931. Seymour, P. (2016). Hadwiger's conjecture. In: *Open Problems in Mathematics*, pages 417–437. Springer.
932. Shabanov, D.A. (2007). On some extremal properties of hypergraph colorings. *Electron. Notes Discrete Math.,* 29:97–100.
933. Shabanov, D.A. (2012). On r-chromatic hypergraphs. *Discrete Math.,* 312:441–458.
934. Shannon, C.E. (1949). A theorem on coloring the lines of a network. *J. Math. Phys.,* 28:148–152.
935. Shapira, A. and Thomas, R. (2011). Color-critical graphs have logarithmic circumference. *Adv. Math.,* 227:2309–2326.
936. Shearer, J.B. (1983). A note on the independence number of triangle-free graphs. *Discrete Math.,* 46:83–87.
937. Shearer, J.B. (1995). A note on the independence number of sparse graphs. *Random Struct. Alg.,* 7:269–271.
938. Shitov, Y. (2019). Counterexamples to Hedetniemi's conjecture. *Ann. of Math.,* 190:663–667.
939. Simonovits, M. (1972). On colour-critical graphs. *Studia Sci. Math. Hungar.,* 7:67–81.
940. Simons, G., Tardif, C. and Wehlau, D. (2017). Generalised Mycielski graphs, signature systems, and bounds on chromatic numbers. *J. Combin. Theory Ser. B,* 122:776–793.
941. Simonyi, G. and Tardos, G. (2006). Local chromatic number, Ky Fan's theorem, and circular colorings. *Combinatorica,* 26:587–626.
942. Simonyi, G. and Tardos, G. (2007). Colorful subgraphs in Kneser-like graphs. *European J. Combin.,* 28:2188–2200.
943. Simonyi, G. and Tardos, G. (2011). On directed local chromatic number, shift graphs, and Borsuk-like graphs. *J. Graph Theory,* 66:65–82.
944. Simonyi, G., Tardos, G. and Vrećica, S. (2009). Local chromatic number and distinguishing the strength of topological obstruction. *Trans. Amer. Math. Soc.,* 361:889–908.
945. Sivaraman, V. (2015). A unified proof of Brooks' theorem and Catlin's theorem. *Discrete Math.,* 338:272–273.
946. Škrekovsky, R. (2002). A theorem on map colorings. *Bull. Inst. Combin. Appl.,* 35:53–60.

947. Škrekovsky, R.(2002). On the critical point-arboricity graphs. *J. Graph Theory,* 39:50–61.
948. Skulrattanakulchai, S. (2006). Δ-List vertex coloring in linear time. *Inform. Process. Lett.,* 98:101–106.
949. Slivnik, T. (1996). Short proof of Galvin's theorem on the list-chromatic index of a bipartite multigraph. *Combin. Probab. Comput.,* 5:91–94.
950. Soifer, A. (2009). *The Mathematical Coloring Book.* Springer
951. Song, Z.-X. (2019). Erdős-Lovász Tihany conjecture for graphs with forbidden holes. *Discrete Math.,* 342:2632–2635.
952. Song, Z.-X. and Thomas, B. (2017). Hadwiger's conjecture for graphs with forbidden holes. *SIAM J. Discrete Math.,* 31:1572–1580.
953. Spanier, E. H. (1966). *Algebraic Topology*. Springer.
954. Spencer, J. (1981). Coloring n-sets red and blue. *J. Combin. Theory Ser. A,* 30:112–113.
955. Stacho, L. (2001). New upper bounds for the chromatic number of a graph. *J. Graph Theory,* 36:117–120.
956. Stahl, S. (1976). n-tuple colorings and associated graphs. *J. Combin. Theory Ser. B,* 20:185–203.
957. Stahl, S. (1998). The multichromatic numbers of some Kneser graphs. *Discrete Math.,* 185:287–291.
958. Stehlík, M. (2003). Critical graphs with connected complements. *J. Combin. Theory Ser. B,* 89:189–194.
959. Stehlík, M. (2006). Minimal connected τ-critical hypergraphs. *Graphs Combin.,* 22:421–426.
960. Stehlík, M. (2019). Critical digraphs with few vertices. Manuscr. arXive:1910.02454
961. Steinberg, R. (1993). The state of the three color problem. In: *Quo Vadis Graph Theory*, pages 211–248. Ann. Discrete Math., Vol. 55, North-Holland.
962. Steiner, R. (2020). A note on coloring digraphs of large girth. *Discrete Appl. Math.,* 287:62–64.
963. Steiner, R. (2022). Asymptotic equivalence of Hadwiger's conjecture and its odd minor-variant. *J. Combin. Theory Ser. B,* 155:45–51.
964. Steiner, R. (2022). Improved lower bound for the list chromatic number of graphs with no K_t minor. *Combin. Probab. Comput.,* 31:1070–1075.
965. Stiebitz, M. (1982). Proof of a conjecture of T. Gallai concerning connectivity properties of colour-critical graphs. *Combinatorica,* 2:315–323.
966. Stiebitz, M. (1985). *Beiträge zur Theorie der Färbungskritischen Graphen.* Habilitation, TU Ilmenau (Germany).
967. Stiebitz, M. (1987). Subgraphs of colour-critical graphs. *Combinatorica,* 7:303–312.
968. Stiebitz, M. (1987). K_5 is the only double-critical 5-chromatic graph. *Discrete Math.,* 64:91–93.
969. Stiebitz, M. (1988). On k-critical n-chromatic graphs. In: *Combinatorics* (Eger, Hungary, 1987), pages 509–514. Coll. Math. Soc. János Bolyai, Vol. 52, North-Holland.
970. Stiebitz, M. (1994). The forest plus stars colouring problem. *Discrete Math.,* 126:385–389.
971. Stiebitz, M. (1996). Decomposing graphs under degree constraints. *J. Graph Theory,* 23:321–324.
972. Stiebitz, M. (2017). A relaxed version of the Erdős-Lovász Tihany conjecture. *J. Graph Theory,* 85:278–287.
973. Stiebitz, M., Scheide, D., Toft, B. and Favrholdt, L. (2012). *Graph Edge Coloring: Vizing's Theorem and Goldberg's Conjecture*. Wiley-Interscience Series in Discrete Mathematics and Optimization, John Wiley & Sons.
974. Stiebitz, M., Storch, P. and Toft, B. (2016). Decomposable and indecomposable critical hypergraphs. *J. Combin.,* 7:423–451.
975. Stiebitz, M. and Toft, B. (1995). An abstract generalization of a map reduction theorem of Birkhoff. *J. Combin. Theory Ser. B,* 65:165–185.
976. Stiebitz, M. and Toft, B. (2015). Brooks's theorem. In: *Topics in Chromatic Graph Theory*, pages 36–55. Cambridge University Press.

