APPENDIX A

VIZING'S TWO FUNDAMENTAL PAPERS

A.1 ON AN ESTIMATE OF THE CHROMATIC CLASS OF A p-GRAPH

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At the moment there is no practical effective algorithm for a minimal edge-coloring of a multigraph, thus it is interesting to estimate the chromatic class using more visible graph parameters. This paper deals with this problem.

A multigraph is a finite nonoriented multigraph without loops [1]. It is called a p-graph if it has at most p parallel edges. A 1-graph is just a graph. A multigraph with colored edges is said to be properly colored if the edges from the same vertex are always colored differently. The smallest number of colors needed to color the multigraph G properly is called the chromatic class of G and denoted $q(G)$.

The maximum degree in G we denote $\sigma(G)$. This is the maximum number of edges from a vertex. Of course, for every multigraph G we have $q(G) \geq \sigma(G)$. There is also a trivial upper bound $q(G) \leq 2\sigma(G) - 1$.

C. E. Shannon [2] proved that any multigraph G satisfies $q(G) \leq \lfloor \frac{3}{2}\sigma(G) \rfloor$, where the parenthesis denotes the lower integer part. It is possible for each m to construct a multigraph G with $\sigma(G) = m$ and $q(G) = \lfloor \frac{3}{2}m \rfloor$, but even so it is possible to get a better upper bound for $q(G)$ by introducing a second parameter for multigraphs.

First we describe some lemmas from Shannon [2].

Let us suppose that the multigraph G has been properly colored, and let s and t denote two different colors. An (s,t)-path is a set of edges of G forming a connected subgraph, each edge colored s or t.[1] We call an (s,t)-path maximal if it is not a proper part of another (s,t)-path. We say that an (s,t)-path is recolored if the colors s and t are interchanged on the edges of the (s,t)-path.

Lemma A.1 *A properly colored multigraph is still properly colored after recoloring a maximal (s,t)-path.*

A color s is missing at vertex x in a properly colored multigraph if no edge at x has color s.

Lemma A.2 *Let x,y and z be three different vertices in a properly colored multigraph G. Suppose that in each of x,y and z either the color s or the color t is missing. Then at least one of x, y and z is not contained in the same (s,t)-path as any of the two other vertices.*

Theorem A.3 *If m is the maximum degree in the p-graph G, then $q(G) \leq m + p$.*

Proof: We denote the colors by numbers from 1 to $m + p$. We shall show how one can properly color the edges of the p-graph G using these colors.

Let (a,b) be an uncolored edge between vertices a and b. Since the degree of any vertex is at most m there are at least p colors missing at each vertex.[2]

Let A and B respectively denote the set of colors missing at a and at b.

If $A \bigcap B \neq \emptyset$, then the edge (a,b) may be colored by a color missing at both a and b.

Suppose therefore that $A \bigcap B = \emptyset$. We associate to each colored edge between a and a neighbor x of a a color missing at x in such a way that different colored edges between a and x are associated with different missing colors at x. This is possible because there are at least p missing colors at each vertex.

Let $s_0, \beta_1, ..., \beta_{p-1}$ be colors not present at b. We may assume that the color s_0 is not associated with any edge between a and b. Since $s_0 \in B$ and $A \bigcap B = \emptyset$ it follows that s_0 does not belong to A, hence there is an edge (a, x_1) colored s_0. Clearly $x_1 \neq b$. Let s_1 be the color related to (a, x_1). If $s_1 \in A$, then the edge (a, x_1) may be recolored by color s_1, and then the edge (a,b) can get the color s_0. If $s_1 \notin A$, then there is an edge (a, x_2) from a colored s_1.

[1] Such a 2-colored (s,t)-path might be a cycle, or it might contain only one edge, or no edges.
[2] From the set of colors $1, 2, ..., m + p$.

Let $(a, x_1), (a, x_2), ..., (a, x_k)$ $(k \geq 2)$ be a sequence of different edges in the multigraph G, all from the vertex a. Let their colors be $s_0, s_1, ..., s_{k-1}$. The colors s_j $(j = 0, 1, 2, ..., k-1)$ are all different and not in A. For each $i = 1, 2, ..., k-1$ we assume that s_i is the color related to the edge (a, x_i).[3]

Let s_k be the color related to the edge (a, x_k). There are two possible cases:

1. $s_k \in A$. Then $s_k \neq s_j$ for all $j = 0, 1, ..., k-1$. We recolor the edge (a, x_k) by s_k, the edge (a, x_{k-1}) by s_{k-1}, ... , the edge (a, x_1) by s_1. This is still a proper coloring, and then the edge (a, b) may be colored by the color s_0.

2. $s_k \notin A$. Then there are the following possibilities (a), (b), and (c):

(a) $s_k = s_0$. Then $x_k \neq x_1$ since the color s_0 is present at x_1 as the edge (a, x_1) is colored s_0. Also $x_k \neq b$ since the color s_0 is not related to edges parallel to (a, b).

We chose a color $t \in A$ and get the following:

- In vertex a the color t is missing.
- In vertex b the color s_0 is missing.
- In vertex x_k the color s_0 is missing.

By Lemma A.2, at least one of the three vertices a, b, x_k is not joined to any of the other two vertices by an (s_0, t)-path. If a and b are not in the same (s_0, t)-path, then recolor the maximal (s_0, t)-path starting with the edge (a, x_1) at a. After the recoloring the color s_0 is missing at both a and b. Then (a, b) may be colored s_0. If x_k is not in the same (s_0, t)-path as a or b, then recolor the maximal (s_0, t)-path starting at x_k. Then the color $t \in A$ is missing at x_k. If we relate this color to the edge (a, x_k), we are back in Case 1 and can recolor and color the edge (a, b).

(b) $s_k = s_i$ for $1 \leq i \leq k-2$. (Note that $s_k = s_{k-1}$ is impossible because (a, x_k) is colored s_{k-1}.) Then $x_i \neq x_k$ since otherwise two parallel edges (a, x_i) and (a, x_k) are related to the same color s_i, which is impossible.

Let $t \in A$. We get the following:

- In vertex a the color t is missing.
- In vertex x_k the color s_i is missing.
- In vertex x_i the color s_i is missing.

We use again Lemma A.2. If x_k is not in the same (s_i, t)-path as a or x_i, then recolor the maximal (s_i, t)-path starting at x_k. Then $t \in A$ is missing at x_k. We relate t with the edge (a, x_k) and get back to Case 1. In the same way we get back to Case 1 if x_i is not in the same (s_i, t)-path as a or x_k.

[3]Note that we may have that $x_\ell = x_r$ for $\ell \neq r$; $1 \leq \ell, r \leq k$. But even then the edges (a, x_ℓ) and (a, x_r) are different parallel edges, because the edges (a, x_i) for $i = 1, 2, ..., k$ are all different.

So assume that a is not connected by an (s_i, t)-path to x_k or x_i. The first edge in a maximal (s_i, t)-path starting in a is (a, x_{i+1}) colored s_i. After recoloring the path the color s_i is missing at a, and s_i is related to the edge (a, x_i). (The color s_i is still missing at the vertex x_i, since a is not connected to x_i by an (s_i, t)-path.) We are again back in Case 1.

(c) $s_k \neq s_j$ for all $j = 0, 1, ..., k-1$. In this case an edge (a, x_{k+1}) is colored by the color s_k and it is different from the edges $(a, x_1), (a, x_2), ..., (a, x_k)$. We now repeat the argument for the longer sequence of edges $(a, x_1), (a, x_2), ..., (a, x_k), (a, x_{k+1})$. Since the graph is finite we get either Case 1 or Case 2 (a or b) after a finite number of steps. ∎

The theorem gives an upper bound for the chromatic class of a ρ-graph G, and in case $p < \lfloor \frac{\sigma(G)}{2} \rfloor$ it is better than Shannon's bound $q(G) \leq \lfloor \frac{3}{2}\sigma(G) \rfloor$. A natural question may be asked: Is it possible for each m and p with $p \leq \lfloor \frac{m}{2} \rfloor$ to obtain a p-graph G with $\sigma(G) = m$ and $q(G) = m + p$? In case $p = 1$ the answer is yes:

Corollary A.4 *If m is the maximum degree in the graph G, then $m \leq q(G) \leq m+1$. Moreover, for each $m \geq 2$ there is a graph G with $\sigma(G) = m$ and $q(G) = m + 1$*

Proof: We shall only prove the second part.

Let $m \geq 2$. Let H be a graph with $2m$ vertices $x_1, x_2, ..., x_m, y_1, y_2, ..., y_m$ and edges (x_i, y_j) for $i = 1, 2, ..., m$ and $j = 1, 2, ..., m$. Add a new vertex z outside H, remove the edge (x_1, y_1) from H and join z to x_1 and y_1 by edges. The new graph G has $2m + 1$ vertices, $m^2 + 1$ edges, and $\sigma(G) = m$. Because of the odd number $(2m + 1)$ of edges, at most m edges may be given the same color in a proper edge coloring of G. This means that $q(G) \geq \frac{m^2+1}{m} = m + \frac{1}{m} > m$, implying that $q(G) = m + 1$. ∎

When $p \geq 2$ our bound is not always the best possible. One can show that for each $p \geq 2$ the chromatic class of a p-graph G with $\sigma(G) = m = 2p + 1$ is at most $m + p - 1$. However, if $m = 2kp$ $(k \geq 1)$ and G has $2k + 1$ vertices for which all pairs of vertices are joined by p edges, then $q(G) = m + p$.

Hence, the best possible upper bound for the chromatic class of a p-graph G with $p \leq \lfloor \frac{\sigma(G)}{2} \rfloor$ depends on the relationship between $\sigma(G)$ and p. Perhaps I will investigate this question later.

Finally, the author would like to express heartfelt thanks to A. A. Zykov for assistance and valuable advise.

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1. Berge, C. (1962). *Graph Theory and its Applications* (in Russian). IL, Moscow.

