**Technische Universität Ilmenau**
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# Optimal control of differential-algebraic systems via Lur'e equations

Masterarbeit zur Erlangung des akademischen Grades

**Master of Science**

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# Kurzfassung

Diese Arbeit ist eine ausführliche Aufbereitung des Papers "The Kalman-Yakubovich-Popov inequality for differential-algebraic systems"von Timo Reis, Olaf Rendel und Matthias Voigt aus dem Jahre 2015. Mit diesen Resultaten ist es unser Ziel, das linear-quadratische Optimalsteuerungsproblem mit differentiell-algebraischen Nebenbedingungen handhabbar zu machen. Dem Vorgehen liegt das Kalman-Yakubovich-Popov Lemma zugrunde, welches die positive Semidefinitheit der Popov-Funktion auf der Imaginärachse mit der Lösbarkeit einer linearen Matrixungleichung verknüpft. Das Auffinden spezieller Lösungen führt zum Konzept der Lur'e Gleichung, welche wiederum mithilfe von abnehmenden Unterräumen gewisser Matrixbüschel gelöst werden kann. Diese Lösungen ermöglichen es, sowohl den optimalen Kostenwert zu bestimmen als auch die Lösung des Optimalsteuerungsproblems zu charakterisieren.

# Abstract

This thesis is an elaboration of the paper "The Kalman-Yakubovich-Popov inequality for differential-algebraic systems" by Timo Reis, Olaf Rendel, and Matthias Voigt from 2015. Based on their results, we aim to handle the linear-quadratic optimal control problem with differential-algebraic constraints. The considered approach uses the Kalman-Yakubovich-Popov lemma, which relates the positive semi-definiteness of the Popov function on the imaginary axis to the solvability of a linear matrix inequality. Particular solutions of this inequality are provided by the Lur'e equation, which in turn can be solved via deflating subspaces of certain matrix pencils. These solutions enable both the calculation of the optimal costs and the characterization of the solution of the optimal control problem.

# Contents

# 1 Introduction

We consider the linear-quadratic optimal control problem for differential-algebraic equations (DAEs) based on the paper

> T. Reis, O. Rendel, and M. Voigt: The Kalman-Yakubovich-Popov inequality for differential algebraic systems, *Linear Algebra Appl.* 485, pp. 153-193, 2015.

The task is to minimize the quadratic cost functional

$$\mathcal{J}(x,u) = \int_0^\infty \begin{pmatrix} x(t) \\ u(t) \end{pmatrix}^* \begin{bmatrix} Q & S \\ S^* & R \end{bmatrix} \begin{pmatrix} x(t) \\ u(t) \end{pmatrix} \, \mathrm{d}t$$

subject to a linear differential-algebraic system

$$\tfrac{\mathrm{d}}{\mathrm{d}\,t} \left( Ex(t) \right) = Ax(t) + Bu(t)$$

and further technical constraints. In this context $E$, $A$, $B$ are the system matrices and $Q$, $R$, $S$ are weighting matrices.

To handle the optimal control problem the elaborated approach is based on the rational matrix-valued Popov function

$$\Phi(s) := \begin{bmatrix} (-\bar{s}E - A)^{-1}B \\ I_m \end{bmatrix}^* \begin{bmatrix} Q & S \\ S^* & R \end{bmatrix} \begin{bmatrix} (sE - A)^{-1}B \\ I_m \end{bmatrix},$$

a matrix inequality called Kalman-Yakubovich-Popov inequality, and the so-called Lur'e equation. These concepts enable us to characterize the cost functional $\mathcal{J}$ and the minimal costs.

## Outline of the thesis

To get an overview of the connections of the main results of this thesis, they are collected and visualized in Figure 1.1.

In Chapter 2 we provide the mathematical preliminaries to understand the following chapters. Besides introducing differential-algebraic systems, we explain pencils and equivalence relations on them. They lead to a special pencil form called the feedback equivalence form, which is fundamental for later proofs. Based on regular pencils/systems we present system properties, like behavioural controllability and behavioural stabilizability. Further, we set them into relation to the well-known properties for ordinary differential equations (ODEs). To study relations on a certain subspace we introduce the system space in Section 2.4.

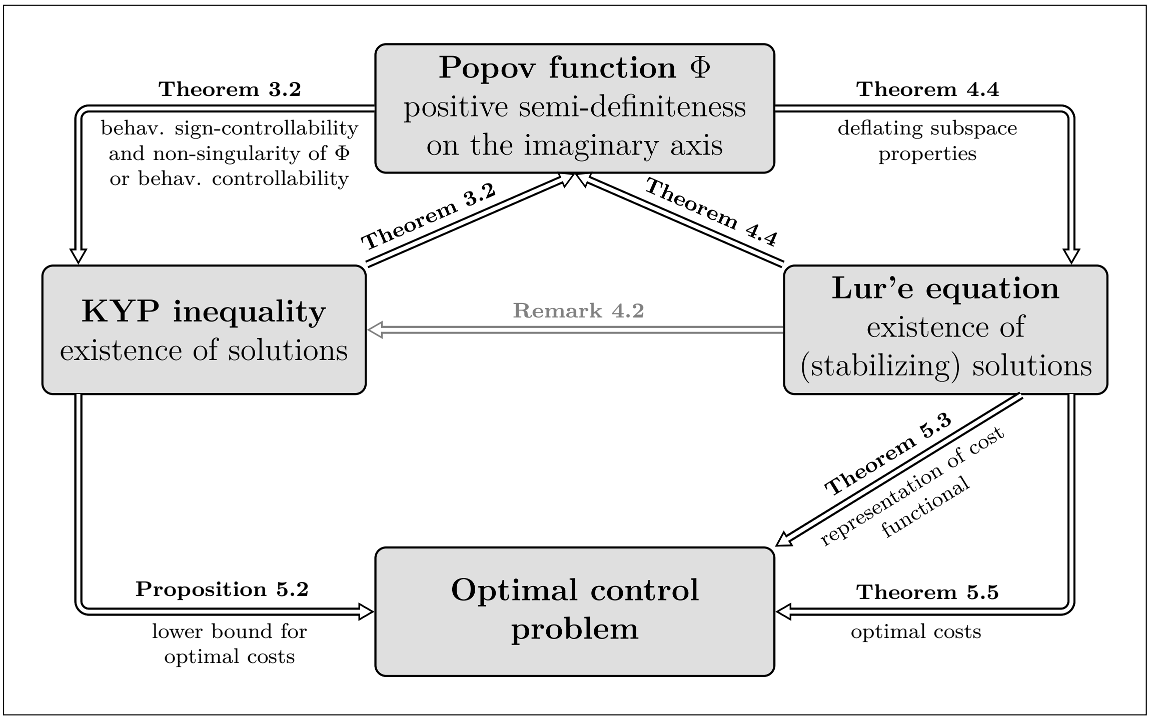


Figure 1.1: Overview of the main results

Chapter 3 is dedicated to the **Kalman-Yakubovich-Popov (KYP) inequality**. We state Theorem 3.2, also known as the KYP lemma for differential-algebraic systems. It provides a relationship between positive semi-definiteness of the Popov function on the imaginary axis and the existence of a solution of the KYP inequality. The theorem is proved at the end of the chapter. For this proof we use the KYP lemma for ODE systems. To apply this ODE result we relate the solvability of the KYP inequality of a DAE system to that of an associated ODE system.

Special solutions of the KYP inequality are studied in Chapter 4. They are, loosely speaking, rank-minimizing and solve the **Lur'e equation**. Theorem 4.4 provides a characterization of the solvability of this equation via deflating subspaces of corresponding even matrix pencils. To prove Theorem 4.4 we use an equivalent result for ODEs again, cf. Section 4.3. This result and some further statements about the existence of solutions are content of Section 4.4. A guideline to construct a solution of the Lur'e equation via deflating subspaces is given and illustrated by means of an example in Section 4.6.

Finally, in Chapter 5 we summarize the results to point out the consequences for the linear-quadratic **optimal control problem**. We introduce the optimal control problem and the associated optimal value function that determines the minimal costs for each admissible initial value. Together with a solution of the KYP inequality, Proposition 5.2 provides that the absolute value of this function is finite for every initial value. A solution of the Lur'e equation provides even more: in Theorem 5.3 an explicit characterization of the optimal value function is derived. If in addition the solution of the Lur'e equation is stabilizing, Theorem 5.5 yields that the optimal costs can be determined without knowing the optimal solution. Furthermore, the optimal solution can be characterized by a differential-algebraic boundary value problem.

Whenever possible we refer to well-known results for ODEs. For this reason, we study the relation between DAEs and ODEs several times, for instance in Lemma 3.12 and Lemma 4.15.

## Our contribution

To offer graduates and non-experts a self-contained version of [RRV15] we carry out intermediate steps more explicitly. For instance in the proof of Theorem 4.4 we illustrate the proofs structure by means of figures. Furthermore, auxiliary calculations are elaborated, e.g. the $\widehat{\mathcal{E}}_{\mathrm{F}}$-neutrality of the $(n+m)$-dimensional subspace $\operatorname{im} \widehat{Y}_{\mathrm{F}}$ and the proof of (4.31), where only literature was provided.

Moreover, we restructure the given proofs by dividing them into several parts to guarantee transparency. In particular, by using the notation of premimages we give the proof of Proposition 2.43 an entirely new structure. Further, we provide some additional findings that make the topic more accessible, such as Proposition 2.10, Proposition 2.17, Proposition 2.19, Remark 2.33, Proposition 2.38, Proposition 2.40, Proposition 2.45, Proposition 3.8 and Lemma 3.9.

Additionally to the contents in [RRV15], we motivate the concept of pencils via Laplace transformation in Section 2.2. Furthermore, we illustrate the findings with several examples and elaborate the guidelines to construct a solution of the Lur'e equation.

# 2 Mathematical preliminaries

This chapter is designed to set a fundamental understanding for differential-algebraic systems (DAE systems). In the first section DAE systems are introduced. Related on this, the meaning of a solution trajectory and the set of admissible initial values is explained. Most of the time we do not refer to the DAE system but to the associated matrix pencil. We illustrate this concept in Section 2.2. A special class of pencils are regular pencils, on which we place strong focus in this thesis. We further define an equivalence relation called feedback equivalence, which leads to a special form of pencils called feedback equivalence form. In Section 2.3 several system properties for DAE systems are characterized and with them properties of the feedback equivalence form are described. This form is an important tool for later proofs. Finally, in Section 2.4 we introduce the system space and discuss some of its attributes. Based on this space the so-called KYP inequality and the Lur'e equation are evaluated in the following chapters.

## 2.1 Introduction to DAE systems

Subject of this work are differential-algebraic systems, also known as descriptor systems, of the form

$$\tfrac{\mathrm{d}}{\mathrm{d}t}\,(Ex(t)) = Ax(t) + Bu(t) \tag{2.1}$$

where $E$, $A \in \mathbb{K}^{\ell\times n}$ and $B \in \mathbb{K}^{\ell\times m}$ (where $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$). Often the system is denoted by $[E, A, B]$. The function $u : \mathbb{R} \to \mathbb{K}^m$ is called *input* of the system and $x(t) \in \mathbb{K}^n$ is called *(generalized) state* of the system at time $t \in \mathbb{R}$.

One could argue that it is not correct to call $u$ input. The common understanding of "input" is that the components are free variables. However, due to the implicit nature of (2.1), it might be that some components of $u$ are uniquely determined, while some others are free. Furthermore, the input function $u$ needs to be sufficiently smooth. The following example illustrates these facts.

**Example 2.1.** Consider the differential-algebraic initial value control problem

$$\tfrac{\mathrm{d}}{\mathrm{d}t}\left(\begin{bmatrix}0&1&0\\0&0&0\\0&0&0\end{bmatrix}\begin{pmatrix}x_1(t)\\x_2(t)\\x_3(t)\end{pmatrix}\right) = \begin{bmatrix}1&0&0\\0&1&0\\0&0&0\end{bmatrix}\begin{pmatrix}x_1(t)\\x_2(t)\\x_3(t)\end{pmatrix} + \begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}\begin{pmatrix}u_1(t)\\u_2(t)\\u_3(t)\end{pmatrix},$$

which is equivalent to the decoupled system

$$\dot{x}_2(t) = x_1(t) + u_1(t), \quad 0 = x_2(t) + u_2(t), \quad 0 = u_3(t).$$

We see that $x_3$ is not restricted at all and the condition $u_3 \equiv 0$ holds true. Thus, the third component of $u$ is no input in the usual sense. Furthermore, $u_2 = -x_2$ holds and hence $\dot{u}_2 = -x_1 - u_1$, i.e. $u_2$ needs to be differentiable. $\Diamond$

As some components of $u$ might be uniquely determined already, we have to enlarge the term of solutions. In the context of DAEs, a *solution trajectory* is a tuple $(x,u) : \mathbb{R} \to \mathbb{K}^n \times \mathbb{K}^m$ of locally $\mathcal{L}^2$-integrable functions $x$ and $u$ that fulfil

$$Ex \in \mathcal{AC}_{\mathrm{loc}}(\mathbb{R} \to \mathbb{K}^n) \quad \text{and} \quad \tfrac{\mathrm{d}}{\mathrm{d}t}(Ex) \in \mathcal{L}^2_{\mathrm{loc}}(\mathbb{R}, \mathbb{K}^n) \tag{2.2}$$

and solve (2.1) for almost all $t \in \mathbb{R}$. The set of solution trajectories $(x,u)$ induces the *behaviour* of (2.1), which is

$$\mathcal{B}_{[E,A,B]} := \left\{ (x,u) \in \mathcal{L}^2_{\mathrm{loc}}(\mathbb{R}, \mathbb{K}^n) \times \mathcal{L}^2_{\mathrm{loc}}(\mathbb{R}, \mathbb{K}^m) \,\middle|\, \begin{array}{c} (x,u) \text{ fulfils (2.2)} \\ \text{and solves (2.1)} \\ \text{for almost all } t \in \mathbb{R} \end{array} \right\}. \tag{2.3}$$

Studying ordinary differential equations (ODEs) often $u \in \mathcal{L}^2_{\mathrm{loc}}(\mathbb{R}_{\geq 0}, \mathbb{K}^m)$ is considered. The extension to the negative time axis in the DAE case works completely analogous. We will see that the concepts of stabilizability and controllability on $\mathbb{R}$ are generalizations of that on $\mathbb{R}_{\geq 0}$ (cf. Remark 2.28).

**Remark 2.2.** For a system $[E,A,B]$ it is sometimes necessary to consider the corresponding *backward system* $[-E,A,B]$. Their behaviours are related as follows:

$$(x(\cdot), u(\cdot)) \in \mathcal{B}_{[E,A,B]} \quad \Leftrightarrow \quad (x(-\cdot), u(-\cdot)) \in \mathcal{B}_{[-E,A,B]}.$$

$\diamondsuit$

Based on the behaviour a space containing admissible initial values is defined.

**Definition 2.3.** For a given system $[E,A,B]$, where $E$, $A \in \mathbb{K}^{\ell \times n}$ and $B \in \mathbb{K}^{\ell \times m}$, the space

$$\mathcal{V}_{\mathrm{diff}} := \left\{ x_0 \in \mathbb{K}^n \,\middle|\, \exists (x,u) \in \mathcal{B}_{[E,A,B]} \text{ such that } Ex(0) = Ex_0 \right\}$$

is called *space of consistent initial differential variables* of $[E,A,B]$. $\diamondsuit$

**Remark 2.4.** Consider the ODE case, i.e. $E = I$. Since for all $x_0 \in \mathbb{K}^n$ the homogeneous initial value problem

$$\tfrac{\mathrm{d}}{\mathrm{d}t} x(t) = Ax(t), \quad x(0) = x_0$$

has a solution $x \in \mathcal{C}^1(\mathbb{R}, \mathbb{K}^n)$, it holds that $\mathcal{V}_{\mathrm{diff}} = \mathbb{K}^n$. $\diamondsuit$

## 2.2 Pencils and equivalences

Instead of differential-algebraic equations we often study an associated matrix pencil. To explain the derivation of this concept we need Laplace transformation.

**Definition 2.5** (Laplace transformation)**.** Denote $\mathbb{C}_{>\alpha} := \{ s \in \mathbb{C} \mid \mathrm{Re}(s) > \alpha \}$ for $\alpha \in \mathbb{R}$ and $\mathcal{D}_\alpha(\mathbb{K}^n) := \{ f \in \mathcal{L}^1_{\mathrm{loc}}(\mathbb{R}_{\geq 0}, \mathbb{K}^n) \mid \mathrm{e}^{-\alpha\cdot} f(\cdot) \in \mathcal{L}^1(\mathbb{R}_{\geq 0}, \mathbb{K}^n) \}$. Then the Laplace transformation is defined by

$$\begin{aligned} \mathcal{L} : \mathcal{D}_\alpha(\mathbb{K}^n) &\to \{ F : \mathbb{C}_{>\alpha} \to \mathbb{C}^n \}, \\ f(\cdot) &\mapsto \left( s \mapsto \mathcal{L}(f)(s) := \int_0^\infty f(t)\mathrm{e}^{-st} dt \right). \end{aligned}$$

◊

Laplace transformation provides some useful properties.

**Lemma 2.6.** *The following statements hold true:*

(i) *The Laplace transformation $\mathcal{L}$ is linear.*

(ii) *If $f \in \mathcal{D}_\alpha(\mathbb{K}^n)$ is differentiable and $f'$ denotes its derivative, it holds that*

$$\mathcal{L}(f')(s) = s\mathcal{L}(f)(s) - f(0), \quad \mathrm{Re}\, s > \alpha.$$

**Proof:** See [LR14, Appendix A.4]. □

Laplace transformation enables us to deduce matrix pencils from a DAE system. Consider system (2.1) and assume that $x$ and $u$ are Laplace transformable. Then we get $sE\,\mathcal{L}(x)(s) - Ex_0 = A\,\mathcal{L}(x)(s) + B\,\mathcal{L}(u)(s)$ and thus

$$(sE - A)\,\mathcal{L}(x)(s) = B\,\mathcal{L}(u)(s) + Ex_0.$$

Define functions $\hat{x}(s) := \mathcal{L}(x(\cdot))(s)$ and $\hat{u}(s) := \mathcal{L}(u(\cdot))(s)$ and set $x_0 = 0$. Then it holds $(sE - A)\,\hat{x}(s) = B\,\hat{u}(s)$, which can be transformed into

$$\begin{bmatrix} sE - A & -B \end{bmatrix} \begin{pmatrix} \hat{x}(s) \\ \hat{u}(s) \end{pmatrix} = 0.$$

To make statements on properties of DAE systems we often refer to their algebraic characterizations. For that reason it is convenient to study associated matrix pencils. This work actually deals with matrices of the form $\begin{bmatrix} sE - A & -B \end{bmatrix}$ instead of the associated DAEs. In case that no input is considered, meaning a homogeneous system, the matrix reduces to $[sE - A]$. Those matrices lead to the concept of pencils. A *(matrix) pencil* is a matrix polynomial of degree at most one. They can be divided into two classes.

**Definition 2.7.** A pencil $sE - A \in \mathbb{K}[s]^{\ell \times n}$ is called

(i) *regular* if $\ell = n$ and $\mathrm{rk}_{\mathbb{K}(s)}(sE - A) = n$,

(ii) *singular* if $\ell \neq n$ or $\mathrm{rk}_{\mathbb{K}(s)}(sE - A) < n$.

The DAE system (2.1) is called *regular (singular)* if, and only if, the corresponding matrix pencil is regular (singular). Further, a complex number $\lambda \in \mathbb{C}$ is called *(generalized) eigenvalue* of the pencil $sE - A$ if

$$\operatorname{rk}(\lambda E - A) < \operatorname{rk}_{\mathbb{K}(s)}(sE - A).$$

If $E$ has one or more eigenvalues $\lambda = 0$, then the pencil is said to have one or more *eigenvalues at infinity.* $\Diamond$

Note that in the ODE case, i.e. $E = I$, there exist no eigenvalues at infinity.

**Remark 2.8.** As we use different types of *ranks* we give a short overview on our notions.

(i) The rank, denoted by rk, of a matrix $H \in \mathbb{K}^{m \times k}$ is defined by the number of linear independent rows/columns, resp.

(ii) The rank $\operatorname{rk}_{\mathbb{K}(s)}$ of $H(s) \in \mathbb{K}(s)^{m \times k}$ over the field of rational functions is defined analogously to (i), where linear independence is considered with respect to $\mathbb{K}(s)$.

(iii) In some papers the rank $\operatorname{rk}_{\mathbb{K}(s)}$ is called *normalrank.* We use this expression when referring to such literature.

(iv) For regular pencils the condition $\operatorname{rk}_{\mathbb{K}(s)}(sE - A) = n$ can be proved by

$$\det(sE - A) \in \mathbb{K}[s] \backslash \{0_{\mathbb{K}[s]}\}.$$

Therefore, these conditions are used equivalently. Moreover, for a polynomial matrix $H(s) \in \mathbb{K}[s]^{\ell \times n}$ the equality

$$\operatorname{rk}_{\mathbb{K}(s)} H(s) = \operatorname{rk}_{\mathbb{K}[s]} H(s)$$

holds, whereby $\operatorname{rk}_{\mathbb{K}[s]} H(s) := \max_{\lambda \in \mathbb{K}} \operatorname{rk} H(\lambda)$. $\Diamond$

On the set of pencils $\left\{ sE - A \;\middle|\; E, A \in \mathbb{K}^{\ell \times n} \right\}$ we define an equivalence relation. Thus, it suffices to show properties for one representative of each equivalence class.

**Definition 2.9.** Two pencils $sE_1 - A_1$, $sE_2 - A_2 \in \mathbb{K}[s]^{\ell \times n}$ are called *pencil equivalent* if

$$\exists\, W \in \mathrm{Gl}_\ell(\mathbb{K})\; \exists\, T \in \mathrm{Gl}_n(\mathbb{K}): \quad W(sE_1 - A_1)T = sE_2 - A_2.$$

We write $sE_1 - A_1 \simeq sE_2 - A_2$ or if required $sE_1 - A_1 \overset{W,T}{\simeq} sE_2 - A_2$. $\Diamond$

**Proposition 2.10.** *The relation $\simeq$ defined above is an equivalence relation on the set of pencils $\left\{ sE - A \;\middle|\; E, A \in \mathbb{K}^{\ell \times n} \right\}$.*

**Proof:** Reflexivity is clear for $W = I_\ell$ and $T = I_n$. Symmetry is also clear because $sE_1 - A_1 \overset{W,T}{\simeq} sE_2 - A_2$ if, and only if, $sE_2 - A_2 \overset{W^{-1},T^{-1}}{\simeq} sE_1 - A_1$. Transitivity holds, as for $sE_1 - A_1 \overset{W_1,T_1}{\simeq} sE_2 - A_2$ and $sE_2 - A_2 \overset{W_2,T_2}{\simeq} sE_3 - A_3$ it is obvious that $sE_1 - A_1 \overset{W_2W_1,T_1T_2}{\simeq} sE_3 - A_3$. □

**Remark 2.11.** Regularity is invariant under pencil equivalence. This statement can be verified easily: if $sE_1 - A_1 \overset{W,T}{\simeq} sE_2 - A_2$ for $W, T \in \mathrm{Gl}_n(\mathbb{R})$ and $sE_1 - A_1$ is regular, then we have

$$\det(sE_1 - A_2) = \det(W(sE_1 - A_1)T) = \det(W)\,\det(sE_1 - A_1)\,\det(T) \neq 0.$$

In other words: $(sE_1 - A_1)$ is regular if, and only if, $(sE_2 - A_2)$ is regular. ◊

In each equivalence class of a regular pencil $sE - A \in \mathbb{K}[s]^{n\times n}$ there is one with a simple form, where the pencil decomposes into two parts: an ODE system and a system of linear (algebraic) equations.

**Definition 2.12.** A pencil $sE - A \in \mathbb{K}[s]^{n\times n}$ is said to be in *quasi Weierstraß form (QWF)* if, and only if, it has the following structure:

$$sE - A = s\begin{bmatrix} I_{\tilde{n}_1} & 0 \\ 0 & E_{22}\end{bmatrix} - \begin{bmatrix} A_{11} & 0 \\ 0 & I_{\tilde{n}_2}\end{bmatrix} = \begin{bmatrix} sI_{\tilde{n}_1} - A_{11} & 0 \\ 0 & sE_{22} - I_{\tilde{n}_2}\end{bmatrix}, \tag{2.4}$$

where $E_{22} \in \mathbb{K}^{\tilde{n}_2\times\tilde{n}_2}$ is nilpotent, $A_{11} \in \mathbb{K}^{\tilde{n}_1\times\tilde{n}_1}$, and $n = \tilde{n}_1 + \tilde{n}_2$ with $\tilde{n}_1, \tilde{n}_2 \in \mathbb{N}_0$. ◊

**Remark 2.13.** In [BIT12, Remark 2.7 (ii)] it is shown how the matrices $A_{11}$ and $E_{22}$ can be calculated by using Wong sequences. Furthermore, [BIT12, Remark 2.7 (iii)] provides that $A_{11}$ and $E_{22}$ are unique up to similarity. ◊

The quasi Weierstraß form with the nilpotent matrix $E_{22}$ motivate the following definition.

**Definition 2.14.** The *index* of a regular pencil $sE - A$ is defined by

$$\mathrm{ind}(sE - A) = \begin{cases} 0, & \text{if } E \text{ is regular} \\ \mathrm{nil\,ind}\, E_{22}, & \text{else}\end{cases},$$

where $\mathrm{nil\,ind}\, H := \min\left\{\, k \in \mathbb{N} \;\middle|\; H^k = 0 \,\right\}$ denotes the nilpotency index of a nilpotent matrix $H \in \mathbb{K}^{\ell\times\ell}$, $\ell \in \mathbb{N}$. ◊

Between the QWF (2.4) and regularity of a pencil there is a crucial connection, more precisely: equivalence to a pencil in QWF characterizes regularity.

**Theorem 2.15.** *Let $sE - A \in \mathbb{K}[s]^{n\times n}$. Then it holds that*

$$sE - A \text{ is regular} \quad \Leftrightarrow \quad sE - A \simeq s\begin{bmatrix} I_{\tilde{n}_1} & 0 \\ 0 & N\end{bmatrix} - \begin{bmatrix} J & 0 \\ 0 & I_{\tilde{n}_2}\end{bmatrix} \text{ in QWF (2.4)},$$

*where $\tilde{n}_1$ is unique.*

**Proof:** In [BIT12, Theorem 2.6] it is proved that regularity of a pencil implies the existence of an equivalent pencil in QWF (2.4) with unique $\tilde{n}_1$. □

Besides pencil equivalence we introduce further equivalence relations, which include the input. Based on feedback equivalence later we will show that in every equivalence class there exists a pencil in feedback-equivalence form. This pencil has a "simple" structure and is used in nearly every proof of the following chapters.

**Definition 2.16.** Let $[E_1, A_1, B_1]$, $[E_2, A_2, B_2] \in \mathbb{K}^{n\times n} \times \mathbb{K}^{n\times n} \times \mathbb{K}^{n\times m}$ be given such that $sE_1 - A_1$ and $sE_2 - A_2$ are regular. The systems are called

(i) *system equivalent* if there exist $W, T \in \mathrm{Gl}_n(\mathbb{K})$ such that

$$W \begin{bmatrix} sE_1 - A_1 & B_1 \end{bmatrix} \begin{bmatrix} T & 0 \\ 0 & I_m \end{bmatrix} = \begin{bmatrix} sE_2 - A_2 & B_2 \end{bmatrix}.$$

We write $[E_1, A_1, B_1] \overset{W,T}{\simeq_{se}} [E_2, A_2, B_2]$.

(ii) *feedback equivalent* if there exist $W, T \in \mathrm{Gl}_n(\mathbb{K})$ and $F \in \mathbb{K}^{m\times n}$ such that

$$W \begin{bmatrix} sE_1 - A_1 & B_1 \end{bmatrix} \begin{bmatrix} T & 0 \\ -FT & I_m \end{bmatrix} = \begin{bmatrix} sE_2 - A_2 & B_2 \end{bmatrix}. \tag{2.5}$$

We write $[E_1, A_1, B_1] \overset{W,T,F}{\simeq_{fe}} [E_2, A_2, B_2]$. ♢

**Proposition 2.17.** *System and feedback equivalence are equivalence relations.*

**Proof:** To prove these two equivalence relations it suffices to show that feedback equivalence is an equivalence relation. The statement for system equivalence then follows by choosing $F = 0_{m\times n}$.
Let regular systems $[E_1, A_1, B_1]$, $[E_2, A_2, B_2] \in \mathbb{K}^{n\times n} \times \mathbb{K}^{n\times n} \times \mathbb{K}^{n\times m}$ be feedback equivalent. According to (2.5) there exist $W, T \in \mathrm{Gl}_n(\mathbb{K})$, $F \in \mathbb{K}^{m\times n}$ such that

$$\begin{bmatrix} W(sE_1 - (A_1 + B_1F))T & WB_1 \end{bmatrix} = \begin{bmatrix} sE_2 - A_2 & B_2 \end{bmatrix}.$$

Reflexivity is clear for matrices $W = I_n$, $T = I_n$ and $F = 0_{m\times n}$. For symmetry we have $[E_1, A_1, B_1] \overset{W,T,F}{\simeq_{fe}} [E_2, A_2, B_2]$. Starting from equation (2.5), inverting matrices yields $\begin{bmatrix} sE_1 - A_1 & B_1 \end{bmatrix} = W^{-1} \begin{bmatrix} sE_2 - A_2 & B_2 \end{bmatrix} \begin{bmatrix} T^{-1} & 0 \\ (FT^{-1})T & I_m \end{bmatrix}$. Thereby for matrices $\tilde{W} = W^{-1}$, $\tilde{T} = T^{-1}$, $\tilde{F} = -FT^{-1}$ the equivalence $[E_2, A_2, B_2] \overset{\tilde{W},\tilde{T},\tilde{F}}{\simeq_{fe}} [E_1, A_1, B_1]$ holds. With $[E_1, A_1, B_1] \overset{W_1,T_1,F_1}{\simeq_{fe}} [E_2, A_2, B_2]$ and $[E_2, A_2, B_2] \overset{W_2,T_2,F_2}{\simeq_{fe}} [E_3, A_3, B_3]$ transitivity can be verified through

$$\begin{bmatrix} sE_3 - A_3 & B_3 \end{bmatrix} = W_2 \begin{bmatrix} sE_2 - A_2 & B_2 \end{bmatrix} \begin{bmatrix} T_2 & 0 \\ -F_2T_2 & \end{bmatrix}$$

$$
\begin{aligned}
&= W_2 W_1 \begin{bmatrix} sE_1 - A_1 & B_1 \end{bmatrix} \begin{bmatrix} T_1 & 0 \\ -F_1 T_1 & I_m \end{bmatrix} \begin{bmatrix} T_2 & 0 \\ -F_2 T_2 & I_m \end{bmatrix} \\
&= W_2 W_1 \begin{bmatrix} sE_1 - A_1 & B_1 \end{bmatrix} \begin{bmatrix} T_1 T_2 & 0 \\ -(F_1 + F_2 T_1^{-1}) T_1 T_2 & I_m \end{bmatrix}.
\end{aligned}
$$

Using matrices $W_3 = W_2 W_1$, $T_3 = T_1 T_2 \in \mathrm{Gl}_n(\mathbb{K})$ and $F_3 = F_1 + F_2 T_1^{-1} \in \mathbb{K}^{m \times n}$ yields $[E_1, A_1, B_1] \overset{W_3,T_3,F_3}{\simeq_{fe}} [E_3, A_3, B_3]$. □

**Remark 2.18.** In general, regularity is not invariant under feedback equivalence. For example, let the regular system $[E, A, B] \in \mathbb{K}^{n \times n} \times \mathbb{K}^{n \times n} \times \mathbb{K}^{n \times m}$ with $\operatorname{rk} E < n$ and $\operatorname{rk} B = n$ be given. Then the transformation matrices $W, T \in \mathrm{Gl}_n(\mathbb{K})$ and $F := -B^+ A$, where $B^+$ denotes the right inverse of $B$, lead to

$$W(A + BF)T = W(A - BB^+ A)T = 0$$

and thus

$$\det(sWET - W(A + BF)T) = \det(sWET) = \det(sE) = 0.$$

Nevertheless, regularity of $[E, A, B] \in \mathbb{K}^{n \times n} \times \mathbb{K}^{n \times n} \times \mathbb{K}^{n \times m}$ and the additional assumption on $sE - (A + BF)$ to be regular guarantee regularity of the system $[WET, W(A + BF)T, WB]$. ♢

We will often consider the solution behaviour and the space of consistent initial differential values. As it is easier to consider pencils with a "simple" structure, it is necessary to show the connections between the original behaviour and the behaviour of the system/feedback equivalent pencil (cf. [Ber14, p. 108]).

**Proposition 2.19.** *Let* $[E_1, A_1, B_1], [E_2, A_2, B_2] \in \mathbb{K}^{n \times n} \times \mathbb{K}^{n \times n} \times \mathbb{K}^{n \times m}$ *be regular.*

(a) *If* $[E_1, A_1, B_1] \overset{W,T}{\simeq_{se}} [E_2, A_2, B_2]$, *then it holds that*

  (a1) $(Tx, u) \in \mathcal{B}_{[E_1,A_1,B_1]} \quad \Leftrightarrow \quad (x, u) \in \mathcal{B}_{[E_2,A_2,B_2]}$ *and*

  (a2) $T^{-1} \mathcal{V}_{\mathrm{diff}}^{[\mathrm{E}_1,\mathrm{A}_1,\mathrm{B}_1]} = \mathcal{V}_{\mathrm{diff}}^{[\mathrm{E}_2,\mathrm{A}_2,\mathrm{B}_2]}$.

(b) *If* $[E_1, A_1, B_1] \overset{W,T,F}{\simeq_{fe}} [E_2, A_2, B_2]$, *then it holds that*

  (b1) $(Tx, FTx + u) \in \mathcal{B}_{[E_1,A_1,B_1]} \quad \Leftrightarrow \quad (x, u) \in \mathcal{B}_{[E_2,A_2,B_2]}$ *and*

  (b2) $T^{-1} \mathcal{V}_{\mathrm{diff}}^{[E_1,A_1,B_1]} = \mathcal{V}_{\mathrm{diff}}^{[E_2,A_2,B_2]}$.

**Proof:** Statement (a) considers system equivalent systems, whereas (b) contains feedback equivalent systems. Pairwise, the proofs run analogously. As feedback equivalence is of greater interest in this work, we prove the implications in (b).

(b1) Let $[E_1, A_1, B_1] \overset{W,T,F}{\simeq_{fe}} [E_2, A_2, B_2]$, i.e.

$$[E_2, A_2, B_2] = [WE_1T, W(A_1 + B_1F)T, WB_1]. \tag{2.6}$$

To show the equivalence the proof is split into two parts.

(A) Let $(Tx, FTx + u) \in \mathcal{B}_{[E_1,A_1,B_1]}$ be given, i.e.

(A1) $(Tx, FTx + u) \in \mathcal{L}^2_{\text{loc}}(\mathbb{R}, \mathbb{K}^n) \times \mathcal{L}^2_{\text{loc}}(\mathbb{R}, \mathbb{K}^m)$,

(A2) $\frac{\mathrm{d}}{\mathrm{d}t}(E_1 Tx) \in \mathcal{L}^2_{\text{loc}}(\mathbb{R}, \mathbb{K}^n)$ and

(A3) $\frac{\mathrm{d}}{\mathrm{d}t}(E_1 Tx) \overset{\text{a.e.}}{=} A_1Tx + B_1(FTx + u)$.

Since $T$ is regular and according to (A1) the function $Tx$ is locally $\mathcal{L}^2$, it holds that $x \in \mathcal{L}^2_{\text{loc}}(\mathbb{R}, \mathbb{K}^n)$. Even $u$ is in $\mathcal{L}^2_{\text{loc}}(\mathbb{R}, \mathbb{K}^n)$ due to linearity and (A1). Because of (A2) together with regularity of $W$ we obtain that $\frac{\mathrm{d}}{\mathrm{d}t}(WE_1 Tx)$ is in $\mathcal{L}^2_{\text{loc}}(\mathbb{R}, \mathbb{K}^n)$. From (A3) it can be inferred that $(x, u)$ fulfils the DAE belonging to the system $[WE_1T, W(A_1 + B_1F)T, WB_1]$. Thus, applying (2.6), we have $(x, u) \in \mathcal{B}_{[E_2,A_2,B_2]}$.

(B) Let $(x, u) \in \mathcal{B}_{[E_2,A_2,B_2]}$ be given, i.e.