977. Stiebitz, M. and Toft, B. (2018). A Brooks type theorem for the maximum local edge connectivity. *Electron. J. Combin.,* 25:#P1.50.
978. Stiebitz, M., Tuza, Zs. and Voigt, M. (2009). On list critical graphs. *Discrete Math.,* 309:4931–4941.
979. Stiebitz, M., Tuza, Zs. and Voigt, M. (2015). Orientations of graphs with prescribed weighted out-degrees. *Graphs Combin.,* 31:265–280.
980. Stiebitz, M. and Voigt, M. (2015). List-colourings. In: *Topics in Chromatic Graph Theory*, pages 114–136. Cambridge University Press.
981. Stockmeyer, L. (1973). Planar 3-colorability is polynomial complete. *ACM SIGACT News,* 5(3):19–25.
982. Su, X.Y. (1995). On complete subgraphs of color-critical graphs. *Discrete Math.,* 143:243–249.
983. Sumner, D.P. (1981). Subtrees of a graph and chromatic number. In: *The Theory and Application of Graphs*, pages 557–576. John Wiley & Sons.
984. Sylvester, J.J. (1878). On an application of the new atomic theory to the graphical representation of the invariants and covariants of binary quantics, with three appendices. *Amer. J. Math.,* 1:64–104.
985. Szekeres, G. and Wilf, H.S. (1968). An inequality for the chromatic number of a graph. *J. Combin. Theory,* 4:1–3.
986. Tait, P.G. (1878-1880). On the colouring of maps. *Proc. Roy. Soc. Edinburgh,* 10:501–503,729.
987. Tardif, C. (2001). Fractional chromatic numbers of cones over graphs. *J. Graph Theory,* 38:87–94.
988. Tardif, C. (2021). The chromatic number of the product of 14-chromatic graphs can be 13. *Combinatorica,* 1–8.
989. Tardif, C. (2023). The chromatic number of the product of 5-chromatic graphs can be 4. *Combinatorica,*
990. Tardif, C. and Zhu, X. (2002). The level of nonmultiplicativity of graphs. *Discrete Math.,* 244:461-471.
991. Tardif, C. and Zhu, X. (2019). A note on Hedetniemi's conjecture, Stahl's conjecture and the Poljak-Rödl function. *Electron. J. Combin.,* 26:#P4.32.
992. Tarjan, R. (1972). Depth-first search and linear graph algorithms. *SIAM J. Comput.,* 1:146–160.
993. Tarsi, M. (1999). The graph polynomial and the number of proper vertex coloring. *Annales de l'institut Fourier,* 49:1089–1093.
994. Tashkinov, V.A. (2000). On an algorithm for the edge coloring of multigraphs (in Russian). *Diskretn. Anal. Issled. Oper. Ser. 1,* 7:72–85.
995. Thomas, R. (1998). An update on the four-color theorem. *Notices Amer. Math. Soc.,* 45:848–859.
996. Thomas, R. and Walls, B. (2004). Three-coloring Klein bottle graphs of girth five. *J. Combin. Theory Ser. B,* 92:115–135.
997. Thomason, A. (1979). Critically partitionable graphs, I.. *J. Combin. Theory Ser. B,* 27:254–259.
998. Thomason, A. (1982). Critically partitionable graphs II.. *Discrete Math.,* 41:67–77.
999. Thomason, A. (1984). An extremal function for contractions of graphs. *Math. Proc. Cambridge Philos. Soc.,* 95:261–265.
1000. Thomassen, C. (1974). Graphs in which every path is contained in a Hamilton path. *J. Reine Angew. Math.,* 268:271–281.
1001. Thomassen, C. (1983). Graph decomposition with constraints on the connectivity and minimum degree. *J. Graph Theory,* 7:165–167.
1002. Thomassen, C. (1983). Disjoint cycles in digraphs. *Combinatorica,* 3:393–396.
1003. Thomassen, C. (1988). Paths, circuits and subdivisions. In: *Selected Topics in Graph Theory*, Vol. III, pages 97–133. Academic Press.
1004. Thomassen, C. (1989). The graph genus problem is NP-complete. *J. Algorithms,* 10:568–576.