2. Shannon, C. E. (1960). A theorem on coloring the lines of a graph (in Russian). IL, Moscow, pages 249–253.

A.2 CRITICAL GRAPHS WITH A GIVEN CHROMATIC CLASS

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In this paper we use the notation from Vizing [1] and the book by C. Berge [2]. In Vizing [1] it was proved that if $\sigma(G)$ is the maximum degree of a vertex of G then the chromatic class $q(G)$ is either $\sigma(G)$ or $\sigma(G) + 1$. Each of the two possibilities depends on the structure of the graph. We do not have criteria that can help us to determine the chromatic class using visible properties of the graph. But we can investigate two directions:

1. What can one say about the properties of a graph if it is known that its chromatic class is one bigger than the maximum degree?
2. If some structural or numerical characteristics of a graph is known, what is its chromatic class?

For the research in the first direction it is natural to introduce the definition of a critical graph.

DEFINITION OF A CRITICAL GRAPH OF DEGREE m AND ITS PROPERTIES

A graph G is called critical of degree m, where $m \geq 2$ is an integer, when:

(a) G is connected;

(b) $\sigma(G) = m$;

(c) $q(G) = m + 1$;

(d) The chromatic class of a graph obtained by removing any edge from G is equal to m.

The following Lemma is of interest because of its use in induction proofs for critical graphs.

Lemma A.5 *Let G be a graph with $\sigma(G) = m$ and $q(G) = m + 1$. Then for any k satisfying $m \geq k \geq 2$ there exists a critical graph of degree k as a subgraph of G.*

Proof: This is obvious for $k = m$. Let $m > k \geq 2$ and G be critical of degree m. Color all edges of G with m colors except an edge (a, b). We have $\delta(a) \neq \emptyset$, $\delta(b) \neq \emptyset$, and since $q(G) = m + 1$, $\delta(a) \cap \delta(b) = \emptyset$. (Here and in what follows, $\delta(x)$ denotes the set of colors missing at the vertex x.) Let the colors s_1 and s_2, respectively, belong to $\delta(a)$ and $\delta(b)$ ($s_1 \neq s_2$). Remove now from G all edges colored with $m - k$ of the colors different from s_1 and s_2. The chromatic class of the remaining graph H of G is $k+1$, therefore $\sigma(H) \geq k$. On the other hand, H does not contain any vertex of degree $k+1$ in H so $\sigma(H) = k$. This means that there is a critical subgraph of H of degree k and therefore also of G.

The Lemma has been proved. ■

In Vizing [3] the following properties of critical graphs were given:

PROPERTY I. A critical graph of degree m cannot have a separating vertex.

PROPERTY II. The sum of the degrees of two adjacent vertices in a critical graph of degree m is $\geq m+2$.

PROPERTY III. In a critical graph of degree m each vertex is adjacent to at least two vertices of degree m.

We shall now obtain a common generalization of Properties II and III. For this we define a fan-sequence of edges.

Let us have a graph with colored edges (it does not matter if all edges are colored or not). A sequence of different edges $(a, x_1), (a, x_2), \ldots, (a, x_n)$ $(n \geq 1)$, at the vertex a and colored $s_1, s_2, \ldots, s_n$, is called a fan-sequence at a, starting with (a, x_1), if the color s_2 is missing at x_1 , the color s_3 is missing at $x_2, \ldots$, the color s_n is missing at x_{n-1}.

REMARK. In a fan-sequence, all edges are incident with the same vertex; hence the colors $s_1, s_2, \ldots, s_n$ are all different.

Theorem A.6 *In a critical graph of degree m each vertex incident with a vertex of degree k is in addition also incident with $m-k+1$ vertices of degree m.*

Proof: Let G be a critical graph of degree m, $\sigma(b) = k \leq m$, where $\sigma(b)$ is the degree of the vertex b and (a, b) is an edge of G. We shall prove that a is adjacent to $m-k+1$ vertices of degree m, different from b.

We color the edges of G with m colors, except the edge (a, b). We have $|\delta(b)| = m-k+1$ and $|\delta(a)| \geq 1$. Since $q(G) = m+1$, we have $\delta(a) \cap \delta(b) = \emptyset$.

Using the same method as in [1], we can show that no fan-sequence at a, starting with an edge having a color from $\delta(b)$, can contain an edge (a, x) such that there is a missing color at x from $\delta(a)$ or there is a missing color at x being the color of an earlier edge of the fan-sequence.

We shall now prove that if $|\delta(b)| \geq 2$, then two fan-sequences at vertex a, starting with two different edges colored with colors from $\delta(b)$ are edge-disjoint. For a contradiction, assume that $(a, x_1), \ldots, (a, x_r)$ and $(a, y_1), \ldots, (a, y_\ell)$ are two fan-sequences at a, colored $s_1, \ldots, s_r$ and $s'_1, \ldots, s'_\ell$, where ($s_1 \in \delta(b)$, $s'_1 \in \delta(b)$, $s_1 \neq s'_1$), and only (a, x_r) and (a, y_ℓ) are equal. Obviously, either $r \geq 2$ or $\ell \geq 2$ (both cases are possible). Let us assume that $r \geq 2$. Then $s_r = s'_\ell$ belongs to $\delta(x_{r-1})$. Let $t \in \delta(a)$ and change the colors in the maximum (s_r, t)-chain starting the vertex x_{r-1} (see Vizing [1]). If the edge (a, x_r) does not get the color t, then the fan-sequence $(a, x_1), \ldots, (a, x_{r-1})$ contains the edge (a, x_{r-1}) with $\delta(x_{r-1}) \cap \delta(a) \neq \emptyset$. If the edge (a, x_r) gets the color t, then when $\ell = 1$ we have $s_r = s'_\ell \in \delta(b) \cap \delta(a)$; and when $\ell \geq 2$ the fan-sequence $(a, y_1), \ldots, (a, y_{\ell-1})$ contains the edge $(a, y_{\ell-1})$ with $\delta(y_{\ell-1}) \cap \delta(a) \neq \emptyset$. This contradicts $q(G) = m + 1$.

We finish the proof as follows.

For each edge with a as end-vertex and having one of the finitely many colors from $\delta(b)$, construct a maximal fan-sequence at the vertex a starting with the edge. Then if (a, x) is the last vertex of the fan, then $\delta(x) = \emptyset$ and $\sigma(x) = m$. Since fan-sequences at a starting with different edges cannot have common edges, it follows that a is adjacent to at least $|\delta(b)| = m - k + 1$ different vertices of degree m, these vertices being also different from b.

Theorem A.6 has been proved. ■

Theorem A.6 obviously generalize properties II and III of critical graphs. Consequently the theorem proves Hypothesis 2 from Vizing [3].

Theorem A.7 *A critical graph of degree m contains an elementary cycle of length $\geq m + 1$.*

Proof: We proceed by contradiction. Let G be a critical graph of degree m for which the length of any elementary cycle is no more than m.

By $\mu = [a_1, a_2, \ldots, a_n]$ we denote a longest elementary path in G. Let a_ℓ be the vertex of μ with the maximum index which is adjacent to a_1 (this means that a_1 and a_ℓ are adjacent, $a_\ell \in \mu$, but for all $j > \ell$ the vertex a_j of the path μ is not adjacent to a_1).

Similarly, let a_r be the vertex of the path μ with the maximum index which is adjacent to a_2.

Since μ is an elementary path of maximum length, all vertices of G adjacent to a_1 belong to μ. Thus if $\sigma(a_1) = m$, then $\ell \geq m + 1$ and the length of the elementary cycle $[a_1, a_2, \ldots, a_\ell, a_1]$ is larger than m. This means that $\sigma(a_1) = k < m$ ($k \geq 2$). By Theorem A.6, the vertex a_2 is adjacent to at least $m - k + 1$ vertices of degree m. If a vertex a' is adjacent to a_2 and $\sigma(a') = m$, then $a' \in \mu$, because otherwise the elementary sequence $[a', a_2, \ldots, a_n]$ would be of maximum length which is not possible since $\sigma(a') = m$. Next, if a_1 is adjacent to a_i ($2 \leq i \leq n$), then $\sigma(a_{i-1}) < m$, because the elementary path $[a_{i-1}, a_{i-2}, \ldots, a_1, a_i, a_{i+1}, \ldots, a_n]$ has maximum length.

Now let us compare ℓ and r. If $\ell \geq r$, then as mentioned above, among the vertices $a_1, a_2, \ldots, a_n$ we can find k vertices of degree $< m$ and $m - k + 1$ vertices of degree m. Consequently the length of the elementary cycle $[a_1, a_2, \ldots, a_\ell, a_1]$ is not smaller than $k + m - k + 1 = m + 1$.

Assume now that $r > \ell$. If a_1 is not adjacent to a_3, then among the vertices $a_3, a_4, \ldots, a_r$ we can find at least $m - k + 1$ vertices of degree m and $k - 1$ vertices of degree $< m$, and consequently the length of the elementary cycle $[a_2, a_3, \ldots, a_r, a_2]$ is at least $m + 1$. If on the other hand a_1 is adjacent to a_3, then among the vertices $a_1, a_2, \ldots, a_r$ we have $m - k + 1$ vertices of degree m and k vertices of degree $< m$. Then the elementary cycle $[a_2, a_1, a_3, a_4, \ldots, a_r, a_2]$ contains at least $m + 1$ vertices.

Theorem A.7 has been proved. ■

It would be of interest to obtain the best possible lower bound for the maximum length of an elementary cycle in a critical graph of degree m, taking into account also the number of vertices of the graph.

Using Theorem A.6, we shall now obtain a lower bound on the number of edges in a critical graph of degree m.

Theorem A.8 *In a critical graph of degree m the number of edges is $\geq (3m^2 + 6m - 1)/8$.*

Proof: Let G be a critical graph of degree m, and let k be the minimum degree of the vertices of G. Then G obviously contains at least $m - k + 2$ vertices of degree m. As the number of vertices of the graph is at least $m + 1$ the number of edges is at least $((m - k + 2)m + (k - 1)k)/2$.