(B1) $(x, u) \in \mathcal{L}^2_{\text{loc}}(\mathbb{R}, \mathbb{K}^n) \times \mathcal{L}^2_{\text{loc}}(\mathbb{R}, \mathbb{K}^m)$,

(B2) $\frac{\mathrm{d}}{\mathrm{d}t}(WE_1T\, x) \in \mathcal{L}^2_{\text{loc}}(\mathbb{R}, \mathbb{K}^n)$ and

(B3) $\frac{\mathrm{d}}{\mathrm{d}t}(WE_1T\, x) \overset{a.e.}{=} W(A_1 + B_1F)Tx + WB_1u = WA_1Tx + WB_1(FTx + u)$

hold. (B1) and linearity yield $(Tx, FTx+u) \in \mathcal{L}^2_{\text{loc}}(\mathbb{R}, \mathbb{K}^n) \times \mathcal{L}^2_{\text{loc}}(\mathbb{R}, \mathbb{K}^m)$. Since (B2) holds and $W$ is regular the function $\frac{\mathrm{d}}{\mathrm{d}t}(E_1T\, x)$ has to be in $\mathcal{L}^2_{\text{loc}}(\mathbb{R}, \mathbb{K}^n)$. Multiplying the inverse of $W$ to the right side of (B3) we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}(E_1Tx) \overset{a.e.}{=} A_1(Tx) + B_1(FTx + u),$$

which means $(Tx, FTx + u)$ fulfils the DAE $\frac{\mathrm{d}}{\mathrm{d}t}(E_1\tilde{x}) = A_1\tilde{x} + B_1\tilde{u}$. Thus, we conclude $(Tx, FTx + u) \in \mathcal{B}_{[E_1,A_1,B_1]}$.

(b2) Let $\hat{x} \in T^{-1}\mathcal{V}_{\text{diff}}^{[\mathrm{E}_1,\mathrm{A}_1,\mathrm{B}_1]}$. There exists some $x_0 \in \mathcal{V}_{\text{diff}}^{[\mathrm{E}_1,\mathrm{A}_1,\mathrm{B}_1]}$ with $\hat{x} = T^{-1}x_0$. In other words, there exists $(x, u) \in \mathcal{B}_{[\mathrm{E}_1,\mathrm{A}_1,\mathrm{B}_1]}$ such that $E_1x(0) = E_1x_0$ and $\hat{x} = T^{-1}x_0$. Applying (ii) first and using the regularity of $W$, leads to

$$\exists\, (x, u) \in \mathcal{B}_{[\mathrm{E}_2,\mathrm{A}_2,\mathrm{B}_2]} : \quad WE_1Tx(0) = WE_1x_0 \text{ and } \hat{x} = T^{-1}x_0.$$

According to the definition of feedback equivalence it is $E_2 = WE_1T$ and we obtain $E_2x(0) = E_2T^{-1}x_0$ and $\hat{x} = T^{-1}x_0$ for a tupel $(x, u) \in \mathcal{B}_{[\mathrm{E}_2,\mathrm{A}_2,\mathrm{B}_2]}$. Substituting $T^{-1}x_0$ with $\hat{x}$ leads to $\hat{x} \in \mathcal{V}_{\text{diff}}^{[\mathrm{E}_2,\mathrm{A}_2,\mathrm{B}_2]}$. Therefore, the proof is completed.

□

**Remark 2.20.** An equivalent form of statement (ii) in Proposition 2.19 is

$$(x,u) \in \mathcal{B}_{[E_1,A_1,B_1]} \quad \Leftrightarrow \quad (T^{-1}x, u - Fx) \in \mathcal{B}_{[E_2,A_2,B_2]}.$$

This can be verified easily by substitutions $\tilde{x} = Tx$ and $\tilde{u} = FTx + u$. ♢

Previously, we considered the QWF (2.4), which is a simple structure of pencils associated with homogeneous systems. Analogously, there exists a simple form for inhomogeneous systems, where again the pencil is decomposed into its ODE part and an algebraic part (cf. QWF). Due to the influence of the input the algebraic part is further divided into two parts, which provides additional information about the structure of the pencil.

**Definition 2.21.** The system $[E, A, B]$ with $E, A \in \mathbb{K}^{n\times n}$ and $B \in \mathbb{K}^{n\times m}$ is said to be in *feedback equivalence form* (FEF) if

$$\begin{bmatrix} sE - A & B \end{bmatrix} = \begin{bmatrix} sI_{n_1} - A_{11} & 0 & 0 & B_1 \\ 0 & -I_{n_2} & sE_{23} & B_2 \\ 0 & 0 & sE_{33} - I_{n_3} & 0 \end{bmatrix}, \tag{2.7}$$

where $E_{33} \in \mathbb{K}^{n_3\times n_3}$ is nilpotent. ♢

Later we will show that this form can be derived by feedback equivalence transformation for every regular pencil.

**Remark 2.22.** Note that the pencil $sE - A$ of a system $[E, A, B]$ in FEF (2.7) is regular: Since $E_{33}$ is nilpotent, it holds

$$\det(sE - A) = (-1)^{n_2+n_3} \det(sI_{n_1} - A_{11}) \neq 0_{\mathbb{K}[s]}.$$

♢

To ensure the existence of some solution of DAE (2.1) with regard to the optimal control problem (see Chapter 5) some additional assumptions are required. In [KM06, Theorem 2.28] it is shown that for a sufficiently smooth inhomogeneity $f$ regularity of the pencil $sE - A$ suffices to guarantee the existence and uniqueness of some $x$ such that

$$\frac{\mathrm{d}}{\mathrm{d}t}\,(Ex(t)) = Ax(t) + f(t) \tag{2.8}$$

holds.

**Theorem 2.23.** *Let $E, A \in \mathbb{K}^{n\times n}$ commute, i.e.*

$$EA = AE, \tag{2.9}$$

*and satisfy*

$$\ker E \cap \ker A = \emptyset. \tag{2.10}$$

*Furthermore, let* $f \in \mathbb{C}^{\nu}(\mathbb{R}, \mathbb{K}^n)$ *with* $\nu = \operatorname{ind}(sE - A)$ *and* $t_0 \in \mathbb{R}$ *be given. Then* $x \in \mathcal{C}^1(\mathbb{R}, \mathbb{C}^n)$ *defined by*

$$x(t) = \int_{t_0}^{t} \mathrm{e}^{E^D A(t-\tau)} E^D f(\tau)\,\mathrm{d}\tau - (I - E^D E) \sum_{i=0}^{\nu-1} (EA^D)^i A^D f^{(i)}(t)$$

*is a particular solution of* (2.8).

**Remark 2.24.** The theorem above requires some explanation.

(a) The matrix $E^D$ denotes the so-called Drazin inverse of $E$, which is uniquely determined (see [KM06, Theorem 2.19]).

(b) "Using a nice trick due to Campbell (see [Cam80]), the general case (i.e., without (2.9)) can be easily reduced to the special case."[KM06, page 32]

(c) Note that for commuting matrices $E$ and $A$, condition (2.10) is equivalent to the regularity of the pencil $sE - A$ (cf. [KM06]). $\diamondsuit$

For this purpose we examine regular systems only. Starting from now we merely consider systems of the kind

$$\tfrac{\mathrm{d}}{\mathrm{d}t}\left(Ex(t)\right) = Ax(t) + Bu(t),$$

where $E,\ A \in \mathbb{K}^{n\times n}$ and $B \in \mathbb{K}^{n\times m}$ such that the pencil $sE - A \in \mathbb{K}[s]^{n\times n}$ is regular. (2.11)

**Notion:** $[E, A, B] \in \Sigma_{n,m}(\mathbb{K})$.

## 2.3 Controllability and stabilizability

Controllability and stabilizability of DAE systems are important concepts for this thesis. As the considered system has a different structure, the concepts differ slightly from those we already know for ODEs. At first, we state the definitions for DAE systems and then, we mention how they correspond to ODE systems. With these concepts we are able to show that FEF can be derived through feedback equivalence transformation. Furthermore, we prove important statements for FEF.

**Definition 2.25.** A system $[E, A, B] \in \Sigma_{n,m}(\mathbb{K})$ with the space of consistent initial differential variables $\mathcal{V}_{\mathrm{diff}}$ is called

(a) *impulse controllable* if

$$\forall\, x_0 \in \mathbb{K}^n \;\exists\, (x,u) \in \mathcal{B}_{[E,A,B]} \text{ with } Ex(0) = Ex_0,$$

which equals $\mathcal{V}_{\text{diff}} = \mathbb{K}^n$;

(b) *behaviourally stabilizable* if

$$\forall\, (x,u) \in \mathcal{B}_{[E,A,B]} \;\exists\, (\hat{x},\hat{u}) \in \mathcal{B}_{[E,A,B]} :$$
$$(x,u)\mid_{(-\infty,0)} = (\hat{x},\hat{u})\mid_{(-\infty,0)} \text{ and } \lim_{t\to\infty} \operatorname{ess\,sup}_{\tau>t} \|(\hat{x}(\tau),\hat{u}(\tau))\| = 0;$$

(c) *behaviourally anti-stabilizable* if

$$\forall\, (x,u) \in \mathcal{B}_{[E,A,B]} \;\exists\, (\hat{x},\hat{u}) \in \mathcal{B}_{[E,A,B]} :$$
$$(x,u)\mid_{(0,\infty)} = (\hat{x},\hat{u})\mid_{(0,\infty)} \text{ and } \lim_{t\to-\infty} \operatorname{ess\,sup}_{\tau<t} \|(\hat{x}(\tau),\hat{u}(\tau))\| = 0;$$

(d) *behaviourally controllable* if

$$\forall\, (x_1,u_1),(x_2,u_2) \in \mathcal{B}_{[E,A,B]} \;\exists\, T>0, (x,u) \in \mathcal{B}_{[E,A,B]}$$
$$\text{with } (x(t),u(t)) = \begin{cases} (x_1(t),u_1(t)) & \text{for } t<0,\\ (x_2(t),u_2(t)) & \text{for } t>T. \end{cases}$$

$\diamondsuit$

**Remark 2.26.** Let a measure space $(\mathcal{X},\mathcal{L},\mu)$ and a Banach space $\mathcal{Y}$ be given. Note that the *essential supremum* of a measurable function $f:\mathcal{X}\to\mathcal{Y}$ on a subset $X\subset\mathcal{X}$ is defined by

$$\operatorname{ess\,sup}_{x\in X}\|f(x)\| := \inf_{N\subseteq X,\mu(N)=0}\ \sup_{x\in X\setminus N}\|f(x)\|.$$

Considering Definition 2.25 (b), (c) we have $\mathcal{X}=\mathbb{R}$, $\mathcal{Y}=\mathbb{K}^{n+m}$ and $\mu=\lambda$, the Lebesgue measure. $\diamondsuit$

The definitions in Definition 2.25 are analytical, but there are algebraic characterizations as well. Actually, these are so-called Hautus criteria, which will turn out to be quite useful later.

**Proposition 2.27.** *Let the system* $[E,A,B]\in\Sigma_{n,m}(\mathbb{K})$ *be given and* $r=\operatorname{rk}(E)$. *Let* $S_\infty\in\mathbb{K}^{n\times(n-r)}$ *be a matrix with* $\operatorname{im}(S_\infty)=\ker(E)$. *Then* $[E,A,B]$ *is*

| | | | | |
|---|---|---|---|---|
| (a) | *impulse controllable* | $\Leftrightarrow$ | | $\operatorname{rk}\begin{bmatrix}E & AS_\infty & B\end{bmatrix} = n,$ |
| (b) | *behaviourally stabilizable* | $\Leftrightarrow$ | $\forall\lambda\in\overline{\mathbb{C}_+}:$ | $\operatorname{rk}\begin{bmatrix}\lambda E-A & B\end{bmatrix} = n,$ |
| (c) | *behaviourally anti-stabilizable* | $\Leftrightarrow$ | $\forall\lambda\in\overline{\mathbb{C}_-}:$ | $\operatorname{rk}\begin{bmatrix}\lambda E-A & B\end{bmatrix} = n,$ |
| (d) | *behaviourally controllable* | $\Leftrightarrow$ | $\forall\lambda\in\mathbb{C}:$ | $\operatorname{rk}\begin{bmatrix}\lambda E-A & B\end{bmatrix} = n.$ |

We need these characterizations, but they are only a technical tool for later proofs. Therefore, we simply state them here and refer to the literature for further information and proofs. Berger proves these criteria of Hautus type in [Ber14, Sec. 3.3] in a more general fashion. In Remark 3.3.5 he discusses the criterion of impulse controllability, while Remark 3.3.6 contains behavioural controllability. The characterization of behavioural controllability is shown in [PW97, Theorem 5.2.10] as well. So is behavioural stabilizability in [PW97, Theorem 5.2.30]. The equivalence for behavioural anti-stabilizability then follows easily through considering the backward system.

**Remark 2.28.** Compare the criteria (b), (c), and (d) in Proposition 2.27 with the well-known Hautus criteria for ODEs. For $E = I$ it is clear that behavioural controllability and behavioural (anti-)stabilizability reduce to the concepts of controllability and (anti-)stabilizability. $\diamondsuit$

**Remark 2.29.** Proposition 2.27 yields that $[E, A, B]$ is behaviourally stabilizable if, and only if, the backward system $[-E, A, B]$ is behaviourally anti-stabilizable. $\diamondsuit$

By Proposition 2.27 (d) a system $[I, A, B] \in \Sigma_{n,m}(\mathbb{K})$ obviously is impulse controllable, i.e. $\mathcal{V}_{\text{diff}} = \mathbb{K}^n$. Note that this does not hold true in general.

**Example 2.30.** Consider the system $[E, A, B]$ with $n = 3$, $m = 1$ and system matrices

$$E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad A = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \quad \text{and} \quad B = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix},$$

which yields

$$\dot{x}_1 \overset{\text{a.e.}}{=} x_3, \quad x_2 \overset{\text{a.e.}}{=} 0, \quad \text{and} \quad \dot{x}_2 \overset{\text{a.e.}}{=} u$$

for any solution $(x, u) \in \mathcal{B}_{[E,A,B]}$.
First, note that for $S_\infty = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}^\top$ we obtain

$$\operatorname{rk} \begin{bmatrix} E & AS_\infty & B \end{bmatrix} = \operatorname{rk} \begin{bmatrix} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} = 2 < n,$$

i.e. $[E, A, B]$ is not impulse controllable.
However, $x_1 \in \mathcal{AC}(\mathbb{R}, \mathbb{K})$ is arbitrary and $x_3 \in \mathcal{L}^2_{\text{loc}}(\mathbb{R}, \mathbb{K})$ with $x_3 \overset{\text{a.e.}}{=} \dot{x}_1$. Further, it is $x_2 \in \mathcal{AC}(\mathbb{R}, \mathbb{K})$ with $x_2 \overset{\text{a.e.}}{=} 0$, i.e. $x_2 = 0$. Hence, $u \in \mathcal{L}^2_{\text{loc}}(\mathbb{R}, \mathbb{K})$ with $u \overset{\text{a.e.}}{=} 0$ holds true. Therefore, we conclude $\mathcal{V}_{\text{diff}} = \mathbb{K} \times \{0\} \times \mathbb{K}$. $\diamondsuit$

Based on the algebraic characterizations we define uncontrollable modes and introduce the concept of behavioural sign-controllability, which is a purely linear algebraic condition. It does not have an interpretation in terms of the behaviour $\mathcal{B}_{[E,A,B]}$. We use this concept later for the KYP lemma, see 3.2.

**Definition 2.31.** Let the system $[E, A, B] \in \Sigma_{n,m}(\mathbb{K})$ be given.

(a) A number $\lambda \in \mathbb{C}$ is called an *uncontrollable mode* of the system $[E, A, B]$ if

$$\operatorname{rk}\begin{bmatrix}\lambda E - A & B\end{bmatrix} < n.$$

(b) The system $[E, A, B]$ is called *behavioural sign-controllable* if for all $\lambda \in \mathbb{C}$ it holds:

$$\operatorname{rk}\begin{bmatrix}\lambda E - A & B\end{bmatrix} = n \quad \text{or} \quad \operatorname{rk}\begin{bmatrix}-\bar{\lambda} E - A & B\end{bmatrix} = n.$$

◊

**Example 2.32.** Consider the DAE system (2.1) with system matrices

$$E = \begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}, \quad A = \begin{bmatrix}1 & 0\\ 1 & 1\end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix}0\\ 1\end{bmatrix},$$

where $n = 2$ and $m = 1$. Since for $\lambda_1 = 1$ it holds that

$$\operatorname{rk}\begin{bmatrix}\lambda_1 E - A & B\end{bmatrix} = \operatorname{rk}\begin{bmatrix}0 & 0 & 0\\ -1 & -1 & 1\end{bmatrix} = 1 < 2 = n,$$

$\lambda_1$ is an uncontrollable mode of the system $[E, A, B]$. Hence, $[E, A, B]$ is not behaviourally controllable. However, due to

$$\operatorname{rk}\begin{bmatrix}-\bar{\lambda}_1 E - A & B\end{bmatrix} = \operatorname{rk}\begin{bmatrix}-2 & 0 & 0\\ -1 & -1 & 1\end{bmatrix} = 2 = n$$

and

$$\operatorname{rk}\begin{bmatrix}\lambda E - A & B\end{bmatrix} = 2 = n \qquad \forall\, \lambda \in \mathbb{C} \setminus \{1\}$$

the system is behaviourally sign-controllable. ◊

**Remark 2.33.** The system properties impulse controllability, behavioural (anti-) stabilizability, behavioural (sign-)controllability, and the set of uncontrollable modes are invariant under feedback equivalence. This follows from Proposition 2.27:

(i) Impulse controllability: Let $[E_1, A_1, B_1] \overset{W,T,F}{\simeq}_{fe} [E_2, A_2, B_2]$ and $S_{\infty,1} \in \mathbb{K}^{n\times(n-r)}$ with $\operatorname{im} S_{\infty,1} = \ker E_1$ and $S_{\infty,2} \in \mathbb{K}^{n\times(n-r)}$ with $\operatorname{im} S_{\infty,2} = \ker E_2$ be given. At first, we show that $\operatorname{im} S_{\infty,2} = \operatorname{im}(T^{-1}S_{\infty,1})$ holds. Due to $E_2 = WE_1T$ with $W \in \mathrm{Gl}_n(\mathbb{K})$ we have $\ker E_2 = \ker WE_1T = \ker E_1T$. Thus, it is

$$\begin{aligned} x \in \ker E_2 \Leftrightarrow x \in \ker E_1T &\Leftrightarrow Tx \in \ker E_1\\ &\Leftrightarrow Tx = S_{\infty,1}y \text{ for some } y \in \mathbb{K}^{n-r}\\ &\Leftrightarrow x = T^{-1}S_{\infty,1}y \in \operatorname{im}(T^{-1}S_{\infty,1}). \end{aligned}$$

Hence, we obtain

$$
\begin{aligned}
n &= \operatorname{rk}\begin{bmatrix} E_1 & A_1 S_{\infty_1} & B_1 \end{bmatrix} \\
\Leftrightarrow n &= \operatorname{rk}\left( W \begin{bmatrix} E_1 & A_1 S_{\infty,1} & B_1 \end{bmatrix} \begin{bmatrix} T & 0 & 0 \\ 0 & I_{n-r} & 0 \\ 0 & FTS_{\infty,2} & I_m \end{bmatrix} \right) \\
&= \operatorname{rk}\begin{bmatrix} WE_1T & W(A_1 S_{\infty,1} + B_1 FTS_{\infty,2}) & WB_1 \end{bmatrix} \\
&= \operatorname{rk}\begin{bmatrix} WE_1T & W(A_1 TS_{\infty,2} + B_1 FTS_{\infty,2}) & WB_1 \end{bmatrix} \\
&= \operatorname{rk}\begin{bmatrix} E_2 & A_2 S_{\infty,2} & B_2 \end{bmatrix}.
\end{aligned}
$$

(ii) From

$$
\operatorname{rk}\begin{bmatrix} \lambda E - A & B \end{bmatrix} = \operatorname{rk} W \begin{bmatrix} \lambda E - A & B \end{bmatrix} \begin{bmatrix} T & 0 \\ FT & I_m \end{bmatrix}
$$

it follows invariance of behavioural (anti-)stabilizability, behavioural (sign-) controllability and the set of uncontrollable modes. $\diamondsuit$

Feedback equivalence, which we introduced in Definition 2.16, enables us to show some statements on FEF (2.7).

**Proposition 2.34.** *Let $[E, A, B] \in \Sigma_{n,m}(\mathbb{K})$ be given. Then there exist matrices $W, T \in \mathrm{Gl}_n(\mathbb{K})$ and $F \in \mathbb{K}^{m \times n}$ such that*

$$
W \begin{bmatrix} sE - A & B \end{bmatrix} \begin{bmatrix} T & 0 \\ FT & I_m \end{bmatrix} = \begin{bmatrix} sI_{n_1} - A_{11} & 0 & 0 & B_1 \\ 0 & -I_{n_2} & sE_{23} & B_2 \\ 0 & 0 & sE_{33} - I_{n_3} & 0 \end{bmatrix}, \tag{2.12}
$$

*where $E_{33} \in \mathbb{K}^{n_3 \times n_3}$ is nilpotent, i.e. $[WET, W(A + BF)T, WB]$ is in FEF (2.7). Moreover, $n_1$ is unique.*
*Furthermore, the following statements hold true:*

(a) $(x, u) \in \mathcal{B}_{[E,A,B]} \Leftrightarrow (x_1, u - Fx) \in \mathcal{B}_{[I_{n_1}, A_{11}, B_1]}$, *where* $x = T \begin{pmatrix} x_1 \\ B_2(Fx - u) \\ 0 \end{pmatrix}$.

(b) *The space of consistent initial differential variables fulfils*

$$
\mathcal{V}_{\mathrm{diff}} = T \left( \mathbb{K}^{n_1 + n_2} \times \ker \begin{bmatrix} E_{23} \\ E_{33} \end{bmatrix} \right).
$$

(c) *It holds that*

$$
\operatorname{rk}\begin{bmatrix} \lambda E - A & B \end{bmatrix} = n_2 + n_3 + \operatorname{rk}\begin{bmatrix} \lambda I_{n_1} - A_{11} & B_1 \end{bmatrix} \quad \forall \lambda \in \mathbb{C}.
$$

*In particular, $\lambda \in \mathbb{C}$ is an uncontrollable mode of $[E, A, B]$ if, and only if, $\lambda$ is an uncontrollable mode of $[I_{n_1}, A_{11}, B_1]$.*

(d) *If* $[E, A, B]$ *is impulse controllable, then* $W$*,* $T$*, and* $F$ *can be chosen such that* $n_3 = 0$*.*

To prove the existence of matrices leading to FEF, we use a result for system equivalence. Similar to FEF, there exists a simple form. This result is provided in [IR17, Proposition 2.9].

**Definition 2.35.** A system $[E, A, B] \in \Sigma_{n,m}(\mathbb{K})$ is said to be in *system equivalence form (SEF)* if the pencil has the structure

$$\begin{bmatrix} sE - A & B \end{bmatrix} = \begin{bmatrix} sE_{11} - A_{11} & sE_{12} - A_{12} & B_1 \\ 0 & sN - I_k & 0 \end{bmatrix}, \tag{2.13}$$

where $N \in \mathbb{K}^{k \times k}$ is nilpotent and $[E_{11}, A_{11}, B_1]$ is impulse controllable. $\Diamond$

**Proposition 2.36.** *Let* $[E, A, B] \in \Sigma_{n,m}(\mathbb{K})$ *be given. Then there exist matrices* $W, T \in \mathrm{Gl}_n(\mathbb{K})$ *such that the system* $[WET, WAT, WB]$ *is in SEF* (2.13)*, i.e.*

$$W \begin{bmatrix} sE - A & B \end{bmatrix} \begin{bmatrix} T & 0 \\ 0 & I_m \end{bmatrix} = \begin{bmatrix} sE_{11} - A_{11} & sE_{12} - A_{12} & B_1 \\ 0 & sN - I_k & 0 \end{bmatrix},$$

*where* $N \in \mathbb{K}^{k \times k}$ *is nilpotent.*

**Proof of Proposition 2.34:** In a first step we show the existence of some matrices $W, T \in \mathrm{Gl}_n(\mathbb{K})$ and $F \in \mathbb{K}^{m \times n}$ such that (2.12) holds. According to [IR17, Prop. 2.12] there exist some matrices $\hat{W}, \hat{T} \in \mathrm{Gl}_n(\mathbb{K})$ and $\hat{F} \in \mathbb{K}^{m \times n}$ fulfilling

$$\hat{W} \begin{bmatrix} sE - A & B \end{bmatrix} \begin{bmatrix} \hat{T} & 0 \\ -\hat{F}\hat{T} & I_m \end{bmatrix} = \begin{bmatrix} sI_{n_1} - A_{11} & 0 & sE_{13} - A_{13} & B_1 \\ 0 & -I_{n_2} & sE_{23} - A_{23} & B_2 \\ 0 & 0 & sN - I_{n_3} & 0 \end{bmatrix} \tag{2.14}$$

for some nilpotent $N \in \mathbb{K}^{n_3 \times n_3}$. To achieve (2.12), a way to eliminate $E_{13}$, $A_{13}$ and $A_{23}$ is shown. Then, in a second step, statements (a) to (d) are proved.

Step 1.1: By Proposition 2.36, for $[E, A, B] \in \Sigma_{n,m}(\mathbb{K})$ there exist $W_1, T_1 \in \mathrm{Gl}_n(\mathbb{K})$ such that

$$W_1(sE - A)T_1 = \begin{bmatrix} s\tilde{E}_{11} - \tilde{A}_{11} & s\tilde{E}_{12} - \tilde{A}_{12} \\ 0 & sN - I_{n_3} \end{bmatrix}, \quad WB = \begin{bmatrix} \tilde{B}_1 \\ 0 \end{bmatrix}, \tag{2.15}$$

where $[\tilde{E}_{11}, \tilde{A}_{11}, \tilde{B}_1]$ is impulse controllable and $N \in \mathbb{K}^{n_3 \times n_3}$ is nilpotent. According to [BR13, Theorem 5.2], the system $[\tilde{E}_{11}, \tilde{A}_{11}, \tilde{B}_1]$ is impulse controllable if, and only if, there exists $\tilde{F}_1 \in \mathbb{K}^{m \times n}$ such that the index of $s\tilde{E}_{11} - (\tilde{A}_{11} + \tilde{B}_1\tilde{F}_1)$ is at most one. In case of $\mathrm{ind}(s\tilde{E}_{11} - (\tilde{A}_{11} + \tilde{B}_1\tilde{F}_1)) = 0$, by definition of index it holds $\det(\tilde{E}_{11}) \neq 0$ and therefore $(s\tilde{E}_{11} - (\tilde{A}_{11} + \tilde{B}_1\tilde{F}_1)) \simeq_{se} (sI - \bar{A})$. In case of $\mathrm{ind}(s\tilde{E}_{11} - (\tilde{A}_{11} + \tilde{B}_1\tilde{F}_1)) = 1$ we know accordingly to QWF that

$$(s\tilde{E}_{11} - (\tilde{A}_{11} + \tilde{B}_1\tilde{F}_1)) \simeq_{se} \begin{bmatrix} sI - J & 0 \\ 0 & sN - I \end{bmatrix},$$

where $N$ is nilpotent with $\operatorname{ind}(N) = 1$, which equals $N = 0$. Note that in both cases mentioned above $n_1$ is unique.
Summarized, due to system equivalence there exist some $\tilde{W}_1, \tilde{T}_1 \in \mathrm{Gl}_{n-n_3}(\mathbb{K})$ such that

$$\tilde{W}_1(s\tilde{E}_{11} - (\tilde{A}_{11} + \tilde{B}_1\tilde{F}_1))\tilde{T}_1 = \begin{bmatrix} sI_{n_1} - \bar{A} & 0 \\ 0 & -I_{n_2} \end{bmatrix}.$$

Finally $\hat{T} := T_1\tilde{T}_1$, $\hat{W} := \tilde{W}_1 W_1$ and $\hat{F} := \tilde{F}_1 T_1^{-1}$ yield (2.14).

Step 1.2: After we arrived at (2.14), we still need to show that $E_{13}$, $A_{13}$ and $A_{23}$ can be eliminated through fitting transformation matrices. In [BT13, Corollary 2.3], Berger and Trenn illustrate the existence of such matrices, where they use a normal form called *quasi-Kronecker form*. Therefore, there exist some matrices $W_2$, $T_2 \in \mathrm{Gl}_n(\mathbb{K})$ such that $W_2\hat{W}B = \hat{W}B$ and the matrices $E_{13}$, $A_{13}$ are eliminated in $W_2\hat{W}(sE - (A + B\hat{F}))\hat{T}T_2$. Further, the remaining matrix $A_{23}$ can be eliminated by a transformation of $W_2\hat{W}(sE - (A + B\hat{F}))\hat{T}T_2$ with $T_3 \in \mathrm{Gl}_n(\mathbb{K})$ from the right. Summarized, $W = W_2\hat{W}$, $T = \hat{T}T_2T_3$, and $F = \hat{F}$ yield (2.12).

Step 2: Now we prove the statements (a) to (d). Let $[E, A, B] \in \Sigma_{n,m}(\mathbb{K})$ be given and let $[E, A, B] \overset{W,T,F}{\simeq_{fe}} [E_{\mathrm{F}}, A_{\mathrm{F}}, B_{\mathrm{F}}]$.

(a) By using Remark 2.20 and $x = T\begin{pmatrix} x_1 \\ B_2(Fx - u) \\ 0 \end{pmatrix}$ we obtain

$$(x, u) \in \mathcal{B}_{[E,A,B]} \Leftrightarrow \left( \begin{pmatrix} x_1 \\ B_2(Fx - u) \\ 0 \end{pmatrix}, \quad u - Fx \right) \in \mathcal{B}_{[E_{\mathrm{F}}, A_{\mathrm{F}}, B_{\mathrm{F}}]}.$$

Calculating straight forward with the matrices $E_{\mathrm{F}}$, $A_{\mathrm{F}}$, $B_{\mathrm{F}}$ and using the definition of the behaviour as in (2.3) then leads to

$$(x, u) \in \mathcal{B}_{[E,A,B]} \quad \Leftrightarrow \quad \dot{x}_1 \in \mathcal{L}^2_{\mathrm{loc}}(\mathbb{R}, \mathbb{K}^n) \text{ and } \dot{x}_1 = A_{11}x_1 + B_1(u - Fx),$$

which is $(x_1, u - Fx) \in \mathcal{B}_{[I_{n_1}, A_{11}, B_1]}$.

(b) This assertion is a direct consequence of (a).

(c) Since $E_{33}$ is nilpotent, we have $\mathrm{rk}(\lambda E_{33} - I_{n_3}) = n_3$ for all $\lambda \in \mathbb{C}$. Therefore,

$$\begin{aligned} \mathrm{rk}\begin{bmatrix} \lambda E - A & B \end{bmatrix} &= \mathrm{rk}\begin{bmatrix} sI_{n_1} - A_{11} & 0 & 0 & B_1 \\ 0 & -I_{n_2} & sE_{23} & B_2 \\ 0 & 0 & sE_{33} - I_{n_3} & 0 \end{bmatrix} \\ &= n_2 + n_3 + \mathrm{rk}\begin{bmatrix} \lambda I_{n_1} - A_{11} & B_1 \end{bmatrix} \quad \forall \lambda \in \mathbb{C}. \end{aligned}$$

(d) Let $[E, A, B] \in \Sigma_{n,m}(\mathbb{K})$ be impulse controllable. Then the system already is in form of (2.15), where $n_3 = 0$. This completes the proof.

□

In the following chapters we study regular pencils with an arbitrary structure. In proofs we refer to pencils in FEF. Therefore, we switch between arbitrary pencils and FEF quite often. To make clear whether we talk about an arbitrary pencil or one in FEF, we use a special notation with an index F for a system in FEF.

Computing the left side of (2.12) leads to a polynomial matrix of the form

$$\begin{bmatrix} sWE_1T - W(A_1 + B_1F)T & WB_1 \end{bmatrix}.$$

Defining matrices

$$E_\mathrm{F} = WE_1T, \quad A_\mathrm{F} = W(A_1 + B_1F)T, \quad B_\mathrm{F} = WB_1$$

and using equation (2.12), we obtain

$$\begin{bmatrix} sE_\mathrm{F} - A_\mathrm{F} & B_\mathrm{F} \end{bmatrix} = \begin{bmatrix} sI_{n_1} - A_{11} & 0 & 0 & B_1 \\ 0 & -I_{n_2} & sE_{23} & B_2 \\ 0 & 0 & sE_{33} - I_{n_3} & 0 \end{bmatrix}.$$

In other words, the matrices of the system $[E_\mathrm{F}, A_\mathrm{F}, B_\mathrm{F}]$ in FEF (2.7) have the following structure:

$$E_\mathrm{F} = \begin{bmatrix} I_{n_1} & 0 & 0 \\ 0 & 0 & E_{23} \\ 0 & 0 & E_{33} \end{bmatrix}, \quad A_\mathrm{F} = \begin{bmatrix} A_{11} & 0 & 0 \\ 0 & I_{n_2} & 0 \\ 0 & 0 & I_{n_3} \end{bmatrix}, \quad B_\mathrm{F} = \begin{bmatrix} B_1 \\ B_2 \\ 0 \end{bmatrix}, \tag{2.16}$$

where $E_{33}$ is nilpotent.

**Remark 2.37.** Let $[E, A, B] \in \Sigma_{n,m}(\mathbb{K})$ in FEF (2.7) be given. Then $[E, A, B]$ is behaviourally stabilizable (anti-stabilizable, sign-controllable) if, and only if, the ODE system $[I_{n_1}, A_{11}, B_1]$ is stabilizable (anti-stabilizable, sign-controllable). This result follows from Proposition 2.34 (c) together with the algebraic characterisations in Proposition 2.27 and Remark 2.28. ◊

## 2.4 The system space

Previously we introduced the behaviour, which is the set that collects the solution trajectories $(x, u) \in \mathcal{L}^2_{\mathrm{loc}}(\mathbb{R}, \mathbb{K}^n) \times \mathcal{L}^2_{\mathrm{loc}}(\mathbb{R}, \mathbb{K}^m)$ of a system. This section is dedicated to the system space, which is a subspace containing vectors of the form $(x(t)^\top, u(t)^\top)^\top \in \mathbb{K}^{n+m}$ instead of functions. Based on this space we will consider linear matrix inequalities/equations in the following chapters.

**Proposition 2.38.** *Let $[E, A, B] \in \Sigma_{n,m}(\mathbb{K})$ be given. Then there exists a subspace $\mathcal{V}^* \subseteq \mathbb{K}^{n+m}$ such that*

$$\forall\, (x, u) \in \mathcal{B}_{[E,A,B]} \text{ and for almost all } t \in \mathbb{R} : \begin{pmatrix} x(t) \\ u(t) \end{pmatrix} \in \mathcal{V}^* \tag{2.17}$$

*and for all subspaces $\mathcal{V} \subseteq \mathbb{K}^{n+m}$ fulfilling* (2.17) *it holds $\mathcal{V}^* \subseteq \mathcal{V}$.*

**Proof:** Define

$$\mathcal{F} := \left\{ \mathcal{V} \subseteq \mathbb{K}^{n+m} \;\middle|\; \mathcal{V} \text{ is a linear subspace of } \mathbb{K}^{n+m} \text{ and fulfils (2.17)} \right\}.$$

Since $\mathbb{K}^{n+m} \in \mathcal{F}$ and hence $\mathcal{F}$ is non-empty, it suffices to show that $\mathcal{F}$ has a minimal element.

Let $\mathcal{V}_1, \mathcal{V}_2 \in \mathcal{F}$ be given. Then $\mathcal{V}_1 \cap \mathcal{V}_2$ is a linear subspace of $\mathbb{K}^{n+m}$. Furthermore, for all $(x, u) \in \mathcal{B}_{[E,A,B]}$ it holds that $(x(t)^\top, u(t)^\top)^\top \in \mathcal{V}_1$ for almost all $t \in \mathbb{R}$ and $(x(t)^\top, u(t)^\top)^\top \in \mathcal{V}_2$ for almost all $t \in \mathbb{R}$. Since the union of two null sets is a null set, we see that $(x(t)^\top, u(t)^\top)^\top \in \mathcal{V}_1 \cap \mathcal{V}_2$ for almost all $t \in \mathbb{R}$. Thus, $(\mathcal{F}, \cap)$ is a commutative semi-group, where all elements are idempotent, i.e. $\mathcal{V} \cap \mathcal{V} = \mathcal{V}$ for all $\mathcal{V} \in \mathcal{F}$.

Since $\mathcal{F} \subseteq \mathbb{K}^{n+m}$ is non-empty, there exists some $\mathcal{V}^* \in \mathcal{F}$ such that $\dim \mathcal{V}^* \leq \dim \mathcal{V}$ for all $\mathcal{V} \in \mathcal{F}$. Furthermore, for all $\mathcal{V} \in \mathcal{F}$ it holds that $\mathcal{V} \cap \mathcal{V}^* \in \mathcal{F}$ and hence

$$\dim \mathcal{V}^* \leq \dim(\mathcal{V} \cap \mathcal{V}^*) \leq \dim \mathcal{V}^*,$$

i.e. $\dim(\mathcal{V} \cap \mathcal{V}^*) = \dim \mathcal{V}^*$. Due to $\mathcal{V} \cap \mathcal{V}^* \subseteq \mathcal{V}^*$ this implies $\mathcal{V} \cap \mathcal{V}^* = \mathcal{V}^*$ and thus $\mathcal{V}^* \subseteq \mathcal{V}$ for all $\mathcal{V} \in \mathcal{F}$, i.e. $\mathcal{V}^*$ is a minimal element of $\mathcal{F}$. □

**Definition 2.39.** For $[E, A, B] \in \Sigma_{n,m}(\mathbb{K})$ the smallest subspace $\mathcal{V}_{\text{sys}} \subseteq \mathbb{K}^{n+m}$ such that (2.17) holds, is called *system space.* ♢

The system space of a system in FEF (2.7) can be characterized as follows.