1005. Thomassen, C. (1992). The even cycle problem for directed graphs. *J. Amer. Math. Soc.,* 5:217–229.
1006. Thomassen, C. (1993). Triangulating a surface with a prescribed graph. *J. Combin. Theory Ser. B,* 57:196–206.
1007. Thomassen, C. (1994). Five-coloring graphs on the torus. *J. Combin. Theory Ser. B,* 62:11–33.
1008. Thomassen, C. (1994). Every planar graph is 5-choosable. *J. Combin. Theory Ser. B,* 62:180–181.
1009. Thomassen, C. (1994). Grötzsch's 3-color theorem and its counterparts for the torus and the projective plane. *J. Combin. Theory Ser. B,* 62:268–279.
1010. Thomassen, C. (1995). 3-list-coloring planar graphs of girth 5. *J. Combin. Theory Ser. B,* 64:101–107.
1011. Thomassen, C. (1995). Embeddings and minors. In: *Handbook of Combinatorics*, Vol. I, pages 301–349. North-Holland.
1012. Thomassen, C. (1997). Color-critical graphs on a fixed surface. *J. Combin. Theory Ser. B,* 70:67–100.
1013. Thomassen, C. (2002). On the chromatic number of triangle-free graphs of large minimum degree. *Combinatorica,* 22:591–596.
1014. Thomassen, C. (2003). The chromatic number of a graph of girth 5 on a fixed surface. *J. Combin. Theory Ser. B,* 87:38–71.
1015. Thomassen, C. (2003). A short list color proof of Grötzsch's theorem. *J. Combin. Theory Ser. B,* 88:189–192.
1016. Thomassen, C. (2005). Some remarks on Hajós' conjecture. *J. Combin. Theory Ser. B,* 93:95–105.
1017. Thomassen, C., Wu, Y. and Zhang, C.-Q. (2016). The 3-flow conjecture, factors modulo k, and the 1-2-3-conjecture. *J. Combin. Theory Ser. B,* 121:308–325.
1018. Tietze, H. (1910). Einige Bemerkungen über das Problem des Kartenfärbens auf einseitigen Flächen. *Jahresber. Dtsch. Math.-Ver.,* 19:155–159.
1019. Toft, B. (1970). *Some Contributions to the Theory of Colour-critical Graphs*. Ph.D. Thesis, University of London. Published as No. 14 in Various Publication Series, Mathematisk Institut, Aarhus Universitet.
1020. Toft, B. (1970). On the maximal number of edges of critical k-chromatic graphs. *Studia Sci. Math. Hungar.,* 5:461–470.
1021. Toft, B. (1972). Two theorems on critical 4-chromatic graphs. *Studia Sci. Math. Hungar.,* 7:83–89.
1022. Toft, B. (1972). On separating set of edges in contraction-critical graphs. *Math. Ann.,* 196:129–147.
1023. Toft, B. (1974). Color-critical graphs and hypergraphs. *J. Combin. Theory Ser. B,* 16:145–161.
1024. Toft, B. (1974). On critical subgraphs of colour-critical graphs. *Discrete Math.,* 7:377–392.
1025. Toft, B. (1975). On colour-critical hypergraphs. In: *Infinite and Finite Sets* (Keszthely, Hungary, 1973), pages 1445–1457. Coll. Math. Soc. János Bolyai, Vol. 10, North-Holland.
1026. Toft, B. (1978). An investigation of colour-critical graphs with complements of low connectivity. In: *Advances in Graph Theory,* pages 279–287. Ann. Discrete Math., Vol. 3, North-Holland.
1027. Toft, B. (1985). Some problems and results related to subgraphs of colour critical graphs. In: *Graphen in Forschung und Unterricht: Festschrift K. Wagner*, pages 178–186., Barbara Franzbecker Verlag.
1028. Toft, B. (1990). 75 graph coloring problems. In: *Graph Colorings*, pages 9–35. Pitman Res. Notes, Vol. 218, Longman.
1029. Toft, B. (1996). A survey of Hadwiger's conjecture. *Congr. Numer.,* 115:249–283.
1030. Tsai, M.-T. and West, D.B. (2011). A new proof of 3-colorability of Eulerian triangulations. *Ars Math. Contemp.,* 4:73–77.
1031. Turan, P. (1941). Eine Extremalaufgabe aus der Graphentheorie. *Mat. Fiz. Lapok,* 48:435–452.

