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Brooks' Theorem

Graph Coloring and Critical Graphs

Springer

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In memory of Gabriel Andrew Dirac, Tibor Gallai and Horst Sachs

Preface

Brooks' Theorem from 1941 is a cornerstone in graph theory. Until then graph coloring theory was centered around planar graphs and the four color problem. Brooks' Theorem was the first nontrivial theorem to relate the chromatic number of general graphs to other graph parameters, in this case to the maximum degree.

Rowland Leonard Brooks (1916–1993) studied mathematics at Cambridge University, where he made friends with Cedric Austin Bardell Smith (1912–2002), Arthur Harold Stone (1916–2000) and William Thomas Tutte (1917–2002). The four together solved the problem of dividing a square into a finite number of smaller squares, all of different sizes, relating it to flow theory for graphs. But it was Brooks alone who took up abstract graph coloring and found the theorem that bears his name. Brooks' paper was communicated to the Cambridge Philosophical Society by Tutte and published in their proceedings in 1941.

The first monograph on abstract graph theory, written in German by the Hungarian Dénes König (1884–1944), had appeared already in 1936. In the academic year 1904/05 König – together with several other brilliant students – attended Minkowski's lectures on Analysis Situs in Göttingen, where Minkowski attempted to solve the four color problem. The book by König did not contain coloring theory for abstract graphs – it did not yet exist. But the lectures in Budapest by König made graph theory – together with number theory – a favorite topic of his students, among them György Hajós (1912–1972), Paul Erdős (1913–1996), Paul Turán (1910–1976) and Tibor Gallai (1912–1992). After the war a gifted young Hungarian student Peter Ungar came to London, where he teamed up with another Hungarian student Gabriel Andrew Dirac (1925–1984), who had moved to England from Budapest in 1937 as a 12 year old boy with his sister, when their mother Margit Wigner (1904–2002) and the world famous physicist Paul Adrian Maurice Dirac (1902–1984) married. Ungar told the young Dirac about graph theory, and Dirac took it as a topic of his ph.d. studies, supervised by Richard Rado (1906–1989), resulting in his thesis *On the colouring of graphs* in 1951. In his studies Dirac rediscovered Brooks' Theorem, without knowing the 1941 paper. He was informed about it by the external examiner Cedric A. B. Smith, resulting in some last minute changes in the thesis. The notion of critical graphs was introduced and studied by Dirac as a key concept in the thesis. During the 1950s Dirac continued to develop the theory, and the study was also taken up by others, in particular by Paul Erdős and Tibor Gallai in Hungary and by Horst Sachs (1927–2016) in East-Germany. In 1963 Dirac spent a year with Sachs as professor at the Technical University of Ilmenau.

In the period 1966–1970 Bjarne Toft had the good fortune to study graph theory with Dirac as supervisor, resulting in his thesis *Some contributions to the theory of colour-critical graphs*. In 1969 he spent a semester in Budapest, where he met Gallai, who in weekly meetings told about his results and problems, in particular relating to Dirac's critical graphs. One such problem asked if the number of connected components in the low-vertex subgraph of a k-critical graph is at least that number for the high-vertex subgraph (where the high vertices are those of degree at least k, and the low vertices those of degree $k-1$). This problem was transferred by Toft to Sachs and his students at the Technical University in Ilmenau, East Germany, where it was solved by Michael Stiebitz and first published in 1985 in his habilitation thesis *Beiträge zur Theorie der Färbungskritischen Graphen*. In 1985 Stiebitz spent a semester in Budapest and also had conversations with Gallai about critical graphs.

We are deeply grateful for the including and open atmosphere we have always met in the graph theory community, to a large extend inspired and fostered through the openness of Erdős and the Hungarian mathematicians in general. Toft is grateful to his fellow student Ivan Tafteberg Jakobsen, who for many years was an important partner, in Denmark, England and Hungary. Also in England, Germany and Denmark we met openness in our relations with Dirac and Sachs. In England Toft got in close contact with Cedric A. B. Smith and via him met Brooks, who worked as a tax-inspector in London.

Our project to write about Brooks' Theorem started in 2015, where Stiebitz and Toft published a book-chapter about Brooks' Theorem. This was a starting point of the project - to dig deeper and create a whole monograph, and we were joined by Stiebitz's ph.d.-student Thomas Schweser. But the literature related to graph coloring has really exploded, and it is impossible, even in a lengthy monograph, to cover everything related to Brooks' Theorem. We had to be selective and have concentrated in particular on three central aspects: the various proofs of Brooks' Theorem, the various extensions of it, and similar theorems for other graph parameters. Our book presents a comprehensive overview of the development and see it in context. It describes results, both early and recent, and explains relations. We hope that it will serve as a reference to a wealth of information, now scattered in journals, proceedings and dissertations. The reader gets easy access to this information, including best known proofs of the results described.

We are grateful to Cambridge University Press for the permission to reproduce the fundamental paper of R. L. Brooks [On colouring the Nodes of a Network, *Proceedings of the Cambridge Philosophical Society* 37 (1941), pp. 194–197]. It is the content of Appendix A1.

Our thanks also go to Tommy R. Jensen, who made us aware of W. T. Tutte's lecture *Fifty Years of Graph Colouring* at the University of Waterloo, June 26, 1992,

and who gave us access to Tutte's original manuscript. In this context our gratitude also goes to Daniel Younger, who acted as executor of Tutte's Will, for the permission to reproduce Tutte's manuscript as Appendix B1. In an e-mail to us in March 2023 Younger wrote: *If I have any authority in this matter, it is my judgement that Tutte would have been pleased to have this lecture-document made public, in a way such as your Monograph.*

We have benefitted from the continued support from our work places, the Mathematical Institute of the Technical University of Ilmenau and the Department of Mathematics and Computer Science and its excellent library at the University of Southern Denmark. Several other individuals provided much help and support, among them Alexander Bock, Thomas Böhme, Thomas Fischer, Florian Hörsch, Christian Klaue, and Till Preuster. And Elena Griniari, Francesca Bonadei, and Francesca Ferrari at Springer have been indispensable in their continued efficiency and support in bringing this monograph to life.

Ilmenau and Odense,
October 2023

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Thomas Schweser
Bjarne Toft

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