**Proposition 2.40.** *Let $[E, A, B] \in \Sigma_{n,m}(\mathbb{K})$ in FEF* (2.7) *be given. Then the system space fulfils*

$$\mathcal{V}_{\text{sys}} = \ker \begin{bmatrix} 0 & I_{n_2} & 0 & B_2 \\ 0 & 0 & I_{n_3} & 0 \end{bmatrix}. \tag{2.18}$$

**Proof:** Since $[E, A, B]$ is in FEF (2.7), Proposition 2.34 (a) yields

$$(x, u) \in \mathcal{B}_{[E,A,B]} \Leftrightarrow (x_1, u) \in \mathcal{B}_{[I_{n_1}, A_{11}, B_1]} \text{ with } x = \begin{pmatrix} x_1 \\ -B_2 u \\ 0 \end{pmatrix},$$

i.e.

$$\mathcal{B}_{[E,A,B]} = \left\{ \left( \begin{pmatrix} x_1 \\ -B_2 u \\ 0 \end{pmatrix}, u \right) \;\middle|\; (x_1, u) \in \mathcal{B}_{[I_{n_1}, A_{11}, B_1]} \right\}.$$

Thus, the definition of $\mathcal{V}_{\text{sys}}$ provides

$$\begin{aligned}\mathcal{V}_{\text{sys}} &= \left\{ \begin{pmatrix} x_1 \\ -B_2 u \\ 0 \\ u \end{pmatrix} \in \mathbb{K}^{n+m} \;\middle|\; x_1 \in \mathbb{K}^{n_1}, u \in \mathbb{K}^m \right\} \\ &= \left\{ \begin{pmatrix} x \\ u \end{pmatrix} \in \mathbb{K}^{n+m} \;\middle|\; \begin{pmatrix} x_2 + B_2 u \\ x_3 \end{pmatrix} = 0 \right\} \\ &= \ker \begin{bmatrix} 0 & I_{n_2} & 0 & B_2 \\ 0 & 0 & I_{n_3} & 0 \end{bmatrix},\end{aligned}$$

which completes the proof. □

In Proposition 2.19 we related the behaviours of two systems, which are feedback equivalent. There is a relation between the system spaces of two feedback equivalent systems as well. This relation is treated in the following proposition.

**Proposition 2.41.** *Let $[E, A, B] \in \Sigma_{n,m}(\mathbb{K})$ with system space $\mathcal{V}_{\text{sys}} \subseteq \mathbb{K}^{n+m}$ and transformation matrices $W, T \in \mathrm{Gl}_n(\mathbb{K})$ and $F \in \mathbb{K}^{m\times n}$ be given such that the pencil $sE - (A + BF)$ is regular. Moreover, let $\mathcal{V}_{\text{sys,fe}}$ be the system space of the system $[WET, W(A+BF)T, WB] \in \Sigma_{n,m}(\mathbb{K})$. Then the system spaces are related as follows*

$$\mathcal{V}_{\text{sys}} = \begin{bmatrix} T & 0 \\ FT & I_m \end{bmatrix} \cdot \mathcal{V}_{\text{sys,fe}}. \tag{2.19}$$

**Proof:** Follows from Proposition 2.19 (b1). □

**Remark 2.42.** Let $[E, A, B] \in \Sigma_{n,m}(\mathbb{K})$ and transformation matrices $W, T \in \mathrm{Gl}_n(\mathbb{K})$ and $F \in \mathbb{K}^{m\times n}$ such that $[E_{\text{F}}, A_{\text{F}}, B_{\text{F}}] := [WET, W(A+BF)T, WB]$ is in FEF (2.7) be given. Then by Remark 2.22 the pencil $sE_{\text{F}} - A_{\text{F}}$ is regular. ◊

We need these results to prove the following proposition, which provides a geometric characterization for the system space.

**Proposition 2.43.** *Let $[E, A, B] \in \Sigma_{n,m}(\mathbb{K})$ with system space $\mathcal{V}_{\text{sys}} \subseteq \mathbb{K}^{n+m}$ be given. Let $(\mathcal{V}_k)$ be a sequence of subspaces defined by*

$$\begin{aligned}\mathcal{V}_0 &:= \mathbb{K}^{n+m} \\ \mathcal{V}_{k+1} &:= \begin{bmatrix} A & B \end{bmatrix}^{-\mathbb{1}} \begin{bmatrix} E & 0 \end{bmatrix} \mathcal{V}_k, \quad k \in \mathbb{N}_0,\end{aligned}$$

*where $\begin{bmatrix} A & B \end{bmatrix}^{-\mathbb{1}}$ denotes the preimage of $\begin{bmatrix} A & B \end{bmatrix}$. Then $\mathcal{V}_{k+1} \subseteq \mathcal{V}_k$ holds for all $k \in \mathbb{N}_0$ and there exists some $k_0 \in \mathbb{N}_0$ such that*

$$\mathcal{V}_{\text{sys}} = \mathcal{V}_{k_0} = \mathcal{V}_{k_0+i} \quad \forall i \in \mathbb{N}_0.$$

**Proof:** Due to Proposition 2.41 together with Remark 2.42 it suffices to consider $[E, A, B]$ in FEF (2.7). We divide the proof into three steps. At first, we show a set equality to characterize $\mathcal{V}_k$ in Step 2. Then the statement is derived via Proposition 2.40.

Step 1: Divided into two steps we show that

$$M_k := \begin{bmatrix} 0 & 0 & E_{23} & 0 \\ 0 & 0 & E_{33} & 0 \end{bmatrix} \begin{bmatrix} 0 & I_{n_2} & 0 & B_2 \\ 0 & 0 & I_{n_3} & 0 \end{bmatrix}^{-\mathbb{1}} \begin{bmatrix} 0 & E_{23} \\ 0 & E_{33} \end{bmatrix}^k \mathbb{K}^{n_2+n_3} = \begin{bmatrix} 0 & E_{23} \\ 0 & E_{33} \end{bmatrix}^{k+1} \mathbb{K}^{n_2+n_3} \tag{2.20}$$

holds for all $k \in \mathbb{N}_0$. First note that the statement is evident for $k = 0$. Further, for all $k \in \mathbb{N}$ it is

$$\begin{bmatrix} 0 & E_{23} \\ 0 & E_{33} \end{bmatrix}^k = \begin{bmatrix} 0 & E_{23}E_{33}^{k-1} \\ 0 & E_{33}^k \end{bmatrix}. \tag{2.21}$$

Step 1.1: We show $M_k \subseteq \begin{bmatrix} 0 & E_{23} \\ 0 & E_{33} \end{bmatrix}^{k+1} \mathbb{K}^{n_2+n_3}$. Let $k \in \mathbb{N}$ and $\begin{pmatrix} y_2 \\ y_3 \end{pmatrix} \in M_k$ be given. Then there exist some $x_2 \in \mathbb{K}^{n_2}$ and $x_3 \in \mathbb{K}^{n_3}$ such that

$$\begin{aligned}
\begin{pmatrix} y_2 \\ y_3 \end{pmatrix} &= \begin{bmatrix} 0 & 0 & E_{23} & 0 \\ 0 & 0 & E_{33} & 0 \end{bmatrix} \begin{bmatrix} 0 & I_{n_2} & 0 & B_2 \\ 0 & 0 & I_{n_3} & 0 \end{bmatrix}^{-\mathbb{1}} \begin{bmatrix} 0 & E_{23} \\ 0 & E_{33} \end{bmatrix}^k \begin{pmatrix} x_2 \\ x_3 \end{pmatrix} \\
&\overset{(2.21)}{=} \begin{bmatrix} 0 & 0 & E_{23} & 0 \\ 0 & 0 & E_{33} & 0 \end{bmatrix} \begin{bmatrix} 0 & I_{n_2} & 0 & B_2 \\ 0 & 0 & I_{n_3} & 0 \end{bmatrix}^{-\mathbb{1}} \begin{pmatrix} E_{23}E_{33}^{k-1}x_3 \\ E_{33}^k x_3 \end{pmatrix} \\
&= \begin{bmatrix} 0 & 0 & E_{23} & 0 \\ 0 & 0 & E_{33} & 0 \end{bmatrix} \begin{pmatrix} 0 \\ E_{23}E_{33}^{k-1}x_3 \\ E_{33}^k x_3 \\ 0 \end{pmatrix} \\
&= \begin{pmatrix} E_{23}E_{33}^k x_3 \\ E_{33}^{k+1} x_3 \end{pmatrix} \overset{(2.21)}{\in} \begin{bmatrix} 0 & E_{23} \\ 0 & E_{33} \end{bmatrix}^{k+1} \mathbb{K}^{n_2+n_3}.
\end{aligned}$$

Step 1.2: We show $\begin{bmatrix} 0 & E_{23} \\ 0 & E_{33} \end{bmatrix}^{k+1} \mathbb{K}^{n_2+n_3} \subseteq M_k$.

Let $k \in \mathbb{N}$ and $\begin{pmatrix} y_2 \\ y_3 \end{pmatrix} \in \begin{bmatrix} 0 & E_{23} \\ 0 & E_{33} \end{bmatrix}^{k+1} \mathbb{K}^{n_2+n_3}$ be given. Then there exist some $x_2 \in \mathbb{K}^{n_2}$ and $x_3 \in \mathbb{K}^{n_3}$ such that

$$\begin{aligned}
\begin{pmatrix} y_2 \\ y_3 \end{pmatrix} &= \begin{bmatrix} 0 & E_{23} \\ 0 & E_{33} \end{bmatrix}^{k+1} \begin{pmatrix} x_2 \\ x_3 \end{pmatrix} \overset{(2.21)}{=} \begin{pmatrix} E_{23}E_{33}^k x_3 \\ E_{33}^{k+1} x_3 \end{pmatrix} \\
&= \begin{bmatrix} 0 & 0 & E_{23} & 0 \\ 0 & 0 & E_{33} & 0 \end{bmatrix} \begin{pmatrix} 0 \\ E_{23}E_{33}^{k-1}x_3 \\ E_{33}^k x_3 \\ 0 \end{pmatrix}
\end{aligned}$$

$$
\begin{aligned}
&= \begin{bmatrix} 0 & 0 & E_{23} & 0 \\ 0 & 0 & E_{33} & 0 \end{bmatrix} \begin{bmatrix} 0 & I_{n_2} & 0 & B_2 \\ 0 & 0 & I_{n_3} & 0 \end{bmatrix}^{-\mathbb{1}} \begin{pmatrix} E_{23}E_{33}^{k-1}x_3 \\ E_{33}^k x_3 \end{pmatrix} \\
&\overset{(2.21)}{=} \begin{bmatrix} 0 & 0 & E_{23} & 0 \\ 0 & 0 & E_{33} & 0 \end{bmatrix} \begin{bmatrix} 0 & I_{n_2} & 0 & B_2 \\ 0 & 0 & I_{n_3} & 0 \end{bmatrix}^{-\mathbb{1}} \begin{bmatrix} 0 & E_{23} \\ 0 & E_{33} \end{bmatrix}^k \begin{pmatrix} x_2 \\ x_3 \end{pmatrix} \in M_k,
\end{aligned}
$$

which completes Step 1.

Step 2: We show that

$$
\mathcal{V}_k = \begin{bmatrix} 0 & I_{n_2} & 0 & B_2 \\ 0 & 0 & I_{n_3} & 0 \end{bmatrix}^{-\mathbb{1}} \begin{bmatrix} 0 & E_{23} \\ 0 & E_{33} \end{bmatrix}^k \mathbb{K}^{n_2+n_3} \tag{2.22}
$$

holds for all $k \in \mathbb{N}_0$. Note that the statement is trivial for $k = 0$. Since $[E, A, B]$ is in FEF (2.7) we get

$$
\begin{aligned}
\mathcal{V}_1 = \begin{bmatrix} A & B \end{bmatrix}^{-\mathbb{1}} \begin{bmatrix} E & 0 \end{bmatrix} \mathbb{K}^{n+m} &= \begin{bmatrix} A_{11} & 0 & 0 & B_1 \\ 0 & I_{n_2} & 0 & B_2 \\ 0 & 0 & I_{n_3} & 0 \end{bmatrix}^{-\mathbb{1}} \begin{bmatrix} I_{n_1} & 0 & 0 & 0 \\ 0 & 0 & E_{23} & 0 \\ 0 & 0 & E_{33} & 0 \end{bmatrix} \mathbb{K}^{n+m} \\
&= \begin{bmatrix} A_{11} & 0 & 0 & B_1 \\ 0 & I_{n_2} & 0 & B_2 \\ 0 & 0 & I_{n_3} & 0 \end{bmatrix}^{-\mathbb{1}} \left( \mathbb{K}^{n_1} \times \begin{bmatrix} 0 & 0 & E_{23} & 0 \\ 0 & 0 & E_{33} & 0 \end{bmatrix} \mathbb{K}^{n+m} \right) \\
&= \begin{bmatrix} 0 & I_{n_2} & 0 & B_2 \\ 0 & 0 & I_{n_3} & 0 \end{bmatrix}^{-\mathbb{1}} \begin{bmatrix} 0 & 0 & E_{23} & 0 \\ 0 & 0 & E_{33} & 0 \end{bmatrix} \mathbb{K}^{n_2+n_3}
\end{aligned}
$$

for $k = 1$. Due to the recursive definition of $\mathcal{V}_{k+1}$ this yields

$$
\mathcal{V}_{k+1} = \begin{bmatrix} 0 & I_{n_2} & 0 & B_2 \\ 0 & 0 & I_{n_3} & 0 \end{bmatrix}^{-\mathbb{1}} \begin{bmatrix} 0 & 0 & E_{23} & 0 \\ 0 & 0 & E_{33} & 0 \end{bmatrix} \mathcal{V}_k
$$

for $k \in \mathbb{N}_0$. By induction we conclude

$$
\begin{aligned}
\mathcal{V}_{k+1} &= \begin{bmatrix} 0 & I_{n_2} & 0 & B_2 \\ 0 & 0 & I_{n_3} & 0 \end{bmatrix}^{-\mathbb{1}} \begin{bmatrix} 0 & 0 & E_{23} & 0 \\ 0 & 0 & E_{33} & 0 \end{bmatrix} \begin{bmatrix} 0 & I_{n_2} & 0 & B_2 \\ 0 & 0 & I_{n_3} & 0 \end{bmatrix}^{-\mathbb{1}} \begin{bmatrix} 0 & E_{23} \\ 0 & E_{33} \end{bmatrix}^k \mathbb{K}^{n_2+n_3} \\
&\overset{(2.20)}{=} \begin{bmatrix} 0 & I_{n_2} & 0 & B_2 \\ 0 & 0 & I_{n_3} & 0 \end{bmatrix}^{-\mathbb{1}} \begin{bmatrix} 0 & E_{23} \\ 0 & E_{33} \end{bmatrix}^{k+1} \mathbb{K}^{n_2+n_3},
\end{aligned}
$$

which completes Step 2.

Step 3: Since $E_{33}$ is nilpotent with nilpotency index $\nu \in \mathbb{N}$, the characterization in (2.22) yields $\mathcal{V}_{k+1} \subseteq \mathcal{V}_k$ for all $k \in \mathbb{N}_0$. Moreover, for $k_0 = \nu + 1$ and all $i \in \mathbb{N}_0$ it holds that

$$
\mathcal{V}_{k_0+i} = \mathcal{V}_{k_0} = \mathcal{V}_{\nu+1} \underset{(2.22)}{\overset{(2.21)}{=}} \ker \begin{bmatrix} 0 & I_{n_2} & 0 & B_2 \\ 0 & 0 & I_{n_3} & 0 \end{bmatrix} \overset{(2.18)}{=} \mathcal{V}_{\text{sys}}.
$$

□

Proposition 2.43 yields further properties of $\mathcal{V}_{\mathrm{sys}}$. The first one is an algebraic characterization of the system space.

**Corollary 2.44.** *Let $[E, A, B] \in \Sigma_{n,m}(\mathbb{K})$ with system space $\mathcal{V}_{\mathrm{sys}} \subseteq \mathbb{K}^{n+m}$ be given. Then the following statements hold true:*

(a) $\mathcal{V}_{\mathrm{sys}} = \bigcap_{k\in\mathbb{N}_0} \mathcal{V}_k$, *where the subspaces $\mathcal{V}_k$ are defined as in Proposition 2.43;*

(b) *The system space $\mathcal{V}_{\mathrm{sys,F}}$ of a system in FEF* (2.7) *fulfils*

$$\mathcal{V}_{\mathrm{sys,F}} = \ker \begin{bmatrix} 0 & I_{n_2} & 0 & B_2 \\ 0 & 0 & I_{n_3} & 0 \end{bmatrix} = \left\{ \left. \begin{pmatrix} x_1 \\ -B_2 u \\ 0_{n_3\times 1} \\ u \end{pmatrix} \right| x_1 \in \mathbb{K}^{n_1}, u \in \mathbb{K}^m \right\}; \tag{2.23}$$

(c) $\dim \mathcal{V}_{\mathrm{sys}} = \dim \mathcal{V}_{\mathrm{sys,F}} = n_1 + m$, *where $n_1$ is given by FEF* (2.7)*;*

(d) *If in addition $[E, A, B]$ is impulse controllable, the system space fulfils*

$$\mathcal{V}_{\mathrm{sys}} = \begin{bmatrix} A & B \end{bmatrix}^{-\mathbb{1}} E\,\mathbb{K}^n;$$

(e) *In the ODE case, i.e. $E = I$, it holds $\mathcal{V}_{\mathrm{sys}} = \mathbb{K}^{n+m}$.*

**Proof:** The first assertion follows immediately from Proposition 2.43 and the second from Proposition 2.40. Further, (b) $\Rightarrow$ (c) and (d) $\Rightarrow$ (e) holds. Thus, it suffices to show assertion (d):
Proposition 2.34 (d) allows to choose $n_3 = 0$ in FEF (2.7). Hence, the matrix $\begin{bmatrix} 0 & E_{23} \\ 0 & E_{33} \end{bmatrix}$ in the proof of Proposition 2.43 simplifies to $\begin{bmatrix} 0 & E_{23} \\ 0 & E_{33} \end{bmatrix} = 0_{n_2\times n_2}$. Thus, equation (2.22) yields

$$\begin{aligned} \mathcal{V}_k &= \begin{bmatrix} 0 & I_{n_2} & 0 & B_2 \end{bmatrix}^{-\mathbb{1}} 0^k_{n_2\times n_2}\,\mathbb{K}^{n_2} \\ &= \begin{bmatrix} A_{11} & 0 & 0 & B_1 \\ 0 & I_{n_2} & 0 & B_2 \end{bmatrix}^{-1} \begin{bmatrix} I_{n_1} & 0 \\ 0 & 0 \end{bmatrix} \mathbb{K}^n \\ &= \begin{bmatrix} A & B \end{bmatrix}^{-1} E\,\mathbb{K}^n \end{aligned}$$

for $k \geq 1$. Then (a) provides (d). $\square$

According to Corollary 2.44 the system space of a system in QWF (2.4) can be characterized as follows.

**Proposition 2.45.** *Let $[E, A, B] \in \Sigma_{n,m}(\mathbb{K})$ be given such that the pencil $sE - A$ is in QWF* (2.4)*. Then the system space fulfils*

$$\mathcal{V}_{\mathrm{sys}} = \ker \begin{bmatrix} 0 & I_{\tilde{n}_2} & \tilde{B}_2 \end{bmatrix}, \quad \textit{where} \quad B = \begin{bmatrix} \tilde{B}_1 \\ \tilde{B}_2 \end{bmatrix}. \tag{2.24}$$

**Remark 2.46.** Compare (2.24) to (2.18). As in our introducing comments on FEF (2.7) one can see again that the FEF provides more information on the structure of the system space than the QWF does. $\diamondsuit$

**Proof of Proposition 2.45:** Since $sE - A$ is in QWF (2.4), we have

$$\begin{bmatrix} sE - A & B \end{bmatrix} = \begin{bmatrix} sI_{n_1} - A_{11} & 0 & \tilde{B}_1 \\ 0 & sE_{22} - I_{\tilde{n}_2} & \tilde{B}_2 \end{bmatrix}.$$

We determine $\mathcal{V}_{\text{sys}}$ via the chain $\mathcal{V}_k$ as in Proposition 2.43. The $\mathcal{V}_k$ are given by

$$\begin{aligned}
\mathcal{V}_0 &= \mathbb{K}^{n+m}, \\
\mathcal{V}_1 &= \begin{bmatrix} A_{11} & 0 & \tilde{B}_1 \\ 0 & I_{\tilde{n}_2} & \tilde{B}_2 \end{bmatrix}^{-\mathbb{1}} \begin{bmatrix} I_{n_1} & 0 & 0 \\ 0 & E_{22} & 0 \end{bmatrix} \mathbb{K}^{n+m} \\
&= \begin{bmatrix} A_{11} & 0 & \tilde{B}_1 \\ 0 & I_{\tilde{n}_2} & \tilde{B}_2 \end{bmatrix}^{-\mathbb{1}} \left\{ \begin{pmatrix} a \\ E_{22} b \end{pmatrix} \,\middle|\, a \in \mathbb{K}^{n_1}, b \in \mathbb{K}^{\tilde{n}_2} \right\} \\
&= \mathbb{K}^{n_1} \times \begin{bmatrix} I_{\tilde{n}_2} & \tilde{B}_2 \end{bmatrix}^{-\mathbb{1}} E_{22}\, \mathbb{K}^{\tilde{n}_2}, \\
\mathcal{V}_2 &= \begin{bmatrix} A_{11} & 0 & \tilde{B}_1 \\ 0 & I_{\tilde{n}_2} & \tilde{B}_2 \end{bmatrix}^{-\mathbb{1}} \begin{bmatrix} I_{n_1} & 0 & 0 \\ 0 & E_{22} & 0 \end{bmatrix} \left( \mathbb{K}^{n_1} \times \begin{bmatrix} I_{\tilde{n}_2} & \tilde{B}_2 \end{bmatrix}^{-\mathbb{1}} E_{22}\, \mathbb{K}^{\tilde{n}_2} \right) \\
&= \mathbb{K}^{n_1} \times \begin{bmatrix} I_{\tilde{n}_2} & \tilde{B}_2 \end{bmatrix}^{-\mathbb{1}} \begin{bmatrix} E_{22} & 0 \end{bmatrix} \left( \begin{bmatrix} I_{\tilde{n}_2} & \tilde{B}_2 \end{bmatrix}^{-\mathbb{1}} E_{22}\, \mathbb{K}^{\tilde{n}_2} \right) \\
&= \mathbb{K}^{n_1} \times \begin{bmatrix} I_{\tilde{n}_2} & \tilde{B}_2 \end{bmatrix}^{-\mathbb{1}} \left( E_{22} I_{\tilde{n}_2}^{-\mathbb{1}} E_{22} \mathbb{K}^{\tilde{n}_2} \right) \\
&= \mathbb{K}^{n_1} \times \begin{bmatrix} I_{\tilde{n}_2} & \tilde{B}_2 \end{bmatrix}^{-\mathbb{1}} E_{22}^2 \mathbb{K}^{\tilde{n}_2}.
\end{aligned}$$

By induction we conclude

$$\mathcal{V}_k = \mathbb{K}^{n_1} \times \begin{bmatrix} I_{\tilde{n}_2} & \tilde{B}_2 \end{bmatrix}^{-\mathbb{1}} E_{22}^k \mathbb{K}^{\tilde{n}_2}.$$

Due to the nilpotency of $E_{22}$, Corollary 2.44 (a) yields the assertion. $\square$

Next we state a lemma which will turn out to be quite useful in context of optimal control (see Chapter 5). It provides a set inclusion of the system space.

**Lemma 2.47.** *Let* $[E, A, B] \in \Sigma_{n,m}(\mathbb{K})$ *with system space* $\mathcal{V}_{\text{sys}} \subseteq \mathbb{K}^{n+m}$ *be given. Then the inclusion*

$$\operatorname{im} \begin{bmatrix} (\lambda E - A)^{-1} B \\ I_m \end{bmatrix} \subseteq \mathcal{V}_{\text{sys}} \tag{2.25}$$

*holds for all* $\lambda \in \mathbb{C}$ *with* $\det(\lambda E - A) \neq 0$.

**Proof:** According to Corollary 2.44 (a), it suffices to show that

$$\operatorname{im} \begin{bmatrix} (\lambda E - A)^{-1} B \\ I_m \end{bmatrix} \subseteq \mathcal{V}_k \quad \forall \lambda \in \mathbb{C} \text{ with } \det(\lambda E - A) \neq 0 \tag{2.26}$$

is fulfilled for all $k \in \mathbb{N}_0$. This is done by induction. For $k = 0$ the subset relationship (2.26) is evident. Let $k \geq 1$ and suppose (2.26) holds for $k-1$. The equality

$$A(sE-A)^{-1}B = (sE-(sE-A))(sE-A)^{-1}B = sE(sE-A)^{-1}B - B$$

leads to

$$\begin{aligned}\begin{bmatrix} A & B \end{bmatrix} \operatorname{im} \begin{bmatrix} (\lambda E - A)^{-1}B \\ I_m \end{bmatrix} &= \operatorname{im} \big(\lambda E(\lambda E - A)^{-1}B\big) \\ &= \begin{bmatrix} E & 0 \end{bmatrix} \operatorname{im} \begin{bmatrix} (\lambda E - A)^{-1}B \\ I_m \end{bmatrix} \\ &\subseteq \begin{bmatrix} E & 0 \end{bmatrix} \mathcal{V}_{k-1}\end{aligned}$$

for all $\lambda \in \mathbb{C}$ with $\det(\lambda E - A) \neq 0$. Thus, for all $\begin{pmatrix} x \\ u \end{pmatrix} \in \operatorname{im} \begin{bmatrix} (\lambda E - A)^{-1}B \\ I_m \end{bmatrix}$ we have $Ax + Bu \in \begin{bmatrix} E & 0 \end{bmatrix} \mathcal{V}_{k-1}$. Due to the definition of $\mathcal{V}_k$, this completes the proof. □

The following example illustrates how to determine the system space for a system that is not impulse controllable.

**Example 2.48.** Consider the DAE system

$$\frac{\mathrm{d}}{\mathrm{d}t}\left( \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix} x(t) \right) = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix} x(t) + \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 0 & 1 \end{bmatrix} u(t).$$

For $S_\infty = (0,0,1)^\top$ we see that

$$\operatorname{rk} \begin{bmatrix} E & AS_\infty & B \end{bmatrix} = \operatorname{rk} \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 \end{bmatrix} = 2 < 3,$$

i.e. the system is not impulse controllable and hence Corollary 2.44 (d) is not applicable. However, to determine the system space, we calculate the chain $(\mathcal{V}_k)$ as in Proposition 2.43.

<u>$k = 0$:</u> $\mathcal{V}_0 = \mathbb{R}^5$ by definition.
<u>$k = 1$:</u>

$$\begin{aligned}\mathcal{V}_1 &= \begin{bmatrix} A & B \end{bmatrix}^{-\mathbb{1}} \begin{bmatrix} E & 0 \end{bmatrix} \mathbb{R}^5 \\ &= \begin{bmatrix} 1 & 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 \end{bmatrix}^{-\mathbb{1}} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{bmatrix} \mathbb{R}^5\end{aligned}$$

$$= \begin{bmatrix} 1 & 0 & 0 & & 1 & 1 \\ 1 & 1 & 1 & & 0 & 1 \\ 1 & 0 & 1 & & 0 & 1 \end{bmatrix}^{-\mathbb{1}} \left\{ \begin{pmatrix} a \\ b \\ b \end{pmatrix} \;\middle|\; a, b \in \mathbb{R} \right\}$$
$$= \left\{ \begin{pmatrix} x_1 & 0 & x_3 & u_1 & u_2 \end{pmatrix}^\top \;\middle|\; x_1, x_3, u_1, u_2 \in \mathbb{R} \right\}$$

<u>$k = 2$:</u>

$$\mathcal{V}_2 = \begin{bmatrix} A & B \end{bmatrix}^{-\mathbb{1}} \begin{bmatrix} E & 0 \end{bmatrix} \mathcal{V}_1$$
$$= \begin{bmatrix} 1 & 0 & 0 & & 1 & 1 \\ 1 & 1 & 1 & & 0 & 1 \\ 1 & 0 & 1 & & 0 & 1 \end{bmatrix}^{-\mathbb{1}} \left\{ \begin{pmatrix} a \\ 0 \\ 0 \end{pmatrix} \;\middle|\; a \in \mathbb{R} \right\}$$
$$= \left\{ \begin{pmatrix} x_1 & 0 & x_3 & u_1 & u_2 \end{pmatrix}^\top \;\middle|\; x_1 + x_3 + u_2 = 0, x_2 = 0, u_1 \in \mathbb{R} \right\}$$

<u>$k = 3$:</u>

$$\mathcal{V}_3 = \begin{bmatrix} A & B \end{bmatrix}^{-\mathbb{1}} \begin{bmatrix} E & 0 \end{bmatrix} \mathcal{V}_2 = \begin{bmatrix} 1 & 0 & 0 & & 1 & 1 \\ 1 & 1 & 1 & & 0 & 1 \\ 1 & 0 & 1 & & 0 & 1 \end{bmatrix}^{-\mathbb{1}} \left\{ \begin{pmatrix} a \\ 0 \\ 0 \end{pmatrix} \;\middle|\; a \in \mathbb{R} \right\} = \mathcal{V}_2$$

Thus, Proposition 2.43 provides

$$\mathcal{V}_{\text{sys}} = \mathcal{V}_2 = \left\{ \begin{pmatrix} x_1 & 0 & x_3 & u_1 & u_2 \end{pmatrix}^\top \;\middle|\; x_1 + x_3 + u_2 = 0, x_2 = 0, u_1 \in \mathbb{R} \right\}.$$

◊

# 3 The Kalman-Yakubovich-Popov inequality

In this chapter we derive a relationship between positive semi-definiteness of the Popov function on the imaginary axis and solvability of the Kalman-Yakubovich-Popov (KYP) inequality. The Popov function and the KYP inequality are introduced in Section 3.1. The relationship mentioned above is content of the KYP lemma for DAE systems. It is stated in Section 3.1 together with an alternative version of the KYP inequality. To prove both, some auxiliary findings are required, which are presented in Section 3.4. They are studied in Section 3.3 which is split into three parts. At first, some general knowledge is presented, followed by the KYP lemma for ODE systems. Lemma 3.12 then provides a relation between a DAE Lur'e equation and its associated ODE Lur'e equation, which enables us to apply ODE results.

## 3.1 Introduction to Popov function and KYP inequality

Let $[E, A, B] \in \Sigma_{n,m}(\mathbb{K})$ and weighting matrices $Q = Q^* \in \mathbb{K}^{n\times n}$, $S \in \mathbb{K}^{m\times n}$ and $R = R^* \in \mathbb{K}^{m\times m}$ be given. The function $\Phi : \mathbb{C} \to \mathbb{K}(s)^{m\times m}$, where

$$\Phi(s) := \begin{bmatrix} (-\bar{s}E - A)^{-1}B \\ I_m \end{bmatrix}^* \begin{bmatrix} Q & S \\ S^* & R \end{bmatrix} \begin{bmatrix} (sE - A)^{-1}B \\ I_m \end{bmatrix}, \tag{3.1}$$

is called *Popov function*. If $\mathrm{rk}_{\mathbb{K}(s)}\, \Phi(s) = m$ holds, we call $\Phi$ *nonsingular*.

Positive semi-definiteness of the Popov function on the imaginary axis can be related to the solvability of the so-called *Kalman-Yakubovich-Popov inequality*

$$\begin{bmatrix} A^*PE + E^*PA + Q & E^*PB + S \\ B^*PE + S^* & R \end{bmatrix} \geq_{\mathcal{V}_{\mathrm{sys}}} 0, \quad P = P^* \tag{3.2}$$

for some $P \in \mathbb{K}^{n\times n}$.

**Remark 3.1.** Note that the relations $=_{\mathcal{V}}$ and $\geq_{\mathcal{V}}$ associated with a linear subspace $\mathcal{V} \subseteq \mathbb{K}^{\ell}$ are defined by

$$M =_{\mathcal{V}} N :\Leftrightarrow x^*Mx = x^*Nx \quad \forall\, x \in \mathcal{V},$$
$$M \geq_{\mathcal{V}} N :\Leftrightarrow x^*Mx \geq x^*Nx \quad \forall\, x \in \mathcal{V}.$$

◊

## 3.2 KYP lemma for DAE systems

The following theorem is a generalization of the KYP lemma for ODE systems (cf. Theorem 3.10 and Theorem 3.11). It states the already mentioned relationship between Popov function (3.1) and KYP inequality (3.2).

**Theorem 3.2** (KYP lemma for DAE systems)**.** *Let* $[E, A, B] \in \Sigma_{n,m}(\mathbb{K})$ *with weighting matrices* $Q = Q^* \in \mathbb{K}^{n\times n}$, $S \in \mathbb{K}^{n\times m}$, $R = R^* \in \mathbb{K}^{m\times m}$ *be given. Let* $\Phi(s) \in \mathbb{K}(s)^{m\times m}$ *be the Popov function as in* (3.1)*. Then the following relations hold true:*

(a) *If there exists some* $P \in \mathbb{K}^{n\times n}$ *such that* (3.2) *holds, then*

$$\Phi(\mathrm{i}\omega) \geq 0 \quad \forall \omega \in \mathbb{R} \textit{ with } \det(\mathrm{i}\omega E - A) \neq 0. \tag{3.3}$$

(b) *If* (3.3) *and at least one of the two properties*

(b1) $[E, A, B]$ *is behaviourally sign-controllable and* $\mathrm{rk}_{\mathbb{K}(s)}\,\Phi(s) = m$*;*

(b2) $[E, A, B]$ *is behaviourally controllable;*

*is satisfied, then there exists some* $P \in \mathbb{K}^{n\times n}$ *that solves the KYP inequality* (3.2).

The following simple example illustrates the use of the KYP lemma.

**Example 3.3.** Consider the DAE (2.1) with $n = 2$, $m = 1$, system matrices

$$E = \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}, \quad A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

and weighting matrices

$$Q = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}, \quad R = 1, \quad S = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

First note that due to

$$\det(sE - A) = \det \begin{bmatrix} s-1 & 2s \\ -1 & -1 \end{bmatrix} = s + 1 \neq 0_{\mathbb{K}[s]}$$

the pencil $sE - A$ is regular.
Since $[E, A, B]$ is impulse controllable by

$$\mathrm{rk}\begin{bmatrix} E & AS_\infty & B \end{bmatrix} = \mathrm{rk}\begin{bmatrix} 1 & 2 & -2 & 1 \\ 0 & 0 & -1 & -1 \end{bmatrix} = 2 = n \quad \text{for} \quad S_\infty = \begin{bmatrix} -2 \\ 1 \end{bmatrix},$$

Corollary 2.44 (d) yields

$$\mathcal{V}_{\mathrm{sys}} = \begin{bmatrix} A & B \end{bmatrix}^{-\mathbb{1}} E\,\mathbb{K}^n = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & -1 \end{bmatrix}^{-\mathbb{1}} \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix} \mathbb{K}^2$$

$$= \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & -1 \end{bmatrix}^{-\mathbb{1}} \left\{ \begin{pmatrix} a \\ 0 \end{pmatrix} \,\middle|\, a \in \mathbb{K} \right\} = \left\{ \begin{pmatrix} a \\ b \\ a+b \end{pmatrix} \,\middle|\, a, b \in \mathbb{K} \right\}.$$

As

$$\operatorname{rk} \begin{bmatrix} \lambda E - A & B \end{bmatrix} = \operatorname{rk} \begin{bmatrix} \lambda - 1 & 2\lambda & 1 \\ -1 & -1 & -1 \end{bmatrix} = 2 = n$$

holds for all $\lambda \in \mathbb{C}$, the system $[E, A, B]$ is behaviourally controllable. Furthermore, the Popov function $\Phi$ fulfils

$$\begin{aligned} \operatorname{rk} \Phi(s) &= \operatorname{rk}_{\mathbb{K}(s)} \left( \begin{bmatrix} (-\bar{s}E - A)^{-1}B \\ 1 \end{bmatrix}^* \begin{bmatrix} Q & S \\ S^* & R \end{bmatrix} \begin{bmatrix} (sE - A)^{-1}B \\ 1 \end{bmatrix} \right) \\ &= \operatorname{rk}_{\mathbb{K}(s)} \left( \begin{bmatrix} \frac{2s+1}{s-1} & \frac{-s-2}{s-1} & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{2s-1}{s+1} \\ \frac{-s+2}{s+1} \\ 1 \end{bmatrix} \right) \\ &= \operatorname{rk}_{\mathbb{K}(s)}(10) = 1 = m. \end{aligned}$$

Thus, Theorem 3.2 provides that the KYP inequality (3.2) has a solution. ◊

As an alternative characterization of the KYP inequality we state the following proposition.

