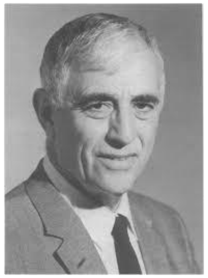
Koopman Operator Theory for Autonomous Systems

Alexandre Mauroy *(University of Namur, Belgium)*

UNIVERSITÉ DE NAMUR

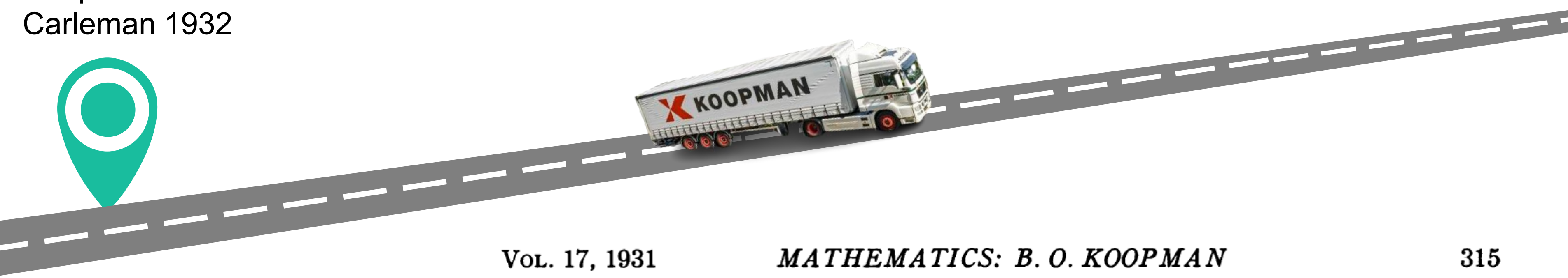
naxys
Namur Institute for Complex Systems



The origins

Koopman 1931

Carleman 1932



HAMILTONIAN SYSTEMS AND TRANSFORMATIONS IN HILBERT SPACE

BY B. O. KOOPMAN

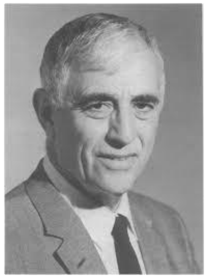
DEPARTMENT OF MATHEMATICS, COLUMBIA UNIVERSITY

23, 1931

APPLICATION DE LA THÉORIE DES ÉQUATIONS INTÉGRALES LINÉAIRES AUX SYSTÈMES D'ÉQUATIONS DIFFÉRENTIELLES NON LINÉAIRES.

PAR

TORSTEN CARLEMAN



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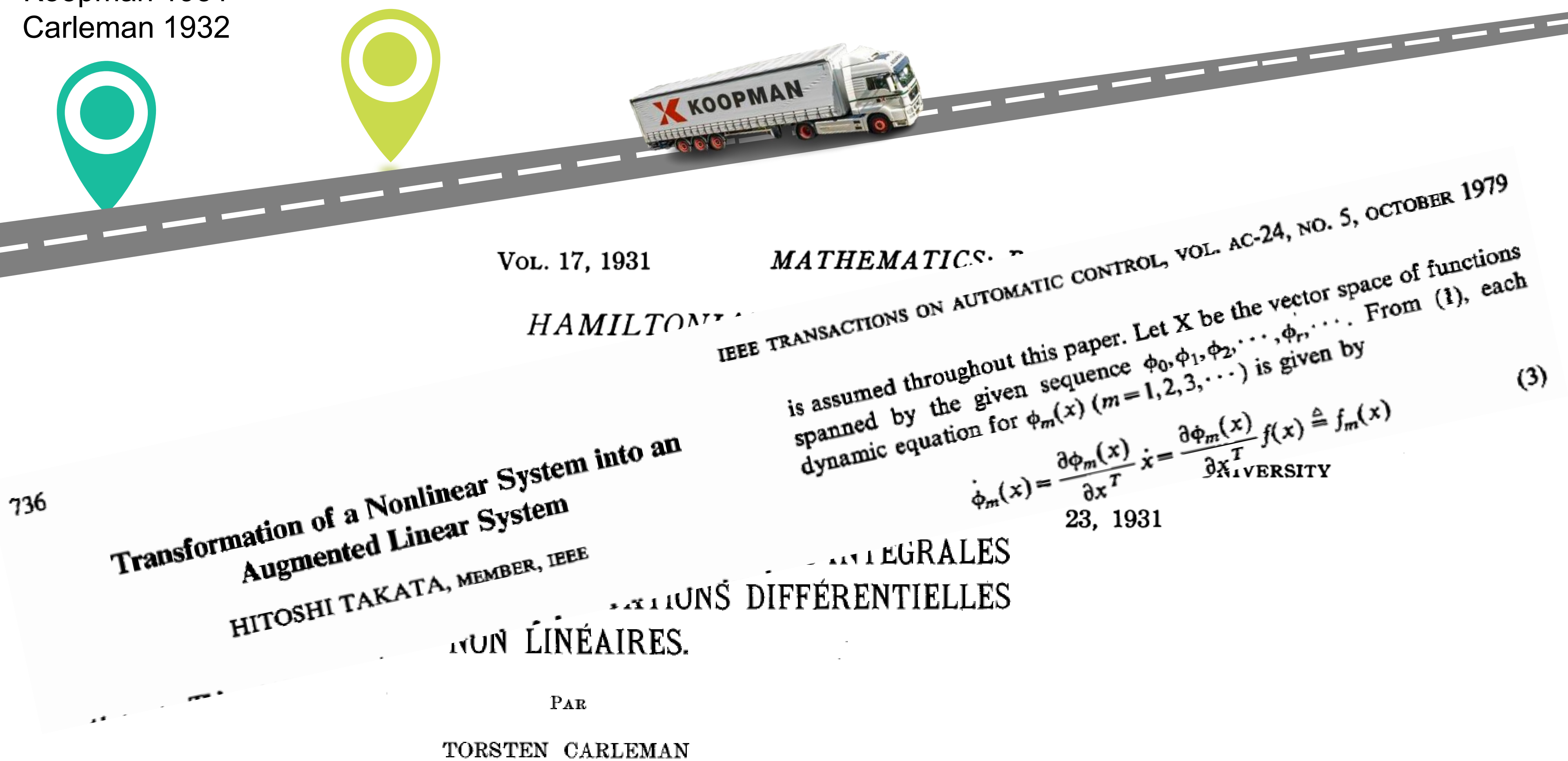
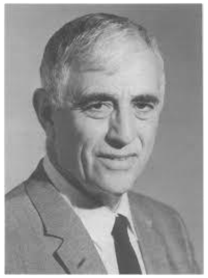
Early works in control theory

Krener 1974
Takata 1979
Banks 1985





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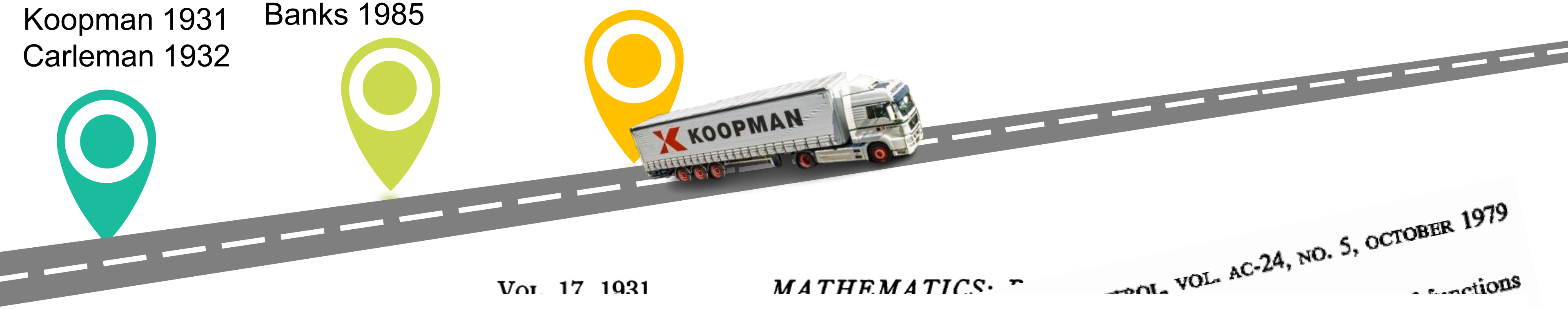
The rebirth: spectral approach