The minimum of this expression is obtained for $k = (m + 1)/2$ (this may be proved by differentiation). If we insert $(m+1)/2$ instead of k we get that the number of edges is at least $(3m^2 + 6m - 1)/8$.

Theorem A.8 has been proved. ■

In Vizing [3] it is conjectured that the number of edges in a critical graph of degree m is $> m^2/2$ (Hypothesis I). The author has been unable to prove or disprove this conjecture. In the particular case when m is even and the number of vertices is $m+1$, the problem may be formulated as follows:

Suppose we have an odd number n of elements. Prove or disprove that every set of $(n - 1)^2/2$ unordered pairs, each consisting of two of the elements, can be divided into $n-1$ groups such that the pairs in each group have no common elements.

A METHOD FOR CLASSIFICATION OF GRAPHS

We define that a graph G belongs to the class L_k, where k is an integer ≥ 0, if every subgraph of G has a vertex of degree at most k. It follows that if $G \in L_k$, then $G \in L'_k$ for $k' \geq k$. On the other hand, a graph of maximum degree m belongs to L_m.

Theorem A.9 *If $G \in L_k$ and $\sigma(G) \geq 2k$, then $q(G) = \sigma(G)$.*

Proof: Suppose the theorem is not true and that $G \in L_k$ $(k \geq 1)$, $\sigma(G) = m \geq 2k$, but $q(G) = m + 1$.

We may assume that G is critical of degree m. Let X denote the set of all vertices of G and S the subset of all vertices of degree $\leq k$. Then since $\sigma(G) \geq 2k$ the set $X \setminus S$ is nonempty. By Theorem A.6, every vertex from $X \setminus S$ with a neighbor from S, also has at least $m - k + 1$ neighbors of degree m. Every vertex of degree m is in $X \setminus S$ and $m - k + 1 \geq k + 1 > k$. Thus the subgraph spanned by $X \setminus S$ has no vertex of degree $\leq k$. This contradicts $G \in L_k$.

Theorem A.9 has been proved. ■

Let $S(k)$ ($k \geq 0$ is an integer) denote the least natural number such that any graph $G \in L_k$ with $\sigma(G) \geq S(k)$ satisfies $q(G) = \sigma(G)$. Theorem A.9 says that $S(k) \leq 2k$. It is easy to show that $S(0) = 0, S(1) = 1, S(2) = 4$, and $S(3) = 6$. But the author conjectures that for k large enough the estimate of Theorem A.9 may be improved. It would be interesting to investigate more thoroughly the class L_k and the function $S(k)$.

Theorem A.10 *If G is planar and $\sigma(G) \geq 8$, then $q(G) = \sigma(G)$.*

Proof: Because Lemma A.5 it is enough to consider the case $\sigma(G) = 8$ to prove Theorem A.10. So let G be a planar graph with $\sigma(G) = 8$ and $q(G) = 9$. We may assume that G is critical of degree 8.

Let n_i ($0 \leq i \leq 8$) denote the number of vertices of G of degree i. Since G is critical, $n_0 = 0$ and $n_1 = 0$. Since G is planar, $\sum_{i=2}^{8} in_i \leq 6(\sum_{i=2}^{8} n_i - 2)$.[4] It follows that

$$2n_8 + n_7 \leq n_5 + 2n_4 + 3n_3 + 4n_2 - 12. \tag{A.1}$$

We denote by $n_8(i_1, i_2, i_3, i_4)$ the number of vertices of degree 8, joined to i_1 vertices of degree 2, i_2 vertices of degree 3, i_3 vertices of degree 4, and i_4 vertices of degree 5. It follows from Theorem A.6 that if $i_1 + i_2 + i_3 + i_4 > 0$, then $i_1 + i_2 + i_3 + i_4 \leq \ell$, where ℓ is the smallest natural number for which $i_\ell > 0$.

We denote by $n_7(j_1, j_2, j_3)$ the number of vertices of degree 7, joined to j_1 vertices of degree 3, j_2 vertices of degree 4, and j_3 vertices of degree 5. It follows again from Theorem A.6 that if $j_1 + j_2 + j_3 > 0$, then $j_1 + j_2 + j_3 \leq r$, where r is the smallest natural number for which $j_r > 0$. We have

$$2n_2 = n_8(1, 0, 0, 0) \tag{A.2}$$

$$\begin{aligned} 3n_3 = n_8(0, 1, 0, 0) + n_8(0, 1, 1, 0) + n_8(0, 1, 0, 1) \\ +2n_8(0, 2, 0, 0) + n_7(1, 0, 0). \end{aligned} \tag{A.3}$$

Since every vertex of degree 5 is joined to at least two vertices of degree 8, we have:

$$\begin{aligned} 2n_5 \leq n_8(0, 1, 0, 1) + n_8(0, 0, 1, 1) + 2n_8(0, 0, 1, 2) + n_8(0, 0, 2, 1) \\ +n_8(0, 0, 0, 1) + 2n_8(0, 0, 0, 2) + 3n_8(0, 0, 0, 3) + 4n_8(0, 0, 0, 4). \end{aligned} \tag{A.4}$$

By Property II, a vertex of degree 4 can only be joined to vertices of degree ≥ 6; and by Theorem A.6, any vertex of degree 4 with a neighbor of degree 6 is also joined to three vertices of degree 8. Therefore, if we denote by n_4' the number of vertices of degree 4 with a neighbor of degree 6, then

$$\begin{aligned} 3n_4' + 4(n_4 - n_4') &= n_8(0, 1, 1, 0) + n_8(0, 0, 1, 0) \\ +n_8(0, 0, 1, 1) + n_8(0, 0, 1, 2) &+ 2n_8(0, 0, 2, 0) + 2n_8(0, 0, 2, 1) \\ +3n_8(0, 0, 3, 0) + n_7(0, 1, 0) &+ n_7(0, 1, 1) + 2n_7(0, 2, 0). \end{aligned}$$

[4]Here we use the fact from Euler's polyhedron formula that the sum of the degrees of a planar graph is at most $6(n - 2)$.

Since every vertex of degree 4 is joined to at most two vertices of degree 7, we have $2(n_4 - n'_4) \geq 2n_7(0,2,0)$, hence $n_4 - n'_4 \geq n_7(0,2,0)$. Therefore

$$\begin{aligned} 3n_4 &\leq n_8(0,1,1,0) + n_8(0,0,1,0) + n_8(0,0,1,1) \\ &+n_8(0,0,1,2) + 2n_8(0,0,2,0) + 2n_8(0,0,2,1) + 3n_8(0,0,3,0) \\ &+n_7(0,1,0) + n_7(0,1,1) + n_7(0,2,0). \end{aligned} \tag{A.5}$$

From (A.2), (A.3), (A.4), and (A.3) we get

$$n_5 + 2n_4 + 3n_3 + 4n_2 \leq 2n_8 + n_7,$$

which contradicts (A.1).

Theorem A.10 has been proved. ■

The author has not solved the problem if there exists a planar graph G with $\sigma(G) = 7$ or $\sigma(G) = 6$ which has $q(G) = \sigma(G) + 1$.

For each integer m, where $2 \leq m \leq 5$, it is easy to obtain a planar graph with $\sigma(G) = m$ and $q(G) = m + 1$.

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Notes

Vadim G. Vizing was born in Kiev, Ukraine, on March 25, 1937. After the war, when he was 10, his family was forced to move to Siberia because his mother was half-German. He began his studies at the University of Tomsk in 1954 and graduated in 1959. From Tomsk he then moved to Moscow to the famous Steklow Institute to study for a Ph.D. He did not finish this program, however, and moved back to Siberia, to work at the Mathematical Institute of the Academy of Sciences in Akademgorodok outside Novosibirsk, where he stayed until 1968. This was an extremely fruitful period, where he was influenced and helped by the leading Russian graph theorist A. A. Zykov, and he obtained a Ph.D. in 1966.

In 1964 Vizing obtained his fundamental results on edge colorings of graphs, published in Vizing [297, 298, 299]. The main theorem soon became known also in the West, as Zykov [327] mentioned it (added in proof), together with other results by Novosibirsk mathematicians, including Vizing, in the proceedings from the meeting in Smolenice in Czechoslovakia in 1963, published jointly by the Czechoslovak Academy of Sciences in Prague and Academic Press in New York and London in

1964/65. Zykov [328] mentioned again the results of Vizing at the meeting in Rome in 1966. The theory was still in its beginnings, and during these years Vizing [300] considered extensions and related unsolved problems.

In 1968 Vizing moved back to Ukraine, where since 1974 he has lived in Odessa. He is now retired from a position as teacher of mathematics at the Academy for Food Technology, and he is still active in research in graph theory, see the interview by Gutin and Toft [122] from 2000.

In his papers Vizing uses the term chromatic class for the edge-chromatic number of a graph. It is unusual to denote a graph parameter by the word class. The terminology seems due to Sainte-Laguë [259, 260, 261], who used the term rank for the vertex-chromatic number and class for the edge-chromatic number. Sainte-Laguë's terminology for the edge-chromatic number was adopted by Claude Berge in his influential first book on graphs [24]. However, in the English translation of Berge's book the translator Alison Doig (now Alison Harcourt), then at the London School of Economics, changed the terminology from class to index. The term index thereafter became commonly used and is now the preferred term by most authors.