1032. Tutte, W.T. (1956). A contribution to the theory of chromatic polynomials. *Canad. J. Math.,* 6:80–91.
1033. Tutte, W.T. (1960). A colloquiun on graph theory. *Canad. Math. Bull.,* 3:102–103.
1034. Tutte, W.T. (1961). On the problem of decomposing a graph into n connected factors. *J. London Math. Soc.,* 36:221–230.
1035. Tutte, W.T. (1966). On the algebraic theory of graph colorings. *J. Combin. Theory,* 1:15–50.
1036. Tutte, W.T. (1979). *Selected Papers by W. T. Tutte, Volumes I and II.* (edited by McCarthy, D. and Stanton, R. G.), Charles Babbage Research Center.
1037. Tutte, W.T. (1998). *Graph Theory As I Have Known It.* Oxford University Press.
1038. Tuza, Zs. (1992). Theorem proving through depth-first test. *Acta Univ. Carol., Math. Phys.,* 33:135–141.
1039. Tuza, Zs. (1992). Graph coloring in linear time. *J. Combin. Theory Ser. B,* 55:236–243.
1040. Tuza, Zs. (1997). Graph coloring with local constraints – a survey. *Discuss. Math., Graph Theory,* 17:161–228.
1041. Tuza, Zs. (2013). Problems on cycle and colorings. *Discrete Math.,* 313:2007–2013.
1042. Tuza, Zs. and Voigt, M. (1996). On a conjecture of Erdős, Rubin and Taylor. *Tatra Mt. Math. Publ.,* 9:69–82.
1043. Tverberg, H. (1983). On Brooks' theorem and some related results. *Math. Scand.,* 52:37–40.
1044. Tverberg, H. (1984). On Schmerl's efecctive version of Brooks' theorem. *J. Combin. Theory Ser. B,* 37:27–30.
1045. Urquhart, A. (1997). The graph constructions of Hajós and Ore. *J. Graph Theory,* 26:211–215.
1046. Vince, A. (1988). Star chromatic number. *J. Graph Theory,* 12:551–559.
1047. Vishwanathan, S (2003). On 2-coloring certain k-uniform hypergraphs. *J. Combin. Theory Ser. A,* 101:168–172.
1048. Vitaver, L.M. (1962). Determination of minimal coloring of vertices of a graph by means of Boolean powers of the incidence matrix (in Russian). *Dokl. Akad. Nauk. SSSR,* 147:758–759.
1049. Vizing, V.G. (1964). On an estimate of the chromatic class of a p-graph (in Russian). *Diskret. Analiz,* 3:25–30.
1050. Vizing, V. G. (1968). Some unsolved problems in graph theory (in Russian). *Uspekhi Mat. Nauk,* 23:117–134. English translation in: *Russian Mathematical Surveys,* 23:125–141.
1051. Vizing, V.G. (1976). Coloring the vertices of a graph in prescribed colors (in Russian). *Diskret. Analiz,* 29:3–10.
1052. Voigt, M. (1993). List colourings of planar graphs. *Discrete Math.,* 120:215–219.
1053. Voigt, M. (1995). A not 3-choosable planar graph without 3-cycles. *Discrete Math.,* 146:325–328.
1054. Voigt, M. (2007). A non-3-choosable planar graph without cycles of length 4 and 5. *Discrete Math.,* 307:1013–1015.
1055. Voss, H.-J. (1977). Graphs with prescribed maximal subgraphs and critical chromatic graphs. *Comment. Math. Univ. Carolinae,* 18:129–142.
1056. Vu, V. (2002). A general upper bound on the list chromatic number of locally sparse graphs. *Combin. Probab. Comput.,* 11:103–111.
1057. Wagner, K. (1937). Über eine Eigenschaft der ebenen Komplexe. *Math. Ann.,* 114:570–590.
1058. Wagner, K. (1964). Beweis einer Abschwächung der Hadwiger-Vermutung. *Math. Ann.,* 153:139–141.
1059. Wagner, K. (1970). *Graphentheorie*. BI Hochschultaschenbücher, Band 248.
1060. Wang, Y. and Wu, R. (2023). Graphs with girth 9 and without larger odd holes are 3-colorable. Manuscritpt arXive:2307.01460
1061. Wanless, I.M. and Wood, D.R. (2022). A general framework for hypergraph coloring. *SIAM J. Discrete Math.,* 36:1663–1677.
1062. Weichsel, P. (1962). The Kronecker product of graphs. *Proc. Amer. Math. Soc.,* 13:47–52.