Mezic 2005



VOL. AC-24, NO. 5, OCTOBER 1979

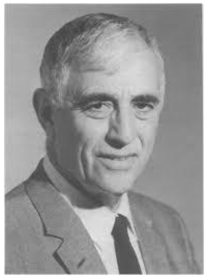
Nonlinear Dynamics (2005) 41: 309–325

736

Transfor

Spectral Properties of Dynamical Systems, Model Reduction and Decompositions

IGOR MEZIĆ
Department of Mechanical and Environmental Engineering and Department of Mathematics, University of California, Santa Barbara, CA 93105-5070, U.S.A. (e-mail: mezic@engineering.ucsb.edu; fax: +1-805-893-8651)



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Journal of Nonlinear Science

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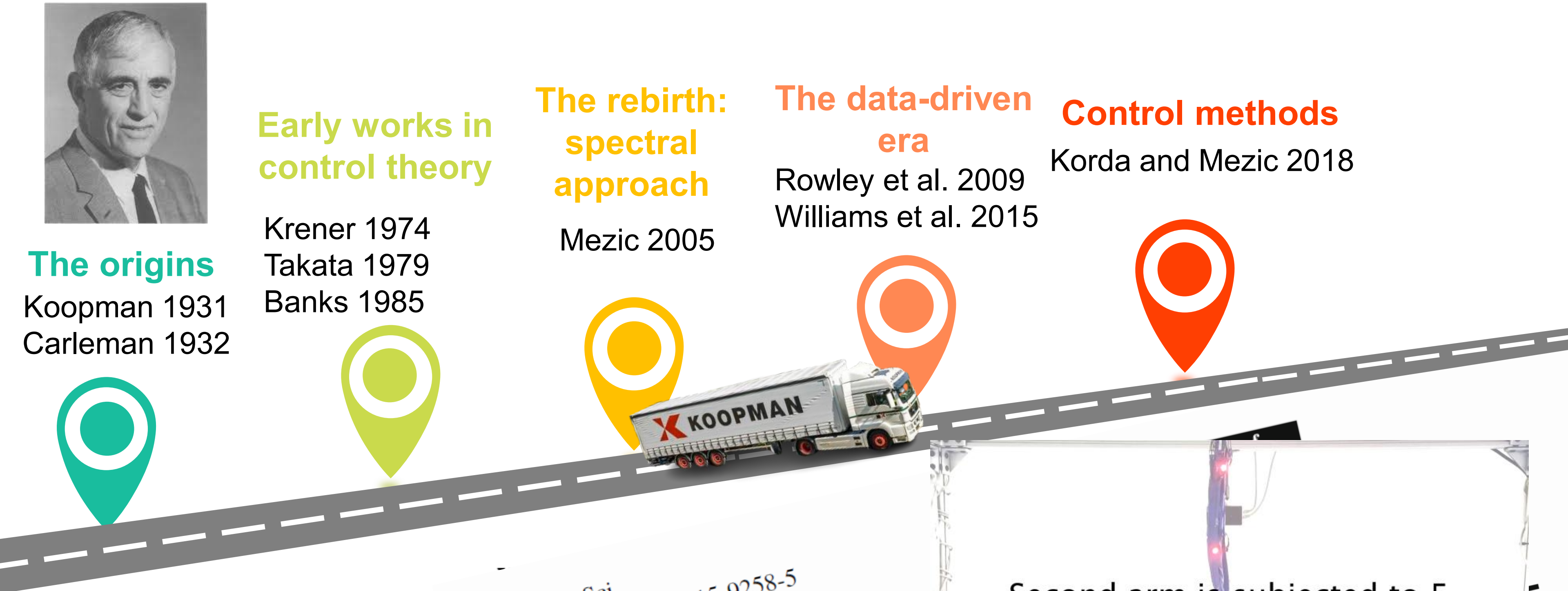
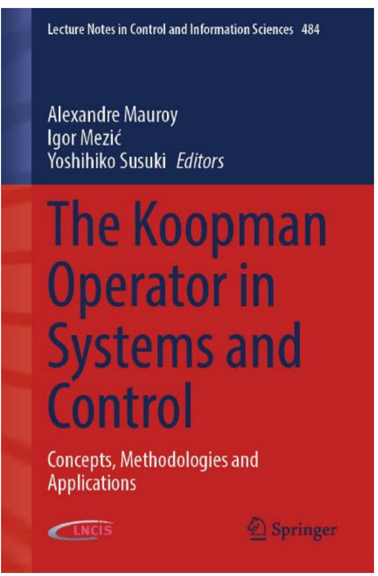
J Nonlinear Sci
DOI 10.1007/s00332-015-9258-5

A Data–Driven Approximation of the Koopman Operator: Extending Dynamic Mode Decomposition

Matthew O. Williams[1] · Ioannis G. Kevrekidis[2] · Clarence W. Rowley[3]





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J Nonlinear Sci
DOI 10.1007/s00332-015-9258-5

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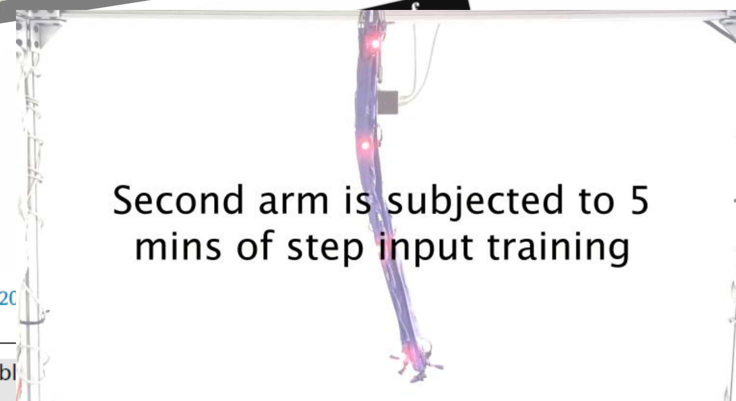
Linear predictors for nonlinear dynamical systems: Koopman operator meets model predictive control☆

Milan Korda *, Igor Mezić

University of California, Santa Barbara, United States



[Haggerty et al., Science Robotics 2023]

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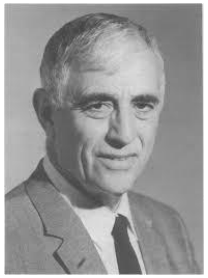