During the fruitful period 1964–1965 in Akademgorodok Vizing produced the three fundamental papers on edge coloring of which the first and last appear above in English translations (we thank the Russian Academy of Sciences in Novosibirsk for permission to publish these translations). In the first paper [297] Vizing dealt with graphs with multiple edges, and he introduced fans to obtain the inequality $\chi' \leq \Delta + \mu$. Moreover he briefly discussed the question for which values of Δ and μ equality is possible. In this connection he mentioned without proof that:

Theorem A.11 *$\chi'(G) \leq \Delta(G) + \mu(G) - 1$ for all graphs with $\mu(G) \geq 2$ and $\Delta(G) = 2\mu(G) - 1$.*

A proof of Theorem A.11 appeared in the second paper [298], which already in 1965 was translated and published in English. In that paper the main result of Vizing [297] and its proof was repeated, critical simple graphs were introduced and the first version of the adjacency lemma was obtained (that each vertex of a critical simple graph has at least two neighbors of degree Δ). Moreover, he treated König's Theorem on bipartite graphs, and Shannon's Theorem was proved together with a characterization of the graphs achieving Shannon's bound with equality. The classification of simple graphs into Class 1 and Class 2 graphs was presented, and the result that all planar simple graphs with $\Delta(G) \geq 10$ are Class 1, i.e., $\chi'(G) = \Delta(G)$, was proved by an elegant simple proof, using just that any subgraph of a planar graph has minimum degree at most 5. Also, in the second paper Vizing considered the problem of repeated recolorings of maximal connected 2-colored subgraphs to get from one edge coloring to another given coloring. And it treated complementary graphs.

In the third paper the adjacency lemma was extended, again considering critical simple graphs, to what we today know as Vizing's Adjacency Lemma. The proof again used fans, now defined in a more general way than in Vizing [297]. In addition, fundamental results about cycle lengths and the number of edges in critical graphs were obtained. In the third paper again the classification of simple graphs into Class

1 and Class 2 graphs were considered, and the result that planar simple graphs with $\Delta(G) \geq 8$ are Class 1 was proved, using a rather complicated analysis based on Euler's Theorem.

In 1968, at the end of Vizing's period in Akademgorodok he published the paper [300] containing a wealth of unsolved graph theory problems, including several about edge colorings. That paper was already the same year translated and published in English.

Theorem A.11 above was also obtained by Gupta [120], who conjectured for exactly which values of $\Delta(G)$ and $\mu(G)$ the inequality $\chi'(G) \leq \Delta(G) + \mu(G) - 1$ holds (see Gupta's Conjecture ♣ 4 in Chap. 9). Without knowing Gupta's work in detail Favrholdt, Stiebitz, and Toft [91] took the first steps in the direction of the present book. This preliminary version appeared as a technical report at the University of Southern Denmark in 2006. That same year Diego Scheide, in his Diplomarbeit at the Technical University of Ilmenau, supervised by Stiebitz, proved that Gupta's Conjecture follows from Goldberg's Conjecture, thus relating these two fundamental questions. Scheide's proof was included in Favrholdt et al. [91].

APPENDIX B

FRACTIONAL EDGE COLORINGS

Fractional graph theory, introduced by Berge [26] in 1978, deals with real-valued analogues of traditional integral graph parameters and concepts. If a graph parameter ρ can be expressed as the optimal value of an integer program, a fractional graph parameter ρ^* associated with ρ can be defined as the linear programming relaxation of this integer program.

B.1 THE FRACTIONAL CHROMATIC INDEX

In what follows, let (G, f) be an arbitrary weighted graph, that is, G is a graph and $f : V(G) \to \mathbb{N}$ a vertex function.

An **f-matching** of G is defined to be an edge set $M \subseteq E(G)$ such that each vertex $v \in V(G)$ satisfies $|M \cap E_G(v)| \leq f(v)$. The set of all f-matchings of G is denoted by $\mathcal{M}_f(G)$. Clearly, if $\varphi \in \mathcal{C}^k_f(G)$ is an f-coloring of G, then the color class $E_{\varphi,\alpha} = \{e \in E(G) \mid \varphi(e) = \alpha\}$ is an f-matching of G for every color $\alpha \in \{1, \dots, k\}$. Hence, an f-coloring of G can be viewed as a partition of $E(G)$ into f-matchings, and the **f-chromatic index** $\chi'_f(G)$ is the least possible number of classes in such a partition.

Graph Edge Coloring: Vizing's Theorem and Goldberg's Conjecture, First Edition. By M. Stiebitz, D. Scheide, B. Toft, and L. M. Favrholdt

A **fractional f-coloring** of G is a function $w : \mathcal{M}_f(G) \to [0,1]$ such that every edge $e \in E(G)$ satisfies

$$\sum_{M \in \mathcal{M}_f(G): e \in M} w(M) = 1.$$

Let $\mathcal{R}_f(G)$ denote the set of all fractional f-colorings of G. For a fractional coloring $w \in \mathcal{R}_f(G)$, we call

$$\sum_{M \in \mathcal{M}_f(G)} w(M)$$

the **value** of w. The minimum value over all fractional f-colorings of G is the **fractional f-chromatic index** of G denoted $\chi_f'^*(G)$, i.e.,

$$\chi_f'^*(G) = \min \left\{ \sum_{M \in \mathcal{M}_f(G)} w(M) \mid w \in \mathcal{R}_f(G) \right\}.$$

The minimum exists, since this is an LP-problem bounded from below. A subset of an f-matching is itself an f-matching, and the f-chromatic index $\chi'_f(G)$ is the smallest number k such that $E(G)$ can be covered by k f-matchings. The linear relaxation of this formulation leads to the set $\mathcal{R}'_f(G)$ of all functions $w : \mathcal{M}_f(G) \to \mathbb{R}^{\geq 0}$ such that every edge $e \in E(G)$ satisfies

$$\sum_{M \in \mathcal{M}_f(G): e \in M} w(M) \geq 1.$$

Theorem B.1 *Every weighted graph (G, f) satisfies*

$$\chi_f'^*(G) = \min \left\{ \sum_{M \in \mathcal{M}_f(G)} w(M) \mid w \in \mathcal{R}'_f(G) \right\}.$$

Proof: Let $\mathcal{R}_f'^o(G)$ denote the set of all functions $w \in \mathcal{R}'_f(G)$ having minimum value $\sum_{M \in \mathcal{M}_f(G)} w(M)$. Since $\mathcal{R}_f(G) \subseteq \mathcal{R}'_f(G)$, it suffices to show that $\mathcal{R}_f(G) \cap \mathcal{R}_f'^o(G) \neq \emptyset$. To this end, define a function $p : \mathcal{R}_f'^o(G) \to \mathbb{N} \cup \{0\}$ by

$$p(w) = \left| \left\{ e \in E(G) \mid \sum_{M \in \mathcal{M}_f(G): e \in M} w(M) = 1 \right\} \right|.$$

Let $w' \in \mathcal{R}_f'^o(G)$ be a function such that $p(w')$ is maximum. If $p(w') = |E(G)|$, then $w' \in \mathcal{R}_f(G)$ and we are done. So suppose that $p(w') < |E(G)|$. Then there exists an edge $e_0 \in E(G)$ such that $\sum_{M \in \mathcal{M}_f(G): e_0 \in M} w'(M) > 1$. Let $\mathcal{R}^0$ be the set of all functions $w \in \mathcal{R}_f'^o(G)$ such that $p(w) = p(w')$ and $\sum_{M \in \mathcal{M}_f(G): e_0 \in M} w(M) > 1$. Clearly, w' belongs to $\mathcal{R}^0$. Now define a function $q : \mathcal{R}^0 \to \mathbb{N} \cup \{0\}$ by

$$q(w) = |\{M \in \mathcal{M}_f(G) \mid e_0 \in M, w(M) = 0\}|.$$

To reach a contradiction, we choose a function $w \in \mathcal{R}^0$ such that $q(w)$ is maximum. Since $w \in \mathcal{R}'^o_f(G)$, there is a f-matching $M_0 \in \mathcal{M}_f(G)$ such that $e_0 \in M_0$ and $w(M_0) > 0$. We now define a new function $\tilde{w} : \mathcal{M}_f(G) \to \mathbb{R}$ by

$$\tilde{w}(M) = \begin{cases} w(M) - \epsilon & \text{if } M = M_0, \\ w(M) + \epsilon & \text{if } M = M_0 \setminus \{e_0\}, \\ w(M) & \text{otherwise,} \end{cases}$$

where $\epsilon := \min\left\{\sum_{M \in \mathcal{M}_f(G): e_0 \in M} w(M) - 1, w(M_0)\right\}$. Clearly, $\tilde{w}(M) \geq 0$ for all $M \in \mathcal{M}_f(G)$ and we have

$$\sum_{M \in \mathcal{M}_f(G)} \tilde{w}(M) = \sum_{M \in \mathcal{M}_f(G)} w(M).$$

Furthermore, every edge $e \neq e_0$ of G satisfies

$$\sum_{M \in \mathcal{M}_f(G): e \in M} w(M) = \sum_{M \in \mathcal{M}_f(G): e \in M} \tilde{w}(M) \geq 1$$

and the edge e_0 satisfies

$$\sum_{M \in \mathcal{M}_f(G): e_0 \in M} \tilde{w}(M) = \sum_{M \in \mathcal{M}_f(G): e_0 \in M} w(M) - \epsilon \geq 1.$$

Consequently, $\tilde{w}$ belongs to the set $\mathcal{R}'^o_f(G)$. If

$$\epsilon = \sum_{M \in \mathcal{M}_f(G): e_0 \in M} w(M) - 1,$$

then $p(\tilde{w}) = p(w) + 1$, a contradiction. Otherwise, $\epsilon = w(M_0)$ and, therefore, $\tilde{w} \in \mathcal{R}^0$ and $q(\tilde{w}) = q(w) + 1$, a contradiction too. This completes the proof of the theorem. ■

Finally, let $\mathcal{R}^*_f(G)$ be the set of all functions $w: \mathcal{M}_f(G) \to [0, 1]$ such that every edge $e \in E(G)$ satisfies

$$\sum_{M \in \mathcal{M}_f(G): e \in M} w(M) \geq 1.$$

Evidently, we have

$$\mathcal{R}_f(G) \subseteq \mathcal{R}^*_f(G) \subseteq \mathcal{R}'_f(G)$$

and Theorem B.1 implies that

$$\chi'^*_f(G) = \min\left\{\sum_{M \in \mathcal{M}_f(G)} w(M) \mid w \in \mathcal{R}^*_f(G)\right\}.$$