**Proposition 3.4.** *Let $[E, A, B] \in \Sigma_{n,m}(\mathbb{K})$ with system space $\mathcal{V}_{\text{sys}} \subseteq \mathbb{K}^{n+m}$, and matrices $Q = Q^* \in \mathbb{K}^{n\times n}, S \in \mathbb{K}^{n\times m}, R = R^* \in \mathbb{K}^{m\times m}$ be given. If $Y \in \mathbb{K}^{n\times n}$ fulfils*

$$\begin{bmatrix} A^*Y + Y^*A + Q & Y^*B + S \\ B^*Y + S^* & R \end{bmatrix} \geq_{\mathcal{V}_{\text{sys}}} 0, \quad E^*Y = Y^*E, \tag{3.4}$$

*then there exists some $P \in \mathbb{K}^{n\times n}$ that solves the KYP inequality* (3.2) *with*

$$E^*PE = E^*Y.$$

*Further, if $P \in \mathbb{K}^{n\times n}$ solves the KYP inequality* (3.2)*, then $Y = PE$ fulfils* (3.4)*.*

As mentioned above, these statements are proved in Section 3.4.

## 3.3 Auxiliary results

This section provides several results to prove the KYP lemma for DAE systems. At first, we collect some general knowledge in Subsection 3.3.1. Then, we state some results for the ODE case in 3.3.2 and connect the ODE to the DAE case in Subsection 3.3.3.

### 3.3.1 General knowledge

In this subsection some well-known findings are collected. For the proof of the first lemma we refer to [Wer11, Proposition II.1.11]. It is about the so-called Neumann series.

**Lemma 3.5** (Neumann series)**.** *Assume $(\mathcal{X}, \|\cdot\|)$ to be an normed vector space and let $X : \mathcal{X} \to \mathcal{X}$ be linear and bounded. If the Neumann series $\sum_{k=0}^{\infty} X^k$ converges with respect to the operator norm, then the inverse of $(I - X)$ exists and the following identity holds:*

$$(I - X)^{-1} = \sum_{k=0}^{\infty} X^k.$$

Another result we need to prove Lemma 3.12 and Theorem 3.2 is the Sherman-Morrison-Woodbury identity.

**Lemma 3.6** (Sherman-Morrison-Woodbury identity)**.** *Let $K \in \mathrm{Gl}_k(\mathbb{K})$, $L \in \mathbb{K}^{k \times j}$, $M \in \mathrm{Gl}_j(\mathbb{K})$ and $N \in \mathbb{K}^{j \times k}$ be given such that $K + LMN \in \mathrm{Gl}_k(\mathbb{K})$. Then it holds*

$$(K + LMN)^{-1} = K^{-1} - K^{-1}L(M^{-1} + NK^{-1}L)^{-1}NK^{-1}.$$

**Proof:** The statement can be verified by a simple matrix multiplication. It is

$$\begin{aligned}
&(K + LMN)\left[K^{-1} - K^{-1}L(M^{-1} + NK^{-1}L)^{-1}NK^{-1}\right] \\
&= I - L(M^{-1} + NK^{-1}L)^{-1}NK^{-1} + LMNK^{-1} - LMNK^{-1}L(M^{-1} + NK^{-1}L)^{-1}NK^{-1} \\
&= I + LMNK^{-1} \\
&\quad - \left[L(M^{-1} + NK^{-1}L)^{-1}NK^{-1} + LMNK^{-1}L(M^{-1} + NK^{-1}L)^{-1}NK^{-1}\right] \\
&= I + LMNK^{-1} - (L + LMNK^{-1}L)(M^{-1} + NK^{-1}L)^{-1}NK^{-1} \\
&= I + LMNK^{-1} - LM(M^{-1} + NK^{-1}L)(M^{-1} + NK^{-1}L)^{-1}NK^{-1} \\
&= I + LMNK^{-1} - LMNK^{-1} \\
&= I.
\end{aligned}$$

□

**Remark 3.7.** Note that the statement holds true for $K = sE - A \in \mathbb{K}[s]^{n \times n}$, where $\mathrm{rk}_{\mathbb{K}(s)}(sE - A) = n$. The proof stays the same. ◊

To prove Proposition 3.4 we use the following result on matrix decomposition.

**Proposition 3.8.** *Let matrices $M \in \mathbb{K}^{k \times \ell}$, $N \in \mathbb{K}^{\ell \times k}$ with $k \leq \ell$ and $MN = (MN)^*$ be given. Then there exists some matrix $G = G^* \in \mathbb{K}^{\ell \times \ell}$ such that*

$$MN = MGM^*.$$

At first, we show the statement for a special case, where the matrices $M$ and $N$ have full rank.

**Lemma 3.9.** *Let matrices* $M \in \mathbb{K}^{m\times k}$ *and* $N \in \mathbb{K}^{k\times m}$ *with* $\operatorname{rk} M = \operatorname{rk} N = k$ *and* $MN = (MN)^*$ *be given. Then there exists some* $G = G^* \in \mathbb{K}^{k\times k}$ *such that*

$$MN = MGM^*. \tag{3.5}$$

**Proof:** Since $M^*$ has full row rank its right inverse is $(M^*)^+ = M(M^*M)^{-1}$. Define $G := N(M^*)^+$. Due to $MGM^* = MN(M^*)^+M^*$ and the full row rank of $M$ it suffices to show that

$$N(M^*)^+M^* = N \tag{3.6}$$

holds true.
Define $D := (M^*)^+M^*$. The proof of equation (3.6) is divided into three steps. In the first two steps we show $\ker D = \ker N = (\operatorname{im} M)^\perp$. Then we complete the proof via simple conclusions.

Step 1: First, note that since $M$ and $N$ have full rank, it is $\ker(MN) = \ker N$ and $\operatorname{im}(MN) = \operatorname{im} M$. Let $x \in \mathbb{K}^m$ be given. Then the following equivalences hold:

$$\begin{array}{lll} x \in \ker D & \Leftrightarrow (M^*)^+M^*x = 0 & \Leftrightarrow M^*x = x^*M = 0 \\ \Leftrightarrow \forall\, y \in \mathbb{K}^k : \, x^*My = 0 & \Leftrightarrow \forall\, z \in \mathbb{K}^m : \, x^*MNz = 0 & \Leftrightarrow x^*MN = 0 \\ \Leftrightarrow (MN)x = (MN)^*x = 0 & \Leftrightarrow x \in \ker(MN) = \ker N, & \end{array}$$

i.e. $\ker D = \ker N$.

Step 2: Analogously to Step 1 we receive that for all $x \in \mathbb{K}^n$

$$\begin{aligned} x \in (\operatorname{im} M)^\perp \Leftrightarrow \forall\, y \in \mathbb{K}^k : \, My \perp x &\Leftrightarrow \forall\, y \in \mathbb{K}^k : \, y^*M^*x = 0 \\ &\Leftrightarrow \forall\, z \in \mathbb{K}^m : \, z^*(MN)^*x = z^*(MN)\, x = 0 \\ &\Leftrightarrow x \in \ker(MN) = \ker(N) \end{aligned}$$

holds, which is $\ker N = (\operatorname{im} M)^\perp$.

Step 3: Now we show (3.6) via $\mathbb{K}^m = \ker N \oplus (\ker N)^\perp$. Let $x \in \mathbb{K}^m$ be arbitrary. We differentiate between two cases. If $x \in \ker N = \ker D$, it is $NDx = 0 = Nx$. If on the other hand $x \in (\ker N)^\perp = \operatorname{im} M$, there exists some $y \in \mathbb{K}^k$ such that $x = My$. Thus, we get

$$NDx = N(M^*)^+M^*My = NM(M^*M)^{-1}M^*My = NMy = Nx.$$

Hence, $NDx = Nx$ holds true for all $x \in \mathbb{K}^n$, i.e. $ND = N$, which completes the proof of (3.6).

Hermiticity of $G$ then follows from (3.5) together with the assumptions $\operatorname{rk} M = k$ and $(MN)^* = MN$. $\square$

**Proof of Proposition 3.8:** Choose some unitary matrix $U \in \mathbb{K}^{\ell\times\ell}$ such that

$$MN = MUU^*N = \begin{bmatrix} M_1 & 0 \end{bmatrix} \begin{bmatrix} N_1 \\ N_2 \end{bmatrix} = \begin{bmatrix} M_1 & 0 \end{bmatrix} \begin{bmatrix} N_1 \\ 0 \end{bmatrix}$$

holds, where $M_1 \in \mathbb{K}^{k\times\ell_1}$ with $\operatorname{rk} M_1 = \ell_1$ and $N_1 \in \mathbb{K}^{\ell_1\times k}$. Then there further exists some unitary $V \in \mathbb{K}^{\ell_1\times\ell_1}$ such that

$$\begin{bmatrix} M_1 & 0 \end{bmatrix} \begin{bmatrix} N_1 \\ 0 \end{bmatrix} = \begin{bmatrix} M_1 & 0 \end{bmatrix} \begin{bmatrix} VV^* & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} N_1 \\ 0 \end{bmatrix} = \begin{bmatrix} M_2 & \tilde{M}_2 & 0 \end{bmatrix} \begin{bmatrix} N_2 \\ 0 \\ 0 \end{bmatrix} = M_2N_2$$

holds, where $M_2 \in \mathbb{K}^{k\times\ell_2}$ and $N_2 \in \mathbb{K}^{\ell_2\times k}$ with $\operatorname{rk} M_2 = \operatorname{rk} N_2 = \ell_2$. Thus, Lemma 3.9 provides some $\tilde{G} = \tilde{G}^* \in \mathbb{K}^{\ell_2\times\ell_2}$ such that

$$\begin{aligned} M_2N_2 = M_2\tilde{G}M_2^* &= \begin{bmatrix} M_2 & \tilde{M}_2 & 0 \end{bmatrix} \begin{bmatrix} \tilde{G} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} M_2 & \tilde{M}_2 & 0 \end{bmatrix}^* \\ &= \begin{bmatrix} M_1V & 0 \end{bmatrix} \begin{bmatrix} \begin{bmatrix} \tilde{G} & 0 \\ 0 & 0 \end{bmatrix} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} M_1V & 0 \end{bmatrix}^* \\ &= \begin{bmatrix} M_1 & 0 \end{bmatrix} \begin{bmatrix} V \begin{bmatrix} \tilde{G} & 0 \\ 0 & 0 \end{bmatrix} V^* & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} M_1 & 0 \end{bmatrix}^* \\ &= M \left( U \begin{bmatrix} V \begin{bmatrix} \tilde{G} & 0 \\ 0 & 0 \end{bmatrix} V^* & 0 \\ 0 & 0 \end{bmatrix} U^* \right) M^* \\ &= MGM^* \end{aligned}$$

holds for $G = G^* = U \begin{bmatrix} V \begin{bmatrix} \tilde{G} & 0 \\ 0 & 0 \end{bmatrix} V^* & 0 \\ 0 & 0 \end{bmatrix} U^*$, which completes the proof. □

### 3.3.2 The KYP lemma for ODE systems

To prove the KYP lemma for DAE systems we will apply two theorems for the ODE case from [CALM97]. Consider $[I, A, B] \in \Sigma_{n,m}(\mathbb{K})$ and the linear matrix inequality

$$\begin{bmatrix} A^*P + PA + Q & PB + S \\ B^*P + S^* & R \end{bmatrix} \geq 0, \quad P = P^*, \tag{3.7}$$

where $Q$ and $R$ are Hermitian. Note that since $E = I$ and hence $\mathcal{V}_{\text{sys}} = \mathbb{K}^{n+m}$, this is simply the KYP inequality (3.2). Then [CALM97] prove the following two theorems, which yield an ODE version of the KYP lemma.

**Theorem 3.10.** *If $(I, A, B)$ is behaviourally sign-controllable and the Popov function $\Phi$ is nonsingular, then the following statements are equivalent:*

(i) *$\Phi$ is positive semi-definite on the imaginary axis;*

(ii) *there exists a Hermitian solution $P$ of* (3.7).

**Theorem 3.11.** *If $(I, A, B)$ is behaviourally controllable, then the following statements are equivalent:*

(i) *$\Phi$ is positive semi-definite on the imaginary axis;*

(ii) *there exists an Hermitian solution $P$ of* (3.7).

### 3.3.3 Relation between DAE and ODE case

Whenever possible, we apply well-known results for ODEs to make statements on DAEs. The following lemma relates the solvability of the KYP inequality (3.2) to the solvability of a KYP inequality for an associated ODE system.

**Lemma 3.12.** *Let $[E, A, B] \in \Sigma_{n,m}(\mathbb{K})$ with the system space $\mathcal{V}_{\mathrm{sys}} \subseteq \mathbb{K}^{n+m}$, weighting matrices $Q = Q^* \in \mathbb{K}^{n\times n}$, $S \in \mathbb{K}^{n\times m}$, $R = R^* \in K^{m\times m}$, and the Popov function $\Phi(s) \in \mathbb{K}(s)^{m\times m}$ be given. Let $W$, $T \in \mathrm{Gl}_n(\mathbb{K})$ and $F \in \mathbb{K}^{m\times n}$ be matrices such that* (2.12) *holds. Define the following matrices*

$$\begin{aligned} &E_{\mathrm{F}} = WET, \quad A_{\mathrm{F}} = W(A + BF)T, \quad B_{\mathrm{F}} = WB,\\ &Q_{\mathrm{F}} = T^*(Q + SF + F^*S^* + F^*RF)T, \quad S_{\mathrm{F}} = T^*(S + F^*R), \quad R_{\mathrm{F}} = R, \end{aligned} \tag{3.8}$$

*where $Q_{\mathrm{F}}$ and $S_{\mathrm{F}}$ are partitioned according to block structure of FEF* (2.7) *as*

$$Q_{\mathrm{F}} = \begin{bmatrix} Q_{11} & Q_{12} & Q_{13}\\ Q_{12}^* & Q_{22} & Q_{23}\\ Q_{13}^* & Q_{23}^* & Q_{33} \end{bmatrix}, \quad S_{\mathrm{F}} = \begin{bmatrix} S_1\\ S_2\\ S_3 \end{bmatrix}.$$

(a) *For $\Theta_{\mathrm{F}}(s) := I_m + FT(sE_{\mathrm{F}} - A_{\mathrm{F}})^{-1}B_{\mathrm{F}} \in \mathbb{K}(s)^{m\times m}$ the rational function*

$$\Phi_{\mathrm{F}}(s) = \begin{bmatrix} (-\bar{s}E_{\mathrm{F}} - A_{\mathrm{F}})^{-1}B_{\mathrm{F}}\\ I_m \end{bmatrix}^* \begin{bmatrix} Q_{\mathrm{F}} & S_{\mathrm{F}}\\ S_F^* & R_{\mathrm{F}} \end{bmatrix} \begin{bmatrix} (sE_{\mathrm{F}} - A_{\mathrm{F}})^{-1}B_{\mathrm{F}}\\ I_m \end{bmatrix}$$

*fulfils*

$$\Phi_{\mathrm{F}}(s) = \Theta_{\mathrm{F}}^*(-\bar{s})\Phi(s)\Theta_{\mathrm{F}}(s). \tag{3.9}$$

*Moreover, it holds that*

$$\begin{aligned} \Phi_{\mathrm{F}}(s) = &\begin{bmatrix} (-\bar{s}I_{n_1} - A_{11})^{-1}B_1\\ I_m \end{bmatrix}^* \begin{bmatrix} Q_{11} & S_1 - Q_{12}B_2\\ S_1^* - B_2^*Q_{12}^* & B_2^*Q_{22}B_2 - B_2^*S_2 - S_2^*B_2 + R \end{bmatrix}\\ &\times \begin{bmatrix} (sI_{n_1} - A_{11})^{-1}B_1\\ I_m \end{bmatrix}. \end{aligned} \tag{3.10}$$

(b) *Let* $P \in \mathbb{K}^{n\times n}$ *and define*

$$P_{\mathrm{F}} = W^{-*}PW^{-1} = \begin{bmatrix} P_{11} & P_{12} & P_{13} \\ P_{12}^* & P_{22} & P_{23} \\ P_{13}^* & P_{23}^* & P_{33} \end{bmatrix} \in \mathbb{K}^{n\times n}, \tag{3.11}$$

*which is partitioned according to the block structure of the FEF* (2.7)*. Then the following holds true:*
$P \in \mathbb{K}^{n\times n}$ *solves the KYP inequality* (3.2) *if, and only if,* $P$ *is Hermitian and*

$$\begin{bmatrix} A_{11}^*P_{11} + P_{11}A_{11} + Q_{11} & P_{11}B_1 + S_1 - Q_{12}B_2 \\ B_1^*P_{11} + S_1^* - B_2^*Q_{12}^* & B_2^*Q_{22}B_2 - B_2^*S_2 - S_2^*B_2 + R \end{bmatrix} \geq 0, \quad P_{11} = P_{11}^*. \tag{3.12}$$

**Remark 3.13.** Note that by definition of the matrices in (3.8) we have the following relationship between the weighting block matrices of $\Phi(s)$ and $\Phi_{\mathrm{F}}(s)$

$$\begin{aligned} &\begin{bmatrix} T^* & T^*F^* \\ 0 & I_m \end{bmatrix} \begin{bmatrix} Q & S \\ S^* & R \end{bmatrix} \begin{bmatrix} T & 0 \\ FT & I_m \end{bmatrix} \\ &= \begin{bmatrix} T^*Q + T^*F^*S^* & T^*S + T^*F^*R \\ S^* & R \end{bmatrix} \begin{bmatrix} T & 0 \\ FT & I_m \end{bmatrix} \\ &= \begin{bmatrix} T^*QT + T^*F^*S^*T + T^*SFT + T^*F^*RFT & T^*S + T^*F^*R \\ S^*T + RFT & R \end{bmatrix} \\ &= \begin{bmatrix} Q_{\mathrm{F}} & S_{\mathrm{F}} \\ S_{\mathrm{F}}^* & R_{\mathrm{F}} \end{bmatrix}. \end{aligned}$$

◊

**Proof of Lemma 3.12:**
(a) We divide the proof of (a) into two steps. At first, we show equation (3.9) and then (3.10).
Step 1: Due to Remark 3.13 we have

$$\begin{aligned} \Phi_{\mathrm{F}}(s) &= \begin{bmatrix} (-\bar{s}E_{\mathrm{F}} - A_{\mathrm{F}})^{-1}B_{\mathrm{F}} \\ I_m \end{bmatrix}^* \begin{bmatrix} Q_{\mathrm{F}} & S_{\mathrm{F}} \\ S_{\mathrm{F}}^* & R_{\mathrm{F}} \end{bmatrix} \begin{bmatrix} (sE_{\mathrm{F}} - A_{\mathrm{F}})^{-1}B_{\mathrm{F}} \\ I_m \end{bmatrix} \\ &= \begin{bmatrix} (-\bar{s}E_{\mathrm{F}} - A_{\mathrm{F}})^{-1}B_{\mathrm{F}} \\ I_m \end{bmatrix}^* \begin{bmatrix} T^* & T^*F^* \\ 0 & I_m \end{bmatrix} \begin{bmatrix} Q & S \\ S^* & R \end{bmatrix} \begin{bmatrix} T & 0 \\ FT & I_m \end{bmatrix} \begin{bmatrix} (sE_{\mathrm{F}} - A_{\mathrm{F}})^{-1}B_{\mathrm{F}} \\ I_m \end{bmatrix}. \end{aligned} \tag{3.13}$$

We show the following equality

$$H(s) := \begin{bmatrix} T & 0 \\ FT & I_m \end{bmatrix} \begin{bmatrix} (sE_{\mathrm{F}} - A_{\mathrm{F}})^{-1}B_{\mathrm{F}} \\ I_m \end{bmatrix} = \begin{bmatrix} (sE - A)^{-1}B \\ I_m \end{bmatrix} \Theta_{\mathrm{F}}(s). \tag{3.14}$$

The bottom block rows in (3.14) coincide by definition of $\Theta_{\mathrm{F}}(s)$. Further,

$$T(sE_{\mathrm{F}} - A_{\mathrm{F}})^{-1}B_{\mathrm{F}} = T(sWET - W(A + BF)T)^{-1}WB$$

$$
\begin{aligned}
&= T(W(sE-(A+BF))T)^{-1}WB \\
&= ((sE-A)-BF)^{-1}B
\end{aligned}
$$

holds. The Sherman-Morrison-Woodbury identity from Lemma 3.6 for $K := sE-A$, $L := -B$, $M := I_m$ and $N := F$ yields

$$((sE-A)-BF)^{-1}B = (sE-A)^{-1}B+(sE-A)^{-1}B(I_m-F(sE-A)^{-1}B)^{-1}F(sE-A)^{-1}B.$$

Applying Lemma 3.5 to $(I_m - F(sE-A)^{-1}B)^{-1}$ we get

$$
\begin{aligned}
& (sE-A)^{-1}B + (sE-A)^{-1}B\sum_{k=1}^{\infty}\left(F(sE-A)^{-1}B\right)^k \\
= \;& (sE-A)^{-1}B + (sE-A)^{-1}B\left(F(sE-A)^{-1}B + \sum_{k=2}^{\infty}\left(F(sE-A)^{-1}B\right)^k\right) \\
= \;& (sE-A)^{-1}B + (sE-A)^{-1}BF(sE-A)^{-1}B + (sE-A)^{-1}B\sum_{k=2}^{\infty}\left(F(sE-A)^{-1}B\right)^k \\
\overset{\text{Lem. 3.5}}{=} & (sE-A)^{-1}B + (sE-A)^{-1}BF(sE-A)^{-1}B \\
& \quad + (sE-A)^{-1}BF(sE-A)^{-1}B(I_m-F(sE-A)^{-1}B)^{-1}F(sE-A)^{-1}B \\
= \;& (sE-A)^{-1}B \\
& \quad \times \left[I_m + F(sE-A)^{-1}B + F(sE-A)^{-1}B(I_m-F(sE-A)^{-1}B)^{-1}F(sE-A)^{-1}B\right].
\end{aligned}
$$

Applying Lemma 3.6 once more yields

$$T(sE_{\mathrm{F}}-A_{\mathrm{F}})^{-1}B_{\mathrm{F}} = (sE-A)^{-1}B\left[I_m + F((sE-A)-BF)^{-1}B\right],$$

which completes the proof of (3.14).

Equation (3.14) provides

$$H^*(-s) = \begin{bmatrix}(-sE_{\mathrm{F}}-A_{\mathrm{F}})^{-1}B_{\mathrm{F}} \\ I_m\end{bmatrix}^* \begin{bmatrix}T^* & T^*F^* \\ 0 & I_m\end{bmatrix} = \Theta_{\mathrm{F}}^*(-s)\begin{bmatrix}(-sE_A)^{-1}B \\ I_m\end{bmatrix}^*.$$

Thus, (3.13) yields (3.9).

Step 2: We verify (3.10). Note that for regular $W, X$ and $Z$ we have

$$\begin{bmatrix}W & 0 & 0 \\ 0 & X & Y \\ 0 & 0 & Z\end{bmatrix}^{-1} = \begin{bmatrix}W^{-1} & 0 & 0 \\ 0 & X^{-1} & -X^{-1}YZ^{-1} \\ 0 & 0 & Z^{-1}\end{bmatrix}.$$

Thus,

$$(sE_{\mathrm{F}}-A_{\mathrm{F}})^{-1}B_{\mathrm{F}} = \begin{bmatrix}(sI_{n_1}-A_{11})^{-1} & 0 & 0 \\ 0 & -I_{n_2} & -sI_{n_2}E_{23}(sE_{23}-I_{n_3})^{-1} \\ 0 & 0 & (sE_{33}-I_{n_3})^{-1}\end{bmatrix}\begin{bmatrix}B_1 \\ B_2 \\ 0\end{bmatrix}$$

$$= \begin{bmatrix} (sI_{n_1} - A_{11})^{-1}B_1 \\ -B_2 \\ 0_{n_3 \times m} \end{bmatrix}. \tag{3.15}$$

Now we can prove the desired result (3.10) by using the definition of $\Phi_{\mathrm{F}}$, the matrices as in (3.8), and (3.15). Hence

$$\begin{aligned}
\Phi_{\mathrm{F}}(s) &= \begin{bmatrix} (-\bar{s}E_{\mathrm{F}} - A_{\mathrm{F}})^{-1}B_{\mathrm{F}} \\ I_m \end{bmatrix}^* \begin{bmatrix} Q_{\mathrm{F}} & S_{\mathrm{F}} \\ S_{\mathrm{F}}^* & R_{\mathrm{F}} \end{bmatrix} \begin{bmatrix} (sE_{\mathrm{F}} - A_{\mathrm{F}})^{-1}B_{\mathrm{F}} \\ I_m \end{bmatrix} \\
&= \begin{bmatrix} (-\bar{s}I_{n_1} - A_{11})^{-1}B_1 \\ -B_2 \\ 0_{n_3 \times m} \\ I_m \end{bmatrix}^* \begin{bmatrix} Q_{\mathrm{F}} & S_{\mathrm{F}} \\ S_{\mathrm{F}}^* & R_{\mathrm{F}} \end{bmatrix} \begin{bmatrix} (sI_{n_1} - A_{11})^{-1}B_1 \\ -B_2 \\ 0_{n_3 \times m} \\ I_m \end{bmatrix} \\
&= \begin{bmatrix} ((-\bar{s}I_{n_1} - A_{11})^{-1}B_1)^* & -B_2^* & 0_{m \times n_3} & I_m \end{bmatrix} \\
&\qquad \times \begin{bmatrix} Q_{11} & Q_{12} & Q_{13} & S_1 \\ Q_{12}^* & Q_{22} & Q_{23} & S_2 \\ Q_{13}^* & Q_{23}^* & Q_{33} & S_3 \\ S_1^* & S_2^* & S_3^* & R \end{bmatrix} \begin{bmatrix} (sI_{n_1} - A_{11})^{-1}B_1 \\ -B_2 \\ 0_{n_3 \times m} \\ I_m \end{bmatrix}.
\end{aligned}$$

Thus,

$$\begin{aligned}
\Phi_{\mathrm{F}}(s) &= \begin{bmatrix} ((-\bar{s}I_{n_1} - A_{11})^{-1}B_1)^* & -B_2^* & I_m \end{bmatrix} \begin{bmatrix} Q_{11} & Q_{12} & S_1 \\ Q_{12}^* & Q_{22} & S_2 \\ S_1^* & S_2^* & R \end{bmatrix} \begin{bmatrix} (sI_{n_1} - A_{11})^{-1}B_1 \\ -B_2 \\ I_m \end{bmatrix} \\
&= \begin{bmatrix} ((-\bar{s}I_{n_1} - A_{11})^{-1}B_1)^* & I_m \end{bmatrix} \begin{bmatrix} Q_{11} & S_1 - Q_{12}B_2 \\ S_1^* - B_2^*Q_{12}^* & B_2^*Q_{22}B_2 - B_2^*S_2 - S_2^*B_2 + R \end{bmatrix} \\
&\quad \times \begin{bmatrix} (sI_{n_1} - A_{11})^{-1}B_1 \\ I_m \end{bmatrix}.
\end{aligned}$$

The last equation holds since one can multiply the second row by $-B_2^*$ from the left, the second column by $-B_2$ from the right, and then combine the second and the forth row/column, resp. This proves (3.10).

(b) We divide the proof into two steps. At first, we relate the solvability of the KYP inequality (3.2) to that of the KYP inequality for the corresponding system in FEF (2.7). Then, we show the equivalence in (b).

Step 1: We show that $P$ solves the KYP inequality for the system $[E, A, B]$ if, and only if, $P_{\mathrm{F}}$ solves the KYP inequality for the system $[E_{\mathrm{F}}, A_{\mathrm{F}}, B_{\mathrm{F}}]$ in FEF (2.7).
By (3.8) the following calculations hold true:

$$\begin{bmatrix} T^* & T^*F^* \\ 0 & I_m \end{bmatrix} \begin{bmatrix} A^*PE + E^*PA + Q & E^*PB + S \\ B^*PE + S^* & R \end{bmatrix} \begin{bmatrix} T & 0 \\ FT & I_m \end{bmatrix} = \begin{bmatrix} X_1 & X_2 \\ X_3 & X_4 \end{bmatrix}, \tag{3.16}$$

where

$$\begin{aligned}
X_1 &:= T^*A^*PET + T^*E^*PAT + T^*(Q + F^*B^*PE + F^*S^* + E^*PBF + SF + F^*RF)T\\
&= T^*A^*PET + T^*E^*PAT + T^*(F^*B^*PE + E^*PBF)T + Q_{\mathrm{F}}\\
&= T^*(T^{-*}A_{\mathrm{F}}^*W^{-*} - F^*B^*)W^*P_{\mathrm{F}}WET + T^*E^*W^*P_{\mathrm{F}}W(W^{-1}A_{\mathrm{F}}T^{-1} - BF)T\\
&\qquad + T^*(F^*B^*PE + E^*PBF)T + Q_{\mathrm{F}}\\
&= A_{\mathrm{F}}^*P_{\mathrm{F}}E_{\mathrm{F}} - T^*F^*B^*PET + E_{\mathrm{F}}^*P_{\mathrm{F}}A_{\mathrm{F}} - T^*E^*PBFT\\
&\qquad + T^*(F^*B^*PE + E^*PBF)T + Q_{\mathrm{F}}\\
&= A_{\mathrm{F}}^*P_{\mathrm{F}}E_{\mathrm{F}} + E_{\mathrm{F}}^*P_{\mathrm{F}}A_{\mathrm{F}} + Q_{\mathrm{F}},\\
X_2 &:= T^*E^*PB + T^*S + T^*F^*R = T^*E^*W^*P_{\mathrm{F}}WB + T^*(S + F^*R) = E_{\mathrm{F}}^*P_{\mathrm{F}}B_{\mathrm{F}} + S_{\mathrm{F}},\\
X_3 &:= B^*W^*P_{\mathrm{F}}WET + (S^* + RF)T = B_{\mathrm{F}}^*P_{\mathrm{F}}E_{\mathrm{F}} + S_{\mathrm{F}}^* \text{ and}\\
X_4 &:= R = R_{\mathrm{F}}.
\end{aligned}$$

In combination with Proposition 2.41 this provides

$$\begin{aligned}
& 0 \le_{\mathcal{V}_{\mathrm{sys,F}}} \begin{bmatrix} X_1 & X_2 \\ X_3 & X_4 \end{bmatrix}\\
\Leftrightarrow\ & 0 \le_{\mathcal{V}_{\mathrm{sys,F}}} \begin{bmatrix} A_{\mathrm{F}}^*P_{\mathrm{F}}E_{\mathrm{F}} + E_{\mathrm{F}}^*P_{\mathrm{F}}A_{\mathrm{F}} + Q_{\mathrm{F}} & E_{\mathrm{F}}^*P_{\mathrm{F}}B_{\mathrm{F}} + S_{\mathrm{F}} \\ B_{\mathrm{F}}^*P_{\mathrm{F}}E_{\mathrm{F}} + S_{\mathrm{F}}^* & R_{\mathrm{F}} \end{bmatrix} && (3.17)\\
\Leftrightarrow\ & 0 \le x^* \begin{bmatrix} A_{\mathrm{F}}^*P_{\mathrm{F}}E_{\mathrm{F}} + E_{\mathrm{F}}^*P_{\mathrm{F}}A_{\mathrm{F}} + Q_{\mathrm{F}} & E_{\mathrm{F}}^*P_{\mathrm{F}}B_{\mathrm{F}} + S_{\mathrm{F}} \\ B_{\mathrm{F}}^*P_{\mathrm{F}}E_{\mathrm{F}} + S_{\mathrm{F}}^* & R_{\mathrm{F}} \end{bmatrix} x && \forall\, x \in \mathcal{V}_{\mathrm{sys,F}}\\
\overset{(3.16)}{\Leftrightarrow}\ & 0 \le x^* \begin{bmatrix} T^* & T^*F^* \\ 0 & I_m \end{bmatrix} \begin{bmatrix} A^*PE + E^*PA + Q & E^*PB + S \\ B^*PE + S^* & R \end{bmatrix} \begin{bmatrix} T & 0 \\ FT & I_m \end{bmatrix} x && \forall\, x \in \mathcal{V}_{\mathrm{sys,F}}\\
\overset{(2.19)}{\Leftrightarrow}\ & 0 \le y^* \begin{bmatrix} A^*PE + E^*PA + Q & E^*PB + S \\ B^*PE + S^* & R \end{bmatrix} y && \forall\, y \in \mathcal{V}_{\mathrm{sys}}\\
\Leftrightarrow\ & 0 \le_{\mathcal{V}_{\mathrm{sys}}} \begin{bmatrix} A^*PE + E^*PA + Q & E^*PB + S \\ B^*PE + S^* & R \end{bmatrix} && (3.18)
\end{aligned}$$

for all Hermitian $P \in \mathbb{K}^{n\times n}$.

Step 2: We prove the equivalence in (b) via two implications.
Step 2.1: Let $P \in \mathbb{K}^{n\times n}$ fulfil the KYP inequality (3.2). In particular, $P$ is Hermitian. Thus,

$$P_{\mathrm{F}}^* = \left(W^{-*}PW^{-1}\right)^* = W^{-*}PW^{-1} = P_{\mathrm{F}}$$

and thereby $P_{\mathrm{F}}$ and $P_{11}$ are Hermitian.
According to the definition of $E_{\mathrm{F}}, A_{\mathrm{F}}, B_{\mathrm{F}}, Q_{\mathrm{F}}, S_{\mathrm{F}}, R_{\mathrm{F}}$ and $P_{\mathrm{F}}$ we obtain

$$\begin{aligned}
& \begin{bmatrix} A_{\mathrm{F}}^*P_{\mathrm{F}}E_{\mathrm{F}} + E_{\mathrm{F}}^*P_{\mathrm{F}}A_{\mathrm{F}} + Q_{\mathrm{F}} & E_{\mathrm{F}}^*P_{\mathrm{F}}B_{\mathrm{F}} + S_{\mathrm{F}} \\ B_{\mathrm{F}}^*P_{\mathrm{F}}E_{\mathrm{F}} + S_{\mathrm{F}}^* & R_{\mathrm{F}} \end{bmatrix}\\
&= \begin{bmatrix} A_{11}^*P_{11} + P_{11}A_{11} + Q_{11} & P_{12} + Q_{12} & M_{13} & P_{11}B_1 + P_{12}B_2 + S_1 \\ P_{12}^* + Q_{12}^* & Q_{22} & M_{23} & S_2 \\ M_{13}^* & M_{23}^* & M_{33} & M_{34} \\ B_1^*P_{11} + B_2^*P_{12}^* + S_1^* & S_2^* & M_{34}^* & R \end{bmatrix} \qquad (3.19)
\end{aligned}$$

for some $M_{13} \in \mathbb{K}^{n_1 \times n_3}$, $M_{23} \in \mathbb{K}^{n_2 \times n_3}$, $M_{33} = M_{33}^* \in \mathbb{K}^{n_3 \times n_3}$, and $M_{34} \in \mathbb{K}^{n_3 \times m}$. In fact, we could determine the missing matrices, but they are irrelevant in the following.

To complete Step 2.1 we use (3.17), (3.18), (3.19) and (2.23), united to $\stackrel{\star}{=}$. Since $P$ solves the KYP inequality (3.2), calculating in an analogue manner as in Step 3 in the proof of (a) we arrive at

$$\begin{aligned}
0 &\leq \begin{pmatrix} x \\ u \end{pmatrix}^* \begin{bmatrix} A^*PE + E^*PA + Q & E^*PB + S \\ B^*PE + S^* & R \end{bmatrix} \begin{pmatrix} x \\ u \end{pmatrix} \\
&\stackrel{\star}{=} \begin{pmatrix} x_1 \\ -B_2u \\ 0_{n_3 \times 1} \\ u \end{pmatrix}^* \begin{bmatrix} A_{11}^*P_{11} + P_{11}A_{11} + Q_{11} & P_{12} + Q_{12} & M_{13} & P_{11}B_1 + P_{12}B_2 + S_1 \\ P_{12}^* + Q_{12}^* & Q_{22} & M_{23} & S_2 \\ M_{13}^* & M_{23}^* & M_{33} & M_{34} \\ B_1^*P_{11} + B_2^*P_{12}^* + S_1^* & S_2^* & M_{34}^* & R \end{bmatrix} \begin{pmatrix} x_1 \\ -B_2u \\ 0_{n_3 \times 1} \\ u \end{pmatrix} \\
&= \begin{pmatrix} x_1 \\ -B_2u \\ u \end{pmatrix}^* \begin{bmatrix} A_{11}^*P_{11} + P_{11}A_{11} + Q_{11} & P_{12} + Q_{12} & P_{11}B_1 + P_{12}B_2 + S_1 \\ P_{12}^* + Q_{12}^* & Q_{22} & S_2 \\ B_1^*P_{11} + B_2^*P_{12}^* + S_1^* & S_2^* & R \end{bmatrix} \begin{pmatrix} x_1 \\ -B_2u \\ u \end{pmatrix} \\
&= \begin{pmatrix} x_1 \\ u \end{pmatrix}^* \begin{bmatrix} A_{11}^*P_{11} + P_{11}A_{11} + Q_{11} & P_{11}B_1 + S_1 - Q_{12}B_2 \\ B_1^*P_{11} + S_1^* - B_2^*Q_{12} & B_2^*Q_{22}B_2 - B_2^*S_2 - S_2^*B_2 + R \end{bmatrix} \begin{pmatrix} x_1 \\ u \end{pmatrix}
\end{aligned}$$

and thus $P_{11}$ fulfils (3.12).