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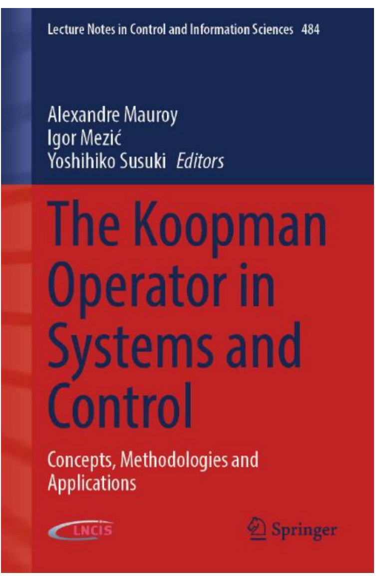
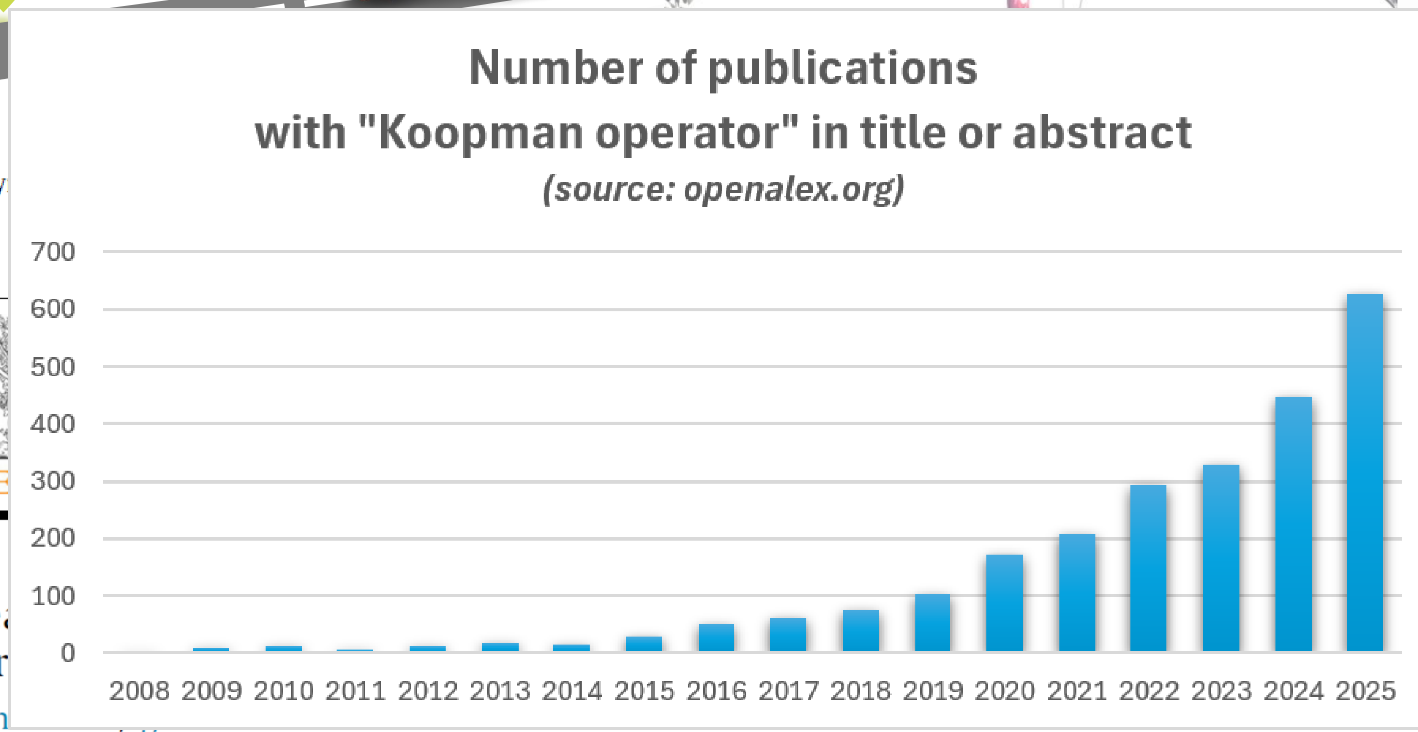
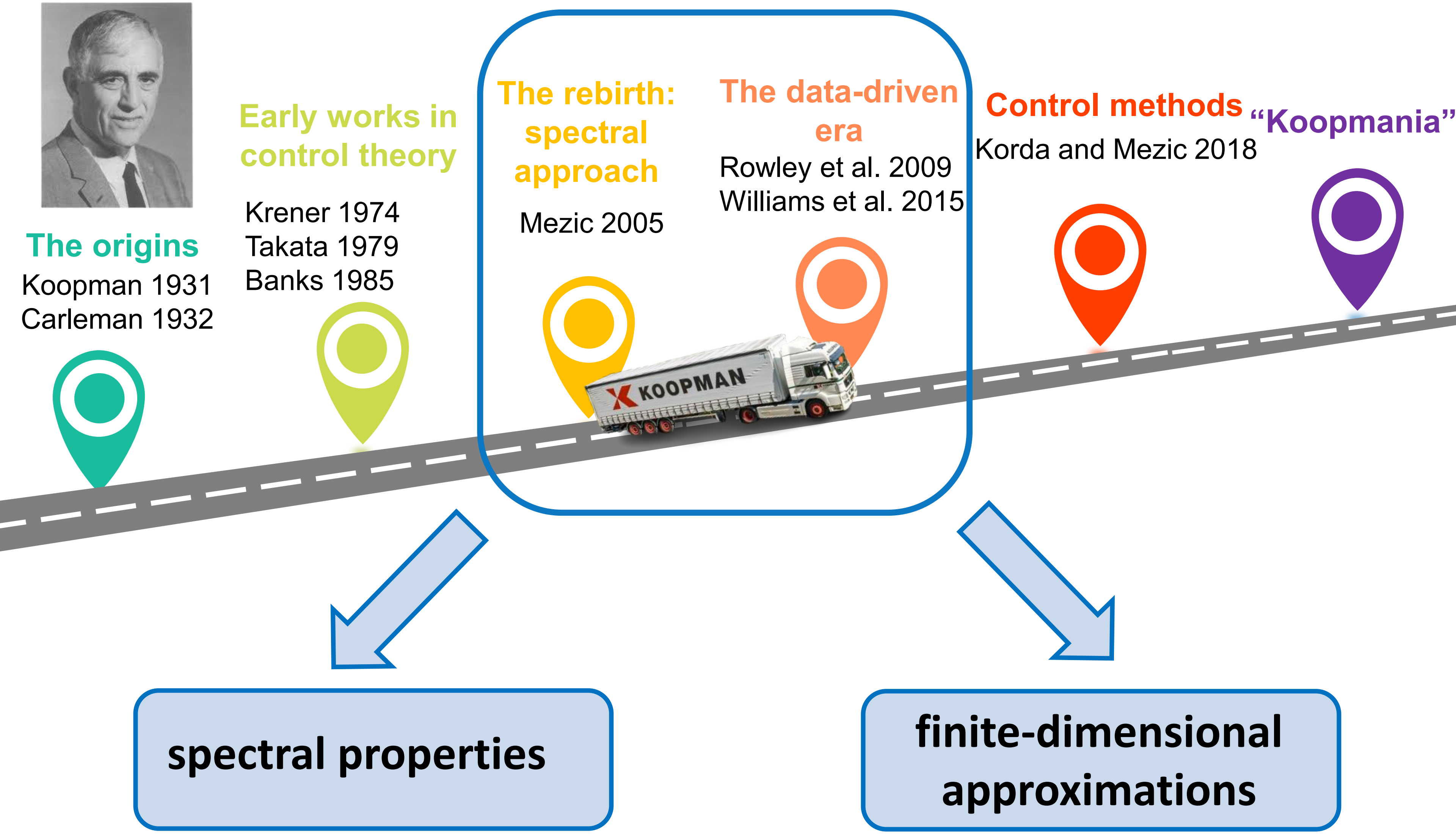
Control methods
Korda and Mezic 2018

“Koopmania”



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KOOPMAN
spectral properties
finite-dimensional approximations