1063. Weinstein, J (1975). Excess in critical graphs. *J. Combin. Theory Ser. B,* 18:24–31.

1064. Wernicke, P. (1904). Über den katographischen Vierfarbensatz. *Math. Ann.,* 58:413–426.

1065. Wessel, W. (1981). *Critical Lines, Critical Graphs, and Odd Cycles.* Report, Akademie der Wissenschaften der DDR, Institut für Mathematik.

1066. West, D.B. (2001). *Introduction to Graph Theory*. Second Edition. Prentice-Hall.

1067. White, A.T. (1973/1984). *Graphs, Groups, and Surfaces.* North-Holland 1973; Mathematics Studies 8, Revised Edition 1984.

1068. Whitney, H. (1932). The coloring of graphs. *Ann. of Math.,* 33:688–718.

1069. Wilf, H.S. (1967). The eigenvalues of a graph and its chromatic number. *J. London Math. Soc.,* 42:330–332.

1070. Wilson, R. (2002). *Four Color Suffice: How the Map Problem Was Solved.* Revised Color Edition, 2014, Princeton Science Library 128.

1071. Woeginger, G.J. and Sgall, J. (2001). The complexity of coloring graphs without long induced paths. *Acta Cybernet.,* 15:107–117.

1072. Woodall, D.R. (1972). Property B and the four-color problem. In: *Combinatorics* (Proc. Oxford Combinatorial Conference, Institute of Mathematics and its Applications, Southend-on-Sea, 1972), pages 322–340.

1073. Woodall, D.R. (1999). Edge-choosability of multicircuits. *Discrete Math.,* 202:271–277.

1074. Woodall, D.R. (2001). List colourings of graphs. In: *Surveys in Combinatorics, 2001,* pages 269–301. London Math. Soc. Lecture Notes Series, Vol. 288, Cambridge University Press.

1075. Wrochna, M. (2019). On inverse powers of graphs and topological implications of Hedetniemi's conjecture. *J. Combin. Theory Ser. B,* 139:267–295.

1076. Wrochna, M. (2020). Smaller counterexamples to Hedetniemi's conjecture. Manuscript arXiv:2012.13558

1077. Wu, Y.Q., Yuan, J.J. and Zhao, Y.C. (1996). Partition a graph into two induced forests. *J. Math. Study,* 29:1–6.

1078. Yang, A. and Yuan, J. (2006). Partition the vertices of a graph into one independent set and one acyclic set. *Discrete Math.,* 306:1207–1216.

1079. Yao, T. and Zhou, G. (2017). Constructing a family of 4-critical planar graphs with high edge density. *J. Graph Theory,* 86:244–249.

1080. Younger, D.H. (1983). Integer flows. *J. Graph Theory,* 7:349–357.

1081. Younger, D.H. (2004). Dedication W. T. Tutte. *J. Combin. Theory Ser. B,* 92:193–198.

1082. Younger, D.H. (2012). Williams Thomas Tutte. *Biogr. Mems Fell. R. Soc.,* 58:283–297.

1083. Youngs, D.A. (1992). Gallai's problem on Dirac's construction. *Discrete Math.,* 101:343–350.

1084. Youngs, D. A. (1995). Minimal orientations of colour critical graphs. *Combinatorica,* 15:289–295.

1085. Youngs, D.A. (1996). 4-chromatic projective graphs. *J. Graph Theory,* 21:219–227.

1086. Youngs, J.W.T. (1963). Minimal imbeddings and the genus of a graph. *J. Math. Mech.,* 12:303–315.

1087. Yuster, R. (1997). Independent transversals in r-partite graphs. *Discrete Math.,* 176:255–261.

1088. Zajac, M. (2018). A short proof of Brooks' theorem. Manuscript arXiv:1805.11176

1089. Zaker, M. (2008). New bounds for the chromatic number of graphs. *J. Graph Theory,* 58:110–122.

1090. Zaslavsky, T. (1982). Signed graphs. *Discrete Applied Math.,* 4:47–74.

1091. Zhou, Q. (2023). Hadwiger's conjecture for some graphs with independence number two. Manuscript arXive:2305.05868

1092. Zhu, X. (1992). Star chromatic numbers and products of graphs. *J. Graph Theory,* 16:557–569.

1093. Zhu, X. (1998). A survey on Hedetniemi's conjecture. *Taiwanese J. Math.,* 2:1–24.

1094. Zhu, X. (2001). Circular chromatic number: a survey. *Discrete Math.,* 229:371–410.

1095. Zhu, X. (2001). An analogue of Hajós' theorem for the circular chromatic number. *Proc. Amer. Math. Soc.,* 129:2845–2852.