Step 2.2: Suppose $P_{11} \in \mathbb{K}^{n_1 \times n_1}$ solves (3.12) and let $P_{12}, P_{13}, P_{22} = P_{22}^*, P_{23}$, and $P_{33} = P_{33}^*$ have appropriate dimensions. Applying (2.23) to $(x^\top, u^\top)^\top \in \mathcal{V}_{\mathrm{sys,F}}$ we find some $x_1 \in \mathbb{K}^{n_1}$ such that $(x_1^\top, -u^\top B_2^\top, 0, u^\top)^\top \in \mathcal{V}_{\mathrm{sys,F}}$. Analogously to Step 2.1 it can be easily verified that $P$ fulfils the KYP inequality (3.2). □

## 3.4 Proof of KYP lemma and KYP alternative

In this section we prove the main results of this chapter mentioned in Section 3.2.

**Proof of Theorem 3.2:**
(a) At first, let $P \in \mathbb{K}^{n \times n}$ be a solution of the KYP inequality (3.2). Using a simple matrix subtraction the inequality is equivalent to

$$\begin{bmatrix} Q & S \\ S^* & R \end{bmatrix} \geq_{\mathcal{V}_{\mathrm{sys}}} - \begin{bmatrix} A^*PE + E^*PA & E^*PB \\ B^*PE & 0 \end{bmatrix}.$$

Further, assume that

$$\begin{bmatrix} (-\bar{s}E - A)^{-1}B \\ I_m \end{bmatrix}^* \begin{bmatrix} A^*PE + E^*PA & E^*PB \\ B^*PE & 0 \end{bmatrix} \begin{bmatrix} (sE - A)^{-1}B \\ I_m \end{bmatrix} = 0. \tag{3.20}$$

With (2.25) and (3.20), for all $\omega \in \mathbb{R}$ with $\det(\mathrm{i}\omega E - A) \neq 0$ we can infer that

$$\Phi(\mathrm{i}\omega) = \begin{bmatrix} (\mathrm{i}\omega E - A)^{-1}B \\ I_m \end{bmatrix}^* \begin{bmatrix} Q & S \\ S^* & R \end{bmatrix} \begin{bmatrix} (\mathrm{i}\omega E - A)^{-1}B \\ I_m \end{bmatrix}$$

$$\geq -\begin{bmatrix}(\mathrm{i}\omega E-A)^{-1}B\\ I_m\end{bmatrix}^* \begin{bmatrix}A^*PE+E^*PA & E^*PB\\ B^*PE & 0\end{bmatrix}\begin{bmatrix}(\mathrm{i}\omega E-A)^{-1}B\\ I_m\end{bmatrix}=0,$$

which is (3.3).

We still need to show that (3.20) holds true. For the left side of the equation it holds

$$\begin{aligned}
&\begin{bmatrix}(-\bar{s}E-A)^{-1}B\\ I_m\end{bmatrix}^* \begin{bmatrix}A^*PE+E^*PA & E^*PB\\ B^*PE & 0\end{bmatrix}\begin{bmatrix}(sE-A)^{-1}B\\ I_m\end{bmatrix}\\
&=\begin{bmatrix}(-\bar{s}E-A)^{-1}B\\ I_m\end{bmatrix}^* \left(\begin{bmatrix}A^*\\ B^*\end{bmatrix}\begin{bmatrix}PE & 0\end{bmatrix}+\begin{bmatrix}E^*P\\ 0\end{bmatrix}\begin{bmatrix}A & B\end{bmatrix}\right)\begin{bmatrix}(sE-A)^{-1}B\\ I_m\end{bmatrix}\\
&=\begin{bmatrix}(-\bar{s}E-A)^{-1}B\\ I_m\end{bmatrix}^* \begin{bmatrix}A^*\\ B^*\end{bmatrix}\begin{bmatrix}PE & 0\end{bmatrix}\begin{bmatrix}(sE-A)^{-1}B\\ I_m\end{bmatrix}\\
&\quad+\begin{bmatrix}(-\bar{s}E-A)^{-1}B\\ I_m\end{bmatrix}^* \begin{bmatrix}E^*P\\ 0\end{bmatrix}\begin{bmatrix}A & B\end{bmatrix}\begin{bmatrix}(sE-A)^{-1}B\\ I_m\end{bmatrix}.
\end{aligned}$$

Applying $A(sE-A)^{-1}B=(sE-(sE-A))(sE-A)^{-1}B=sE(sE-A)^{-1}B-B$ at $\overset{\star}{=}$ yields

$$\begin{aligned}
H(s)&:=\begin{bmatrix}(-\bar{s}E-A)^{-1}B\\ I_m\end{bmatrix}^* \begin{bmatrix}E^*P\\ 0\end{bmatrix}\begin{bmatrix}A & B\end{bmatrix}\begin{bmatrix}(sE-A)^{-1}B\\ I_m\end{bmatrix}\\
&\overset{\star}{=} s\begin{bmatrix}(-\bar{s}E-A)^{-1}B\\ I_m\end{bmatrix}^* \begin{bmatrix}E^*P\\ 0\end{bmatrix}\begin{bmatrix}E & 0\end{bmatrix}\begin{bmatrix}(sE-A)^{-1}B\\ I_m\end{bmatrix}\\
&= s\begin{bmatrix}B^*(-\bar{s}E-A)^{-*} & I_m\end{bmatrix}\begin{bmatrix}E^*PE(sE-A)^{-1}B\\ 0\end{bmatrix}\\
&= sB^*(-\bar{s}E-A)^{-*}E^*PE(sE-A)^{-1}B.
\end{aligned}$$

So we have an alternative representation of the second summand. Note that the first summand from above equals to $H^*(-s)$. Since $P\in\mathbb{K}^{n\times n}$ fulfils the KYP inequality (3.2) and thus in particular is Hermitian, we have

$$\begin{aligned}
H^*(-s)&=\left(-\bar{s}\left(B^*(sE-A)^{-*}E^*PE(-\bar{s}E-A)^{-1}B\right)\right)^*\\
&=-sB^*(-\bar{s}E-A)^{-*}E^*PE(sE-A)^{-1}B.
\end{aligned}$$

Altogether we obtain $H^*(-s)+H(s)=0$ and therefore (3.20) holds true, which completes the proof of (a).

(b) Let (3.3) and at least one of the properties (b1) and (b2) be fulfilled. According to Proposition 2.34 there exist transformation matrices $W,T$ and $F$ such that FEF (2.12) holds. Let $\omega\in\mathbb{R}$ with $\det(\mathrm{i}\omega E-A)\neq 0$. Then (3.3) together with (3.9) yields (b) Let (3.3) and at least one of the properties (b1) and (b2) be fulfilled. According to Proposition 2.34 there exist transformation matrices $W,T$ and $F$ such

that FEF (2.12) holds. Let $\omega \in \mathbb{R}$ with $\det(\mathrm{i}\omega E - A) \neq 0$. Then (3.3) together with (3.9) yields

$$\begin{aligned}
& 0 \le \Phi(\mathrm{i}\omega) \\
\Leftrightarrow \quad & 0 \le \eta^* \Phi(\mathrm{i}\omega)\eta^* && \forall\, \eta \in \mathbb{K}^m \\
\overset{(3.9)}{\Leftrightarrow} \quad & 0 \le (\Theta_{\mathrm{F}}(\mathrm{i}\omega)\xi)^* \, \Phi(\mathrm{i}\omega)\Theta_{\mathrm{F}}(\mathrm{i}\omega)\xi && \forall\, \xi \in \mathbb{K}^m \\
\Leftrightarrow \quad & 0 \le \xi^* \left(\Theta_{\mathrm{F}}(-\overline{\mathrm{i}\omega})\right)^* \Phi(\mathrm{i}\omega)\Theta_{\mathrm{F}}(\mathrm{i}\omega)\xi && \forall\, \xi \in \mathbb{K}^m \\
\overset{(3.9)}{\Leftrightarrow} \quad & 0 \le \xi^* \Phi_{\mathrm{F}}(\mathrm{i}\omega)\xi && \forall\, \xi \in \mathbb{K}^m \\
\Leftrightarrow \quad & 0 \le \Phi_{\mathrm{F}}(\mathrm{i}\omega).
\end{aligned}$$

Regarding the representation of $\Phi_{\mathrm{F}}$ as in (3.10) we get $0 \le \Phi_{\mathrm{F}}(\mathrm{i}\omega)$ for all $\omega \in \mathbb{R}$ with $\mathrm{i}\omega \notin \sigma(A_{11})$, i.e. (i) in Theorem 3.10 and in Theorem 3.11 holds.
For the rest of the proof we differ whether (b1) or (b2) holds.

<u>Case 1:</u> Assume that (b1) is true. The idea is to apply Theorem 3.10 to the ODE system $[I_{n_1}, A_{11}, B_1]$ and the corresponding Popov function $\Phi_{\mathrm{F}}$ as in (3.10).
As (b1) holds, the system $[E, A, B]$ is behaviourally sign-controllable and $\Phi$ is nonsingular. Using Remark 2.37 we see that the ODE system $[I_{n_1}, A_{11}, B_1]$ is also behaviourally sign-controllable. To make use of Theorem 3.10 we have to show that even $\Phi_{\mathrm{F}}$ is nonsingular.

By using (3.9) and since $\mathrm{rk}_{\mathbb{K}(s)}(\Phi(s)) = m$, it suffices to verify that $\Theta_{\mathrm{F}}(s)$ has full rank. Due to the equivalence

$$sE_{\mathrm{F}} - A_{\mathrm{F}} \text{ is regular } \Leftrightarrow sWET - W(A + BF)T \text{ is regular},$$

which holds by definition of $[E_{\mathrm{F}}, A_{\mathrm{F}}, B_{\mathrm{F}}]$, the matrix $(sE_{\mathrm{F}} - A_{\mathrm{F}}) + B_{\mathrm{F}}FT$ is invertible in $\mathrm{Gl}_n(\mathbb{K}(s))$. Thus, we can compute the inverse of $\Theta_{\mathrm{F}}(s)$ according to Lemma 3.6 as follows:

$$\Theta_{\mathrm{F}}(s)^{-1} = I_m - FT((sE_{\mathrm{F}} - A_{\mathrm{F}}) + B_{\mathrm{F}}FT)^{-1}B_{\mathrm{F}},$$

whereby $\mathrm{rk}_{\mathbb{K}(s)}\, \Theta_{\mathrm{F}}(s) = m$ obviously holds true. Hence, $\Phi_{\mathrm{F}}$ is nonsingular.

It remains to show that the corresponding matrices $B_2^* Q_{22} B_2 - B_2^* S_2 - S_2^* B_2 + R$ and $Q_{11}$ of (3.7) are Hermitian. Therefore, note that $Q$ and $R$ are Hermitian already. Due to its construction as in (3.8), $Q_{\mathrm{F}}$ and thereby $Q_{11}$ and $Q_{22}$ are Hermitian as well. This implies

$$(B_2^* Q_{22} B_2 - B_2^* S_2 - S_2^* B_2 + R)^* = B_2^* Q_{22} B_2 - S_2^* B_2 - B_2^* S_2 + R.$$

Thus, Theorem 3.10 can be applied to the ODE system $[I_{n_1}, A_{11}, B_1]$ and the corresponding Popov function $\Phi_{\mathrm{F}}$ as in (3.10). Then (3.7) leads to

$$\begin{bmatrix} A_{11}^* P_{11} + P_{11} A_{11} + Q_{11} & P_{11} B_1 + S_1 - Q_{12} B_2 \\ S_1^* - B_2^* Q_{12}^* + B_1^* P_{11} & B_2^* Q_{22} B_2 - B_2^* S_2 - S_2^* B_2 + R \end{bmatrix} \ge 0$$

for some $P_{11} = P_{11}^*$. According to Lemma 3.12 (b) we see that

$$P = W^* \begin{bmatrix} P_{11} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} W$$

solves the KYP inequality (3.2) and this proof is complete.

Case 2: If (b2) holds, then $[I_{n_1}, A_{11}, B_1]$ is behaviourally controllable and we can apply Theorem 3.11. The remaining part works analogously to the last paragraph in Case 1 and therefore it is omitted. $\square$

The other important result in this chapter is the existence of an alternative version of the KYP inequality in Proposition 3.4, which is proved next.

**Proof of Proposition 3.4:** At first, note that the second statement is evident. To prove the first implication the proof is divided into three steps. Those three steps are preparations to apply Lemma 3.12 (b). In Step 1 we define a matrix $Y_{\mathrm{F}}$, show that $E_{\mathrm{F}}^* Y_{\mathrm{F}}$ is Hermitian and arrive at a KYP inequality like in (3.12). The second step provides an auxiliary result that we need for Step 3. In this last step a matrix $P$ is defined with help of $Y_{\mathrm{F}}$ matrices of Step 2 such that $E^*PE = E^*Y$. The implication in Proposition 3.4 follows then with Lemma 3.12 (b).

Step 1:
Let $Y \in \mathbb{K}^{n\times n}$ fulfil (3.4). By Proposition 2.34 there exist some transformation matrices $W, T$ and $F$ such that (2.12) holds. We define

$$Y_{\mathrm{F}} := W^{-*} Y T = \begin{bmatrix} Y_{11} & Y_{12} & Y_{13} \\ Y_{21} & Y_{22} & Y_{23} \\ Y_{31} & Y_{32} & Y_{33} \end{bmatrix} \in \mathbb{K}^{n\times n}.$$

Since $E^*Y = Y^*E$ holds, we obtain

$$E_{\mathrm{F}}^* Y_{\mathrm{F}} = (WET)^*(W^{-*}YT) = T^*E^*YT = T^*Y^*ET = (T^*Y^*W^{-1})(WET) = Y_{\mathrm{F}}^* E_{\mathrm{F}}, \tag{3.21}$$

which is equivalent to

$$\begin{bmatrix} Y_{11} & Y_{12} & Y_{13} \\ 0 & 0 & 0 \\ E_{23}^*Y_{21} + E_{33}^*Y_{31} & E_{23}^*Y_{22} + E_{33}^*Y_{32} & E_{23}^*Y_{23} + E_{33}^*Y_{33} \end{bmatrix} = \begin{bmatrix} Y_{11}^* & 0 & Y_{21}^*E_{23} + Y_{31}^*E_{33} \\ Y_{12}^* & 0 & Y_{22}^*E_{23} + Y_{32}^*E_{33} \\ Y_{13}^* & 0 & Y_{23}^*E_{23} + Y_{33}^*E_{33} \end{bmatrix}.$$

Since (3.21) the matrix $E_{\mathrm{F}}^* Y_F$ is Hermitian, we conclude

(i) $Y_{11} = Y_{11}^*$,

(ii) $0 = Y_{12} = Y_{12}^*$,

(iii) $Y_{13} = Y_{21}^* E_{23} + Y_{31}^* E_{33}$,

(iv) $0 = Y_{22}^* E_{23} + Y_{32}^* E_{33}$ and

(v) $E_{23}^* Y_{23} + E_{33}^* Y_{33}$ is Hermitian.

With an argumentation analogous to the proof of Lemma 3.12 (b) we arrive at

$$\begin{bmatrix} A_{11}^* Y_{11} + Y_{11} A_{11} + Q_{11} & Y_{11} B_1 + S_1 - Q_{12} B_2 \\ B_1^* Y_{11} + S_1^* - B_2^* Q_{12}^* & B_2^* Q_{22} B_2 - B_2^* S_2 - S_2^* B_2 + R \end{bmatrix} \geq 0, \quad Y_{11} = Y_{11}^*. \tag{3.22}$$

Step 2: Note that $\begin{bmatrix} E_{23}^* & E_{33}^* \end{bmatrix} \in \mathbb{K}^{n_3 \times (n_2+n_3)}$ and $\begin{bmatrix} Y_{23} \\ Y_{33} \end{bmatrix} \in \mathbb{K}^{(n_2+n_3) \times n_3}$ holds and the product of these matrices is Hermitian. Thus, Proposition 3.8 yields

$$E_{23}^* Y_{23} + E_{33}^* Y_{33} = \begin{bmatrix} E_{23}^* & E_{33}^* \end{bmatrix} \begin{bmatrix} Y_{23} \\ Y_{33} \end{bmatrix} = \begin{bmatrix} E_{23}^* & E_{33}^* \end{bmatrix} \begin{bmatrix} P_{22} & P_{23} \\ P_{23}^* & P_{33} \end{bmatrix} \begin{bmatrix} E_{23} \\ E_{33} \end{bmatrix} \tag{3.23}$$

for some $P_{22} = P_{22}^* \in \mathbb{K}^{n_2 \times n_2}$, $P_{23} \in \mathbb{K}^{n_2 \times n_3}$, $P_{33} = P_{33}^* \in \mathbb{K}^{n_3 \times n_3}$.

Step 3: Now we are able to define a solution matrix for the KYP inequality. Let $P \in \mathbb{K}^{n \times n}$ be defined as

$$P = W^* P_{\mathrm{F}} W = W^* \begin{bmatrix} Y_{11} & Y_{21}^* & Y_{31}^* \\ Y_{21} & P_{22} & P_{23} \\ Y_{31} & P_{23}^* & P_{33} \end{bmatrix} W.$$

To prove the statement of the proposition, we still need to show that the constructed matrix $P$ fulfils $E^* P E = E^* Y$ or alternatively $E_{\mathrm{F}}^* P_{\mathrm{F}} E_{\mathrm{F}} = E_{\mathrm{F}}^* Y_{\mathrm{F}}$, which is equivalent:

$$\begin{aligned} E^* P E = E^* Y \quad &\Leftrightarrow \quad E^* W^* P_{\mathrm{F}} W E = (W^{-1} E_{\mathrm{F}} T^{-1})^* \left( W^* Y_{\mathrm{F}} T^{-1} \right) \\ &\Leftrightarrow \quad T^{-*} E_{\mathrm{F}}^* P_{\mathrm{F}} E_{\mathrm{F}} T^{-1} = T^{-*} E_{\mathrm{F}}^* Y_{\mathrm{F}} T^{-1} \\ &\Leftrightarrow \quad E_{\mathrm{F}}^* P_{\mathrm{F}} E_{\mathrm{F}} = E_{\mathrm{F}}^* Y_{\mathrm{F}}. \end{aligned}$$

Calculating straight forward and using (3.23) we get

$$\begin{aligned} E_{\mathrm{F}}^* P_{\mathrm{F}} E_{\mathrm{F}} &= \begin{bmatrix} Y_{11} & Y_{21}^* & Y_{31}^* \\ 0 & 0 & 0 \\ E_{23}^* Y_{21} + E_{33}^* Y_{31} & E_{23}^* P_{22} + E_{33}^* P_{23}^* & E_{23}^* P_{23} + E_{33}^* P_{33} \end{bmatrix} E_{\mathrm{F}} \\ &= \begin{bmatrix} Y_{11} & 0 & Y_{21}^* E_{23} + Y_{31}^* E_{33} \\ 0 & 0 & 0 \\ E_{23}^* Y_{21} + E_{33}^* Y_{31} & 0 & E_{23}^* P_{22} E_{23} + E_{33}^* P_{23}^* E_{23} + E_{23}^* P_{23} E_{33} + E_{33}^* P_{33} E_{33} \end{bmatrix} \\ &\overset{(3.23)}{\underset{\&(\mathrm{iii})}{=}} \begin{bmatrix} Y_{11} & 0 & Y_{13} \\ 0 & 0 & 0 \\ E_{23}^* Y_{21} + E_{33}^* Y_{31} & 0 & E_{23}^* Y_{23} + E_{33}^* Y_{33} \end{bmatrix} = E_{\mathrm{F}}^* Y_{\mathrm{F}}. \end{aligned}$$

Thus, we have $E_{\mathrm{F}}^* P_{\mathrm{F}} E_{\mathrm{F}} = E_{\mathrm{F}}^* Y_{\mathrm{F}}$ or equally $E^* P E = E^* Y$.

The desired result then follows from applying Lemma 3.12 (b) to (3.22). □

# 4 Lur'e equations

In this chapter we study particular solutions of the KYP inequality (3.2). They solve the so-called Lur'e equation, where equality to a Hermitian matrix on the right hand side of (3.2) is claimed.

To perform the studies on Lur'e equations and their solutions, this chapter is structured as follows. At first, we introduce the Lur'e equation (4.1). In Section 4.2 we present the main result - Theorem 4.4, which states a relationship between the solvability of the Lur'e equation and the positive semi-definiteness of the Popov function on the imaginary axis and deflating subspaces for an associated even matrix pencil. The proof of the main result is extensive and requires some results on Lur'e equations. For this reason, in Section 4.3 we collect some findings on Lur'e equations associated with ODEs and in Section 4.4 for the differential-algebraic case. Section 4.5 is dedicated to the proof of Theorem 4.4. In Section 4.6 we finally show how to construct a Lur'e solution using deflating subspaces.

## 4.1 Introduction to Lur'e equations

Consider a system $[E, A, B] \in \Sigma_{n,m}(\mathbb{K})$ with system space $\mathcal{V}_{\text{sys}} \subseteq \mathbb{K}^{n+m}$, weighting matrices $Q = Q^* \in \mathbb{K}^{n\times n}$, $S \in \mathbb{K}^{n\times m}$, and $R = R^* \in \mathbb{K}^{m\times m}$. The equation

$$\begin{bmatrix} A^*XE + E^*XA + Q & E^*XB + S \\ B^*XE + S^* & R \end{bmatrix} =_{\mathcal{V}_{\text{sys}}} \begin{bmatrix} K^* \\ L^* \end{bmatrix} \begin{bmatrix} K & L \end{bmatrix}, \quad X = X^*. \tag{4.1a}$$

is the associated *Lur'e equation.*

A triple $(X, K, L) \in \mathbb{K}^{n\times n} \times \mathbb{K}^{q\times n} \times \mathbb{K}^{q\times m}$ is called *solution of the Lur'e equation* if it satisfies (4.1a) and

$$\operatorname{rk}_{\mathbb{K}(s)} \begin{bmatrix} -sE + A & B \\ K & L \end{bmatrix} = n + q. \tag{4.1b}$$

Note that $q \in \mathbb{N}_0$ is part of the solution and specified in Proposition 4.5.

The concept of Lur'e equations can be applied on ODEs as well. This is the content of the following remark. We also show that with additional assumptions on the weighting matrix $R$ the algebraic Riccati equation can be derived from the Lur'e equation.

**Remark 4.1.** In the ODE case, i.e. $E = I_n$, and thus $\mathcal{V}_{\text{sys}} = \mathbb{K}^{n+m}$, equation (4.1a) simplifies to

$$\begin{bmatrix} A^*X + XA + Q & XB + S \\ B^*X + S^* & R \end{bmatrix} = \begin{bmatrix} K^* \\ L^* \end{bmatrix} \begin{bmatrix} K & L \end{bmatrix}, \quad X = X^*, \tag{4.2}$$

which is equivalent to

$$A^*X + XA + Q = K^*K, \quad XB + S = K^*L, \quad R = L^*L, \quad X = X^*. \tag{4.3}$$

If in addition $R$ is invertible, then the equations in (4.3) simplify even more.
In Proposition 4.5 we will see that $q = \operatorname{rk}_{\mathbb{K}(s)} \Phi(s) \leq m$ holds true. Thus, $R = L^*L$ implies that $L$ is invertible and $K^*K = K^*LL^{-1}L^{-*}L^*K = K^*LR^{-1}L^*K$ holds. Hence, we obtain the algebraic Riccati equation

$$A^*X + XA - (XB + S)R^{-1}(XB + S)^* + Q = 0.$$

Note that for $E \neq I_n$ the assumption $\mathcal{V}_{\text{sys}} = \mathbb{K}^{n+m}$ does not necessarily hold. Therefore, these transformations can not be applied on a DAE Lur'e equation in general. $\Diamond$

To study Lur'e equations, the weighting matrix $R$ does not necessarily need to be invertible. In contrast to the algebraic Riccati equation we can study Lur'e equations for a singular weighting matrix $R = R^*$ as well.

So we know that the concept of Lur'e equations is a generalization of algebraic Riccati equations. Now we can put our focus on the connection between solutions of Lur'e equation and solutions of the KYP inequality.

**Remark 4.2.** If $(X, K, L)$ solves the Lur'e equation (4.1), then $X$ is a solution of the KYP inequality (3.2), i.e.

$$\begin{bmatrix} A^*XE + E^*XA + Q & E^*XB + S \\ B^*XE + S^* & R \end{bmatrix} \geq_{\mathcal{V}_{\text{sys}}} 0.$$

$\Diamond$

## 4.2 Solutions of Lur'e equations via deflating subspaces

To formulate the main result of the chapter we first need to introduce *deflating subspaces*, *neutrality* of a subspace with respect to a matrix and *even* matrix pencils.

**Definition 4.3.**

(a) A matrix $Y \in \mathbb{K}^{n \times k}$ is called *basis matrix* for a subspace $\mathcal{Y} \subseteq \mathbb{K}^n$ of dimension $k$ if $\operatorname{rk} Y = k$ and $\operatorname{im} Y = \mathcal{Y}$.

(b) Let $\mathcal{Y} \subseteq \mathbb{K}^n$ be a subspace with basis matrix $Y \in \mathbb{K}^{n \times k}$. We call $\mathcal{Y}$ a *(right) deflating subspace* for the pencil $sE - A \in \mathbb{K}[s]^{\ell \times n}$ if there exists some $j \in \mathbb{N}$, a matrix $Z \in \mathbb{K}^{\ell \times j}$ and a pencil $s\tilde{E} - \tilde{A} \in \mathbb{K}[s]^{j \times k}$ with $\operatorname{rk}_{\mathbb{K}(s)} \left(s\tilde{E} - \tilde{A}\right) = j$ such that

$$(sE - A)Y = Z(s\tilde{E} - \tilde{A}).$$

(c) Let $H \in \mathbb{K}^{n\times n}$ be given. A subspace $\mathcal{Y} \subseteq \mathbb{K}^n$ is called *$H$-neutral* if

$$y_1^* H y_2 = 0 \quad \forall\, y_1, y_2 \in \mathcal{Y}.$$

(d) A pencil $sE - A \in \mathbb{K}[s]^{n\times n}$ is called *even* if

$$sE - A = -sE^* - A^*.$$

◊

Theorem 4.4 further requires a special even matrix pencil. Associated to a system $[E, A, B] \in \Sigma_{n,m}(\mathbb{K})$ and weighting matrices $Q = Q^* \in \mathbb{K}^{n\times n}$, $R = R^* \in \mathbb{K}^{m\times m}$, and $S \in \mathbb{K}^{n\times m}$ it is denoted by

$$s\mathcal{E} - \mathcal{A} \;:=\; \begin{bmatrix} 0 & -s\Pi E + A & B \\ sE^*\Pi^* + A^* & Q & S \\ B^* & S^* & R \end{bmatrix} \in \mathbb{K}[s]^{(2n+m)\times(2n+m)}. \tag{4.4}$$

The projector $\Pi$ is defined as

$$\Pi := W^{-1} \begin{bmatrix} I_{n_1} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} W \in \mathbb{K}^{n\times n} \tag{4.5}$$

with matrices $W, T \in \mathrm{Gl}_n(\mathbb{K})$ and $F \in \mathbb{K}^{m\times n}$ such that (2.12) holds.

Now we are able to state the main result of this chapter. It describes a relationship between the existence of a Lur'e solution and deflating subspaces. In Section 4.6 we show how a Lur'e solution can be reconstructed based on this theorem.

**Theorem 4.4.** *Let $[E, A, B] \in \Sigma_{n,m}(\mathbb{K})$ with system space $\mathcal{V}_{\mathrm{sys}}$ and weighting matrices $Q = Q^* \in \mathbb{K}^{n\times n}, S \in \mathbb{K}^{n\times m}$ and $R = R^* \in \mathbb{K}^{m\times m}$ be given. Let $W, T \in \mathrm{Gl}_n(\mathbb{K})$ and $F \in \mathbb{K}^{m\times n}$ be given such that* (2.12) *holds. Further, let the projector $\Pi$ be defined as in* (4.5)*, and $s\mathcal{E} - \mathcal{A}$ as in* (4.4)*. Then the following statements are equivalent:*

(a) *The Lur'e equation* (4.1) *has a solution $(X, K, L) \in \mathbb{K}^{n\times n} \times \mathbb{K}^{q\times n} \times \mathbb{K}^{q\times m}$.*

(b) *The Popov function $\Phi$ fulfils $\Phi(\mathrm{i}\omega) \geq 0$ for all $\omega \in \mathbb{R}$ with $\det(\mathrm{i}\omega E - A) \neq 0$ and there exist some matrices $Y_\mu$, $Y_x \in \mathbb{K}^{n\times(n+m)}$, $Y_u \in \mathbb{K}^{m\times(n+m)}$, and $Z_\mu$, $Z_x \in \mathbb{K}^{n\times(n+q)}$, $Z_u \in \mathbb{K}^{m\times(n+q)}$ such that*

$$Y = \begin{bmatrix} Y_\mu \\ Y_x \\ Y_u \end{bmatrix} \in \mathbb{K}^{(2n+m)\times(n+m)}, \qquad Z = \begin{bmatrix} Z_\mu \\ Z_x \\ Z_u \end{bmatrix} \in \mathbb{K}^{(2n+m)\times(n+q)} \tag{4.6}$$

*satisfy the following:*

(b1) *the space* $\operatorname{im} Y$ *is* $(n+m)$*-dimensional and* $\mathcal{E}$*-neutral;*

(b2) $\mathcal{V}_{\mathrm{sys}} \subseteq \operatorname{im} \begin{bmatrix} Y_x \\ Y_u \end{bmatrix}$*;*

(b3) $\operatorname{rk} \Pi E Y_x = n_1$*;*

(b4) *there exist* $\tilde{\mathcal{E}}$, $\tilde{\mathcal{A}} \in \mathbb{K}^{(n+q)\times(n+m)}$ *with* $\operatorname{rk}_{\mathbb{K}(s)}(s\tilde{\mathcal{E}} - \tilde{\mathcal{A}}) = n+q$ *such that*

$$(s\mathcal{E} - \mathcal{A})Y = Z(s\tilde{\mathcal{E}} - \tilde{\mathcal{A}}).$$

We now specify the number $q \in \mathbb{N}_0$ of rows of the matrices $K$ and $L$ belonging to a Lur'e solution $(X, K, L)$ as in (4.1).

**Proposition 4.5.** *Let* $[E, A, B] \in \Sigma_{n,m}(\mathbb{K})$ *with system space* $\mathcal{V}_{\mathrm{sys}} \subseteq \mathbb{K}^{n+m}$ *and weighting matrices* $Q = Q^* \in \mathbb{K}^{n\times n}$, $S \in \mathbb{K}^{n\times m}$, $R = R^* \in \mathbb{K}^{m\times m}$ *be given and let the Popov function* $\Phi(s) \in \mathbb{K}(s)^{m\times m}$ *be defined as in* (3.1)*. Furthermore, let* $(X, K, L) \in \mathbb{K}^{n\times n} \times \mathbb{K}^{q\times n} \times \mathbb{K}^{q\times m}$ *be a solution of the Lur'e equation* (4.1)*. Then it holds*

$$q = \operatorname{rk}_{\mathbb{K}(s)} \Phi(s).$$

To prove this proposition we split a matrix into a product of some other matrix and its conjugate transpose. For this reason we need the following lemma on diagonalization.

**Lemma 4.6.** *Let* $M = M^* \in \mathbb{K}^{\ell\times\ell}$ *be given.*

(a) *There exists some orthogonal* ($\mathbb{K} = \mathbb{R}$) */ unitary* ($\mathbb{K} = \mathbb{C}$) *matrix* $\tilde{\Lambda} \in \mathbb{K}^{\ell\times\ell}$ *such that*

$$\tilde{\Lambda}^* M \tilde{\Lambda} = \operatorname{diag}(\lambda_1, \ldots, \lambda_\ell),$$

*where* $\lambda_1, \ldots, \lambda_\ell \in \mathbb{R}$ *denote the eigenvalues of* $M$*.*

(b) *If in addition* $M$ *is positive semi-definite, i.e.* $\lambda_i \geq 0$ *for all* $i$*, then* $M = \Lambda\Lambda^*$ *for some* $\Lambda \in \mathbb{K}^{\ell\times\ell}$*.*

**Proof:** For the proof of (a) see [Fis10, Corollary in 5.6.2]. Then (b) follows from (a) as

$$M = \tilde{\Lambda} \operatorname{diag}(\lambda_1, \ldots, \lambda_\ell)\, \tilde{\Lambda}^* = \tilde{\Lambda} \operatorname{diag}(\sqrt{\lambda_1}, \ldots, \sqrt{\lambda_\ell}) \operatorname{diag}(\sqrt{\lambda_1}, \ldots, \sqrt{\lambda_\ell})^* \tilde{\Lambda}^* = \Lambda\Lambda^*,$$

for $\Lambda = \tilde{\Lambda} \operatorname{diag}(\sqrt{\lambda_1}, \ldots, \sqrt{\lambda_\ell})$. □

In the same step of the proof of Proposition 4.5 we need to show a rank inequality and apply Sylvester's rank inequality there.

**Lemma 4.7** (Sylvester's rank inequality)**.** *For* $G \in \mathbb{K}^{j\times k}$ *and* $H \in \mathbb{K}^{k\times\ell}$ *it holds that*

$$\operatorname{rk} G + \operatorname{rk} H - k \leq \operatorname{rk}(GH) \leq \min\{\operatorname{rk} G, \operatorname{rk} H\}. \tag{4.7}$$

**Proof:** See [Fis10, Chap. 2.5]. $\square$

**Proof of Proposition 4.5:** To show that $q = \operatorname{rk}_{\mathbb{K}(s)} \Phi(s)$, the proof is divided into three steps. In Step 1 it is shown that for a KYP solution $P$ the left hand side of (3.2) can be decomposed such that $P$ fulfils the Lur'e equation (4.1a). This result is applied in Step 2 to show an upper boundary for the rank of $\Phi(s)$. Step 3 then finally shows the statement through a decomposition of $\Phi(s)$ using Lur'e solutions.

Step 1: Let $P \in \mathbb{K}^{n\times n}$ fulfil the KYP inequality (3.2). We show that there exist some matrices $M \in \mathbb{K}^{\ell\times n}$ and $N \in \mathbb{K}^{\ell\times m}$ such that

$$\begin{bmatrix} A^*PE + E^*PA + Q & E^*PB + S \\ B^*PE + S^* & R \end{bmatrix} =_{\mathcal{V}_{\text{sys}}} \begin{bmatrix} M^* \\ N^* \end{bmatrix} \begin{bmatrix} M & N \end{bmatrix} \tag{4.8}$$

holds.
At first, note that Proposition 2.41 along with (2.23) yields

$$\Gamma^* \begin{bmatrix} A^*PE + E^*PA + Q & E^*PB + S \\ B^*PE + S^* & R \end{bmatrix} \Gamma \geq 0,$$

where the projector $\Gamma : \mathbb{K}^{n+m} \to \mathcal{V}_{\text{sys}} \subseteq \mathbb{K}^{n+m}$ is defined by

$$\Gamma := \begin{bmatrix} T & 0 \\ FT & I_m \end{bmatrix} \begin{bmatrix} I_{n_1} & 0 & 0 & 0 \\ 0 & 0 & 0 & -B_2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & I_m \end{bmatrix}.$$

Thus, Lemma 4.6 provides the existence of some $\Lambda \in \mathbb{K}^{n+m}$ such that

$$\Gamma^* \begin{bmatrix} A^*PE + E^*PA + Q & E^*PB + S \\ B^*PE + S^* & R \end{bmatrix} \Gamma = \Lambda\Lambda^*.$$

Since $\Gamma$ fulfils the projector condition $\Gamma^2 = \Gamma$, this again yields

$$\Gamma^* \begin{bmatrix} A^*PE + E^*PA + Q & E^*PB + S \\ B^*PE + S^* & R \end{bmatrix} \Gamma = \Gamma^*\Lambda\Lambda^*\Gamma,$$

which implies

$$\begin{bmatrix} A^*PE + E^*PA + Q & E^*PB + S \\ B^*PE + S^* & R \end{bmatrix} =_{\mathcal{V}_{\text{sys}}} \Lambda\Lambda^*.$$

Decomposing $\Lambda$ into $\Lambda = \begin{bmatrix} M^* \\ N^* \end{bmatrix}$ provides (4.8), which was to show.