Outline

A few definitions and general properties

Let’s first focus on spectral properties

Back to finite dimension

Application #1: stability analysis

Application #2: state estimation

Outline

A few definitions and general properties

Let’s first focus on spectral properties

Back to finite dimension

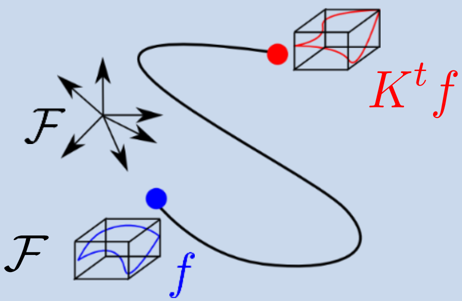
Application #1: stability analysis

Application #2: state estimation

The main philosophy…

Koopman operator description

Operator $f \mapsto K^t f = f$
acting on an observable space $\mathcal{F}$



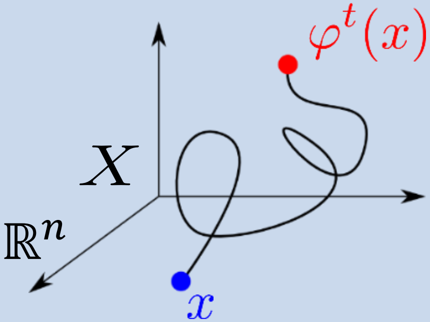
linear infinite-dimensional operator

A concept that is not so new!
e.g. Lyapunov functions,
HJB equation, Lie derivatives,…

LIFTING

Trajectory-oriented description

Flow map $x \mapsto \varphi^t(x)$
acting on the state space $X \subset \mathbb{R}^n$

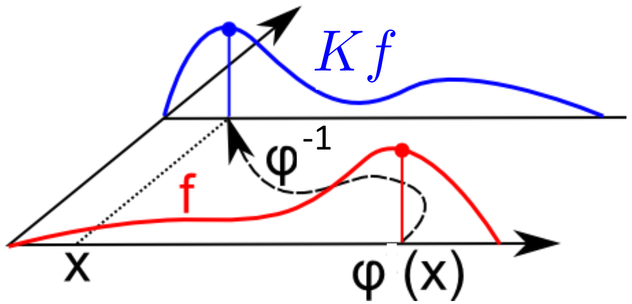


nonlinear finite-dimensional dynamics

The Koopman operator is a composition operator

$f : X \to \mathbb{C}$ is an « observable »

with $f \in \mathcal{F}$



Discrete-time system $\varphi : X \to X$

Koopman operator = composition operator $Kf = f \circ \varphi$ $\quad K : \mathcal{F} \to \mathcal{F}$

Continuous-time system $\varphi : \mathbb{R}^+ \times X \to X$ $\quad \varphi(t, \cdot) \equiv \varphi^t(\cdot)$

semigroup of Koopman operators : $(K^t)_{t \geq 0} : K^t f = f \circ \varphi^t$ $\quad K^t : \mathcal{F} \to \mathcal{F}$

$$K^0 = I, \quad K^t K^s = K^{t+s}, \, t, s \geq 0$$

[Koopman, PNAS 1931]

Finite state space $X = \{1, \ldots, N\}$ ➔ observable $f : \{1, \ldots, N\} \to \mathbb{C}$ $(f \in \mathbb{C}^N)$

$K^T : \mathbb{C}^N \to \mathbb{C}^N$ is the transition matrix of the Markov chain

[Budisic, Mohr and Mezic, Chaos 2012]

A few properties of the semigroup of Koopman operators (continuous time)

Linearity: $K^t(a_1 f_1 + a_2 f_2) = a_1 K^t f_1 + a_2 K^t f_2$

Infinitesimal generator (Koopman generator)

[Lasota and Mackey, Chaos, Fractals, and Noise, Springer 1998]

$$Af = \lim_{t \downarrow 0} \frac{K^t f - f}{t}$$

$$A : \mathcal{D}(A) \to \mathcal{F}$$

$$\dot{x} = F(x) \quad \Rightarrow \quad Af = F \cdot \nabla f$$

Lie derivative

The semigroup must be strongly continuous: $\lim_{t \downarrow 0} \|K^t f - f\| = 0 \;\; \forall f \in \mathcal{F}$

Koopman dynamical system

$\dot{f} = Af, \;\; f(0) = f_0 \in \mathcal{F}$ ➔ solution $f(t) = K^t f_0$

No free lunch: advantages and limitations

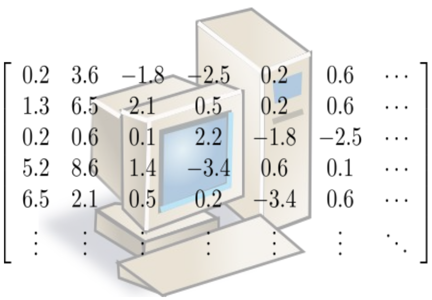
Linearity

➔ systematic methods

➔ well-developed theory

➔ spectral analysis

➔ data-driven approach



Infinite dimension

➔ inherent difficulties and pitfalls in functional analysis

➔ scalability issues

➔ approximations and errors

Outline

A few definitions and general properties

Let's first focus on spectral properties

Back to finite dimension

Application #1: stability analysis

Application #2: state estimation

The Koopman operator is linear, so let's focus on its spectral properties

Koopman eigenfunction $\phi_\lambda \in \mathcal{F}$ $\quad\Longleftrightarrow\quad$ $A\phi_\lambda = \lambda\phi_\lambda$

Koopman eigenvalue $\lambda \in \sigma(A)$ $\quad\Longleftrightarrow\quad$ $K^t\phi_\lambda = e^{\lambda t}\phi_\lambda$

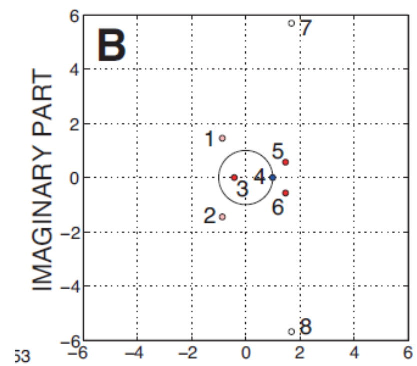
[Mezic, Nonlinear Dynamics, 2005]

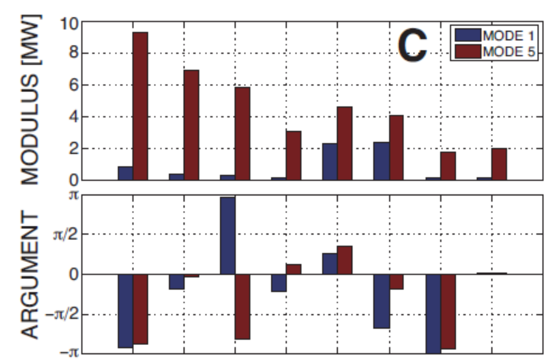
Koopman mode decomposition

regular part

$$K^t f(x) = \sum_{j=1}^{\infty} \overbrace{v_j}^{\text{Koopman mode}} \underbrace{\phi_{\lambda_j}(x)}_{\text{eigenfunction}} e^{\lambda_j t} + K_r^t f(x)$$

eigenvalue



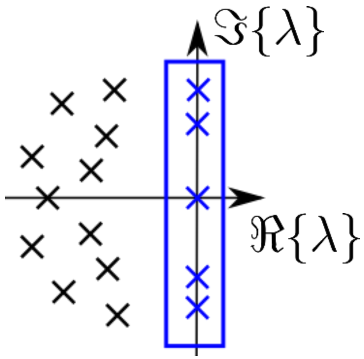


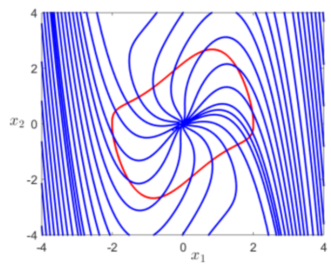
Pitfall #1: The spectral properties depend on the function space $\mathcal{F}$!