1096. Zhu, X. (2003). An analogue of Hajós' theorem for the circular chromatic number. II.. *Graphs Combin.,* 19:419–432.
1097. Zhu, X. (2006). Recent developments in circular colouring of graphs. In: *Topics in Discrete Mathematics*. Algorithms and Combinatorics, Vol 26, pages 497–550. Springer.
1098. Zhu, X. (2008). Refined activation strategy for the marking game. *J. Combin. Theory Ser. B,* 98:1–18.
1099. Zhu, X. (2009). On-line list colouring of graphs. *Electron. J. Combin.,* 16:#R127.
1100. Zhu, X.. (2011). The fractional version of Hedetniemi's conjecture is true. *European J. Combin.,* 32:1168–1175.
1101. Zhu, X. (2020). A note on the Poljak-Rödl function. *Electron. J. Combin.,* 27:#P3.2.
1102. Zhu, X. (2019). The Alon-Tarsi number of planar graphs. *J. Combin. Theory Ser. B,* 134:354–358.
1103. Zhu, X. (2021). Relatively small counterexamples to Hedetniemi's conjecture. *J. Combin. Theory Ser. B,* 146:141–150.
1104. Ziegler, G.M. (2002). Generalized Kneser coloring theorems with combinatorial proofs. *Invent. Math.,* 147:671–691.
1105. Zykov, A.A. (1949). On some properties of linear complexes (in Russian). *Mat. Sbornik N. S.,* 24:163–188. English translation in: *Amer. Math. Soc. Transl.,* (1952), 79:418–449.

Graph and Hypergraph Parameters

$\alpha(G)$	independence number of G	2, 562
$\alpha'(G)$	matching number of G	571
$\mathrm{ad}(G)$	average degree of G	569
$\chi(G)$	chromatic number of G	1, 356
$\chi(G,g)$	chromatic number of (G,g)	40
$\chi(G:\mathcal{P})$	$\mathcal{P}$-chromatic number of G	90
$\chi'(G)$	$=\chi(\mathrm{L}(G))$, chromatic index of G	37
$\chi''(G)$	$=\chi(\mathrm{T}(G))$, total chromatic number of G	39
$\chi^{=}(G)$	equitable chromatic number of G	31
$\chi_{\mathrm{DP}}(G)$	DP-chromatic number of G	16
$\chi_\ell(G)$	list chromatic number of G	10, 357
$\chi_\ell(G,g)$	list chromatic number of (G,g)	41
$\chi_\ell(G:\mathcal{P})$	$\mathcal{P}$-list chromatic number of G	90
$\chi'_\ell(G)$	$=\chi_\ell(\mathrm{L}(G))$, list chromatic index of G	38
$\chi''_\ell(G)$	$=\chi_\ell(\mathrm{T}(G))$, total list chromatic number of G	39
$\chi^*(G)$	fractional chromatic number of G	404
$\chi^\circ(G)$	circular chromatic number of G	413
$\vec{\chi}(D)$	dichromatic number of D	161
$\vec{\chi}_\ell(D)$	list dichromatic number of D	161
$\mathrm{c}(G)$	$=\max CL(G)$, circumference of G	78
$\mathrm{c_o}(G)$	$=\max CL_o(G)$, odd circumference of G	78
$\mathrm{c_e}(G)$	$=\max CL_e(G)$, even circumference of G	78
$\mathrm{col}(G)$	coloring number of G	11
$\delta(G)$	minimum degree of G	554
$\Delta(G)$	maximum degree of G	554
$\Delta^+_{\mathrm{fr}}(G)$	fractional maximum out-degree of G	123
$\delta^-(D)$	minimum in-degree of D	572

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Graph and Hypergraph Families