B.2 THE MATCHING POLYTOPE

Let (G, f) be an arbitrary vertex-weighted graph, i.e., $f : V(G) \to \mathbb{N}$. For a set $U \subseteq V(G)$, define $f(U) = \sum_{u \in U} f(u)$. The set $\mathcal{V}(G)$ of all functions $\mathbf{x} : E(G) \to \mathbb{R}$ form a real vector space with respect to the addition of functions and the multiplication of a function by a real number. If $m = |E(G)|$, then $\mathcal{V}(G)$ has dimension m and is isomorphic to the standard vector space $\mathbb{R}^m$. For a vector $\mathbf{x} \in \mathcal{V}(G)$ and a set $F \subseteq E(G)$, let $\mathbf{x}(F) = \sum_{e \in F} \mathbf{x}(e)$. A **polytope** in $\mathcal{V}(G)$ is the convex hull of finitely many vectors of $\mathcal{V}(G)$. For an edge set $F \subseteq E(G)$, let $\mathbf{x}_F : E(G) \to \mathbb{R}$ be the function defined by

$$\mathbf{x}_F(e) = \begin{cases} 1 & \text{if } e \in F, \\ 0 & \text{if } e \notin F. \end{cases}$$

This function is usually called the **incidence vector** or the **characteristic function** of F. The f**-matching polytope** $\mathcal{P}_f(G)$ of G is then defined as the convex hull of the incidence vectors of all f-matchings in G, i.e.,

$$\mathcal{P}_f(G) = \mathrm{conv}(\{\mathbf{x}_F \mid F \in \mathcal{M}_f(G)\}).$$

Note that $\mathcal{P}_f(G)$ contains the null-vector $\mathbf{x}_\emptyset = \mathbf{0}$. If $f(v) = 1$ for all $v \in V(G)$, we write $\mathcal{P}(G)$ rather than $\mathcal{P}_f(G)$. So $\mathcal{P}(G)$ is the ordinary **matching polytope** of G. Furthermore, let $\mathcal{P}_{\mathrm{perf}}(G)$ denote the **perfect matching polytope** of G, that is, the convex hull of the incidence vectors of all perfect matchings of G. Note that $\mathcal{P}_{\mathrm{perf}}(G) \neq \emptyset$ if and only if G has a perfect matching.

In his pioneering work of 1965, Edmonds [76] gave a full description of the matching polytope of a given graph by a finite system of linear inequalities. Even if the number of constraints is exponential in the size of the graph, the description of the matching polytope has become a very useful tool in combinatorial optimization. Over the years, several different proofs and enhancements of the matching polytope theorem has been given; we refer the reader to the book by Schrijver [277]. The matching polytope theorem follows from a description of the perfect matching polytope, also given by Edmonds [76].

Theorem B.2 (Edmonds' perfect matching polytope theorem [76] 1965) *For any graph G, a vector $\mathbf{x} \in \mathcal{V}(G)$ belongs to $\mathcal{P}_{\mathrm{perf}}(G)$ if and only if $\mathbf{x}$ satisfies the following systems of linear inequalities:*

(a) $\forall e \in E(G) : 0 \leq \mathbf{x}(e) \leq 1$ *(capacity constraint)*

(b) $\forall v \in V(G)$*:* $\mathbf{x}(E_G(v)) = 1$ *(degree equation)*

(c) $\forall U \subseteq V(G)$, $|U|$ *is odd:* $\mathbf{x}(\partial_G(U))) \geq 1$ *(odd cut constraint)*

Theorem B.3 (Edmonds' matching polytope theorem [76] 1965) *For any graph G, a vector $\mathbf{x} \in \mathcal{V}(G)$ belongs to $\mathcal{P}(G)$ if and only if $\mathbf{x}$ satisfies the following systems of linear inequalities:*

(a) $\forall e \in E(G) : 0 \leq \mathbf{x}(e) \leq 1$ *(capacity constraint)*

(b) $\forall v \in V(G)$: $\mathbf{x}(E_G(v)) \leq 1$ *(degree constraint)*

(c) $\forall U \subseteq V(G)$, *where* $|U|$ *is odd:* $\mathbf{x}(E(G[U])) \leq \frac{|U|-1}{2}$ *(blossom constraint)*

A description of the f-matching polytope by a system of linear inequalities was given by Edmonds and Johnson [77] in 1970.

Theorem B.4 (Edmonds and Johnson [77] 1970) *Let* (G, f) *be a weighted graph. A vector* $\mathbf{x} \in \mathcal{V}(G)$ *belongs to* $\mathcal{P}_f(G)$ *if and only if* $\mathbf{x}$ *satisfies the following systems of linear inequalities:*

(a) $\forall e \in E(G)$: $0 \leq \mathbf{x}(e) \leq 1$ *(capacity constraint)*

(b) $\forall v \in V(G)$: $\mathbf{x}(E_G(v)) \leq f(v)$ *(weighted degree constraint)*

(c) $\forall U \subseteq V(G), \forall F \subseteq \partial_G(U)$, *where* $f(U) + |F|$ *is odd:* $\mathbf{x}(E(G[U])) + \mathbf{x}(F) \leq \frac{f(U)+|F|-1}{2}$ *(weighted blossom constraint)*

Note that this result does not immediately imply the matching polytope theorem. For $f \equiv 1$, some of the weighted blossom constraints are redundant, while a blossom constraint for a subset U of $V(G)$ with odd cardinality corresponds to a weighted blossom constraint for the set U and the empty edge set $F \subseteq \partial_G(U)$. Whether in general all weighted blossom constrains are really necessary seems not clear.

From Edmonds' matching polytope theorem, a combinatorial characterization of the fractional edge chromatic index can be easily derived. For a graph G, define

$$\kappa^*(G) = \max \left\{ \Delta(G), \max_{H \subseteq G, |V(H)| \geq 2} \frac{|E(H)|}{\lfloor \frac{1}{2}|V(H)| \rfloor} \right\}. \tag{B.1}$$

Observe that a graph with at most two vertices satisfies $\kappa^*(G) = \Delta(G)$. In searching for a subgraph $H \subseteq G$ achieving the maximum in (B.1) we can clearly restrict H to be an induced subgraph and to have odd order, since for a subgraph H of even order we have $|E(H)|/\lfloor |V(H)|/2 \rfloor \leq \Delta(H) \leq \Delta(G)$. So (B.1) reduces to

$$\kappa^*(G) = \max \left\{ \Delta(G), \max_{X \subseteq V(G), |X| \geq 3 \text{ odd}} \frac{2|E(G[X])|}{|X| - 1} \right\}. \tag{B.2}$$

Theorem B.5 (Seymour [280] 1979, Stahl [286] 1979) *Every graph* G *satisfies* $\chi'^*(G) = \kappa^*(G)$.

Proof: The statement is evident if G is edgeless. So suppose $E(G) \neq \emptyset$. Let $\mathcal{M}$ denote the set of all matchings of G.

To see that $\chi'^*(G) \geq \kappa^*(G)$, choose a fractional edge coloring w of G with minimum value. For a vertex v of maximum degree in G, we then obtain

$$\begin{aligned}\chi'^*(G) &= \sum_{M\in\mathcal{M}} w(M) \geq \sum_{M\in\mathcal{M}} w(M)\,|M\cap E_G(v)| \\ &= \sum_{e\in E_G(v)} \sum_{M\in\mathcal{M}:e\in M} w(M) = \sum_{e\in E_G(v)} 1 = \Delta(G).\end{aligned}$$

Moreover, for each $H \subseteq G$ with $|V(H)| \geq 2$, we obtain

$$\begin{aligned}\chi'^*(G) &= \sum_{M\in\mathcal{M}} w(M) \geq \sum_{M\in\mathcal{M}} w(M)\frac{|M\cap E(H)|}{\lfloor\frac{1}{2}|V(H)|\rfloor} \\ &= \frac{1}{\lfloor\frac{1}{2}|V(H)|\rfloor}\sum_{e\in E(H)} \sum_{M\in\mathcal{M}:e\in M} w(M) = \frac{|E(H)|}{\lfloor\frac{1}{2}|V(H)|\rfloor}.\end{aligned}$$

This proves $\chi'^*(G) \geq \kappa^*(G)$.

To see that $\chi'^*(G) \leq \kappa^*(G)$, let $\mathbf{x} \in \mathcal{V}(G)$ be the function with $\mathbf{x}(e) = 1/\kappa^*(G)$ for all $e \in E(G)$. Obviously, $\mathbf{x}(E_G(v)) \leq 1$ for each $v \in V(G)$ and $\mathbf{x}(E(G[U])) \leq \lfloor\frac{1}{2}|U|\rfloor$ for each $U \subseteq V(G)$ with $|U| \geq 2$. Hence $\mathbf{x}$ belongs to $\mathcal{P}(G)$, that is, $\mathbf{x}$ is a convex combination of incidence vectors of matchings. Let $\mathbf{1} \in \mathcal{V}(G)$ be the all-one vector. Then $\mathbf{1} = \kappa^*(G)\cdot\mathbf{x} = \sum_{M\in\mathcal{M}}\lambda_M\mathbf{x}_M$ for some $\lambda_M \geq 0$ with $\sum_{M\in\mathcal{M}}\lambda_M = \kappa^*(G)$, implying that the function $w : \mathcal{M} \to [0,1]$ with $w(M) = \lambda_M$ for each $M \in \mathcal{M}$ is a fractional edge coloring of G. Hence, we obtain $\chi'^*(G) \leq \sum_{M\in\mathcal{M}} w(M) = \kappa^*(G)$. ■

Theorem B.6 *Let G be a graph and let $\chi'^*(G) = r$. Then r is rational and for every positive integer t we have $\chi'(tG) \geq tr$, where equality holds if and only if tr is an integer and there exist a family of tr matchings in G using each edge exactly t times. Furthermore, there are infinitely many positive integers t such that $\chi'(tG) = tr$.*

Proof: If G is edgeless, the theorem is obviously true. So assume that $E(G) \neq \emptyset$. By (B.1) and Theorem B.5, we obtain that r is rational and $\chi'(tG) \geq \chi'^*(tG) = \kappa^*(tG) = t\kappa^*(G) = t\chi'^*(G) = tr$. Clearly, $\chi'(tG) = tr$ if and only if tr is an integer and the edge set of tG can be partitioned into tr matchings. This is equivalent to the statement that there exists a family of tr matchings in G using each edge exactly t times, because tG is obtained from G by replacing each edge of G by t parallel edges and no matching of tG contains two parallel edges.