[Susuki and Mezic, IEEE Trans. Power Syst. 2014]

For dissipative systems, the spectrum can be decomposed into two parts

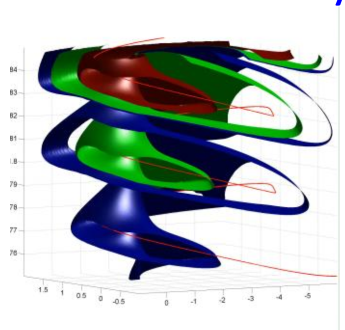
Assume that the system admits an attractor $\mathcal{A}$



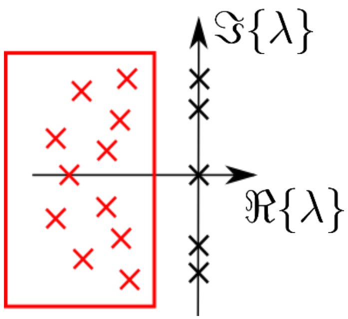


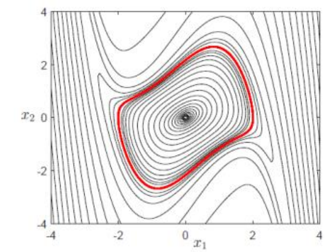
→ captures the ergodic motion on $\mathcal{A}$

→ e.g. isochrons of limit cycles



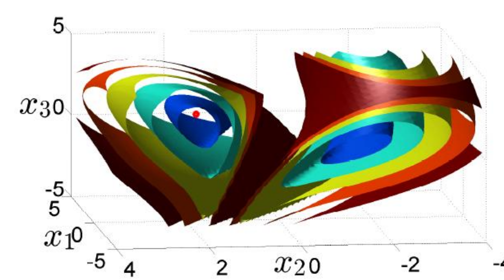
[AM et Mezic, Chaos 2012]





→ captures the asymptotic convergence to $\mathcal{A}$

→ e.g. « isostables » of equilibria

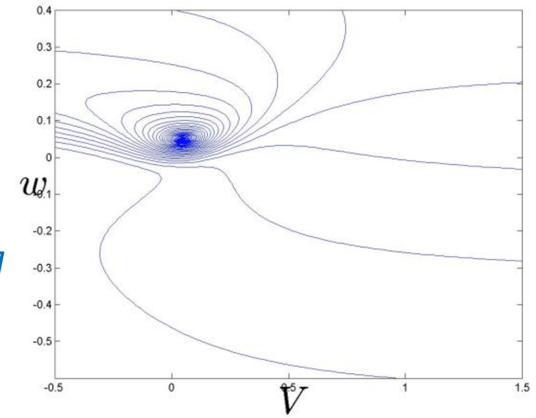


[AM, Mezic and Moehlis, Physica D 2013]

Spectral properties reveal geometric properties of the dynamics: the case of equilibria

Isostables of an equilibrium: level sets of the dominant eigenfunction

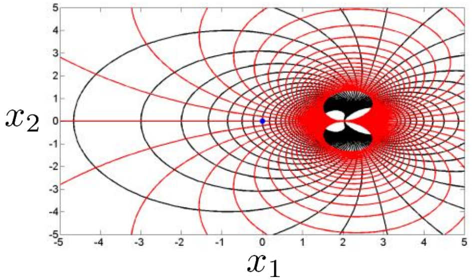
➔ Application to optimal control



Optimal escape - convergence *[AM, CDC 2014]*
Switching problem *[Sootla, AM, and Ernst, Automatica 2018]*
Cardiac defibrillation techniques *[Wilson and Moehlis, SIAM review 2015]*
Deep brain stimulation *[Duchet et al., J. Neural Eng. 2021]*

Global linearization

$$y = (\phi_{\lambda_1}(x), \dots, \phi_{\lambda_n}(x)) \quad \Rightarrow \quad \dot{y} = \mathbf{\Lambda}\, y$$



➔ Poincaré linearization theorem / global extension of Harman-Grobman theorem

[Lan and Mezic, Physica D 2013]

Outline

A few definitions and general properties

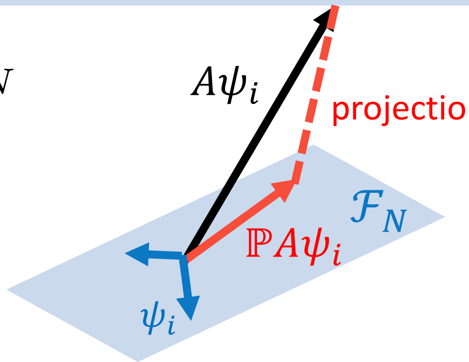
Let’s first focus on spectral properties

Back to finite dimension

Application #1: stability analysis

Application #2: state estimation

A finite-dimensional approximation is obtained by « projecting » the operator onto a subspace



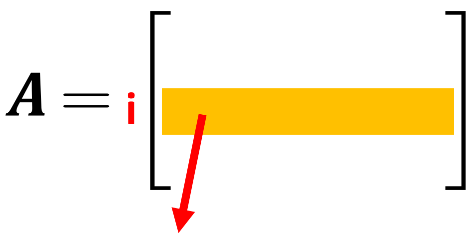
choose: 1. a basis of functions $\{\psi_j\}_{j=1}^N$ that spans a subspace $\mathcal{F}_N$

2. a projection operator $\mathbb{P} : \mathcal{F} \to \mathcal{F}_N$

Koopman matrix:

Compute the matrix representation $\boldsymbol{A} \in \mathbb{R}^{N\times N}$ of the projected generator $\mathbb{P}A|_{\mathcal{F}_N} = \mathbb{P}(\boldsymbol{F}\cdot\nabla)|_{\mathcal{F}_N}$ or the matrix representation $\boldsymbol{K} \in \mathbb{R}^{N\times N}$ of the projected semigroup $\mathbb{P}K^{\Delta t}|_{\mathcal{F}_N}$

Rem.: $\boldsymbol{K} \approx e^{\boldsymbol{A}\Delta t}$



components of $\mathbb{P}(A\psi_i)$
in the basis $\{\psi_j\}_{j=1}^N$

Example: $\dot{x} = ax + bx^2$ $\quad \{\psi_j\}_{j=1}^3 = \{x, x^2, x^3\}$

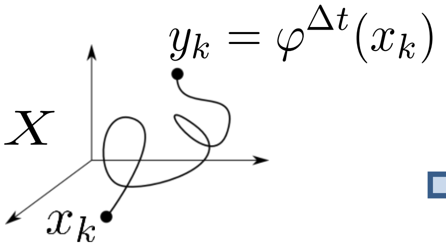
$$\boldsymbol{A} = \begin{array}{c} \begin{array}{ccc} x & x^2 & x^3 \end{array} \\ \begin{bmatrix} a & b & 0 \\ 0 & 2a & 2b \\ 0 & 0 & 3a \end{bmatrix} \end{array} \begin{array}{l} \leftrightarrow A\psi_1(x) = ax + bx^2 \\ \leftrightarrow A\psi_2(x) = 2ax^2 + 2bx^3 \\ \leftrightarrow A\psi_3(x) = 3ax^3 + 3bx^4 \end{array}$$

spectral properties: eigenvalues of $\boldsymbol{A}$ ➔ (approximation of) Koopman eigenvalues

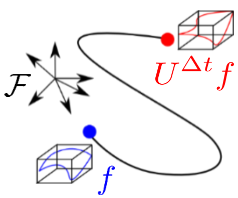
eigenvectors of $\boldsymbol{A}$ ➔ (approximation of) Koopman eigenfunctions