Step 2: Let $\lambda \in \mathbb{C}$ with $\det(\lambda E - A) \neq 0$. Then

$$\Phi(\lambda) \overset{(3.1)}{=} \begin{bmatrix} (\lambda E - A)^{-1}B \\ I_m \end{bmatrix}^* \begin{bmatrix} Q & S \\ S^* & R \end{bmatrix} \begin{bmatrix} (\lambda E - A)^{-1}B \\ I_m \end{bmatrix}$$

$$\begin{aligned}
&\overset{(4.8)}{\underset{(2.25)}{=}} \begin{bmatrix} (\lambda E - A)^{-1}B \\ I_m \end{bmatrix}^* \begin{bmatrix} M^* \\ N^* \end{bmatrix} \begin{bmatrix} M & N \end{bmatrix} \begin{bmatrix} (\lambda E - A)^{-1}B \\ I_m \end{bmatrix} \\
&\qquad - \begin{bmatrix} (\lambda E - A)^{-1}B \\ I_m \end{bmatrix}^* \begin{bmatrix} A^*PE + E^*PA & E^*PB \\ B^*PE & 0 \end{bmatrix} \begin{bmatrix} (\lambda E - A)^{-1}B \\ I_m \end{bmatrix} \\
&\overset{(3.20)}{=} \left( \begin{bmatrix} M & N \end{bmatrix} \begin{bmatrix} (\lambda E - A)^{-1}B \\ I_m \end{bmatrix} \right)^* \left( \begin{bmatrix} M & N \end{bmatrix} \begin{bmatrix} (\lambda E - A)^{-1}B \\ I_m \end{bmatrix} \right) \\
&= Z^*(\lambda) Z(\lambda),
\end{aligned}$$

where

$$Z(\lambda) := N + M(\lambda E - A)^{-1}B \in \mathbb{K}^{\ell \times m} \text{ for all } \lambda \in \mathbb{C} \text{ with } \det(\lambda E - A) \neq 0.$$

Further, it holds that

$$\operatorname{rk}_{\mathbb{K}(s)} \Phi(s) \overset{(4.7)}{\leq} \operatorname{rk}_{\mathbb{K}(s)} Z(s) \leq \min\{\ell, m\} \leq \ell.$$

Step 3: Let $(X, K, L)$ be a solution of the Lur'e equation (4.1). An argumentation analogous to that in Step 2 shows

$$\Phi(\lambda) = W^*(\lambda) W(\lambda), \tag{4.9}$$

where

$$W(\lambda) := L + K(\lambda E - A)^{-1}B \in \mathbb{K}^{q \times m} \text{ for all } \lambda \in \mathbb{C} \text{ with } \det(\lambda E - A) \neq 0.$$

Since $(X, K, L)$ solves the Lur'e equation (4.1) it holds

$$\begin{aligned}
n + q &\overset{(4.1b)}{=} \operatorname{rk}_{\mathbb{K}(s)} \begin{bmatrix} -sE + A & B \\ K & L \end{bmatrix} \\
&= \operatorname{rk}_{\mathbb{K}(s)} \begin{bmatrix} -sE + A & B \\ K & L \end{bmatrix} \begin{bmatrix} (-sE + A)^{-1} & (-sE + A)^{-1}B \\ 0 & I_m \end{bmatrix} \\
&= \operatorname{rk}_{\mathbb{K}(s)} \begin{bmatrix} I_n & 0 \\ K(-sE + A)^{-1} & -W(s) \end{bmatrix}
\end{aligned}$$

and thus $\operatorname{rk}_{\mathbb{K}(s)} W(s) = q$.
Applying Sylvester's rank inequality as in Lemma 4.7

$$\begin{aligned}
q = {} & \operatorname{rk}_{\mathbb{K}(s)} W^*(s) + \operatorname{rk}_{\mathbb{K}(s)} W(s) - q \overset{(4.7)}{\leq} \operatorname{rk}_{\mathbb{K}(s)} (W^*(s) W(s)) \\
&\overset{(4.9)}{=} \operatorname{rk}_{\mathbb{K}(s)} \Phi(s) \leq \operatorname{rk}_{\mathbb{K}(s)} W(s) = q
\end{aligned}$$

we can infer

$$q = \operatorname{rk}_{\mathbb{K}(s)} W(s) = \operatorname{rk}_{\mathbb{K}(s)} \Phi(s) \leq \ell.$$

$\square$

**Remark 4.8.** The proof of Proposition 4.5 provides some notes on Lur'e solutions.

(i) The triple $(P, M, N)$ as in (4.8) is not supposed to fulfil (4.1b). Therefore, in general $(P, M, N)$ is not a solution of the Lur'e equation (4.1).

(ii) Solutions of the Lur'e equation (4.1) are rank-minimizing in the sense that the rank of the matrix on the right hand side of (4.8) is minimized. $\Diamond$

## 4.3 The ODE case

In this section we present some findings from [Rei11] for Lur'e equations in the ODE case. Theorem 4.9 considers the existence of Lur'e solutions in the ODE case. It is applied to prove Theorem 4.4 in Section 4.5. Lemma 4.11 and Lemma 4.12 consider the ODE case as well. Together with Theorem 4.9 they are used in Section 4.4 to prove the existence of Lur'e solutions and that Lur'e solutions are extremal solutions of the KYP inequality.

In the ODE case $[I, A, B] \in \Sigma_{n,m}(\mathbb{K})$, the even matrix pencil (4.4) reads

$$(s\mathcal{E} - \mathcal{A})_{\mathrm{ODE}} = \begin{bmatrix} 0 & -sI_n + A & B \\ sI_n + A^* & Q & S \\ B^* & S^* & R \end{bmatrix}. \tag{4.10}$$

Below we state [Rei11, Theorem 11], which is an ODE version of Theorem 4.4.

**Theorem 4.9.** *Let the pencil $(s\mathcal{E} - \mathcal{A})_{\mathrm{ODE}}$ be defined as in (4.10) and let the Popov function as in (3.1) satisfy $\Phi(i\omega) \geq 0$ for all $\omega \in \mathbb{R}$ with $i\omega \notin \sigma(A)$. Moreover, let $q = \operatorname{normalrank} \Phi(s)$. Then the following two statements are equivalent:*

(i) *For any Hermitian $X \in \mathbb{K}^{n\times n}$ there exist $K \in \mathbb{K}^{q\times n}, L \in \mathbb{K}^{q\times m}$ such that $(X, K, L)$ is a solution of Lur'e equation (4.2).*

(ii) *There exist $V_\mu$, $V_x \in \mathbb{K}^{n\times(n+m)}$, $V_u \in \mathbb{K}^{m\times(n+m)}$, and $W_\mu$, $W_x \in \mathbb{K}^{n\times(n+q)}$, $W_u \in \mathbb{K}^{m\times(n+q)}$ and $\tilde{\mathcal{E}}$, $\tilde{\mathcal{A}} \in \mathbb{K}^{(n+q)\times(n+m)}$ such that*

$$V = \begin{bmatrix} V_\mu \\ V_x \\ V_u \end{bmatrix} \in \mathbb{K}^{(2n+m)\times(n+m)}, \quad W = \begin{bmatrix} W_\mu \\ W_x \\ W_u \end{bmatrix} \in \mathbb{K}^{(2n+m)\times(n+q)} \tag{4.11}$$

*satisfy*

- *the space $\mathcal{V} = \operatorname{im} V$ is maximally $\mathcal{E}$-neutral with $\operatorname{rk} V_x = n$;*
- *$X = V_\mu V_x^+$ for some arbitrary right inverse $V_x^+$ of $V_x$;*
- *$(s\mathcal{E} - \mathcal{A})\, V = W\, (s\tilde{\mathcal{E}} - \tilde{\mathcal{A}})$.*

**Remark 4.10.** Let $\mathcal{E} \in \mathbb{K}^{(2n+m)\times(2n+m)}$ as in (4.10) be given. According to [Rei11] a subspace $\mathcal{V} \subseteq \mathbb{K}^{2n+m}$ is *maximally $\mathcal{E}$-neutral* if it is $\mathcal{E}$-neutral and $\dim \mathcal{V} = n+m$ holds. $\Diamond$

To prove Theorem 4.4 we need Theorem 4.9 and the following auxiliary results that can be found in [Rei11, Theorem 14/15/16].

**Lemma 4.11.** *Let the Lur'e equation* (4.2) *be given with the associated even matrix pencil* $(s\mathcal{E} - \mathcal{A})_{ODE}$ *as in* (4.10). *Assume that the KYP inequality* (3.7) *is feasible. Moreover, let a maximally $\mathcal{E}$-neutral space* $\operatorname{im} V$ *with* $V$ *as in* (4.11) *be given such that* $(s\,\mathcal{E} - \mathcal{A})\,V = W\,(s\,\tilde{\mathcal{E}} - \tilde{\mathcal{A}})$ *holds true for some* $W \in \mathbb{K}^{(2n+m)\times(n+p)}$, *and* $\tilde{\mathcal{E}}$, $\tilde{\mathcal{A}} \in \mathbb{K}^{(n+p)\times(n+m)}$. *Furthermore, assume that for all generalized eigenvalues $\lambda$ of the pencil $s\tilde{\mathcal{E}} - \tilde{\mathcal{A}}$ the number $-\bar{\lambda}$ is not an uncontrollable mode of $[I, A, B]$. Then* $\operatorname{rk} V_x = n$.

**Lemma 4.12.** *Let the Lur'e equation* (4.2) *be given with the associated even matrix pencil* $(s\mathcal{E} - \mathcal{A})_{ODE}$ *as in* (4.10). *Assume that the KYP inequality* (3.7) *is feasible and $[I, A, B]$ is behaviourally stabilizable (anti-stabilizable). Moreover, let $V, W$ be given as in* (4.11) *such that* $\operatorname{im} V$ *is maximally $\mathcal{E}$-neutral and* $\operatorname{rk} V_x = n$ *and it holds* $(s\mathcal{E} - \mathcal{A})V = W(s\tilde{\mathcal{E}} - \tilde{\mathcal{A}})$ *for some* $\tilde{\mathcal{E}}$, $\tilde{\mathcal{A}} \in \mathbb{K}^{(n+p)\times(n+m)}$ *with the property that all generalized eigenvalues of $s\tilde{\mathcal{E}} - \tilde{\mathcal{A}}$ have non-positive (non-negative) real part. Let* $X_+ = V_\mu V_x^+$ *for some right inverse $V_x^+$ of $V_x$. Then for all Hermitian $Y \in \mathbb{K}_+$ solving the KYP inequality* (3.7) *it holds:*

$$Y \le X_+ \quad (Y \ge X_+).$$

## 4.4 Results for DAE Lur'e equations

This section collects results for DAE Lur'e equations. We specify the existence of Lur'e solutions in Theorem 4.14 and prove it with several auxiliary results. Within Theorem 4.18 we show that in terms of definiteness Lur'e solutions are extremal solutions of the KYP inequality. The last result of this section presents a method to remodel a given system into an impulse controllable one with equal behaviours.

In the preliminaries we introduced different concepts of stabilizability. To state a relationship between stabilizability of a DAE system and properties of solutions of the associated Lur'e equation we need the following notions.

**Definition 4.13.** Let $[E, A, B] \in \Sigma_{n,m}(\mathbb{K})$ with system space $\mathcal{V}_{\mathrm{sys}} \subseteq \mathbb{K}^{n+m}$ and matrices $Q = Q^* \in \mathbb{K}^{n\times n}$, $S \in \mathbb{K}^{n\times m}$, and $R = R^* \in \mathbb{K}^{m\times m}$ be given. A solution $(X, K, L) \in \mathbb{K}^{n\times n} \times \mathbb{K}^{q\times n} \times \mathbb{K}^{q\times m}$ of the Lur'e equation (4.1) is called

(i) *stabilizing* if $\qquad \operatorname{rk} \begin{bmatrix} -\lambda E + A & B \\ K & L \end{bmatrix} = n + q \quad \forall\, \lambda \in \overline{\mathbb{C}_+};$

(ii) *anti-stabilizing* if $\quad \operatorname{rk}\begin{bmatrix} -\lambda E + A & B \\ K & L \end{bmatrix} = n + q \quad \forall \lambda \in \overline{\mathbb{C}_-}.$ $\Diamond$

Solvability of Lur'e equations requires conditions to be fulfilled. In the following we provide some sufficient conditions for the existence of (stabilizing/anti-stabilizing) solutions of the Lur'e equation (4.1).

**Theorem 4.14** (Existence of solutions of Lur'e equations)**.** *Let $[E, A, B] \in \Sigma_{n,m}(\mathbb{K})$ with system space $\mathcal{V}_{\text{sys}} \subseteq \mathbb{K}^{n+m}$ and matrices $Q = Q^* \in \mathbb{K}^{n\times n}$, $S \in \mathbb{K}^{n\times m}$, and $R = R^* \in \mathbb{K}^{m\times m}$ be given. Further, let $P \in \mathbb{K}^{n\times n}$ be a solution of the KYP inequality* (3.2)*.*

(a) *If $[E, A, B]$ has no uncontrollable modes on the imaginary axis, then the Lur'e equation* (4.1) *has a solution.*

(b) *If $[E, A, B]$ is behaviourally stabilizable, then the Lur'e equation* (4.1) *has a stabilizing solution.*

(c) *If $[E, A, B]$ is behaviourally anti-stabilizable, then the Lur'e equation* (4.1) *has an anti-stabilizing solution.*

To prove this theorem we need some auxiliary results. The first one can be seen as a version of Lemma 3.12 (b) for Lur'e equations.

**Lemma 4.15.** *Let $[E, A, B] \in \Sigma_{n,m}(\mathbb{K})$ with system space $\mathcal{V}_{\text{sys}} \subseteq \mathbb{K}^{n+m}$, weighting matrices $Q = Q^* \in \mathbb{K}^{n\times n}$, $S \in \mathbb{K}^{n\times m}$, $R = R^* \in \mathbb{K}^{m\times m}$ and matrices $W, T \in \mathrm{Gl}_n(\mathbb{K})$ and $F \in \mathbb{K}^{m\times n}$ be given such that* (2.12) *holds. Define the matrices $E_{\mathrm{F}}$, $A_{\mathrm{F}}$, $B_{\mathrm{F}}$, $Q_{\mathrm{F}}$, $S_{\mathrm{F}}$, and $R_{\mathrm{F}}$ as in* (3.8)*. Then for $(X, K, L) \in \mathbb{K}^{n\times n} \times \mathbb{K}^{q\times n} \times \mathbb{K}^{q\times m}$ and*

$$\begin{aligned} X_{\mathrm{F}} &= W^{-*} X W^{-1} = \begin{bmatrix} X_{11} & X_{12} & X_{13} \\ X_{12}^* & X_{22} & X_{23} \\ X_{13}^* & X_{23}^* & X_{33} \end{bmatrix} \in \mathbb{K}^{n\times n}, \\ K_{\mathrm{F}} &= (K + LF)T = \begin{bmatrix} K_1 & K_2 & K_3 \end{bmatrix} \in \mathbb{K}^{q\times n} \end{aligned}$$

*partitioned according to the block structure of FEF* (2.7) *it holds:*
*$(X, K, L)$ is*

| | |
|---|---|
| (a) | *a solution,* |
| (b) | *a stabilizing solution,* |
| (c) | *an anti-stabilizing solution* |

*of the Lur'e equation* (4.1) *if, and only if, $X_{11}$ is Hermitian with*

$$\begin{aligned} &\begin{bmatrix} A_{11}^* X_{11} + X_{11} A_{11} + Q_{11} & X_{11} B_1 + S_1 - Q_{12} B_2 \\ B_1^* X_{11} + S_1^* - B_2^* Q_{12}^* & B_2^* Q_{22} B_2 - B_2^* S_2 - S_2^* B_2 + R \end{bmatrix} \\ &\qquad = \begin{bmatrix} K_1^* \\ (L - K_2 B_2)^* \end{bmatrix} \begin{bmatrix} K_1 & L - K_2 B_2 \end{bmatrix} \end{aligned} \tag{4.12a}$$

*and*

$$\begin{aligned}
&\text{(a)} \qquad \operatorname{rk}_{\mathbb{K}(s)} \begin{bmatrix} -sI_{n_1} + A_{11} & B_1 \\ K_1 & L - K_2 B_2 \end{bmatrix} = n_1 + q, \\
&\text{(b)} \qquad \operatorname{rk} \begin{bmatrix} -\lambda I_{n_1} + A_{11} & B_1 \\ K_1 & L - K_2 B_2 \end{bmatrix} = n_1 + q \quad \forall \lambda \in \mathbb{C}_+, \\
&\text{(c)} \qquad \operatorname{rk} \begin{bmatrix} -\lambda I_{n_1} + A_{11} & B_1 \\ K_1 & L - K_2 B_2 \end{bmatrix} = n_1 + q \quad \forall \lambda \in \mathbb{C}_-.
\end{aligned} \tag{4.12b}$$

**Proof:** To prove the statement we follow an argumentation which is analogous to that in Lemma 3.12 (b). At first, we show that the Lure'e equation (4.1a) holds for a system if, and only if, it holds for an equivalent system in FEF (2.7). In Step 2 the equivalence of the Lur'e equation (4.1a) and (4.12a) is proved. With Step 3 we finish the proof by showing that the rank condition (4.1b) holds true if, and only if, (4.12b) holds true.

Step 1: Calculations as in (3.16) yield

$$\begin{aligned}
&\begin{bmatrix} T^* & T^*F^* \\ 0 & I_m \end{bmatrix} \begin{bmatrix} A^*XE + E^*XA + Q & E^*XB + S \\ B^*XE + S^* & R \end{bmatrix} \begin{bmatrix} T & 0 \\ FT & I_m \end{bmatrix} \\
&\overset{(3.16)}{=} \begin{bmatrix} A_F^* X_{\mathrm{F}} E_{\mathrm{F}} + E_F^* X_{\mathrm{F}} A_{\mathrm{F}} + Q_{\mathrm{F}} & E_F^* X_{\mathrm{F}} B_{\mathrm{F}} + S_{\mathrm{F}} \\ B_F^* X_{\mathrm{F}} E_{\mathrm{F}} + S_F^* & R_{\mathrm{F}} \end{bmatrix}.
\end{aligned} \tag{4.13}$$

Since $K_{\mathrm{F}}^* = T^*(K^* + F^*L^*)$, we have

$$\begin{bmatrix} T^* & T^*F^* \\ 0 & I_m \end{bmatrix} \begin{bmatrix} K^* \\ L^* \end{bmatrix} \begin{bmatrix} K & L \end{bmatrix} \begin{bmatrix} T & 0 \\ FT & I_m \end{bmatrix} = \begin{bmatrix} K_{\mathrm{F}}^* \\ L^* \end{bmatrix} \begin{bmatrix} K_{\mathrm{F}} & L \end{bmatrix} \tag{4.14}$$

for the right hand side of the Lur'e equation (4.1). Analogous to the calculations leading from (3.17) to (3.18) we obtain

$$\begin{aligned}
&\begin{bmatrix} A^*XE + E^*XA + Q & E^*XB + S \\ B^*XE + S^* & R \end{bmatrix} =_{\mathcal{V}_{\mathrm{sys}}} \begin{bmatrix} K^* \\ L^* \end{bmatrix} \begin{bmatrix} K & L \end{bmatrix} \\
\overset{(4.13)}{\underset{(4.14)}{\Leftrightarrow}} &\begin{bmatrix} A_F^* X_{\mathrm{F}} E_{\mathrm{F}} + E_F^* X_{\mathrm{F}} A_{\mathrm{F}} + Q_{\mathrm{F}} & E_F^* X_{\mathrm{F}} B_{\mathrm{F}} + S_{\mathrm{F}} \\ B_F^* X_{\mathrm{F}} E_{\mathrm{F}} + S_F^* & R_{\mathrm{F}} \end{bmatrix} =_{\mathcal{V}_{\mathrm{sys,F}}} \begin{bmatrix} K_{\mathrm{F}}^* \\ L^* \end{bmatrix} \begin{bmatrix} K_{\mathrm{F}} & L \end{bmatrix}
\end{aligned}$$

for all Hermitian $X \in \mathbb{K}^{n \times n}$. Therefore, the Lur'e equation (4.1a) holds for the system $[E, A, B]$ if, and only if, it holds for $[E_{\mathrm{F}}, A_{\mathrm{F}}, B_{\mathrm{F}}]$ in FEF (2.7).

Step 2: The proof of the equivalence of (4.1a) and (4.12a) is split into two parts.
Step 2.1: Let $X \in \mathbb{K}^{n \times n}$ fulfil the Lur'e equation (4.1). In particular, $X$ is Hermitian. Thus,

$$X_{\mathrm{F}}^* = \left(W^{-*} X W^{-1}\right)^* = W^{-*} X W^{-1} = X_{\mathrm{F}}$$

and thereby $X_{11}$ are Hermitian.

Analogous to the calculations in Step 2.1 in the proof of Lemma 3.12 (b) with

$$\begin{aligned}
&\begin{pmatrix} x \\ u \end{pmatrix}^* \begin{bmatrix} A^*XE + E^*XA + Q & E^*XB + S \\ B^*XE + S^* & R \end{bmatrix} \begin{pmatrix} x \\ u \end{pmatrix} \\
\overset{(3.19)}{=} &\begin{pmatrix} x_1 \\ u \end{pmatrix}^* \begin{bmatrix} A_{11}^*X_{11} + X_{11}A_{11} + Q_{11} & X_{11}B_1 + S_1 - Q_{12}B_2 \\ B_1^*X_{11} + S_1^* - B_2^*Q_{12} & B_2^*Q_{22}B_2 - B_2^*S_2 - S_2^*B_2 + R \end{bmatrix} \begin{pmatrix} x_1 \\ u \end{pmatrix}
\end{aligned}$$

and

$$\begin{aligned}
&\begin{pmatrix} x \\ u \end{pmatrix}^* \begin{bmatrix} K^* \\ L^* \end{bmatrix} \begin{bmatrix} K & L \end{bmatrix} \begin{pmatrix} x \\ u \end{pmatrix} \\
\overset{(3.19)}{=} &\begin{pmatrix} x_1 \\ -B_2u \\ 0_{n_3\times 1} \\ u \end{pmatrix}^* \begin{bmatrix} K_1^* \\ K_2^* \\ K_3^* \\ L^* \end{bmatrix} \begin{bmatrix} K_1 & K_2 & K_3 & L \end{bmatrix} \begin{pmatrix} x_1 \\ -B_2u \\ 0_{n_3\times 1} \\ u \end{pmatrix} \\
= &\begin{pmatrix} x_1 \\ -B_2u \\ u \end{pmatrix}^* \begin{bmatrix} K_1^* \\ K_2^* \\ L^* \end{bmatrix} \begin{bmatrix} K_1 & K_2 & L \end{bmatrix} \begin{pmatrix} x_1 \\ -B_2u \\ u \end{pmatrix} \\
= &\begin{pmatrix} x_1 \\ u \end{pmatrix}^* \begin{bmatrix} K_1^*K_1 & K_1^*\left(L - K_2B_2\right) \\ \left(L^* - B_2^*K_2^*\right)K_1 & \left(L^* - B_2^*K_2^*\right)\left(L - K_2B_2\right) \end{bmatrix} \begin{pmatrix} x_1 \\ u \end{pmatrix} \\
= &\begin{pmatrix} x_1 \\ u \end{pmatrix}^* \begin{bmatrix} K_1^* \\ (L - K_2B_2)^* \end{bmatrix} \begin{bmatrix} K_1 & L - K_2B_2 \end{bmatrix} \begin{pmatrix} x_1 \\ u \end{pmatrix}
\end{aligned}$$

we arrive at

$$\begin{aligned}
&\begin{bmatrix} A_{11}^*X_{11} + X_{11}A_{11} + Q_{11} & X_{11}B_1 + S_1 - Q_{12}B_2 \\ B_1^*X_{11} + S_1^* - B_2^*Q_{12} & B_2^*Q_{22}B_2 - B_2^*S_2 - S_2^*B_2 + R \end{bmatrix} \\
= &\begin{bmatrix} K_1^* \\ (L - K_2B_2)^* \end{bmatrix} \begin{bmatrix} K_1 & L - K_2B_2 \end{bmatrix}
\end{aligned}$$

and thus $X_{11}$ fulfils (4.12a).

Step 2.2: Suppose $X_{11} \in \mathbb{K}^{n_1\times n_1}$ fulfils (4.12a). With exactly the same argumentation as in Step 2.2 in the proof of Lemma 3.12 (b) we obtain that $X$ fulfils the Lur'e equation (4.1).

Step 3: We still need to show

$$\text{(4.1b) holds} \quad \Leftrightarrow \quad \operatorname{rk}_{\mathbb{K}(s)} \begin{bmatrix} -sI_{n_1} + A_{11} & B_1 \\ K_1 & L - K_2B_2 \end{bmatrix} = n_1 + q.$$

Let arbitrary $\lambda \in \mathbb{C}$ be given. Calculating straight forward yields

$$n + q \overset{(4.1b)}{=} \operatorname{rk} \begin{bmatrix} -\lambda E + A & B \\ K & L \end{bmatrix} = \operatorname{rk} \begin{bmatrix} W & 0 \\ 0 & I_q \end{bmatrix} \begin{bmatrix} -\lambda E + A & B \\ K & L \end{bmatrix} \begin{bmatrix} T & 0 \\ FT & I_m \end{bmatrix}$$

$$= \operatorname{rk}\begin{bmatrix} W & 0 \\ 0 & I_q \end{bmatrix}\begin{bmatrix} (-\lambda E + (A+BF))T & B \\ (K+LF)T & L \end{bmatrix} = \operatorname{rk}\begin{bmatrix} -\lambda WET + W(A+BF)T & WB \\ (K+LF)T & L \end{bmatrix}$$

$$\overset{(3.8)}{=} \operatorname{rk}\begin{bmatrix} -\lambda E_{\mathrm{F}} + A_{\mathrm{F}} & B_{\mathrm{F}} \\ K_{\mathrm{F}} & L \end{bmatrix} \overset{(2.16)}{=} \operatorname{rk}\begin{bmatrix} -\lambda I_{n_1} + A_{11} & 0 & 0 & B_1 \\ 0 & I_{n_2} & -\lambda E_{23} & B_2 \\ 0 & 0 & -\lambda E_{33} + I_{n_3} & 0 \\ K_1 & K_2 & K_3 & L \end{bmatrix}$$

which equals

$$\begin{aligned} n+q &= \operatorname{rk}\left(\begin{bmatrix} -\lambda I_{n_1} + A_{11} & 0 & 0 & B_1 \\ 0 & I_{n_2} & -\lambda E_{23} & B_2 \\ 0 & 0 & -\lambda E_{33} + I_{n_3} & 0 \\ K_1 & K_2 & K_3 & L \end{bmatrix}\begin{bmatrix} I_{n_1} & 0 & 0 & 0 \\ 0 & -B_2 & I_{n_2} & 0 \\ 0 & 0 & 0 & I_{n_3} \\ 0 & I_m & 0 & 0 \end{bmatrix}\right) \\ &= \operatorname{rk}\begin{bmatrix} -\lambda I_{n_1} + A_{11} & B_1 & 0 & 0 \\ 0 & 0 & I_{n_2} & -\lambda E_{23} \\ 0 & 0 & 0 & -\lambda E_{33} + I_{n_3} \\ K_1 & L - K_2 B_2 & K_2 & K_3 \end{bmatrix} \\ &\overset{(\star)}{=} \operatorname{rk}\begin{bmatrix} -\lambda I_{n_1} + A_1 1 & B_1 \\ K_1 & L - K_2 B_2 \end{bmatrix} + n_2 + n_3, \end{aligned}$$

where we have used

$$\operatorname{rk}\left(\lambda E_{33} - I_{n_3}\right) = n_3 \text{ for all } \lambda \in \mathbb{C}. \tag{$\star$}$$

Due to $n = n_1 + n_2 + n_3$ it follows

$$n_1 + q = \operatorname{rk}\begin{bmatrix} -\lambda I_{n_1} + A_{11} & B_1 \\ K_1 & L - K_2 B_2 \end{bmatrix}.$$

□

Further, we use [Voi15, Theorem 3.5.3] which is stated below.

**Theorem 4.16.** *Let the system $[E, A, B] \in \Sigma_{n,m}$ with the system space $\mathcal{V}_{\mathrm{sys}}$ and weighting matrices $Q = Q^* \in \mathbb{K}^{n\times n}$, $S \in \mathbb{K}^{n\times m}$, and $R = R^* \in \mathbb{K}^{m\times m}$ be given. Assume that the KYP inequality* (3.2) *is solvable.*

(a) *If $[E, A, B]$ is strongly stabilizable, then the Lur'e equation* (4.1) *has a stabilizing solution.*

(b) *If $[E, A, B]$ is strongly anti-stabilizable, then the Lur'e equation* (4.1) *has an anti-stabilizing solution.*

**Remark 4.17.** Note that by [Voi15, Proposition 2.2.6] a system $[E, A, B] \in \Sigma_{n,m}$ is *strongly (anti-)stabilizable* if, and only if, it is behaviourally (anti-)stabilizable and impulse controllable. ◊

Based on these results we now prove Theorem 4.14.

**Proof of Theorem 4.14:** For $[E, A, B] \in \Sigma_{n,m}(\mathbb{K})$ assume that $W, T \in \mathrm{Gl}_n(\mathbb{K})$ and $F \in \mathbb{K}^{m\times n}$ fulfil (2.12). Define $Q_{\mathrm{F}}, S_{\mathrm{F}}$ and $R_{\mathrm{F}}$ as in (3.8) and suppose that $P$ fulfils the KYP inequality (3.2). Consider $P_{\mathrm{F}} = W^{-*}PW^{-1}$ partitioned according to block structure of FEF (2.7). By Lemma 3.12 we know that $P_{11} \in \mathbb{K}^{n_1\times n_1}$ fulfils the standard KYP inequality (3.12).

(a) If $[E, A, B]$ has no uncontrollable modes on the imaginary axis, it follows from Proposition 2.34 (c) that the system $[I_{n_1}, A_{11}, B_1]$ has no uncontrollable modes on the imaginary axis. Lemma 4.11 and Theorem 4.9 imply the existence of an Hermitian $X_{11} \in \mathbb{K}^{n_1\times n_1}$ and matrices $K_1 \in \mathbb{K}^{q\times n_1}$, $L_1 \in \mathbb{K}^{q\times m}$ such that the triple $(X_{11}, K_1, L_1)$ is a solution of

$$\begin{bmatrix} A_{11}^*X_{11} + X_{11}A_{11} + Q_{11} & X_{11}B_1 + S_1 - Q_{12}B_2 \\ B_1^*X_{11} + S_1^* - B_2^*Q_{12}^* & B_2^*Q_{22}B_2 - B_2^*S_2 - S_2^*B_2 + R \end{bmatrix} = \begin{bmatrix} K_1^* \\ L_1^* \end{bmatrix} \begin{bmatrix} K_1 & L_1 \end{bmatrix}. \tag{4.15}$$

Define matrices

$$X = W^* \begin{bmatrix} X_{11} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} W, \quad K = \begin{bmatrix} K_1 & 0 & 0 \end{bmatrix} T^{-1} - LF, \quad L = L_1. \tag{4.16}$$

Lemma 4.15 with $K_2 = 0$ yields that $(X, K, L)$ solves the Lur'e equation (4.1).

(b) Let $[E, A, B]$ behaviourally stabilizable be given in FEF (2.7). Applying Remark 2.37 the ODE system $[I_{n_1}, A_{11}, B_1]$ is (behaviourally) stabilizable. Further, the ODE system is impulse controllable by Remark 2.4 and therefore, according to Remark 4.17, it is strongly stabilizable. Theorem 4.16 (a) then yields the existence of a stabilizing solution $(X_{11}, K_1, L_1) \in \mathbb{K}^{n_1\times n_1} \times \mathbb{K}^{q\times n_1} \times \mathbb{K}^{q\times m}$ for the ODE Lur'e equation (4.2). Define the triple $(X, K, L)$ as in (4.16) and let $\lambda \in \mathbb{C}_+$. Then, Lemma 4.15 implies that $(X, K, L)$ is a stabilizing solution of the Lur'e equation (4.1).

(c) Statement (c) is proved analogously to (b) by using Theorem 4.16 (b) instead of (a), where strongly anti-stabilizable systems are considered.

□

The following theorem in combination with Remark 4.2 (b) states that (anti-)stabilizing solutions $(X, K, L)$ of Lur'e equations are extremal solutions of the KYP inequality (3.2) in terms of definiteness.

**Theorem 4.18.** *Let $[E, A, B] \in \Sigma_{n,m}(\mathbb{K})$ with system space $\mathcal{V}_{\mathrm{sys}} \subseteq \mathbb{K}^{n+m}$, space of consistent differential variables $\mathcal{V}_{\mathrm{diff}} \subseteq \mathbb{K}^n$, and matrices $Q = Q^* \in \mathbb{K}^{n\times n}$, $S \in \mathbb{K}^{n\times m}$, $R = R^* \in \mathbb{K}^{m\times m}$ be given. Then any solution $P \in \mathbb{K}^{n\times n}$ of the KYP inequality* (3.2) *satisfies the following implications.*

(a) *If* $(X, K, L)$ *is a stabilizing solution of the Lur'e equation* (4.1)*, then*

$$E^* X E \geq_{\mathcal{V}_{\text{diff}}} E^* P E.$$

(b) *If* $(X, K, L)$ *is an anti-stabilizing solution of the Lur'e equation* (4.1)*, then*

$$E^* P E \geq_{\mathcal{V}_{\text{diff}}} E^* X E.$$

**Proof:** Let $P$ solve the KYP inequality. Choose matrices $W$, $T \in \mathrm{Gl}_n(\mathbb{K})$ and $F \in \mathbb{K}^{m \times n}$ such that (2.12) holds and define $P_{\mathrm{F}} := W^{-*} P W^{-1}$ according to the block structure of (3.11). By Lemma 3.12 (b) the matrix $P_{11}$ fulfils the KYP inequality (3.12).

At first, we show that (a) holds. Let $(X, K, L)$ be a stabilizing solution of the Lur'e equation (4.1). Define

$$(X_{\mathrm{F}}, K_{\mathrm{F}}, L_{\mathrm{F}}) := (W^{-*} X W^{-1}, KT + LFT, L)$$

partitioned as in Lemma 4.15. Then Lemma 4.15 yields that $(X_{11}, K_1, L - K_2 B_2)$ is a stabilizing solution of (4.12).
Since $(X_{11}, K_1, L - K_2 B_2)$ is a solution of the Lur'e equation (4.12), the matrix $X_{11}$ solves the ODE KYP inequality (3.7) and hence by Theorem 3.2 the Popov function $\Phi_F$ is positive semi-definite on the imaginary axis. Thus, Theorem 4.9 provides the remaining presuppositions of Lemma 4.12 for the ODE system $[I_{n_1}, A_{11}, B_1]$ with $X_+ = X_{11}$. Applying Lemma 4.12 we get $X_{11} \geq P_{11}$.
Now let $x \in \mathcal{V}_{\text{diff}}$ be arbitrary. Due to Proposition 2.34 (b) this is equivalent to

$$x \in T \left( \mathbb{K}^{n_1 + n_2} \times \ker \begin{bmatrix} E_{23} \\ E_{33} \end{bmatrix} \right)$$

which again implies that there exist some $y_1 \in \mathbb{K}^{n_1}, y_2 \in \mathbb{K}^{n_2}$ and $y_3 \in \ker \begin{bmatrix} E_{23} \\ E_{33} \end{bmatrix}$ such that $T^{-1} x = (y_1, y_2, y_3)^\top$. Further, it holds that

$$x^* E^* X E x = x^* T^{-*} T^* E^* W^* W^{-*} X W^{-1} W E T T^{-1} x = (T^{-1} x)^* E_{\mathrm{F}}^* X_{\mathrm{F}} E_{\mathrm{F}} (T^{-1} x).$$

Since $y_3 \in \ker \begin{bmatrix} E_{23} \\ E_{33} \end{bmatrix}$, we get

$$\begin{aligned} & (T^{-1} x)^* E_{\mathrm{F}}^* X_{\mathrm{F}} E_{\mathrm{F}} (T^{-1} x) \\ = & \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}^* \begin{bmatrix} I_{n_1} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & E_{23}^* & E_{33}^* \end{bmatrix} \begin{bmatrix} X_{11} & X_{12} & X_{13} \\ X_{12}^* & X_{22} & X_{23} \\ X_{13}^* & X_{23}^* & X_{33} \end{bmatrix} \begin{bmatrix} I_{n_1} & 0 & 0 \\ 0 & 0 & E_{23} \\ 0 & 0 & E_{33} \end{bmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} \\ = & y_1^* X_{11} y_1. \end{aligned}$$

By an analogous argumentation we arrive at $x^* E^* P E x = y_1^* P_{11} y_1$. Hence, the choice of $x$ and the inequality $X_{11} \geq P_{11}$ imply

$$x^* E^* X E x \geq x^* E^* P E x \quad \forall\, x \in \mathcal{V}_{\mathrm{diff}}$$

and the proof of (a) is completed.

The proof of (b) works analogously (by using the part of Lemma 4.12 in brackets). □

Theorem 4.18 provides the following conclusion on two solutions of the Lur'e equation.

**Corollary 4.19.** *If $(X_1, K_1, L_1)$ and $(X_2, K_2, L_2)$ are stabilizing (anti-stabilizing) solutions of the Lur'e equation* (4.1)*, then*

$$E^* X_1 E =_{\mathcal{V}_{\mathrm{diff}}} E^* X_2 E.$$

**Proof:** Let $(X_1, K_1, L_1)$ and $(X_2, K_2, L_2)$ be stabilizing solutions of (4.1). Due to Remark 4.2 (b) both $X_1$ and $X_2$ solve the KYP inequality (3.2). Thus, Theorem 4.18 yields

$$E^* X_1 E \geq_{\mathcal{V}_{\mathrm{diff}}} E^* X_2 E \geq_{\mathcal{V}_{\mathrm{diff}}} E^* X_1 E.$$

The proof for the anti-stabilizing case works analogously. □

In the last theorem of this section we deduce a method to remodel the given system into an impulse controllable one such that the behaviour and even the sets of solutions for the corresponding KYP inequality and Lur'e equation stay the same.