The fractional edge chromatic index is the optimal value of an linear program with integer coefficients, hence there is an optimal fractional edge coloring w of G such that $w(M)$ is rational for every matching M, where $\chi'^*(G) = r = \sum_M w(M)$. Let s denote the least common multiple of the denominators of all the values $w(M)$. Then s is a positive integer and $k_M = sw(M)$ is a nonnegative integer for all matching M of G. Since w is an optimal fractional edge coloring, we conclude that

$$\sum_M k_M\mathbf{x}_M = s\mathbf{1} \text{ and } \sum_M k_M = sr.$$

This means that there is a family of sr matchings in G using each edge s times. Clearly, this is equivalent to $\chi'(sG) = sr$. If $t = ks$ for a positive integer k, then there is a family of $tr = ksr$ matchings in G using each edge $t = ks$ times, and so $\chi'(tG) = tr$. This completes the proof. ■

As a corollary of Theorem B.6 we obtain the following characterization of the fractional chromatic index of an arbitrary graph G:

$$\chi'^*(G) = \min_{t \geq 1} \frac{\chi'(tG)}{t} = \lim_{t \to \infty} \frac{\chi'(tG)}{t}. \tag{B.3}$$

Because of Theorem B.6 it is sufficient to show that the limit in (B.3) exists. This follows from the fact that $\chi'((s+t)G) \leq \chi'(sG) + \chi'(tG)$ for all $s, t \in \mathbb{N}$ and Fekete's Lemma [93]. This lemma says that if the sequence $g : \mathbb{N} \to \mathbb{R}^{\geq 0}$ is **subadditive**, that is, $g(s+t) \leq g(s) + g(t)$ for all $s, t \in \mathbb{N}$, then the limit $\lim_{n\to\infty} g(n)/n$ exists and is equal to the infimum of $g(n)/n$ for $n \in \mathbb{N}$. A proof of Fekete's Lemma can be found in the book by Scheinerman and Ullman [276, Lemma A.4.1], see also the book of Jensen and Toft [157, Problem 5.3].

In what follows we will discuss three interesting applications of Edmonds' matching polytope theorem and Edmonds' perfect matching polytope theorem. The first application is a result due to O. Marcotte. This result is used in the proof of Theorem 6.24.

Theorem B.7 (Marcotte [208] 1990) *Every graph G with $w(G) \leq \Delta(G)$ and $\Delta(G) \geq 1$ contains a matching M such that $\Delta(G - M) = \Delta(G) - 1$.*

Proof: Let $\Delta = \Delta(G)$ and let $\mathbf{x} = \frac{1}{\Delta}\mathbf{1}$. Clearly, $\mathbf{x}$ satisfies the capacity constraints and the degree constrains of Theorem B.3, where for each major vertex the degree constraint holds with equality. Since $\Delta \geq w(G)$, the vector $\mathbf{x}$ also satisfy the blossoms constraints. So $\mathbf{x}$ belongs to the matchin polytope $\mathcal{P}(G)$ and is therefore a convex combination of the matching vectors. If a matching vector $\mathbf{x}_M$ occurs in such a convex combination with a positive coefficient, then $\mathbf{x}_M$ also satisfies the degree constraints for each major vertex with equality, that is, each major vertex of G is an endvertex of some edge belonging to the matching M. So $\Delta(G-M) = \Delta(G) - 1$. ■

Let G be a graph. The standard **scalar product** in the vector space $\mathcal{V}(G)$ is denoted by $\circ$, and the set of perfect matchings of G is denoted by $\mathcal{M}_{\text{perf}}(G)$. Recall that an odd set means a set with odd cardinality. The proof of the following statement uses Theorem B.2 and is based on standard arguments from convex analysis.

Proposition B.8 (Kaiser, Král', and Norine [162] 2005) *Let G be a graph with at least one edge and let $\mathbf{x} \in \mathcal{P}_{\text{perf}}(G)$ be a vector. Then there are $\ell \geq 1$ perfect matchings $M_1, \ldots, M_\ell \in \mathcal{M}_{\text{perf}}(G)$ and positive integers $\lambda_1, \ldots, \lambda_\ell$ such that*

$$\mathbf{x} = \lambda_1 \mathbf{x}_{M_1} + \cdots \lambda_\ell \mathbf{x}_{M_\ell} \text{ and } \lambda_1 + \cdots \lambda_\ell = 1,$$

where every such convex combination of perfect matchings satisfies the following statements:

(a) *If* $U \subseteq V(G)$ *is an odd set such that* $\mathbf{x}(\partial_G(U)) = 1$*, then* $\mathbf{x}_{M_i}(\partial_G(U)) = 1$ *for* $i = 1, \ldots, \ell$.

(b) *For every vector* $\mathbf{c} \in \mathcal{V}(G)$ *there is a perfect matching* $M \in \{M_1, \ldots, M_\ell\}$ *such that* $\mathbf{c} \circ \mathbf{x}_M \geq \mathbf{c} \circ \mathbf{x}$.

Let G be a cubic bridgeless graph. Observe that G is a 3-graph, that is, G is 3-regular and $|\partial_G(U)| \geq 3$ for every odd set $U \subseteq V(G)$. If $U \subseteq V(G)$ satisfies $|\partial_G(U)| = 3$, then a simple parity argument shows that U is odd, and we then call $F = \partial_G(U)$ a 3-edge-cut of G. By Petersen's theorem [241], G has a perfect matching. For an integer $k \geq 1$, define

$$m_k(G) = \max\left\{\frac{|\bigcup_{i=1}^k M_i|}{|E(G)|} \mid M_1, \ldots, M_k \in \mathcal{M}_{\mathrm{perf}}(G)\right\}.$$

Evidently, $m_1(G) = \frac{1}{3}$ and a conjecture of C. Berge suggests that $m_5(G) = 1$. Lower bounds for $m_k(G)$ for $2 \leq k \leq 5$ were established by Kaiser, Král', and Norine [162].

Theorem B.9 (Kaiser, Král', and Norine [162] 2005) *Every cubic bridgeless graph* G *satisfies* $m_2(G) \geq \frac{3}{5}$*, where equality holds for the Petersen graph.*

Proof: That the Petersen graph P satisfies $m_2(P) = 3/5$ follows from the fact that any two perfect matchings of P have exactly one edge in common.

So let G be an arbitrary cubic bridgeless graph and let $\mathbf{x} = \frac{1}{3}\mathbf{1} \in \mathcal{V}(G)$. By Theorem B.2, $\mathbf{x} \in \mathcal{P}_{\mathrm{perf}}(G)$ and $\mathbf{x}(F) = 1$ for each 3-edge-cut F of G. Hence, by Proposition B.8, there is a perfect matching M_1 of G intersecting each 3-edge-cut in a single edge. We now define a vector $\mathbf{y} \in \mathcal{V}(G)$ by $\mathbf{y}(e) = 1/5$ if $e \in M_1$ and $\mathbf{y}(e) = 2/5$ otherwise. Since M_1 contains exactly one edge of each 3-edge-cut, it follows that $\mathbf{y} \in \mathcal{P}_{\mathrm{perf}}(G)$. Let $\mathbf{c} = \mathbf{1} - \mathbf{x}_{M_1}$. Again, by Proposition B.8, there is a perfect matching M_2 of G such that

$$\mathbf{c} \circ \mathbf{x}_{M_2} \geq \mathbf{c} \circ \mathbf{y} = \frac{2}{5} \cdot \frac{2}{3}|E(G)| = \frac{4}{15}|E(G)|.$$

Since $\mathbf{c} \circ \mathbf{x}_{M_2} = |M_2 \setminus M_1|$, it follows that $|M_1 \cup M_2| = (\frac{1}{3} + \frac{4}{15})|E(G)| = \frac{3}{5}|E(G)|$. This shows that $m_2(G) \geq 3/5$ as required. ■

Using a similar approach, Kaiser, Král', and Norine [162] proved that

$$m_k(G) \geq 1 - \prod_{i=1}^{k} \frac{i+1}{2i+1}$$

for every cubic bridgeless graph G and every integer $k \geq 1$. For the Petersen graph P, we have

$$m_2(P) = \frac{3}{5}, m_3(P) = \frac{4}{5}, m_4(P) = \frac{14}{15}, m_5(P) = 1.$$

The third application of the perfect matching polytope theorem deals with the Petersen graph. In 2007 V. V. Mkrtchyan posted as an open question on the Open Problem Garden webpage whether any cubic bridgeless graph different from the Petersen graph contains a 2-factor such that at least one of its cycles is not a 5-cycle, that is, a cycle of length 5. Within one day M. DeVos found an affirmative answer and put an outline of a proof on the Open Problem Garden. His proof is based on standard arguments, particularly on Tutte's 1-factor theorem, and was published by DeVoss, Mkrtchyan, and Petrosyan [68]. Shortly after the problem was posed by Mkrtchyan, D. Král' (oral communication) also found a solution, but with a shorter proof. The secret of Král's proof is to show, by means of the perfect matching polytope theorem, that a connected cubic bridgeless graph without a desired 2-factor has so many 5-cycles that it can be only the Petersen graph.