The finite-dimensional approximation can be computed from data

Extended dynamic mode decomposition (EDMD) *[Williams et al., JNLS 2015]*



$$(x_k, \varphi^{\Delta t}(x_k)) = (x_k, y_k)$$
$$k = 1, \ldots, K$$



Lifting of the data

$$\mathbf{\Psi_X} = \begin{pmatrix} \psi_1(x_1) & \cdots & \psi_1(x_K) \\ \vdots & & \vdots \\ \psi_N(x_1) & \cdots & \psi_N(x_K) \end{pmatrix}$$

$$\mathbf{\Psi_Y} = \begin{pmatrix} \psi_1(y_1) & \cdots & \psi_1(y_K) \\ \vdots & & \vdots \\ \psi_N(y_1) & \cdots & \psi_N(y_K) \end{pmatrix}$$

discrete L^2 projection $\boxed{\mathbf{K} = \mathbf{\Psi_Y}\mathbf{\Psi_X}^{\dagger}}$

Many variants *[The multiverse of DMD algorithms, Colbrook, Handbook of Num. Anal. 2024]*

Basis functions

Monomials, radial basis functions, neural networks *[Li et al., Chaos 2017] [Yeung et al., ACC 2019]*

Projection

L^2 orthogonal, kernel-based EDMD *[Williams et al., JNLS 2015]*,
Analytic EDMD (Taylor projection) *[AM and Mezic, SIADS 2026]*
Bernstein operator *[Yadav and AM, CNSNS 2025]*

Approximation errors and convergence rates can be obtained

Pitfall #2: Approximation errors and convergence

Methods	Results	References
EDMD	L^2 convergence	[Korda and Mezic, JNLS 2018]
kernel-based EDMD	convergence rates in Sobolev space	[Nathan et al., CDC 2023]
EDMD	convergence rates for finite-element approximation	[Zhang and Zuazua, Comptes Rendus Mécanique 2023]
EDMD	probabilistic error bounds	[Nüske et al., JNLS 2023]
kernel-based EDMD	probabilistic error bounds	[Philipp et al., Appl. Comp. Anal. 2024]
kernel-based EDMD	L^∞ convergence and bounds	[Köhne et al., SIADS 2025]
Bernstein operator	L^∞ convergence and bounds	[Yadav and AM, CNSNS 2025]
Analytic EDMD	Analytic RKHS norm / pointwise convergence	[AM and Mezic, SIADS 2026]

The Koopman approximation yields linear lifted dynamics

With the lifted state $\mathbf{z} = \boldsymbol{\Psi}(\mathbf{x}) = [\psi_1(\mathbf{x}) \ \cdots \ \psi_N(\mathbf{x})]$

nonlinear dynamics

$$\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x})$$

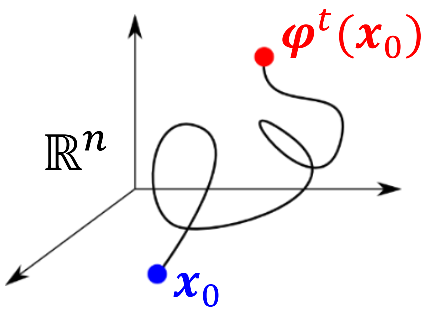
$$\mathbf{x} \in \mathbb{R}^n$$

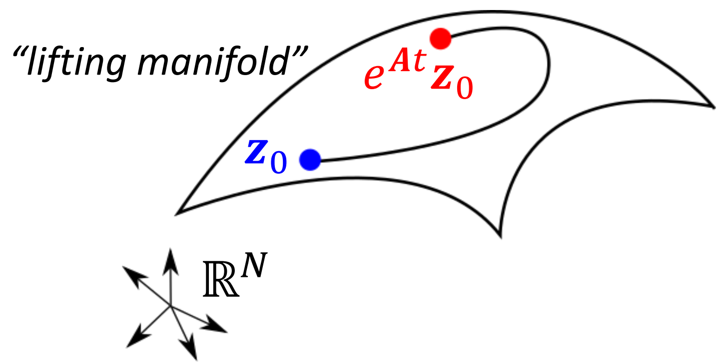
lifting →

linear lifted dynamics

$$\dot{\mathbf{z}} = \mathbf{A}\mathbf{z}$$

$$\mathbf{z} \in \mathbb{R}^N$$





<u>Pitfall #4:</u> The lifted dynamics is an approximation, unless $\mathcal{F}_N = \mathrm{span}\{\psi_i\}$ is invariant (but this is rarely the case in practice!)

<u>Pitfall #5:</u> Reprojecting the lifted state onto the lifting manifold at each step is equivalent to least-squares regression of the flow map

Outline

A few definitions and general properties

Let’s first focus on spectral properties

Back to finite dimension

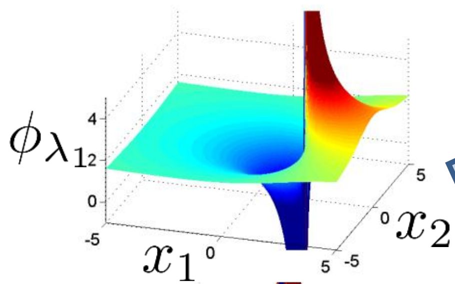
Application #1: stability analysis

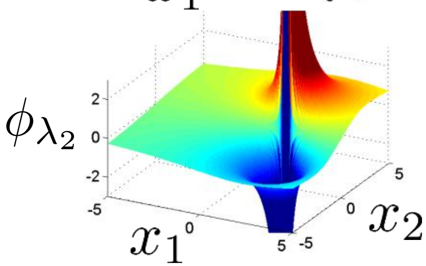
Application #2: state estimation

Global stability is characterized in terms of Koopman eigenfunctions

Theorem: Assume that $X \subset \mathbb{R}^n$ is a forward invariant connected set.
The (hyperbolic) fixed point x^* is globally asympt. stable in X iff
(i) $\Re\{\lambda_j\} < 0$
(ii) there exist n eigenfunctions $\phi_{\lambda_j} \in C^1(X)$ with $\nabla \phi_{\lambda_j}(x^*) \neq 0$

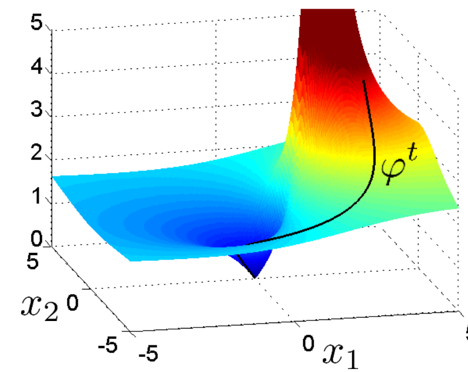
Eigenfunctions





Lyapunov function

$$\mathcal{V} = \sqrt{\phi_{\lambda_1}^2 + \phi_{\lambda_2}^2}$$



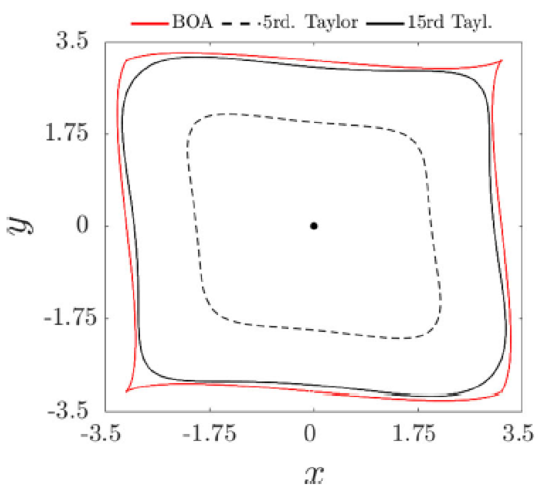
Contracting metric

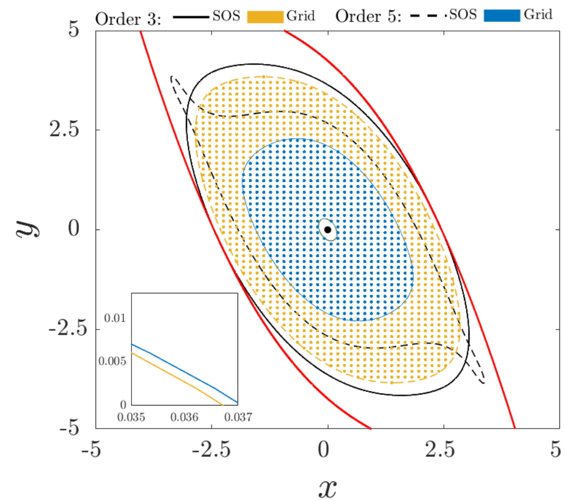
$$\mathcal{M}(x, y) = \sqrt{|\phi_{\lambda_1}(x) - \phi_{\lambda_1}(y)|^2 + |\phi_{\lambda_2}(x) - \phi_{\lambda_2}(y)|^2}$$