$BG(n,\alpha)$	Borsuk graph, also $BG'(n,\alpha)$	454
$\mathcal{BF}(n)$	$=\{BG(n,\alpha) \mid 0<\alpha<2\}$	457
$\mathcal{BF}'(n)$	$=\{BG'(n,\alpha) \mid 0<\alpha<2\}$	457
$\mathrm{Crit}(k)$	k-critical graphs	33
$\mathrm{Crit}(k,n)$	$=\{G\in \mathrm{Crit}(k) \mid \lvert G\rvert=n\}$	167
$\mathrm{Crit}_{\trianglelefteq}(\mathcal{P})$	$\trianglelefteq$-minimal graphs not in $\mathcal{P}$	568
$\mathrm{Crit}^*(k,n)$	graphs in $\mathrm{Crit}(k,n)$ with no dominating vertex	186
$\mathrm{Crit}_\ell(k)$	χ_ℓ-critical graphs with $\chi_\ell=k$	281
$\mathrm{Crit}_\ell(k,n)$	$=\{G\in \mathrm{Crit}_\ell(k) \mid \lvert G\rvert=n\}$	281
$\mathrm{Crit}_{\mathrm{DP}}(k)$	χ_{DP}-critical graphs with $\chi_{\mathrm{DP}}=k$	489
$\mathrm{Crit}_{\mathrm{DP}}(k,n)$	$=\{G\in \mathrm{Crit}_{\mathrm{DP}}(k) \mid \lvert G\rvert=n\}$	490
$\mathrm{Crit}(\mathbf{S},k)$	$=\mathrm{Crit}(k)\cap\mathcal{G}(\hookrightarrow \mathbf{S})$	487
$\mathrm{Crit}(\mathbf{S},k,g)$	$=\{G\in \mathrm{Crit}(\mathbf{S},k) \mid g(G)\ge g\}$	513
$\mathrm{Crit}_\ell(\mathbf{S},k)$	$=\mathrm{Crit}_\ell(k)\cap\mathcal{G}(\hookrightarrow \mathbf{S})$	526
$\mathrm{Crit}_\ell(\mathbf{S},k,g)$	$=\{G\in \mathrm{Crit}_\ell(\mathbf{S},k) \mid g(G)\ge g\}$	526
$\mathrm{Crit}_{\mathrm{DP}}(\mathbf{S},k)$	$=\mathrm{Crit}_{\mathrm{DP}}(k)\cap\mathcal{G}(\hookrightarrow \mathbf{S})$	503
$\mathrm{Crit}_{\mathrm{DP}}(\mathbf{S},k,g)$	$=\{G\in \mathrm{Crit}_{\mathrm{DP}}(\mathbf{S},k) \mid g(G)\ge g\}$	513
$\mathrm{Crih}(k)$	k-critical hypergraphs	362
$\mathrm{Crih}(k,n)$	k-critical hypergraphs of order n	362
$CG(n,d)$	circular graph	414
$C\mathcal{T}(n)$	$=\{G\in\mathcal{G} \mid \mathrm{coind}(\mathbf{T}_H(G))\ge n\}$	461
$\mathcal{D}$	digraph property	121
$\mathcal{DG}(k)$	subfamily of $\mathrm{Crit}(k,2k-1)$	51
$\mathcal{DG}'(k)$	subfamily of $\mathrm{Crit}(k,2k-1)$	232
$\mathcal{DG}_k$	k-degenerate graphs	99
$\mathcal{EG}(k)$	subfamily of $\mathrm{Crit}(k,2k)$	51
$\mathrm{Ext}(k,n)$	$=\{G\in \mathrm{Crit}(k,n) \mid \lvert E(G)\rvert=\mathrm{ext}(k,n)\}$	231

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Operations and Relations

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$H \div X$	$= H\langle V(H) \setminus X\rangle$ with $X \subseteq V(H)$	582
$H - F$	$= (V(H), E(G) \setminus F)$ with $F \subseteq 2^{V(H)}$	583
$H + F$	$= (V(H), E(G) \cup F)$ with $F \subseteq 2^{V(H)}$	583
$\mathrm{L}(G)$	line graph of G	37
$\mathrm{M}_r(G)$	generalized Mycielski graph of G	418
$\mathrm{M}_{r_1,r_2,\ldots,r_k}(G)$	iterated Mycielski graph of G	418
$\mathcal{P} \to \mathcal{P}'$	$\mathcal{P}'$ hom-bounded from below by $\mathcal{P}$	453
$\mathcal{P} \leftrightarrow \mathcal{P}'$	$\mathcal{P}$ hom-equivalent to $\mathcal{P}'$	453
$\mathrm{T}(G)$	total graph of G	39
tH	multigraph G with $\mu_G(v,w) = t\mu_H(v,w)$	17