Theorem B.10 *Let G be a connected cubic bridgeless graph. Then every 2-factor of G is the disjoint union of 5-cycles if and only if G is the Petersen graph.*

Proof: It is not hard to check that each 2-factor of the Petersen graph is the disjoint union of two 5-cycles. So assume that G is a connected cubic bridgeless graph of order n such that every 2-factors of G is the disjoint union of 5-cycles. We first show that the number of 5-cycles in G is at least $6n/5$, from which we then conclude that G must be the Petersen graph.

To count the number of 5-cycles in G we apply Edmonds' perfect matching polytope theorem (Theorem B.2) to the vector $\mathbf{x} = \frac{1}{3}\mathbf{1} \in \mathcal{V}(G)$. By Theorem B.2, $\mathbf{x} \in \mathcal{P}_{\text{perf}}(G)$. Then there are $\ell \geq 1$ perfect matchings $M_1, \ldots, M_\ell \in \mathcal{M}_{\text{perf}}(G)$ and positive integers $\lambda_1, \ldots, \lambda_\ell$ such that

$$\mathbf{x} = \lambda_1 \mathbf{x}_{M_1} + \cdots \lambda_\ell \mathbf{x}_{M_\ell} \text{ and } \lambda_1 + \cdots \lambda_\ell = 1. \tag{B.4}$$

For an edge set $M \subseteq E(G)$ define $\overline{M} = E(G) \setminus M$. If M is a perfect matching of G, then $\overline{M}$ is (the edge set of) a 2-factor of G and the hypothesis implies therefore that $\overline{M}$ is the disjoint union of 5-cycles, where the number of these 5-cycles is $n/5$.

For each i ($i = 1, \ldots, \ell$) the $n/5$ 5-cycles of $\overline{M}_i$ are each given the weight λ_i. The total weight w given to all 5-cycles is then

$$w = (\lambda_1 + \cdots + \lambda_\ell)\frac{n}{5} = \frac{n}{5}.$$

Consider a fixed 5-cycle C of G. The total weight given to C is

$$p = \sum_{i \in I} \lambda_i,$$

where $I = \{i | E(C) \subseteq \overline{M}_i\}$. To estimate p, consider $F = \partial_G(V(C))$, which is a set of three or five edges. If $|F| = 3$, then we deduce from (B.4)

$$\frac{3}{3} = \mathbf{x}(F) = \sum_{i=1}^{\ell} \lambda_i \mathbf{x}_{M_i}(F) \geq \sum_{i \in I} 3\lambda_i + \sum_{i \notin I} \lambda_i \geq 3p + (1-p) = 2p + 1,$$

which gives $p \le 0$, i.e., $p = 0$. If, on the other hand, $|F| = 5$ then we deduce from (B.4) that

$$\frac{5}{3} = \mathbf{x}(F) = \sum_{i=1}^{\ell} \lambda_i \mathbf{x}_{M_i}(F) \ge \sum_{i \in I} 5\lambda_i + \sum_{i \notin I} \lambda_i \ge 5p + (1-p) = 4p + 1,$$

which gives $p \le 1/6$. Let c_n denote the number of 5-cycles of G. Then $(1/6)c_n \ge w = n/5$, i.e., $c_n \ge 6n/5$.

In a connected bridgeless graph G any vertex v is contained in at most six 5-cycles, with equality only if there are exactly six vertices of distance 2 from v joined to each other by six edges, implying that there are no vertices of distance 3 from v, i.e., $|V(G)| = 10$. The total number of 5-cycles is therefore $\le 6n/5$. Since we deduced above that $c_n \ge 6n/5$, it follows that $c_n = 6n/5$ and that all vertices v are contained in exactly six 5-cycles. Then $|V(G)| = 10$ and it follows easily that G is the Petersen graph. ■

B.3 A FORMULA FOR $\chi_f'^*$

Let (G, f) be an arbitrary vertex-weighted graph. Recall from Chap. 8 that there are two lower bounds for the f-chromatic index $\chi'_f(G)$, namely $\Delta_f(G)$ and $\mathcal{W}_f(G)$. The **fractional maximum f-degree** of G is defined by

$$\Delta_f^*(G) = \max_{v \in V(G)} \frac{d_G(v)}{f(v)}$$

and the **fractional f-density** of G is defined by

$$\mathcal{W}_f^*(G) = \max_{U \subseteq V(G), f(U) \ge 3 \text{ odd}} \frac{2|E(G[U])|}{f(U) - 1},$$

where $\mathcal{W}_f^*(G) = 0$ if $f(U) = 1$ or $f(U)$ is even for all $U \subseteq V(G)$. Furthermore, let us define

$$\kappa_f^*(G) = \max\{\Delta_f^*(G), \mathcal{W}_f^*(G)\}.$$

Clearly, $\lceil \kappa_f^*(G) \rceil = \max\{\Delta_f(G), \mathcal{W}_f(G)\} \le \chi'_f(G)$ and $\chi_f'^*(G) \le \chi'_f(G)$. Next, let us introduce a variation of the f-density. Let $T(G)$ denote the set of all tuples (U, F) satisfying $\emptyset \ne U \subseteq V(G)$, $F \subseteq \partial_G(U)$, and $f(U) + |F|$ is odd and ≥ 3. We then define

$$\tilde{\mathcal{W}}_f^*(G) = \max_{(U,F) \in T(G)} \frac{2(|E(G[U])| + |F|)}{f(U) + |F| - 1}$$

if $T(G) \ne \emptyset$ and $\tilde{\mathcal{W}}_f^*(G) = 0$ otherwise. Furthermore, we define

$$\tilde{\kappa}_f^*(G) = \max\{\Delta_f^*(G), \tilde{\mathcal{W}}_f^*(G)\}.$$

Clearly, $\mathcal{W}_f^*(G) \le \tilde{\mathcal{W}}_f^*(G)$ and so $\kappa_f^*(G) \le \tilde{\kappa}_f^*(G)$. If $\Delta_f^*(G) \le 1$, then obviously $\chi_f'^*(G) = \chi'_f(G) = \Delta_f(G)$.

Theorem B.11 *Every graph G with $\Delta_f^*(G) \geq 1$ satisfies $\chi_f'^*(G) = \tilde{\kappa}_f^*(G)$.*

Proof: First we show that $\chi_f'^*(G) \geq \tilde{\kappa}_f^*(G)$. To this end, we consider a fractional f-coloring $w \in \mathcal{R}_f(G)$ with minimum value. For every f-matching $M \in \mathcal{M}_f(G)$ and every vertex $v \in V(G)$, we have $|M \cap E_G(v)| \leq f(v)$. Let $v \in V(G)$ be a vertex such that $\Delta_f^*(G) = d_G(v)/f(v)$. Then we obtain

$$\begin{aligned}
\chi_f'^*(G) &= \sum_{M \in \mathcal{M}_f} w(M) \geq \sum_{M \in \mathcal{M}_f} w(M) \frac{|M \cap E_G(v)|}{f(v)} \\
&= \sum_{e \in E_G(v)} \sum_{M \in \mathcal{M}_f : e \in M} \frac{w(M)}{f(v)} \\
&= \sum_{e \in E_G(v)} \frac{1}{f(v)} = \frac{d_G(v)}{f(v)} = \Delta_f^*(G).
\end{aligned}$$

To see that $\chi_f'^*(G) \geq \tilde{w}_f^*(G)$, we may assume that $T(G) \neq \emptyset$, since otherwise we have $\tilde{w}_f^*(G) = 0 \leq \chi_f'^*(G)$. Then we choose a tuple $(U, F) \in T(G)$ such that

$$\tilde{w}_f^*(G) = \frac{2(|E(G[U])| + |F|)}{f(U) + |F| - 1}.$$

For the edge set $E = E(G[U]) \cup F$ and an arbitrary f-matching $M \in \mathcal{M}_f(G)$ we have

$$2|M \cap E| \leq \sum_{u \in U} f(u) + |M \cap F| \leq f(U) + |F|.$$

Since $2|M \cap E|$ is even and $f(U) + |F|$ is odd (because of $(U, F) \in T(G)$), we have $2|M \cap E| \leq f(U) + |F| - 1$. Then we deduce that

$$\begin{aligned}
\chi_f'^*(G) &= \sum_{M \in \mathcal{M}} w(M) \geq \sum_{M \in \mathcal{M}} w(M) \frac{2|M \cap E|}{f(U) + |F| - 1} \\
&= \sum_{e \in E} \sum_{M \in \mathcal{M} : e \in M} w(M) \frac{2}{f(U) + |F| - 1} \\
&= \sum_{e \in E} \frac{2}{f(U) + |F| - 1} = \frac{2|E(G[U])| + |F|}{f(U) + |F| - 1} = \tilde{w}_f^*(G).
\end{aligned}$$