[AM and Mezic, IEEE TAC 2016]

But validation methods are needed…

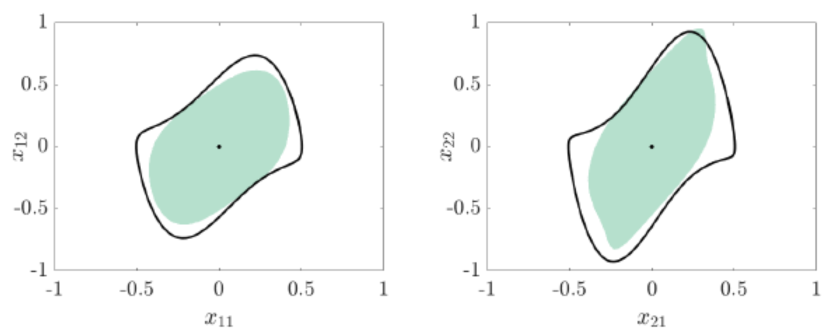
SOS methods or adaptative grids can be used *[Bierwart and AM, Nonlinear Dynamics 2025]*





KOSTA toolbox (Matlab): https://github.com/FgBierwart/KOSTA-Toolbox

Kernel-based (RKHS) methods for high-dimensional systems (+ probabilistic guarantees)



➔ F.-G. Bierwart's talk on Friday at 11:40am (session FrAT6 - Árna 3)

The lifting dynamics can be used for global stability analysis

Nonlinear dynamics

$$\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x})$$

Linear lifted dynamics

$$\dot{\mathbf{z}} = \mathbf{A}\mathbf{z}$$

$$\mathbf{z} = \mathbf{\Psi}(\mathbf{x})$$

Quadratic Lyapunov function

$$V(\mathbf{z}) = \mathbf{z}^T \mathbf{P} \mathbf{z}$$

Nonlinear Lyapunov function

$$\bar{V}(\mathbf{x}) = \mathbf{\Psi}(\mathbf{x})^T \mathbf{P} \mathbf{\Psi}(\mathbf{x})$$

Approximation error ≡ perturbation ➔ robust control (e.g. small gain theorem)

[AM, Sootla and Mezic, Koopman operator in Systems and Control, Chapter 2]

Other related approaches

Lyapunov equation in infinite dimension *[Breiten and Höveler, SICON 2023]*

Zubov-Koopman approach *[Meng and Liu, IEEE TAC 2025]*

Outline

A few definitions and general properties

Let’s first focus on spectral properties

Back to finite dimension

Application #1: stability analysis

Application #2: state estimation

Observability analysis can be performed with the lifted dynamics or in infinite dimension

Nonlinear dynamics

$$\dot{x} = F(x), \qquad x(0) = x_0 \in \mathbb{R}^n$$
$$y = h(x) \in \mathbb{R}^m$$

➔ estimation of the **state x**

$\mathbf{z} = \mathbf{\Psi}(\mathbf{x})$

Linear lifted dynamics

$$\dot{\boldsymbol{z}} = \boldsymbol{A}\, \boldsymbol{z}, \qquad \boldsymbol{z}(0) = \boldsymbol{z}_0 \in \mathbb{R}^{\mathrm{N}}$$
$$\boldsymbol{y} = \boldsymbol{C}^{\boldsymbol{T}}\, \boldsymbol{z}$$

➔ estimation of the **lifted state z**

Linear observability analysis (approximation)

Linear dual Koopman system

$$\dot{f} = A^* f, \qquad f(0) = k_{x_0} \in \mathcal{F}$$
$$y = C_h f = \langle f, h \rangle$$

➔ estimation of an **evaluation functional** k_x s.t. $\langle k_x, f \rangle = f(x)\ \forall f$

Sufficient spectral observability condition

$$\langle h, \psi_{\lambda_k} \rangle \neq 0 \quad \forall k \in \{1, \dots, n\}$$

where ψ_{λ_k} are the principal eigenfunctions of A^*

[Mohet, AM, and Winkin, arXiv:2503.08345]

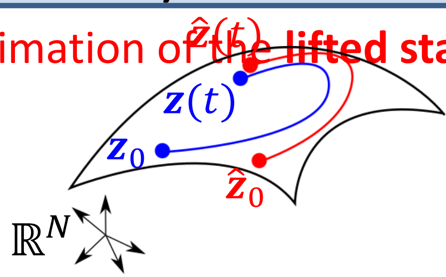
State estimation can be performed with the lifted dynamics or in infinite dimension

Linear lifted dynamics

$$\dot{\boldsymbol{z}} = \boldsymbol{A}\,\boldsymbol{z}, \qquad \boldsymbol{z}(0) = \boldsymbol{z}_0 \in \mathbb{R}^N$$
$$\boldsymbol{y} = \boldsymbol{C}^T \boldsymbol{z}$$

Finite-dimensional Luenberger observer (or Kalman filter)

➔ estimation of the **lifted state z**



[Surana and Banaszuk, NOLCOS 2016]
[Surana, CDC 2016]

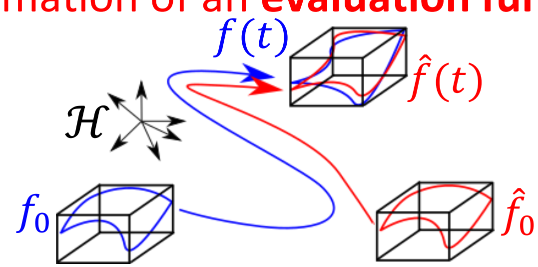
Linear dual Koopman system

$$\dot f = A^* f, \qquad f(0) = k_{x_0} \in \mathcal{F}$$
$$y = C_h f = \langle f, h \rangle$$