**Theorem 4.20.** *Let $[E, A, B] \in \Sigma_{n,m}(\mathbb{K})$ with system space $\mathcal{V}_{\mathrm{sys}} \subseteq \mathbb{K}^{n+m}$, space of consistent differential variables $\mathcal{V}_{\mathrm{diff}} \subseteq \mathbb{K}^n$, and weighting matrices $Q = Q^* \in \mathbb{K}^{n \times n}$, $S \in \mathbb{K}^{n \times m}$, $R = R^* \in \mathbb{K}^{m \times m}$ be given. Further, let $W, T \in \mathrm{Gl}_n(\mathbb{K})$ and $F \in \mathbb{K}^{m \times n}$ be transformation matrices such that* (2.12) *holds. Define the projector $\Pi$ as in* (4.5)*. Then we have*

$$\operatorname{im} \Pi = E \mathcal{V}_{\mathrm{diff}} \tag{4.17}$$

*and the following statements hold true:*

(a) $[\Pi E, A, B] \in \Sigma_{n,m}(\mathbb{K})$ *is impulse controllable and*

$$\mathcal{B}_{[E,A,B]} = \mathcal{B}_{[\Pi E,A,B]}.$$

*In particular, the system space of $[\Pi E, A, B]$ is $\mathcal{V}_{\mathrm{sys}}$.*

(b) $P \in \mathbb{K}^{n \times n}$ *fulfils the KYP inequality* (3.2) *if, and only if,*

$$\begin{bmatrix} A^* P \Pi E + E^* \Pi^* P A + Q & E^* \Pi^* P B + S \\ B^* P \Pi E + S^* & R \end{bmatrix} \geq_{\mathcal{V}_{\mathrm{sys}}} 0, \quad P = P^*. \tag{4.18}$$

(c) $(X,K,L)\in\mathbb{K}^{n\times n}\times\mathbb{K}^{q\times n}\times\mathbb{K}^{q\times m}$ *fulfils the Lur'e equation* (4.1a) *if, and only if,*

$$\begin{bmatrix} A^*X\Pi E+E^*\Pi^*XA+Q & E^*\Pi^*XB+S\\ B^*X\Pi E+S^* & R\end{bmatrix} =_{\mathcal{V}_{\mathrm{sys}}} \begin{bmatrix} K^*\\ L^*\end{bmatrix}\begin{bmatrix} K & L\end{bmatrix},\quad X=X^*. \tag{4.19}$$

*Furthermore, it holds that*

$$\operatorname{rk}\begin{bmatrix} -\lambda E+A & B\\ K & L\end{bmatrix} = \operatorname{rk}\begin{bmatrix} -\lambda \Pi E+A & B\\ K & L\end{bmatrix}\quad \forall\lambda\in\mathbb{C}.$$

**Proof:** Let $E_{\mathrm{F}},A_{\mathrm{F}},B_{\mathrm{F}},Q_{\mathrm{F}},S_{\mathrm{F}}$ and $R_{\mathrm{F}}$ as in (3.8).

At first, we show that (4.17) holds. Therefore, note that Proposition 2.34 (b) yields

$$\begin{aligned} E\mathcal{V}_{\mathrm{diff}} = ET\left(\mathbb{K}^{n_1+n_2}\times\ker\begin{bmatrix}E_{23}\\E_{33}\end{bmatrix}\right) &= W^{-1}E_{\mathrm{F}}\left(\mathbb{K}^{n_1+n_2}\times\ker\begin{bmatrix}E_{23}\\E_{33}\end{bmatrix}\right)\\ &= W^{-1}\begin{bmatrix} I_{n_1} & 0 & 0\\ 0 & 0 & E_{23}\\ 0 & 0 & E_{33}\end{bmatrix}\left(\mathbb{K}^{n_1+n_2}\times\ker\begin{bmatrix}E_{23}\\E_{33}\end{bmatrix}\right)\\ &= W^{-1}\left(\mathbb{K}^{n_1}\times\{0_{(n_2+n_3)\times 1}\}\right).\end{aligned}$$

Together with

$$\operatorname{im}\Pi = W^{-1}\begin{bmatrix} I_{n_1} & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix} W\,\mathbb{K}^n = W^{-1}\left(\mathbb{K}^{n_1}\times\{0_{(n_2+n_3)\times 1}\}\right)$$

this completes the proof of (4.17).

Now we verify the statements (a), (b) and (c) step by step.

(a) Since

$$[\Pi_{\mathrm{F}}E_{\mathrm{F}},A_{\mathrm{F}},B_{\mathrm{F}}] = \left[\begin{bmatrix} I_{n_1} & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix},\begin{bmatrix} A_{11} & 0 & 0\\ 0 & I_{n_2} & 0\\ 0 & 0 & I_{n_3}\end{bmatrix},\begin{bmatrix} B_1\\ B_2\\ 0\end{bmatrix}\right]$$

and

$$\ker\Pi_{\mathrm{F}}E_{\mathrm{F}} = \{0_{n_1}\}\times\mathbb{K}^{n_2+n_3}$$

holds, Proposition 2.27 and Remark 2.33 yield that $[\Pi E,A,B]$ is impulse controllable. Further, we have

$$\left(\begin{pmatrix} x_1\\ x_2\\ x_3\end{pmatrix},u\right)\in\mathcal{B}_{[\Pi_{\mathrm{F}}E_{\mathrm{F}},A_{\mathrm{F}},B_{\mathrm{F}}]}\;\Leftrightarrow\;\begin{cases}\dot{x}_1 &= A_{11}x_1+B_1u\\ 0 &= x_2+B_2u\\ 0 &= x_3\end{cases}$$

$$\begin{aligned}&\Leftrightarrow x_2 = -B_2 u, x_3 = 0 \text{ and } (x_1, u) \in \mathcal{B}_{[I_{n_1}, A_{11}, B_1]}\\ &\overset{\text{Prop.}}{\underset{2.34\ (a)}{\Leftrightarrow}} \left( \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}, u \right) \in \mathcal{B}_{[E_{\mathrm{F}}, A_{\mathrm{F}}, B_{\mathrm{F}}]}.\end{aligned}$$

This equivalence together with Remark 2.20 gives

$$\begin{aligned}(x,u) \in \mathcal{B}_{[E,A,B]} &\overset{\text{Rem.2.20}}{\Leftrightarrow} \big(T^{-1}x, u - Fx\big) \in \mathcal{B}_{[E_{\mathrm{F}}, A_{\mathrm{F}}, B_{\mathrm{F}}]}\\ &\Leftrightarrow \big(T^{-1}x, u - Fx\big) \in \mathcal{B}_{[\Pi_{\mathrm{F}} E_{\mathrm{F}}, A_{\mathrm{F}}, B_{\mathrm{F}}]}\\ &\overset{\text{Rem.2.20}}{\Leftrightarrow} (x,u) \in \mathcal{B}_{[\Pi E, A, B]},\end{aligned}$$

which completes the proof of (a).

(b) Let $P \in \mathbb{K}^{n \times n}$, its transformation into $P_{\mathrm{F}} = W^{-*} P W^{-1}$ with block structure as in (3.11), and matrices $Q_{\mathrm{F}}$ and $S_{\mathrm{F}}$ as in (3.8) be given. The idea of the proof is to show that $P$ fulfils the KYP inequality (4.18) if, and only if, $P_{11}$ fulfils the ODE KYP inequality (3.12). Then we can apply Lemma 3.12 (b) and hence we know that $P$ fulfils the original KYP inequality (3.2).
The proof of the first equivalence mentioned above runs completely analogously to the Proof of Lemma 3.12 (b). One should keep in mind that we use $\Pi E$ instead of $E$ and therefore calculate with

$$E_{\Pi,F} = \begin{bmatrix} I_{n_1} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \text{ instead of } E_{\mathrm{F}} = \begin{bmatrix} I_{n_1} & 0 & 0 \\ 0 & 0 & E_{23} \\ 0 & 0 & E_{33} \end{bmatrix}.$$

However, that only changes the unspecified matrices in equation (3.19) and has no impact on the proof itself.

(c) The equivalence between the Lur'e equation (4.1a) for $[E, A, B]$ and (4.19) for $[\Pi E, A, B]$ can be verified analogously to (b). Note that we could also refer to the proof of Lemma 4.15 as it is an adopted version of the proof of Lemma 3.12 (b) for Lur'e equations.
Further, we show that the rank condition holds. Therefore, let $\lambda \in \mathbb{C}$. Then, analogous to the rank calculations in the proof of Lemma 4.15 and using the structure of $E_{\Pi,F}$, it holds

$$\begin{aligned}\operatorname{rk} \begin{bmatrix} -\lambda E + A & B \\ K & L \end{bmatrix} &= \operatorname{rk} \begin{bmatrix} -\lambda E_{\mathrm{F}} + A_{\mathrm{F}} & B_{\mathrm{F}} \\ K_{\mathrm{F}} & L \end{bmatrix}\\ = \operatorname{rk} &\begin{bmatrix} -\lambda I_{n_1} + A_{11} & 0 & 0 & B_1 \\ 0 & I_{n_2} & -\lambda E_{23} & B_2 \\ 0 & 0 & -\lambda E_{33} + I_{n_3} & 0 \\ K_1 & K_2 & K_3 & L \end{bmatrix}\end{aligned}$$

$$
\begin{aligned}
&= \mathrm{rk} \begin{bmatrix} -\lambda I_{n_1} + A_{11} & 0 & 0 & B_1 \\ 0 & I_{n_2} & 0 & B_2 \\ 0 & 0 & I_{n_3} & 0 \\ K_1 & K_2 & K_3 & L \end{bmatrix} \\
&= \mathrm{rk} \begin{bmatrix} -\lambda E_{\Pi,F} + A_{\mathrm{F}} & B_{\mathrm{F}} \\ K_{\mathrm{F}} & L \end{bmatrix} = \mathrm{rk} \begin{bmatrix} -\lambda \Pi E + A & B \\ K & L \end{bmatrix},
\end{aligned}
$$

which finishes the proof.

□

## 4.5 Proof of Theorem 4.4

In this section we apply the findings of the previous ones to prove Theorem 4.4.

**Proof of Theorem 4.4:** Note that by Theorem 4.20 (a) the system $[\Pi E, A, B]$ is impulse controllable. By Proposition 2.34 (d) we can choose $W,\ T \in \mathrm{Gl}_n(\mathbb{K})$ and $F \in \mathbb{K}^{m\times n}$ such that $n_3 = 0$. Thereby, the block structure of the matrices simplifies as follows

$$
\left.
\begin{aligned}
&E_{\mathrm{F}} = W\Pi ET = \begin{bmatrix} I_{n_1} & 0 \\ 0 & 0 \end{bmatrix}, \\
&A_{\mathrm{F}} = W(A+BF)T = \begin{bmatrix} A_{11} & 0 \\ 0 & I_{n_2} \end{bmatrix}, \quad B_{\mathrm{F}} = WB = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}, \\
&Q_{\mathrm{F}} = T^*(Q + SF + F^*S^* + F^*RF)T = \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{12}^* & Q_{22} \end{bmatrix}, \\
&S_{\mathrm{F}} = T^*(S + F^*R) = \begin{bmatrix} S_1 \\ S_2 \end{bmatrix}, \quad R_{\mathrm{F}} = R, \\
&X_{\mathrm{F}} = W^{-*}XW^{-1} = \begin{bmatrix} X_{11} & X_{12} \\ X_{12}^* & X_{22} \end{bmatrix} \text{ and } K_{\mathrm{F}} = (K+LF)T = \begin{bmatrix} K_1 & K_2 \end{bmatrix}.
\end{aligned}
\right\} \tag{4.20}
$$

The proof is split into two parts. At first, the implication (a) follows (b) is shown. Afterwards, we prove (b) follows (a).

(a) $\Rightarrow$ (b):

The idea to prove (b) is to construct matrices $Y$ and $Z$ as in (4.6). Therefore, we will first determine some matrices $Y_{\mathrm{F}}$ and $Z_F$ fulfilling the properties in (b1) to (b4) for an associated system and then by a transformation we will receive (b). For a better understanding the structure of the proof is given in Figure 4.1.

Let statement (a) hold true, i.e. $(X, K, L) \in \mathbb{K}^{n\times n} \times \mathbb{K}^{q\times n} \times \mathbb{K}^{q\times m}$ is a solution of the Lur'e equation (4.1). Since $X$ solves the KYP inequality (3.2), Theorem 3.2 (a) provides that $\Phi$ is positive semi-definite on the imaginary axis. Further, Theorem 4.20 (c) implies that $(X, K, L)$ even solves the Lur'e equation (4.19)

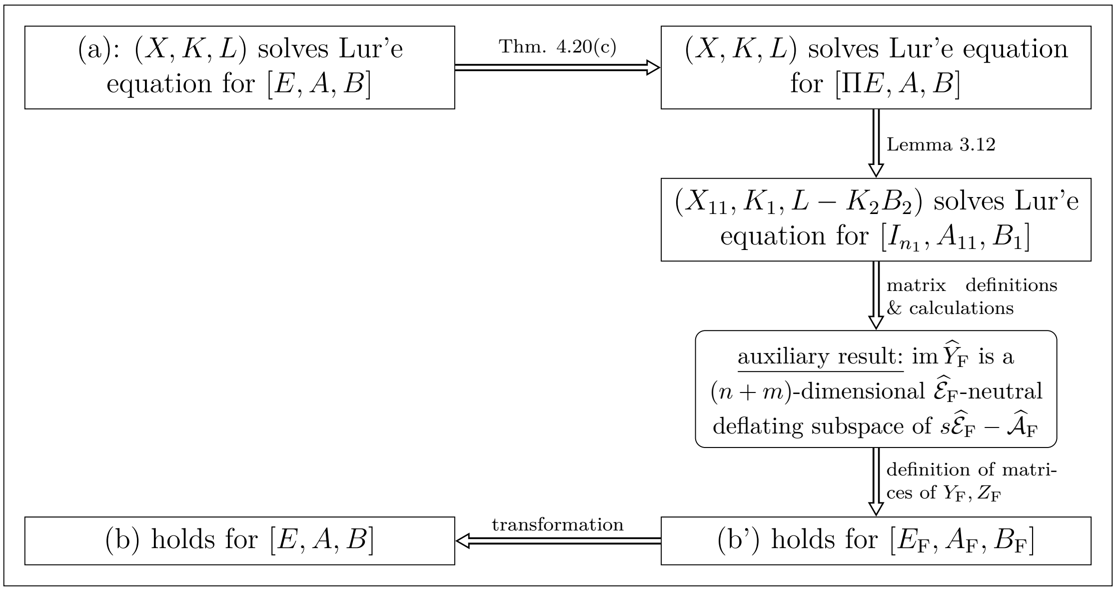


Figure 4.1: Structure of proof of "(a) $\Rightarrow$ (b)" in Theorem 4.4

for the impulse controllable system $[\Pi E, A, B]$. Thus, Lemma 4.15 again provides that $(X_{11}, K_1, L - K_2 B_2)$ fulfils the ODE Lur'e equation (4.12) for the subsystem $[I_{n_1}, A_{11}, B_1]$ of $[\Pi E, A, B]$.

To obtain some auxiliary results define

$$s\widehat{\mathcal{E}}_{\mathrm{F}} - \widehat{\mathcal{A}}_{\mathrm{F}} := \begin{bmatrix} 0 & -sI_{n_1} + A_{11} & B_1 & 0 & 0 \\ sI_{n_1} + A_{11}^* & Q_{11} & -Q_{12}B_2 + S_1 & 0 & 0 \\ B_1^* & -B_2^* Q_{12}^* + S_1^* & B_2^* Q_{22} B_2 - B_2^* S_2 - S_2^* B_2 + R & 0 & 0 \\ 0 & 0 & 0 & 0 & I_{n_2} \\ 0 & 0 & 0 & I_{n_2} & 0 \end{bmatrix}, \tag{4.21}$$

$$s\tilde{\mathcal{E}}_{\mathrm{F}} - \tilde{\mathcal{A}}_{\mathrm{F}} := \begin{bmatrix} -sI_{n_1} + A_{11} & B_1 & 0 \\ K_1 & L - K_2 B_2 & 0 \\ 0 & 0 & I_{n_2} \end{bmatrix}, \tag{4.22}$$

$$\widehat{Y}_{\mathrm{F}} := \begin{bmatrix} X_{11} & 0 & 0 \\ I_{n_1} & 0 & 0 \\ 0 & I_m & 0 \\ 0 & 0 & I_{n_2} \\ 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad \widehat{Z}_{\mathrm{F}} := \begin{bmatrix} I_{n_1} & 0 & 0 \\ -X_{11} & K_1^* & 0 \\ 0 & (L - K_2 B_2)^* & 0 \\ 0 & 0 & 0 \\ 0 & 0 & I_{n2} \end{bmatrix}, \tag{4.23}$$

where the number of rows in the zero blocks of $\widehat{Y}_{\mathrm{F}}$ and $\widehat{Z}_{\mathrm{F}}$ equals $n_2$. Note that by (4.12b) we have

$$\mathrm{rk}_{\mathbb{K}(s)} \left( s\tilde{\mathcal{E}}_{\mathrm{F}} - \tilde{\mathcal{A}}_{\mathrm{F}} \right) = n + q. \tag{4.24}$$

Calculating straight forward we get

$$
\begin{aligned}
&\left(s\widehat{\mathcal{E}}_{\mathrm{F}}-\widehat{\mathcal{A}}_{\mathrm{F}}\right)\widehat{Y}_{\mathrm{F}} \\
&= \begin{bmatrix} -sI_{n_1}+A_{11} & B_1 & 0 \\ (sI_{n_1}+A_{11}^*)X_{11}+Q_{11} & -Q_{12}B_2+S_1 & 0 \\ B_1X_{11}-B_2^*Q_{12}^*+S_1^* & B_2^*Q_{22}B_2-B_2^*S_2-S_2^*B_2+R & 0 \\ 0 & 0 & 0 \\ 0 & 0 & I_{n_2} \end{bmatrix} \\
&\overset{(4.12a)}{=} \begin{bmatrix} -sI_{n_1}+A_{11} & B_1 & 0 \\ -X_{11}(-sI_{n_1}+A_{11})+K_1^*K_1 & -X_{11}B_1+K_1^*(L-K_2B_2) & 0 \\ (L-K_2B_2)^*K_1 & (L-K_2B_2)^*(L-K_2B_2) & 0 \\ 0 & 0 & 0 \\ 0 & 0 & I_{n_2} \end{bmatrix} \\
&= \widehat{Z}_{\mathrm{F}}\left(s\tilde{\mathcal{E}}_{\mathrm{F}}-\tilde{\mathcal{A}}_{\mathrm{F}}\right).
\end{aligned} \tag{4.25}
$$

Hence, $\operatorname{im}\widehat{Y}_{\mathrm{F}}$ is a deflating subspace for $s\widehat{\mathcal{E}}_{\mathrm{F}}-\widehat{\mathcal{A}}_{\mathrm{F}}$. Further, let $z_1, z_2 \in \operatorname{im}\widehat{Y}_{\mathrm{F}}$ be arbitrary, i.e. there exist some $x = (x_1^\top, x_2^\top, x_3^\top)^\top, y = (y_1^\top, y_2^\top, y_3^\top)^\top \in \mathbb{K}^{n_1+m+n_2}$ such that

$$
z_1 = \begin{pmatrix}(X_{11}x_1)^\top & x_1^\top & x_2^\top & x_3^\top & 0^\top\end{pmatrix}^\top \quad \text{and} \quad z_2 = \begin{pmatrix}(X_{11}y_1)^\top & y_1^\top & y_2^\top & y_3^\top & 0^\top\end{pmatrix}^\top.
$$

Since $X_{11}$ is Hermitian, it holds that

$$
z_1^*\widehat{\mathcal{E}}_{\mathrm{F}}z_2 = \begin{pmatrix} x_1^* & -x_1^*X_{11}^* & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} X_{11}y_1 \\ y_1 \\ y_2 \\ y_3 \\ 0 \end{pmatrix} = 0.
$$

Thus, $\operatorname{im}\widehat{Y}_{\mathrm{F}}$ is an $(n+m)$-dimensional $\widehat{\mathcal{E}}_{\mathrm{F}}$-neutral deflating subspace for $s\widehat{\mathcal{E}}_{\mathrm{F}}-\widehat{\mathcal{A}}_{\mathrm{F}}$.

Following on this auxiliary result, we define transformation matrices

$$
\widehat{U} := \begin{bmatrix} I_{n_1} & 0 & 0 & 0 & 0 \\ 0 & -Q_{12}^* & Q_{22}B_2-S_2 & -\frac{1}{2}Q_{22} & I_{n_2} \\ 0 & I_{n_1} & 0 & 0 & 0 \\ 0 & 0 & -B_2 & I_{n_2} & 0 \\ 0 & 0 & I_m & 0 & 0 \end{bmatrix}, \tag{4.26}
$$
$$
\widehat{V} := \begin{bmatrix} I_{n_1} & 0 & 0 \\ 0 & 0 & I_m \\ 0 & I_{n_2} & B_2 \end{bmatrix} \quad \text{and} \quad \widehat{W} := \begin{bmatrix} I_{n_1} & 0 & 0 \\ 0 & 0 & I_q \\ 0 & I_{n_2} & K_2 \end{bmatrix}.
$$

Then

$$\widehat{U}^{-1} = \begin{bmatrix} I_{n_1} & 0 & 0 & 0 & 0 \\ 0 & 0 & I_{n_1} & 0 & 0 \\ 0 & 0 & 0 & 0 & I_m \\ 0 & 0 & 0 & I_{n_2} & B_2 \\ 0 & I_{n_2} & Q_{12}^* & \frac{1}{2}Q_{22} & S_2 - \frac{1}{2}Q_{22}B_2 \end{bmatrix} \quad \text{and} \quad \widehat{W}^{-1} = \begin{bmatrix} I_{n_1} & 0 & 0 \\ 0 & -K_2 & I_{n_2} \\ 0 & I_q & 0 \end{bmatrix}$$

and thus

$$\widehat{U}^{-*} = \begin{bmatrix} I_{n_1} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & I_{n_2} \\ 0 & I_{n_1} & 0 & 0 & Q_{12} \\ 0 & 0 & 0 & I_{n_2} & \frac{1}{2}Q_{22} \\ 0 & 0 & I_m & B_2^* & S_2^* - \frac{1}{2}B_2^*Q_{22} \end{bmatrix}.$$

Based on these transformation matrices we are able to define pencils and matrices which belong to a corresponding system in FEF (2.7):

$$\left.\begin{array}{c} s\mathcal{E}_{\mathrm{F}} - \mathcal{A}_{\mathrm{F}} := \widehat{U}^{-*}\left(s\widehat{\mathcal{E}}_{\mathrm{F}} - \widehat{\mathcal{A}}_{\mathrm{F}}\right)\widehat{U}^{-1}, \quad s\tilde{\mathcal{E}} - \tilde{\mathcal{A}} := \widehat{W}\left(s\tilde{\mathcal{E}}_{\mathrm{F}} - \tilde{\mathcal{A}}_{\mathrm{F}}\right)\widehat{V}, \\ Y_{\mathrm{F}} := \widehat{U}\widehat{Y}_{\mathrm{F}}\widehat{V} \quad \text{and} \quad Z_{\mathrm{F}} := \widehat{U}^{-*}\widehat{Z}_{\mathrm{F}}\widehat{W}^{-1}. \end{array}\right\} \tag{4.27}$$

For those pencils and matrices we can show that the following properties (b1') to (b4') hold true. Note that they are an adopted version of the properties in (b) for a feedback equivalent system:

(b1') $\operatorname{im} Y_{\mathrm{F}}$ is $(n+m)$-dimensional and $\mathcal{E}_F$-neutral;

(b2') $\mathcal{V}_{\mathrm{sys,F}} \subseteq \operatorname{im} \begin{bmatrix} Y_{x,\mathrm{F}} \\ Y_{u,\mathrm{F}} \end{bmatrix}$;

(b3') $\operatorname{rk} E_{\mathrm{F}} Y_{x,\mathrm{F}} = n_1$;

(b4') $(s\mathcal{E}_{\mathrm{F}} - \mathcal{A}_{\mathrm{F}})Y_{\mathrm{F}} = Z_{\mathrm{F}}(s\tilde{\mathcal{E}} - \tilde{\mathcal{A}})$ with $\operatorname{rk}_{\mathbb{K}(s)}(s\tilde{\mathcal{E}} - \tilde{\mathcal{A}}) = n + q$.

At first, we study the structure of the matrices in (4.27), i.e.

$$\begin{aligned} &s\mathcal{E}_{\mathrm{F}} - \mathcal{A}_{\mathrm{F}} \\ &= \begin{bmatrix} 0 & -sI_{n_1} + A_{11} & B_1 & 0 & 0 \\ 0 & 0 & 0 & I_{n_2} & 0 \\ sI_{n_1} + A_{11}^* & Q_{11} & -Q_{12}B_2 + S_1 & Q_{12} & 0 \\ 0 & 0 & 0 & \frac{1}{2}Q_{22} & I_{n_2} \\ B_1^* & -B_2^*Q_{12}^* + S_1^* & B_2^*Q_{22}B_2 - B_2^*S_2 - S_2^*B_2 + R & S_2^* - \frac{1}{2}B_2^*Q_{22} & B_2^* \end{bmatrix} \widehat{U}^{-1} \\ &= \begin{bmatrix} 0 & 0 & -sI_{n_1} + A_{11} & 0 & B_1 \\ 0 & 0 & 0 & I_{n_2} & B_2 \\ sI_{n_1} + A_{11}^* & 0 & Q_{11} & Q_{12} & S_1 \\ 0 & I_{n_2} & Q_{12}^* & Q_{22} & S_2 \\ B_1^* & B_2^* & S_1^* & S_2^* & R \end{bmatrix} \end{aligned}$$

and

$$Y_{\mathrm{F}} = \begin{bmatrix} X_{11} & 0 & 0 \\ -Q_{12}^* & Q_{22}B_2 - S_2 & -\frac{1}{2}Q_{22} \\ I_{n_1} & 0 & 0 \\ 0 & -B_2 & I_{n_2} \\ 0 & I_m & 0 \end{bmatrix} \widehat{V} = \begin{bmatrix} X_{11} & 0 & 0 \\ -Q_{12}^* & -\frac{1}{2}Q_{22} & \frac{1}{2}Q_{22}B_2 - S_2 \\ I_{n_1} & 0 & 0 \\ 0 & I_{n_2} & 0 \\ 0 & 0 & I_m \end{bmatrix} =: \begin{bmatrix} Y_{\mu,1} \\ Y_{\mu,2} \\ Y_{x,1} \\ Y_{x,2} \\ Y_{u,\mathrm{F}} \end{bmatrix}.$$

Note that $\dim(\operatorname{im} Y_{\mathrm{F}}) = n + m$. To show the $\mathcal{E}_{\mathrm{F}}$-neutrality let $\tilde{z}_1, \tilde{z}_2 \in \operatorname{im} Y_{\mathrm{F}}$, i.e. there exist some $\tilde{x} = (\tilde{x}_1^\top, \tilde{x}_2^\top, \tilde{x}_3^\top)^\top, \tilde{y} = (\tilde{y}_1^\top, \tilde{y}_2^\top, \tilde{y}_3^\top)^\top \in \mathbb{K}^{n+m}$ such that

$$\tilde{z}_1 = Y_{\mathrm{F}}\tilde{x} = \widehat{U}\widehat{Y}_{\mathrm{F}}\widehat{V}\tilde{x} = \widehat{U}\widehat{Y}_{\mathrm{F}}x \quad \text{and} \quad \tilde{z}_2 = Y_{\mathrm{F}}\tilde{y} = \widehat{U}\widehat{Y}_{\mathrm{F}}y,$$

where $x = \widehat{V}\tilde{x} \in \mathbb{K}^{n+m}$ and $y = \widehat{V}\tilde{y} \in \mathbb{K}^{n+m}$. Since $\operatorname{im} \widehat{Y}_{\mathrm{F}}$ is $\widehat{\mathcal{E}}_{\mathrm{F}}$-neutral, we have

$$\tilde{z}_1^* \mathcal{E}_{\mathrm{F}} \tilde{z}_2 = \left(\widehat{U}\widehat{Y}_{\mathrm{F}}x\right)^* \widehat{U}^{-*}\widehat{\mathcal{E}}_{\mathrm{F}}\widehat{U}^{-1}\left(\widehat{U}\widehat{Y}_{\mathrm{F}}y\right) = \left(\widehat{Y}_{\mathrm{F}}x\right)^* \widehat{\mathcal{E}}_{\mathrm{F}}\left(\widehat{Y}_{\mathrm{F}}y\right) = 0,$$

which completes the proof of (b1'). Denoting $Y_{x,\mathrm{F}} := \begin{bmatrix} Y_{x,1} \\ Y_{x,2} \end{bmatrix}$ we find

$$\mathcal{V}_{\mathrm{sys,F}} \subseteq \mathbb{K}^{n+m} = \operatorname{im} \begin{bmatrix} Y_{x,\mathrm{F}} \\ Y_{u,\mathrm{F}} \end{bmatrix},$$

i.e. (b2') holds.
By (4.20) it is $\operatorname{rk} E_{\mathrm{F}} Y_{x,\mathrm{F}} = \operatorname{rk} Y_{x,1} = n_1$ and hence (b3') holds. Using (4.27) we obtain $\widehat{Z}_{\mathrm{F}} = \widehat{U}^* Z_{\mathrm{F}} \widehat{W}$. Furthermore, we achieve

$$\begin{aligned} (s\mathcal{E}_{\mathrm{F}} - \mathcal{A}_{\mathrm{F}})Y_{\mathrm{F}} &= \widehat{U}^{-*}\left(s\widehat{\mathcal{E}}_{\mathrm{F}} - \widehat{\mathcal{A}}_{\mathrm{F}}\right)\widehat{Y}_{\mathrm{F}}\widehat{V} \\ &\overset{(4.25)}{=} U^{-*}\widehat{Z}_{\mathrm{F}}\left(s\tilde{\mathcal{E}}_{\mathrm{F}} - \tilde{\mathcal{A}}_{\mathrm{F}}\right)\widehat{V} \\ &= Z_{\mathrm{F}}\widehat{W}\left(s\tilde{\mathcal{E}}_{\mathrm{F}} - \tilde{\mathcal{A}}_{\mathrm{F}}\right)\widehat{V} \\ &\overset{(4.27)}{=} Z_{\mathrm{F}}(s\tilde{\mathcal{E}} - \tilde{\mathcal{A}}). \end{aligned}$$

Since we have $\operatorname{rk}_{\mathbb{K}(s)}(s\tilde{\mathcal{E}}_{\mathrm{F}} - \tilde{\mathcal{A}}_{\mathrm{F}}) = n + q$ by (4.24), the second equation in (4.27) provides that $\operatorname{rk}_{\mathbb{K}(s)}(s\tilde{\mathcal{E}} - \tilde{\mathcal{A}}) = n + q$. Altogether, we have shown that (b1') to (b4') hold for the Lur'e equation associated with the system in FEF (2.7).

In the end, we are going to transform the system from FEF (2.7) to its initial form to receive (b). Define matrices

$$U := \begin{bmatrix} W^* & 0 & 0 \\ 0 & T & 0 \\ 0 & FT & I_m \end{bmatrix}, \quad Y = UY_{\mathrm{F}}, \quad \text{and} \quad Z = U^{-*}Z_{\mathrm{F}}. \tag{4.28}$$

It can be verified that

$$s\mathcal{E} - \mathcal{A} = U^{-*}(s\mathcal{E}_{\mathrm{F}} - \mathcal{A}_{\mathrm{F}})U^{-1} \tag{4.29}$$

holds. This follows from extensive calculations using simple matrix multiplications. Thus, we have

$$(s\mathcal{E} - \mathcal{A})Y = U^{-*}(s\mathcal{E}_{\mathrm{F}} - \mathcal{A}_{\mathrm{F}})Y_{\mathrm{F}} = U^{-*}Z_{\mathrm{F}}(s\tilde{\mathcal{E}} - \tilde{\mathcal{A}}) = Z(s\tilde{\mathcal{E}} - \tilde{\mathcal{A}}),$$

which is (b4). Since $\dim(\operatorname{im} Y) = \operatorname{rk} Y = \operatorname{rk} UY_{\mathrm{F}} = \operatorname{rk} Y_{\mathrm{F}} = n+m$ and $Y$ is $\mathcal{E}$-neutral by an argumentation analogous to above, property (b1) holds as well. Further, due to (4.28) we have $\begin{bmatrix} T & 0 \\ FT & I_m \end{bmatrix} \begin{bmatrix} Y_{x,\mathrm{F}} \\ Y_{u,\mathrm{F}} \end{bmatrix} = \begin{bmatrix} Y_x \\ Y_u \end{bmatrix}$ and hence, the subset relation

$$\mathcal{V}_{\mathrm{sys}} \overset{(2.19)}{=} \begin{bmatrix} T & 0 \\ FT & I_m \end{bmatrix} \mathcal{V}_{\mathrm{sys,F}} \overset{(\mathrm{b2}')}{\subseteq} \operatorname{im} \begin{bmatrix} T & 0 \\ FT & I_m \end{bmatrix} \begin{bmatrix} Y_{x,\mathrm{F}} \\ Y_{u,\mathrm{F}} \end{bmatrix} = \operatorname{im} \begin{bmatrix} Y_x \\ Y_u \end{bmatrix}$$

implies (b2). Finally,

$$\operatorname{rk}(\Pi E Y_x) = \operatorname{rk}\left((W^{-1}E_{\mathrm{F}}T^{-1})TY_{x,\mathrm{F}}\right) = \operatorname{rk}\left(E_{\mathrm{F}}Y_{x,\mathrm{F}}\right) \overset{(\mathrm{b3}')}{=} n_1$$

provides (b3) and the first part of the proof is finished.

(b) $\Rightarrow$ (a):
The idea to show (a) is to apply the ODE case Theorem 4.9. Therefore, we need to transfer the given system into a suitable ODE system. To make clear, how the ODE case Theorem and our findings fit together, Figure 4.2 is given. Using the ODE case Theorem and some previous findings the statement follows easily.

Let (b) hold true. Consider the matrices $E_{\mathrm{F}}, A_{\mathrm{F}}, B_{\mathrm{F}}$ and $Q_{\mathrm{F}}$, $S_{\mathrm{F}}$, and $R_{\mathrm{F}}$ as in (4.20) and $\widehat{U}$ and $U$ as in (4.26) and (4.28), resp.. Furthermore, define

$$\widehat{Y}_{\mathrm{F}} := \widehat{U}^{-1}U^{-1}\begin{bmatrix} Y_\mu \\ Y_x \\ Y_u \end{bmatrix} = \widehat{U}^{-1}\begin{bmatrix} Y_{\mu,1} \\ Y_{\mu,2} \\ Y_{x,1} \\ Y_{x,2} \\ Y_{u,\mathrm{F}} \end{bmatrix} =: \begin{bmatrix} Y_{\mu,1} \\ Y_{x,1} \\ Y_{u,\mathrm{F}} \\ \widehat{Y}_{x,2} \\ \widehat{Y}_{\mu,2} \end{bmatrix} \quad \text{and} \quad \widehat{Z}_{\mathrm{F}} := \widehat{U}^*U^*Z. \tag{4.30}$$

Thus, for $s\widehat{\mathcal{E}}_{\mathrm{F}} - \widehat{\mathcal{A}}_{\mathrm{F}}$ as in (4.21) we can show that $\operatorname{im}\widehat{Y}_{\mathrm{F}}$ is a deflating subspace:

$$\begin{aligned} \widehat{Z}_{\mathrm{F}}(s\tilde{\mathcal{E}} - \tilde{\mathcal{A}}) &= \widehat{U}^*U^*(s\mathcal{E} - \mathcal{A})Y \\ &\underset{(4.27)}{\overset{(4.29)}{=}} \widehat{U}^*U^*\left(U^{-*}\widehat{U}^{-*}(s\widehat{\mathcal{E}}_{\mathrm{F}} - \widehat{\mathcal{A}}_{\mathrm{F}})\widehat{U}^{-1}U^{-1}\right)Y \\ &= (s\widehat{\mathcal{E}}_{\mathrm{F}} - \widehat{\mathcal{A}}_{\mathrm{F}})\widehat{Y}_{\mathrm{F}}. \end{aligned}$$

Next we prove that $\operatorname{im}\widehat{Y}_{\mathrm{F}}$ is $\widehat{\mathcal{E}}_{\mathrm{F}}$-neutral. Therefore, consider $z_1, z_2 \in \operatorname{im}\widehat{Y}_{\mathrm{F}}$, i.e. there exist some $x = (x_1, \ldots, x_5)^\top, y = (y_1, \ldots, y_5)^\top$ such that

$$z_1^* \widehat{\mathcal{E}}_{\mathrm{F}} z_2 = \left(\widehat{Y}_{\mathrm{F}} x\right)^* \widehat{\mathcal{E}}_{\mathrm{F}} \left(\widehat{Y}_{\mathrm{F}} y\right) = \left(\widehat{U}^{-1} U^{-1} Y x\right)^* \left(\widehat{U}^* U \mathcal{E} U \widehat{U}\right) \left(\widehat{U}^{-1} U^{-1} Y y\right)$$
$$= (Yx)^* \mathcal{E} (Yy) = 0$$

since $\operatorname{im} Y$ is $\mathcal{E}$-neutral by (b1).