Other Notation

$\mathrm{Aut}(G)$	automorphism group of G	566
$\mathfrak{B}(G)$	set of blocks of G	558
$\mathfrak{B}_v(G)$	set of blocks B of G with $v \in V(B)$	558
$\mathrm{bro}(\Delta,\omega)$	$= \max\{\chi(G) \mid \Delta(G) = \Delta$ and $\omega(G) = \omega\}$	293
$\mathcal{C}(G)$	maximum cliques of G	295
$\mathrm{CL}(G)$	cycle lengths in G	76
$\mathrm{CL}_e(G)$	even cycle lengths in G	76
$\mathrm{CL}_o(G)$	odd cycle lengths in G	76
$\mathcal{CO}(G,C)$	colorings of G with color set C	61
$\mathcal{CO}(G,k)$	colorings of G with color set $C = [1,k]$	61
$d_G(v)$	degree of v in G	554
$d_G(f)$	degree of f in G	575
$\mathrm{dist}_G(v,w)$	distance of v and w in G	558
$d_D^-(v)$	in-degree of v in D	572
$d_D^+(v)$	out-degree of v in D	572
$D[X]$	subdigraph of D induced by $X \subseteq V(D)$	572
D/F	reverse the orientation of $e \in F \subseteq E(D)$	132
$D(F)$	$= (V(D),F)$, subdigraph of D with $F \subseteq E(D)$	132
$E(G)$	edge set of G	554
$E_G(v)$	set of edges incident with vertex v in G	554
$E_G(X,Y)$	set of edges incident joining X and Y in G	554
$E_D^-(v)$	edges e of D with $v_D^-(e) = v$	571
$E_D^+(v)$	edges e of D with $v_D^+(e) = v$	571
$\varepsilon(\mathbf{S})$	Euler characteristic of $\mathbf{S}$	573
$\mathrm{eg}(\mathbf{S})$	$= 2 - \varepsilon(\mathbf{S})$, Euler genus of $\mathbf{S}$	573
$\mathrm{ext}(k,n)$	$= \min\{\lvert E(G)\rvert \mid G \in \mathrm{Crit}(k,n)\}$	231

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$\mathrm{ext}_\ell(k,n)$	$= \min\{\lvert E(G)\rvert \mid G \in \mathrm{Crit}_\ell(k,n)\}$	281
$\mathrm{ext}_{\mathrm{DP}}(k,n)$	$= \min\{\lvert E(G)\rvert \mid G \in \mathrm{Crit}_{\mathrm{DP}}(k,n)\}$	490
$\mathrm{exth}(k,n)$	$= \min\{\lvert E(H)\rvert \mid H \in \mathrm{Crih}(k,n)\}$	380
$\lvert G\rvert$	$= \lvert V(G)\rvert$, order of G	554
$G[X]$	subgraph of G induced by $X \subseteq V(G)$	555
G_D	underlying graph of a multidigraph D	121
G_H	high vertex subgraph of a critical graph G	35
G_L	low vertex subgraph of a critical graph G	35
(G,g)	weighted graph with $g : V(G) \to \mathbb{N}_0$	40
$g_k(n,c)$	Gallai-function	237
$\mathrm{Hom}(G,H)$	homomorphisms from G to H	439
$H[X]$	subhypergraph of H induced by $X \subseteq V(H)$	582
$H\langle X\rangle$	hypergraph H shrinked to $X \subseteq V(H)$	582
$H(\mathbf{S})$	Heawood number of $\mathbf{S}$	578
h_Δ	$= \max\{h \in \mathbb{N}_0 \mid (h+2)(h+1) \leq \Delta\}$	328
$\mathcal{I}(G)$	independent sets of G	335
$\mathcal{I}(G,v)$	independent sets of G containing v	404
$\mathcal{K}_p(G)$	subgraphs K of G isomorphic to K_p	188
K^G	the map graph of G w.r.t. K	441
$\mathrm{koy}(k,n)$	Kostochka–Yancey function	247
$L_k(n)$	$= \min\{\mathrm{c}(G) \mid G \in \mathrm{Crit}(k,n)\}$	206
$\mu_G(v,w)$	multiplicity of two distinct vertex v and w in G	570
$N_G(v)$	set of neighbors of v in G	554
$N_G(X)$	set of neighbors of X in G	554
$N_G^p(X)$	pth distance class of X in G	426
$N_D^-(v)$	set of in-neighbors of v in D	571
$N_D^+(v)$	set of out-neighbors of v in D	571
$N(\mathbf{S},k)$	$= \max\{\lvert G\rvert \mid G \in \mathrm{Crit}(\mathbf{S},k)\}$	487
$\mathbf{N}_h$	nonorientable surface with $h \geq 1$	573
$\mathrm{ore}(k,n)$	Ore-function	233
$\mathcal{OC}(k,\ell)$	odd (k,ℓ)-colorings of G	426
(Ω,Pr)	probability space	586
P_G, P_D	polynomials associated with G and D, respectively	128
$\partial_G(X)$	$= E_G(X, V(G) \setminus X)$ coboundary of X in G	554
$\rho(\mathbf{S})$	$= \max\{\rho(G) \mid G \in \mathcal{G}(\hookrightarrow \mathbf{S})\}$	579

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