Infinite-dimensional Luenberger observer

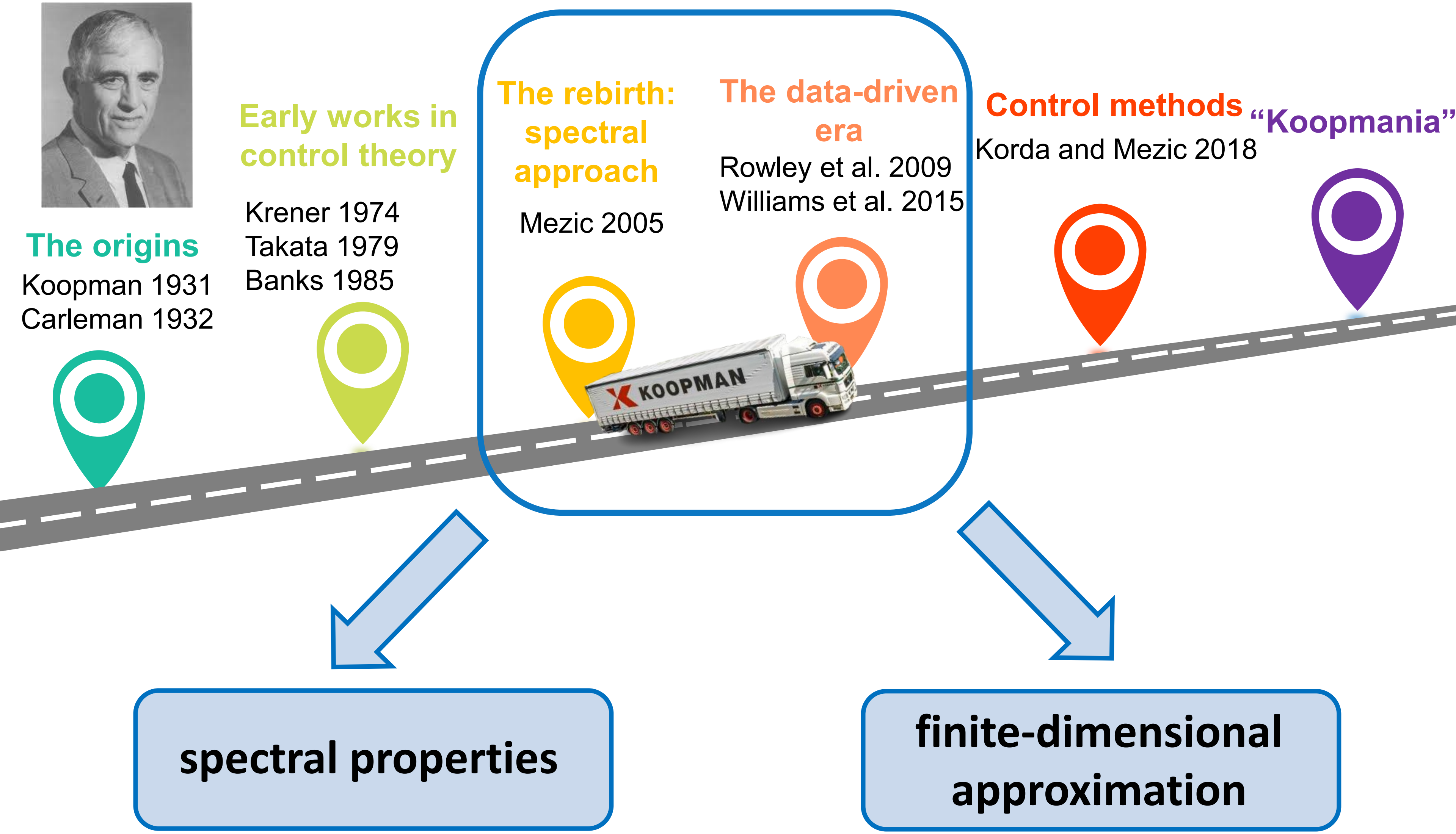
$$\dot{\hat f} = A^* \hat f + L\big(C_h \hat f - y\big), \qquad \hat f(0) = k_{\hat x_0}$$

➔ estimation of an **evaluation functional k_x**



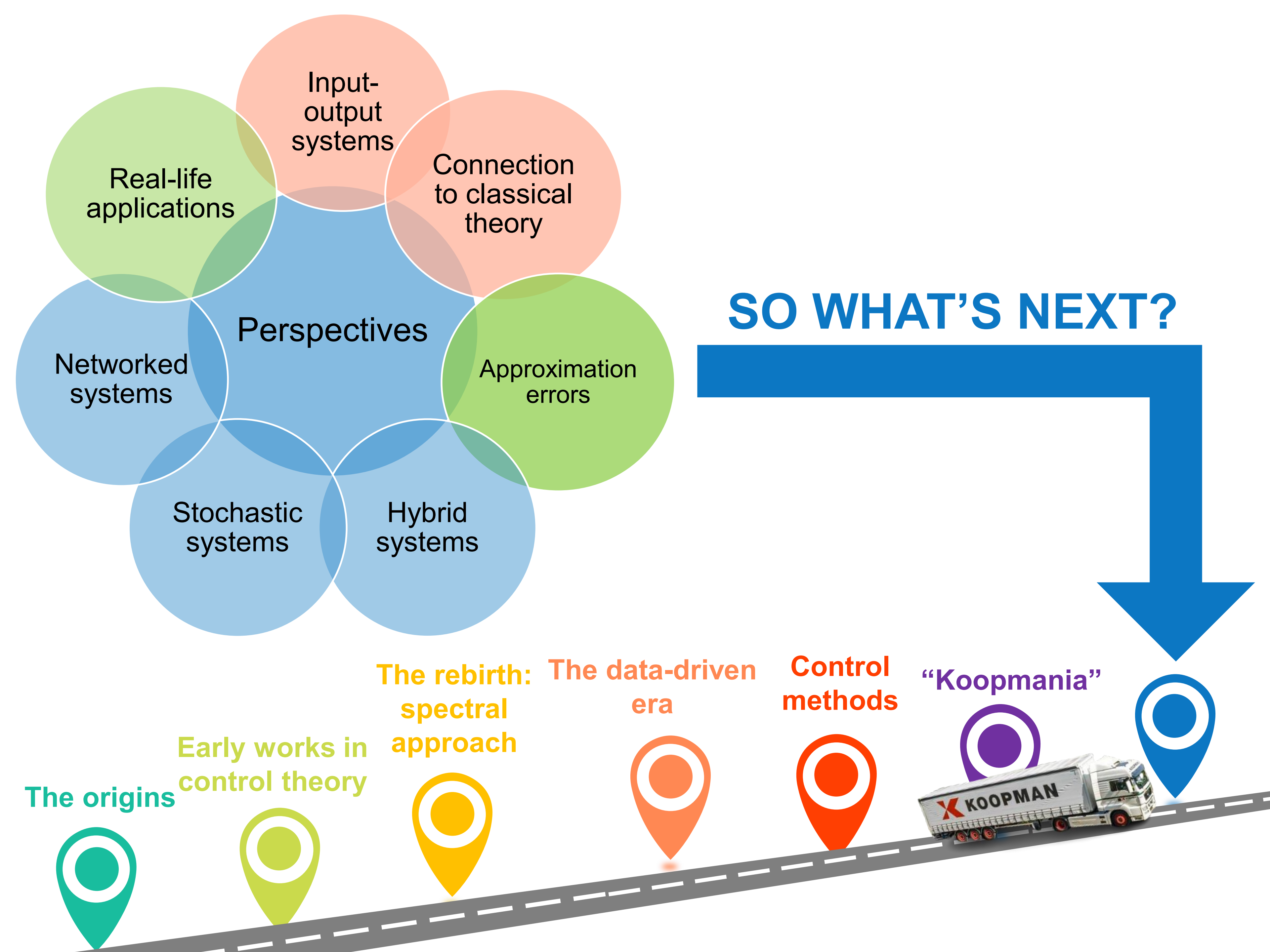
[Mohet, AM, and Winkin, arXiv:2503.08345]

➔ J. Mohet's talk today at 15:30 (session WeB7 - Árna 4)



Common mistakes and pitfalls

Applications to stability analysis and state estimation



Input-output systems
Real-life applications
Connection to classical theory
Perspectives
Networked systems
Approximation errors
Stochastic systems
Hybrid systems
SO WHAT'S NEXT?
The origins
Early works in control theory
The rebirth: spectral approach
The data-driven era
Control methods
"Koopmania"
KOOPMAN

Koopman Operator Theory for Autonomous Systems

Alexandre Mauroy *(University of Namur, Belgium)*

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