This proves that $\chi_f'^*(G) \geq \tilde{\kappa}_f^*(G)$. It remains to show that $\chi_f'^*(G) \leq \tilde{\kappa}_f^*(G)$. To see this, let $\mathbf{x} \in \mathcal{V}(G)$ be the function with $\mathbf{x}(e) = 1/\tilde{\kappa}_f^*(G)$ for all $e \in E(G)$. We claim that $\mathbf{x} \in \mathcal{P}_f(G)$. Clearly, $\mathbf{x}$ satisfies the capacity constraints. From $\tilde{\kappa}_f^*(G) \geq \Delta_f^*(G)$ we obtain

$$\mathbf{x}(E_G(v)) = \frac{d_G(v)}{\tilde{\kappa}_f^*(G)} \leq \frac{\Delta_f^*(G) f(v)}{\tilde{\kappa}_f^*(G)} \leq f(v)$$

for all $v \in V(G)$. So $\mathbf{x}$ satisfies the weighted degree constraints. For any tuple $(U, F) \in T(G)$ we obtain

$$\mathbf{x}(E(G[U])) + \mathbf{x}(F) = \frac{|E(G[U])| + |F|}{\tilde{\kappa}_f^*(G)} \leq \frac{f(U) + |F| - 1}{2},$$

since $\tilde{\kappa}_f^*(G) \geq \tilde{w}_f^*(G)$. So $\mathbf{x}$ satisfies the weighted blossom constraints. Then Theorem B.4 implies that $\mathbf{x}$ belongs to $\mathcal{P}_f(G)$, that is, $\mathbf{x}$ is a convex combination of incidence vectors of f-matchings. This gives

$$\mathbf{1} = \tilde{\kappa}_f^*(G) \cdot \mathbf{x} = \sum_{M \in \mathcal{M}_f} \lambda_M \mathbf{x}_M$$

for some $\lambda_M \geq 0$ with

$$\sum_{M \in \mathcal{M}} \lambda_M = \tilde{\kappa}_f^*(G).$$

This implies that the function $w : \mathcal{M}_f(G) \to [0, 1]$ with $w(M) = \lambda_M$ for each $M \in \mathcal{M}_f$ is a fractional f-coloring of G. Then we obtain $\chi_f'^*(G) \leq \sum_{M \in \mathcal{M}} w(M) = \tilde{\kappa}_f^*(G)$ and the proof is complete. ■

Every graph G satisfies $\kappa_f^*(G) \leq \tilde{\kappa}_f^*(G) = \chi_f'^*(G)$ and we are now interested in conditions for equality. To find at least a partial answer the following inequality can be used. Let $a_1, \ldots, a_n, b_1, \ldots, b_n$ be positive real numbers with $n \geq 1$. Then

$$\frac{a_1 + \cdots + a_n}{b_1 + \cdots + b_n} \leq \max_{1 \leq i \leq n} \frac{a_i}{b_i} \tag{B.5}$$

With $B = b_1 + \cdots + b_n$ and $M = \max_{1 \leq i \leq n} \frac{a_i}{b_i}$ we obtain

$$\frac{\sum_{i=1}^n a_i}{\sum_{i=1}^n b_i} = \sum_{i=1}^n \frac{a_i}{b_i} \frac{b_i}{B} \leq \sum_{i=1}^n M \frac{b_i}{B} = M,$$

which proves (B.5).

In what follows, let (G, f) be an arbitrary vertex-weighted graph. If $T(G) = \emptyset$, then we clearly have $\tilde{\kappa}_f^*(G) = \Delta_f^*(G)$ and, therefore, $\tilde{\kappa}_f^*(G) = \kappa_f^*(G)$. So assume that $T(G) \neq \emptyset$. Then there is a tuple $(U, F) \in T(G)$ such that

$$\tilde{w}_f^*(G) = \frac{2(|E(G[U])| + |F|)}{f(U) + |F| - 1}.$$

By the definition of $T(G)$, $\emptyset \neq U \subseteq V(G)$, $F \subseteq \partial_G(U)$, and $f(U) + |F|$ is odd and ≥ 3. Then we distinguish two cases.

Case 1: *$f(U)$ is odd.* Then $|F|$ is even. If $|F| = 0$, we obtain

$$\tilde{w}_f^*(G) = \frac{2|E(G[U])|}{f(U) - 1} \leq w_f^*(G).$$

Otherwise, $|F| \geq 2$ and, using (B.5), we obtain

$$\tilde{w}_f^*(G) \leq \max\{\frac{2|E(G[U])|}{f(U) - 1}, 2\} \leq \max\{w_f^*(G), 2\}.$$

Case 2: *$f(U)$ is even.* Then $|F|$ is odd. If $|F| \geq 3$, then based on (B.5) we deduce that

$$\begin{aligned}\tilde{w}_f^*(G) &= \frac{2(|E(G[U]| + |F|)}{f(U) + |F| - 1} = \frac{\sum_{u\in U} d_{G[U]}(u) + 2|F|}{f(U) + |F| - 1} \\ &= \frac{\sum_{u\in U} d_G(u) + |F|}{f(U) + |F| - 1} \leq \max\left\{\frac{\sum_{u\in U} d_G(u)}{\sum_{u\in U} f(u)}, \frac{|F|}{|F|-1}\right\} \\ &\leq \max\left\{\max_{u\in U}\frac{d_G(u)}{f(u)}, \frac{|F|}{|F|-1}\right\} \leq \max\left\{\Delta_f^*(G), \frac{3}{2}\right\}.\end{aligned}$$

It remains to consider the case $|F| = 1$. Then we have

$$\tilde{w}_f^*(G) = \frac{2|E(G[U])| + 2}{f(U)}.$$

If $|\partial_G(U)| \geq 2$, then

$$2|E(G[U])| + 2 = \sum_{u\in U} d_{G[U]}(u) + 2 \leq \sum_{u\in U} d_G(u).$$

By (B.5), this gives

$$\tilde{w}_f^*(G) \leq \frac{\sum_{u\in U} d_G(u)}{\sum_{u\in U} f(u)} \leq \Delta_f^*(G).$$

Otherwise, $|\partial_G(U)| = 1$ and the only edge in F is a cut-edge (or bridge) of G. Since $f(U)$ is even, we have $f(U) \geq 2$. Based on (B.5) we then obtain that

$$\begin{aligned}\tilde{w}_f^*(G) &= \frac{\sum_{u\in U} d_{G[U]}(u) + 2}{f(U)} = \frac{\sum_{u\in U} d_G(u) + 1}{\sum_{u\in U} f(u)} \\ &\leq \Delta_f^*(G) + \frac{1}{f(U)} \leq \Delta_f^*(G) + \frac{1}{2}.\end{aligned}$$

Hence, the following result is proved.

Proposition B.12 *If (G, f) is a vertex-weighted graph, then*

$$\tilde{w}_f^*(G) \leq \max\{\Delta_f^*(G) + \frac{1}{2}, w_f^*(G), 2\}.$$

Furthermore, $\tilde{w}_f^*(G) \leq \max\{\Delta_f^*(G), w_f^*(G), 2\}$ provided that G is bridgeless.

Since any χ_f'-critical graph with $\chi_f' \geq \Delta_f + 1$ is bridgeless (Proposition 8.30), Proposition B.12 implies the following result:

Corollary B.13 *Let (G, f) be a vertex-weighted graph with $\chi_f'(G) \geq \Delta_f(G) + 1$ and $\Delta_f^*(G) \geq 2$. If G is χ_f'-critical, i.e., $\chi_f'(H) < \chi_f'(G)$ for every proper subgraph H of G, then $\chi_f'^*(G) = \kappa_f^*(G)$.*

Notes

Fractional graph theory has developed into an important and powerful method in combinatorics and combinatorial optimization. The first monograph on the subject was written by Berge [26] in 1978. Another comprehensive monograph, providing a rational, rather than an integral, approach to the theory of graphs, was written by Scheinerman and Ullman [276] in 1997.

The formula for the fractional chromatic index in Theorem B.5 can be easily derived from Edmonds' matching polytope theorem. This was first noticed by Stahl [286] in 1979. However, Seymour [281] found a different proof of the formula and used this result to give a purely combinatorial proof of Edmonds' matching polytope theorem.

Pulleyblank and Edmonds [250] characterized the facets of the matching polytope. In particular, they proved that a blossom constraint for a set U is a facet of $\mathcal{P}(G)$ if and only if $G[U]$ is **2-connected** (i.e., $G[U]$ is connected and has no cut vertex) and **factor-critical** (i.e., $G[U]-v$ has a perfect matching for every vertex $v \in U$). As observed by Marcotte [210], the above characterization implies the following strengthening of Lemma 6.28 by Fernandes and Thomas [94]: Let G be a graph with $\Delta(G) \leq k$ for some integer k. Then $\kappa(G) = \max\{\Delta(G), \mathcal{W}(G)\} \leq k$ if and only if $2|E(G[U])| \leq k(|U|-1)$ for every vertex set $U \subseteq V(G)$ such that $G[U]$ is 2-connected and factor-critical.

Corollary B.13 can easily be extended to arbitrary χ'_f-critical graphs with $\Delta_f^* \geq 2$. A χ'_f-critical graph G with $\chi'^*(G) = \Delta_f(G) \geq 2$ satisfies $\chi'^*_f(G) = \Delta^*_f(G) = \kappa^*_f(G)$ and $\chi'_f(G) = \lceil \chi'^*_f(G) \rceil$. It is also easy to show that a vertex weighted graph (G,f) with $\Delta_f(G) \leq 2$ satisfies $\chi'_f(G) = \max\{\Delta_f(G), \mathcal{W}_f(G)\}$. Combining these results with Theorem 8.29, it follows that every weighted graph (G,f) satisfies

$$\chi'_f(G) \leq \lceil \chi'^*_f(G) \rceil + \sqrt{\lceil \chi'^*_f(G) \rceil / 2}.$$

Zhang, Yu, and Liu [320] proved a strengthening of Theorem B.4 characterizing the f-matching polytope of a graph G. They proved that if $f(v) \geq d_G(v)$ for every vertex v of G, then a vector $\mathbf{x} \in \mathcal{V}(G)$ belongs to $\mathcal{P}_f(G)$ if and only if $\mathbf{x}$ satisfies the capacity constraints, the weighted degree constraints for all $v \in V(G)$, and the weighted blossom constraints for all tuples $(U,F) \in T(G)$ with $F = \emptyset$. As a consequence, they deduced that $\chi'^*_f(G) = \kappa^*_f(G)$ for every weighted graph (G,f) with $f(v) \geq d_G(v)$ for all $v \in V(G)$.

A fractional chromatic index for the fg-color problem can be also defined by relaxation of the integer program defining $\chi'_{f,g}$. However, a combinatorial characterization of the fractional chromatic index seems not known.

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