Besides $\widehat{\mathcal{E}}_{\mathrm{F}}$-neutrality of $\operatorname{im}\widehat{Y}_{\mathrm{F}}$ we are able to show that for the rank of the two blocks $\begin{bmatrix} Y_{\mu,1} \\ Y_{x,1} \end{bmatrix}$ of $\widehat{Y}_{\mathrm{F}}$ is bounded above, more precisely,

$$\dim\left(\operatorname{im}\begin{bmatrix} Y_{\mu,1} \\ Y_{x,1} \end{bmatrix}\right) \leq n_1 \tag{4.31}$$

holds true. Due to the block structure of $\widehat{\mathcal{E}}_{\mathrm{F}}$, the $\widehat{\mathcal{E}}_{\mathrm{F}}$-neutrality of $\operatorname{im}\widehat{Y}_{\mathrm{F}}$ implies that $\operatorname{im}\begin{bmatrix} Y_{\mu,1} \\ Y_{x,1} \end{bmatrix}$ is $\begin{bmatrix} 0 & -I_{n_1} \\ I_{n_1} & 0 \end{bmatrix}$-neutral. Thus, we have $0 = v^* \begin{bmatrix} Y_{\mu,1} \\ Y_{x,1} \end{bmatrix}^* \begin{bmatrix} 0 & -I_{n_1} \\ I_{n_1} & 0 \end{bmatrix} \begin{bmatrix} Y_{\mu,1} \\ Y_{x,1} \end{bmatrix} w$ for all $v, w \in \mathbb{K}^{n+m}$ and therefore

$$0 = \begin{bmatrix} Y_{\mu,1} \\ Y_{x,1} \end{bmatrix}^* \begin{bmatrix} 0 & -I_{n_1} \\ I_{n_1} & 0 \end{bmatrix} \begin{bmatrix} Y_{\mu,1} \\ Y_{x,1} \end{bmatrix} = \begin{bmatrix} Y_{\mu,1} \\ Y_{x,1} \end{bmatrix}^* \tilde{Y},$$

where

$$\tilde{Y} = \begin{bmatrix} -y_{n_1+1} \\ \vdots \\ -y_{2n_1} \\ y_1 \\ \vdots \\ y_{n_1} \end{bmatrix} \quad \text{for} \quad \begin{bmatrix} Y_{\mu,1} \\ Y_{x,1} \end{bmatrix} = \begin{bmatrix} y_1 \\ \vdots \\ y_{2n_1} \end{bmatrix}.$$

Assume $\dim\left(\operatorname{im}\begin{bmatrix} Y_{\mu,1} \\ Y_{x,1} \end{bmatrix}\right) > n_1$, i.e. $\operatorname{rk}\begin{bmatrix} Y_{\mu,1} \\ Y_{x,1} \end{bmatrix}^* = \operatorname{rk}\tilde{Y} = \operatorname{rk}\begin{bmatrix} Y_{\mu,1} \\ Y_{x,1} \end{bmatrix} > n_1$. Hence, Lemma 4.7 provides the contradiction

$$0 < \operatorname{rk}\begin{bmatrix} Y_{\mu,1} \\ Y_{x,1} \end{bmatrix}^* + \operatorname{rk}\tilde{Y} - 2n_1 \leq \operatorname{rk}\left(\begin{bmatrix} Y_{\mu,1} \\ Y_{x,1} \end{bmatrix}^* \tilde{Y}\right) = 0,$$

which completes the proof of (4.31).

Additionally to the rank condition in (4.31) we can calculate the rank of the matrix $\begin{bmatrix} Y_{x,1}^\top & Y_{u,\mathrm{F}}^\top \end{bmatrix}^\top$. Since (b2) holds, Corollary 2.41 yields

$$\mathcal{V}_{\mathrm{sys,F}} \subseteq \operatorname{im}\begin{bmatrix} Y_{x,\mathrm{F}} \\ Y_{u,\mathrm{F}} \end{bmatrix} = \operatorname{im}\begin{bmatrix} Y_{x,1} \\ Y_{x,2} \\ Y_{u,\mathrm{F}} \end{bmatrix}.$$

By Corollary 2.44 (b) we have $\dim \mathcal{V}_{\mathrm{sys,F}} = n_1 + m$. According to the characterization in (2.23) it is

$$\mathcal{V}_{\mathrm{sys,F}} = \left\{ \begin{pmatrix} x_1 \\ -B_2 u \\ u \end{pmatrix} \;\middle|\; x_1 \in \mathbb{K}^{n_1}, u \in \mathbb{K}^m \right\}$$

which implies that the $x_2$-components of $\mathcal{V}_{\mathrm{sys,F}}$ linearly depend on the $u$-components. We get $\operatorname{rk} \begin{bmatrix} Y_{x,1} \\ Y_{u,\mathrm{F}} \end{bmatrix} = \dim \left( \operatorname{im} \begin{bmatrix} Y_{x,1} \\ Y_{u,\mathrm{F}} \end{bmatrix} \right) \geq \dim \mathcal{V}_{\mathrm{sys,F}} = n_1 + m$ and thus

$$\operatorname{rk} \begin{bmatrix} Y_{x,1} \\ Y_{u,\mathrm{F}} \end{bmatrix} = n_1 + m. \tag{4.32}$$

Furthermore, due to (4.20) we find that $\Pi E = \begin{bmatrix} M & 0 \\ 0 & 0 \end{bmatrix}$, where $M \in \mathrm{Gl}_{n_1}(\mathbb{K})$. Thus, by (b3) we can infer $n_1 = \operatorname{rk} \Pi E Y_x = \operatorname{rk} \begin{bmatrix} M Y_{x,1} \\ 0 \end{bmatrix} = \operatorname{rk} Y_{x,1}$ and hence, by (4.31) and (4.32) we conclude

$$\operatorname{rk} \begin{bmatrix} Y_{\mu,1} \\ Y_{x,1} \\ Y_{u,\mathrm{F}} \end{bmatrix} = n_1 + m. \tag{4.33}$$

Due to (b1) and (4.30) we have $n + m = \operatorname{rk} Y = \operatorname{rk} \widehat{Y}_{\mathrm{F}}$. Together with (4.33) we obtain the only missing rank condition

$$\operatorname{rk} \begin{bmatrix} \widehat{Y}_{x,2} \\ \widehat{Y}_{\mu,2} \end{bmatrix} = n_2.$$

Combining all facts that we have obtained on the rank conditions earlier we can find some matrix $V \in \mathrm{Gl}_{n+m}(\mathbb{K})$ such that

$$\begin{bmatrix} Y_{\mu,1} \\ Y_{x,1} \\ Y_{u,\mathrm{F}} \\ \widehat{Y}_{x,2} \\ \widehat{Y}_{\mu,2} \end{bmatrix} V = \begin{bmatrix} X_{11} & 0 & 0 \\ I_{n_1} & 0 & 0 \\ 0 & I_m & 0 \\ 0 & 0 & \tilde{Y}_{x,2} \\ 0 & 0 & \tilde{Y}_{\mu,2} \end{bmatrix}, \text{ where } \operatorname{rk} \begin{bmatrix} \tilde{Y}_{x,2} \\ \tilde{Y}_{\mu,2} \end{bmatrix} = n_2. \tag{4.34}$$

For the following application of the ODE case Theorem 4.9, define $s\mathcal{E}' - \mathcal{A}'$ as the upper left part of $s\widehat{\mathcal{E}}_{\mathrm{F}} - \widehat{\mathcal{A}}_{\mathrm{F}}$, i.e.

$$\begin{aligned} s\mathcal{E}' - \mathcal{A}' &= \begin{bmatrix} I_{n_1} & 0 & 0 & 0 & 0 \\ 0 & I_{n_1} & 0 & 0 & 0 \\ 0 & 0 & I_m & 0 & 0 \end{bmatrix} (s\widehat{\mathcal{E}}_{\mathrm{F}} - \widehat{\mathcal{A}}_{\mathrm{F}}) \begin{bmatrix} I_{n_1} & 0 & 0 & 0 & 0 \\ 0 & I_{n_1} & 0 & 0 & 0 \\ 0 & 0 & I_m & 0 & 0 \end{bmatrix}^\top \\ &= \begin{bmatrix} 0 & -sI_{n_1} + A_{11} & B_1 \\ sI_{n_1} + A_{11}^* & Q_{11} & -Q_{12}B_2 + S_1 \\ B_1^* & -B_2^* Q_{12}^* + S_1^* & B_2^* Q_{22} B_2 - B_2^* S_2 - S_2^* B_2 + R \end{bmatrix} \end{aligned} \tag{4.35}$$

and $s\tilde{\mathcal{E}}' - \tilde{\mathcal{A}}'$ analogously.

Now we want to apply Theorem 4.9. To see how the theorem can be applied, we collect the required presuppositions and our results in Figure 4.2, following on the next page, and show how they fit together.

Then, along with the given presuppositions, Theorem 4.9 provides the existence of some $K_1 \in \mathbb{K}^{q \times n_1}$ and $L_1 \in \mathbb{K}^{q \times m}$ such that (4.15) holds. Denoting $(X, K, L)$ as in (4.16) we receive a solution of the Lur'e equation (4.19) for the impulse controllable system $[\Pi E, A, B]$. Hence, Theorem 4.20 (c) completes the proof. $\square$

| **Presuppositions for ODE Theorem** | | **Used to apply the ODE Theorem** | | **Constructions of the proof** |
|---|---|---|---|---|
| $V = \begin{bmatrix} V_\mu \\ V_x \\ V_u \end{bmatrix}$, $W = \begin{bmatrix} W_\mu \\ W_x \\ W_u \end{bmatrix}$ | $\longrightarrow$ | $V = Y' = \begin{bmatrix} Y_{\mu,1} \\ Y_{x,1} \\ Y_{u,\mathrm{F}} \end{bmatrix}$, $W = Z'$ (defined anal. from $\widehat{Z}_\mathrm{F}$) | $\longleftarrow$ | $\widehat{Y}_\mathrm{F} = \begin{bmatrix} Y_{\mu,1} \\ Y_{x,1} \\ Y_{u,\mathrm{F}} \\ \widehat{Y}_{x,2} \\ \widehat{Y}_{\mu,2} \end{bmatrix}$, $\widehat{Z}_\mathrm{F}$ as in (4.30) |
| $(s\mathcal{E} - \mathcal{A})_{\mathrm{ODE}}$ | $\longrightarrow$ | $(s\mathcal{E} - \mathcal{A})_{\mathrm{ODE}} = s\mathcal{E}' - \mathcal{A}'$ as in (4.35) | $\longleftarrow$ | $s\widehat{\mathcal{E}}_\mathrm{F} - \widehat{\mathcal{A}}_\mathrm{F}$ |
| $\operatorname{im} V$ is maximally $\mathcal{E}$-neutral, $\operatorname{rk} V_x = n$ | $\longrightarrow$ | $\operatorname{im} Y'$ is $\mathcal{E}'$-neutral, $\operatorname{rk} Y' = n_1 + m$, $\operatorname{rk} Y_{x,1} = n_1$ | $\longleftarrow$ | $\operatorname{im} \widehat{Y}_\mathrm{F}$ is $\widehat{\mathcal{E}}_\mathrm{F}$-neutral, $\operatorname{rk} \widehat{Y}_\mathrm{F} = n + m$ |
| $X = V_\mu V_x^+$ | $\longrightarrow$ | $X = X_{11}$, $V_x^+ = V'$, $X_{11} = Y_{\mu,1} V'$ | $\longleftarrow$ | see equation (4.34) |
| $(s\mathcal{E} - \mathcal{A})_{\mathrm{ODE}} V = W(s\tilde{\mathcal{E}} - \tilde{\mathcal{A}})$ | $\longrightarrow$ | $(s\mathcal{E}' - \mathcal{A}')Y' = Z'(s\tilde{\mathcal{E}}' - \tilde{\mathcal{A}}')$ | $\longleftarrow$ | $(s\widehat{\mathcal{E}}_\mathrm{F} - \widehat{\mathcal{A}}_\mathrm{F})\widehat{Y}_\mathrm{F} = \widehat{Z}_\mathrm{F}(s\tilde{\mathcal{E}} - \tilde{\mathcal{A}})$ |

Figure 4.2: Presuppositions for application of Theorem 4.9

## 4.6 Importance of Theorem 4.4

This section explains, why Theorem 4.4 is the chapters main result. It can be used to allocate a plan to construct a solution of the Lur'e equation (4.1). A guideline on how to calculate a Lur'e solution is given and verified at first. Then, an example follows.

### 4.6.1 Guideline to construct a Lur'e solution

Let a system $[E, A, B] \in \Sigma_{n,m}(\mathbb{K})$ and some weighting matrices $Q = Q^* \in \mathbb{K}^{n\times n}$, $S \in \mathbb{K}^{n\times m}$, and $R = R^* \in \mathbb{K}^{m\times m}$ be given such that the Popov function fulfils $\Phi(\mathrm{i}\omega) \geq 0$ for all $\omega \in \mathbb{R}$ with $\det(\mathrm{i}\omega E - A) \neq 0$. Let $W, T \in \mathrm{Gl}_n(\mathbb{K})$ and $F \in \mathbb{K}^{m\times n}$ be matrices such that (2.12) holds. Furthermore, let the projector $\Pi \in \mathbb{K}^{n\times n}$ be defined as in (4.5) and the even matrix pencil $s\mathcal{E} - \mathcal{A}$ as in (4.4).

**Conclusion 4.21.** *Let matrices $Y \in \mathbb{K}^{(2n+m)\times(n+m)}$, $Z \in \mathbb{K}^{(2n+m)\times(n+q)}$, and a pencil $s\tilde{\mathcal{E}} - \tilde{\mathcal{A}} \in \mathbb{K}[s]^{(n+q)\times(n+m)}$ be given such that Theorem 4.4 (b) holds true. Then a solution $(X, K, L)$ of the Lur'e equation (4.1) can be calculated as follows:*

$$X = W^* \begin{bmatrix} I_{n_1} & 0 \\ 0 & 0 \end{bmatrix} W^{-*} Y_\mu \begin{bmatrix} I_n \\ -FT \end{bmatrix} W, \tag{4.36a}$$

*and*

$$K = \begin{bmatrix} K_1 & 0 \end{bmatrix} T^{-1} - LF \quad \textit{and} \quad L = \tilde{L} - K_2 B_2, \tag{4.36b}$$

*where* $\begin{bmatrix} K_1 & K_2 & \tilde{L} \end{bmatrix} = \begin{bmatrix} 0 & 0 & I_q \end{bmatrix} \left( s\tilde{\mathcal{E}} - \tilde{\mathcal{A}} \right)$.

**Proof:** We divide the proof into four steps. At first, we deduce a transformation of $Y$, from which a solution matrix $X_{11}$ for the ODE Lur'e equation (4.12) is reconstructed in a second step. Then we derive the formula for $K_1, K_2$ and $\tilde{L}$. Finally, we combine our findings to receive (4.36).

<u>Step 1:</u> We show

$$\begin{bmatrix} Y_\mu \\ Y_x \\ Y_u \end{bmatrix} \begin{bmatrix} T^{-1} & 0 \\ -F & I_m \end{bmatrix} = \begin{bmatrix} X\Pi E + G_1 & G_2 \\ I_n & 0 \\ 0 & I_m \end{bmatrix}, \tag{4.37}$$

where $\operatorname{im} G_1 \subseteq \ker E^*\Pi^*$ and $\operatorname{im} G_2 \subseteq \ker E^*\Pi^*$.

According to the proof of "(a)⇒(b)" in Theorem 4.4 the definitions of $\widehat{Y}_F$, $Y_{\mathrm{F}}$ and $Y$ (see (4.23), (4.27), and (4.28)) yield

$$\begin{bmatrix} Y_\mu \\ Y_x \\ Y_u \end{bmatrix} = Y = UY_{\mathrm{F}} = \begin{bmatrix} W^* & 0 & 0 \\ 0 & T & 0 \\ 0 & FT & I_m \end{bmatrix} \begin{bmatrix} X_{11} & 0 & 0 \\ -Q_{12}^* & -\frac{1}{2}Q_{22} & \frac{1}{2}Q_{22}B_2 - S_2 \\ I_{n_1} & 0 & 0 \\ 0 & I_{n_2} & 0 \\ 0 & 0 & I_m \end{bmatrix}.$$

Thus, we get $\begin{bmatrix} Y_x \\ Y_u \end{bmatrix} = \begin{bmatrix} T & 0 \\ FT & I_m \end{bmatrix} \in \mathrm{Gl}_{n+m}(\mathbb{K})$ and hence $\begin{bmatrix} Y_x \\ Y_u \end{bmatrix}^{-1} = \begin{bmatrix} T^{-1} & 0 \\ -F & I_m \end{bmatrix}$. We further conclude

$$\begin{bmatrix} Y_\mu \\ Y_x \\ Y_u \end{bmatrix} \begin{bmatrix} T^{-1} & 0 \\ -F & I_m \end{bmatrix} = \begin{bmatrix} Y_1 T^{-1} - Y_2 F & Y_2 \\ I_n & 0 \\ 0 & I_m \end{bmatrix},$$

where

$$\begin{bmatrix} Y_1 & Y_2 \end{bmatrix} := Y_\mu = W^* \begin{bmatrix} X_{11} & 0 & 0 \\ -Q_{12}^* & -\frac{1}{2}Q_{22} & \frac{1}{2}Q_{22}B_2 - S_2 \end{bmatrix}. \tag{4.38}$$

Due to the definition of $E_{\mathrm{F}}$ in (4.20) it is

$$E^*\Pi^* = T^{-*} \begin{bmatrix} I_{n_1} & 0 \\ 0 & 0 \end{bmatrix} W^{-*}, \tag{4.39}$$

and hence

$$E^*\Pi^* Y_2 = T^{-*} \begin{bmatrix} I_{n_1} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ \frac{1}{2}Q_{22}B_2 - S_2 \end{bmatrix} = 0, \tag{4.40}$$

i.e. $\operatorname{im} Y_2 \subseteq \ker(E^*\Pi^*)$ and obviously $\operatorname{im}(Y_2 F) \subseteq \ker(E^*\Pi^*)$. This, together with (4.38), (4.39), and $X$ as in (4.20), yields

$$\begin{aligned}
&E^*\Pi^*(Y_1 T^{-1} - Y_2 F - X\Pi E) \\
\overset{(4.40)}{=} &E^*\Pi^*(Y_1 T^{-1} - X\Pi E) \\
= \ &T^{-*} \begin{bmatrix} I_{n_1} & 0 \\ 0 & 0 \end{bmatrix} W^{-*} \left( W^* \begin{bmatrix} X_{11} & 0 \\ -Q_{12}^* & -\frac{1}{2}Q_{22} \end{bmatrix} T^{-1} - W^* \begin{bmatrix} X_{11} & X_{12} \\ X_{12}^* & X_{22} \end{bmatrix} \begin{bmatrix} I_{n_1} & 0 \\ 0 & 0 \end{bmatrix} T^{-1} \right) \\
= \ &T^{-*} \begin{bmatrix} I_{n_1} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} X_{11} & 0 \\ -Q_{12}^* & -\frac{1}{2}Q_{22} \end{bmatrix} T^{-1} - T^{-*} \begin{bmatrix} I_{n_1} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} X_{11} & X_{12} \\ X_{12}^* & X_{22} \end{bmatrix} \begin{bmatrix} I_{n_1} & 0 \\ 0 & 0 \end{bmatrix} T^{-1} \\
= \ &0,
\end{aligned}$$

i.e. $\operatorname{im}(Y_{\mu,1} T^{-1} - Y_{\mu,2} F - X\Pi E) \subseteq \ker(E^*\Pi^*)$.
Thus, we achieve (4.37) for $G_1 = Y_1 T^{-1} - Y_2 F - X\Pi E$ and $G_2 = Y_2$.

Step 2: We calculate $X_{11}$. Denoting $Y_x^+ := \begin{bmatrix} T^{-1} \\ -F \end{bmatrix}$ equation (4.37) yields

$$E^*\Pi^* X \Pi E = E^*\Pi^* Y_\mu Y_x^+,$$

from which a solution matrix $X_{11}$ of the Lur'e equation (4.12) can be reconstructed:

$$\begin{aligned}
&E^*\Pi^* X \Pi E &&= E^*\Pi^* Y_\mu Y_x^+ \\
\underset{(4.39)}{\overset{(4.20)}{\Leftrightarrow}} \ &T^{-*} \begin{bmatrix} X_{11} & 0 \\ 0 & 0 \end{bmatrix} T^{-1} &&= T^{-*} \begin{bmatrix} I_{n_1} & 0 \\ 0 & 0 \end{bmatrix} W^{-*} Y_\mu \begin{bmatrix} T^{-1} \\ -F \end{bmatrix} \\
\Leftrightarrow \ &\begin{bmatrix} X_{11} & 0 \\ 0 & 0 \end{bmatrix} &&= \begin{bmatrix} I_{n_1} & 0 \\ 0 & 0 \end{bmatrix} W^{-*} Y_\mu \begin{bmatrix} I_n \\ -FT \end{bmatrix}.
\end{aligned}$$

Step 3: We determine $K_1$, $K_2$ and $\tilde{L}$.
Regarding the equations (4.22) and (4.27) we obtain

$$s\tilde{\mathcal{E}} - \tilde{\mathcal{A}} = \widehat{W}(s\tilde{\mathcal{E}}_F - \tilde{\mathcal{A}}_F)\widehat{V} = \begin{bmatrix} -sI_{n_1} + A_{11} & 0 & B_1 \\ 0 & I_{n_2} & B_2 \\ K_1 & K_2 & L \end{bmatrix},$$

from where $K_1$, $K_2$ and $\tilde{L} = L$ can be read out.

Step 4: According to the proof of Theorem 4.4 the triple $(X_{11}, K_1, \tilde{L})$ fulfils the ODE Lur'e equation (4.15). This equals that $(X_{11}, K_1, \tilde{L} - K_2B_2)$ solves the Lur'e equation (4.15). Hence,

$$X = W^* \begin{bmatrix} X_{11} & 0 \\ 0 & 0 \end{bmatrix} W, \quad \begin{bmatrix} K_1 & 0 \end{bmatrix} T^{-1} - LF, \quad L = \tilde{L} - K_2B_2$$

together with Lemma 4.15 (a) yields the statement. □

Note that the construction of a Lur'e solution is simple for given matrices $Y$, $Z$ and an existing even matrix pencil $s\tilde{\mathcal{E}} - \tilde{\mathcal{A}}$. In general they are not known, and therefore it remains difficult to construct a solution of the Lur'e equation.

### 4.6.2 Example: Construction of a Lur'e solution

The following example shows how to deduce a solution for the Lur'e equation (4.1) with help of Conclusion 4.21.

Consider the DAE system

$$\tfrac{\mathrm{d}}{\mathrm{d}t}\,(Ex(t)) = Ax(t) + Bu(t)$$

$$\text{where } E = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \; A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \; B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \; \text{ and } \; n = 2,\, m = 1.$$

The associated pencil is given by

$$\begin{bmatrix} sE - A & B \end{bmatrix} = \begin{bmatrix} -1 & s & 0 \\ 0 & -1 & 1 \end{bmatrix}.$$

Further, let $Q = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$, $R = 0$ and $S = 0_{2\times 1}$ be the weighting matrices. With matrices

$$W = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}, \; T = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \; F = \begin{bmatrix} 1 & -1 \end{bmatrix}$$

the system can be transformed into FEF (2.7), i.e.

$$E_\mathrm{F} = WET = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \; A_\mathrm{F} = W(A + BF)T = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \; B_\mathrm{F} = WB = \begin{bmatrix} -1 \\ 1 \end{bmatrix}.$$

Note that for the given system the following properties hold true:

(i) The pencil is regular as $\mathrm{rk}_{\mathbb{K}(s)}(sE - A) = 2 = n$.

(ii) The system is impulse controllable as

$$\mathrm{rk}\begin{bmatrix} E & AS_\infty & B \end{bmatrix} = \mathrm{rk}\begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = 2, \text{ where } S_\infty = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

(iii) By Corollary 2.44 (b) the system space is

$$\mathcal{V}_{\mathrm{sys}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix}^{-\mathbb{1}} \left\{ \begin{pmatrix} y \\ 0 \end{pmatrix} \middle| \, y \in \mathbb{R} \right\} = \left\{ \begin{pmatrix} a \\ b \\ -b \end{pmatrix} \middle| \, a, b \in \mathbb{R} \right\}. \tag{4.41}$$

We want to calculate a solution of the Lur'e equation (4.1). For that purpose we use Conclusion 4.21. According to its definition in (4.5), we consider the projector

$$\Pi = W^{-1}EW = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix},$$

and the even matrix pencil $s\mathcal{E} - \mathcal{A}$ as in (4.4):

$$s\mathcal{E} - \mathcal{A} = \begin{bmatrix} 0 & 0 & 1 & -s & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 2 & 1 & 0 \\ s & 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{bmatrix} \quad \text{where } \mathcal{E} = \begin{bmatrix} 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}. \tag{4.42}$$

Now let the matrices $Y \in \mathbb{R}^{5\times 3}$ and $Z \in \mathbb{R}^{5\times 3}$ be given as

$$Y = \begin{bmatrix} Y_\mu \\ Y_x \\ Y_u \end{bmatrix} = \begin{bmatrix} \sqrt{2}-1 & 0 & 0 \\ -\sqrt{2} & -1 & 1 \\ \hline 0 & 1 & 0 \\ 1 & 0 & 0 \\ \hline -1 & 1 & 1 \end{bmatrix}, \quad Z = \begin{bmatrix} Z_\mu \\ Z_x \\ Z_u \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ \hline 0 & 0 & -\sqrt{2} \\ 1-\sqrt{2} & 2-\sqrt{2} & \sqrt{2}-1 \\ \hline 0 & 1 & \sqrt{2} \end{bmatrix} \tag{4.43}$$

**Verification of the properties in Theorem 4.4 (b):**

To apply Conclusion 4.21, we need to verify the properties of Theorem 4.4 (b). At first, we take a look at the Popov-function as in (3.1).

$$\Phi(\mathrm{i}\omega) = \left[\begin{pmatrix} -1 & -\bar{s} \\ 0 & -1 \end{pmatrix}^{-1}\begin{pmatrix} 0 \\ 1 \end{pmatrix} \atop 1\right]^* \begin{bmatrix} 2 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \left[\begin{pmatrix} -1 & s \\ 0 & -1 \end{pmatrix}^{-1}\begin{pmatrix} 0 \\ 1 \end{pmatrix} \atop 1\right]$$

$$
\begin{aligned}
&= \left[\begin{pmatrix} -1 & \bar{s} \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \atop 1 \right]^* \begin{bmatrix} 2 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \left[\begin{pmatrix} -1 & -s \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \atop 1 \right] \\
&= \begin{bmatrix} \bar{s} \\ -1 \\ 1 \end{bmatrix}^* \begin{bmatrix} 2 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} -s \\ -1 \\ 1 \end{bmatrix} \quad = \quad -2s^2 + 1.
\end{aligned}
$$

It holds $\operatorname{rk}_{\mathbb{K}(s)} \Phi(s) = 1$ and by Proposition 4.5 therefore it is $q = 1$. Evaluating the Popov-function on the imaginary axis, i.e. $s = \mathrm{i}\omega$ with $\omega \in \mathbb{R}$, yields

$$\Phi(\mathrm{i}\omega) = -2\mathrm{i}^2\omega^2 + 1 = 2\omega^2 + 1 \geq 0,$$

which is positive semi-definiteness of the Popov-function on the imaginary axis.

Further, it is necessary to show that the matrices $Y$ and $Z$ as in (4.43) fulfil the properties (b1) to (b4):

(b1) The space $\operatorname{im} Y$ is 3-dimensional and $\mathcal{E}$-neutral:
Let $y, \tilde{y} \in \operatorname{im} Y$, i.e.

$$y = \begin{bmatrix} Y_{11}x_1 + Y_{12}x_2 + Y_{13}x_3 \\ Y_{21}x_1 + Y_{22}x_2 + Y_{23}x_3 \\ x_2 \\ x_1 \\ -x_1 + x_2 + x_3 \end{bmatrix} \text{ and } \tilde{y} = \begin{bmatrix} Y_{11}\tilde{x}_1 + Y_{12}\tilde{x}_2 + Y_{13}\tilde{x}_3 \\ Y_{21}\tilde{x}_1 + Y_{22}\tilde{x}_2 + Y_{23}\tilde{x}_3 \\ \tilde{x}_2 \\ \tilde{x}_1 \\ -\tilde{x}_1 + \tilde{x}_2 + \tilde{x}_3 \end{bmatrix}.$$

for some $x, \tilde{x} \in \mathbb{K}^3$. Then for $\mathcal{E}$ as in (4.42) we have

$$
\begin{aligned}
& 0 = y^* \mathcal{E} \tilde{y} \\
\Leftrightarrow \quad & 0 = x_1(Y_{11}\tilde{x}_1 + Y_{12}\tilde{x}_2 + Y_{13}\tilde{x}_3) - \tilde{x}_1(Y_{11}x_1 + Y_{12}x_2 + Y_{13}x_3)
\end{aligned}
$$

which holds true as $Y_{12} = Y_{13} = 0$.

(b2) With $\mathcal{V}_{\text{sys}}$ as in (4.41) it holds

$$\mathcal{V}_{\text{sys}} \subseteq \operatorname{im} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ -1 & 1 & 1 \end{bmatrix}.$$

(b3) With $\Pi E = \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ we obtain

$$\operatorname{rk} \Pi E Y_x = \operatorname{rk} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} = 1 = n_1.$$

(b4) For

$$s\tilde{\mathcal{E}} - \tilde{\mathcal{A}} = \begin{bmatrix} -s & 0 & | & -1 \\ 0 & 1 & | & 1 \\ -- & -- & | & -- \\ -1 & -\sqrt{2} & | & 0 \end{bmatrix}, \text{ where } \operatorname{rk}_{\mathbb{K}(s)}(s\tilde{\mathcal{E}} - \tilde{\mathcal{A}}) = 3 = n + q, \quad (4.44)$$

it holds

$$(s\mathcal{E} - \mathcal{A})\, Y = \begin{bmatrix} -s & 1 & 0 \\ 0 & 1 & 1 \\ \sqrt{2} & 2 & 0 \\ (\sqrt{2}-1)s - \sqrt{2} + 1 & 0 & 1 \\ -\sqrt{2} & -1 & 1 \end{bmatrix} = Z\,(s\tilde{\mathcal{E}} - \tilde{\mathcal{A}}).$$

Thus, Theorem 4.4 yields that there exists a solution $(X, K, L) \in \mathbb{K}^{2\times 2} \times \mathbb{K}^{1\times 2} \times \mathbb{K}$ for the Lur'e equation (4.1) and we can apply Conclusion 4.21.

### Construction of a Lur'e solution:

By using (4.36a) we obtain $X$ as follows:

$$\begin{aligned} X &= \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}^* \begin{bmatrix} I_{n_1} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}^{-*} \begin{bmatrix} \sqrt{2}-1 & 0 & 0 \\ -\sqrt{2} & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}^* \begin{bmatrix} \sqrt{2}-1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} \sqrt{2}-1 & 1-\sqrt{2} \\ 1-\sqrt{2} & \sqrt{2}-1 \end{bmatrix}. \end{aligned}$$

Applying (4.36b) on the pencil in (4.44) yields

$$\begin{bmatrix} K_1 & K_2 & \tilde{L} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -s & 0 & -1 \\ 0 & 1 & 1 \\ -1 & -\sqrt{2} & 0 \end{bmatrix} = \begin{bmatrix} -1 & -\sqrt{2} & 0 \end{bmatrix}$$

and therefore

$$L = 0 + \sqrt{2} = \sqrt{2}, \text{ and } K = \begin{bmatrix} -1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}^{-1} - \sqrt{2}\begin{bmatrix} 1 & -1 \end{bmatrix} = \begin{bmatrix} -\sqrt{2} & \sqrt{2}-1 \end{bmatrix}.$$

Summarized we have

$$(X, K, L) \;=\; \left( \begin{bmatrix} \sqrt{2}-1 & 1-\sqrt{2} \\ 1-\sqrt{2} & \sqrt{2}-1 \end{bmatrix}, \begin{bmatrix} -\sqrt{2} & \sqrt{2}-1 \end{bmatrix}, \sqrt{2} \right). \qquad (4.45)$$

**Proof that $(X, K, L)$ is a Lur'e solution:**

We show that (4.1a) and (4.1b) are fulfilled. Using $(X, K, L)$ as in (4.45) we obtain

$$\begin{bmatrix} A^*XE + E^*XA + Q & E^*XB + S \\ B^*XE + S^* & R \end{bmatrix} = \begin{bmatrix} 2 & \sqrt{2} & 0 \\ \sqrt{2} & 3 - 2\sqrt{2} & 1 - \sqrt{2} \\ 0 & 1 - \sqrt{2} & 0 \end{bmatrix},$$

and

$$\begin{bmatrix} K^* \\ L^* \end{bmatrix} \begin{bmatrix} K & L \end{bmatrix} = \begin{bmatrix} 2 & \sqrt{2} - 2 & -2 \\ \sqrt{2} - 2 & 3 - 2\sqrt{2} & 2 - \sqrt{2} \\ -2 & 2 - \sqrt{2} & 2 \end{bmatrix}.$$

Evaluating the Lur'e equation on $\mathcal{V}_{\text{sys}}$ as in (4.41), for $a$, $b \in \mathbb{R}$ this is

$$\begin{aligned} & \begin{bmatrix} a & b & -b \end{bmatrix} \begin{bmatrix} 2 & \sqrt{2} & 0 \\ \sqrt{2} & 3 - 2\sqrt{2} & 1 - \sqrt{2} \\ 0 & 1 - \sqrt{2} & 0 \end{bmatrix} \begin{bmatrix} a \\ b \\ -b \end{bmatrix} \\ = \ & 2a^2 + 2\sqrt{2}ab + b^2 \\ = \ & \begin{bmatrix} a & b & -b \end{bmatrix} \begin{bmatrix} 2 & \sqrt{2} - 2 & -2 \\ \sqrt{2} - 2 & 3 - 2\sqrt{2} & 2 - \sqrt{2} \\ -2 & 2 - \sqrt{2} & 2 \end{bmatrix}, \begin{bmatrix} a \\ b \\ -b \end{bmatrix}, \end{aligned}$$

i.e. (4.1a) is fulfilled. Moreover, it holds (4.1b):

$$\operatorname{rk}_{\mathbb{K}(s)} \begin{bmatrix} -sE + A & B \\ K & L \end{bmatrix} = \operatorname{rk}_{\mathbb{K}(s)} \begin{bmatrix} 1 & -s & 0 \\ 0 & 1 & 1 \\ -\sqrt{2} & \sqrt{2} - 1 & \sqrt{2} \end{bmatrix} = 3 = n + q.$$

Consequently, $(X, K, L)$ as in (4.45) is a solution of the Lur'e equation (4.1).

# 5 The linear-quadratic optimal control problem

In this chapter we merge the so far revised theory of [RRV15] to discuss the main application, namely the linear-quadratic optimal control problem. With help of a solution of the KYP inequality (3.2) we are able to introduce a lower bound for the optimal value function in Section 5.1. If in addition the Lur'e equation (4.1) is solvable, we can evaluate the cost functional for given solution trajectory and initial value. To characterize the existence and structure of solutions of the examined optimal control problem we require a stabilizing solution of (4.1). Section 5.2 is dedicated to illustrate the results by means of an example.

Throughout the whole chapter we consider the *linear-quadratic optimal control problem (OCP)*

$$\begin{aligned} \min_{(x,u)} \quad & \mathcal{J}(x,u) = \int_0^\infty \begin{pmatrix} x(\tau) \\ u(\tau) \end{pmatrix}^* \begin{bmatrix} Q & S \\ S^* & R \end{bmatrix} \begin{pmatrix} x(\tau) \\ u(\tau) \end{pmatrix} \mathrm{d}\tau \\ \text{s.t.} \quad & (x,u) \in \mathcal{B}_{[E,A,B]} \text{ with } Ex(0) = Ex_0 \text{ and } \lim_{t\to\infty} Ex(t) = 0 \end{aligned} \tag{5.1}$$

for a system $[E, A, B] \in \Sigma_{n,m}(\mathbb{K})$ and some weighting matrices $Q = Q^* \in \mathbb{K}^{n\times n}$, $S \in \mathbb{K}^{n\times m}$, $R = R^* \in \mathbb{K}^{m\times m}$.

To study (5.1), we consider the *optimal value function* $V^+ : \mathcal{V}_{\mathrm{diff}} \to \mathbb{R} \cup \{\pm\infty\}$ defined as

$$V^+(x_0) := \inf \left\{ \mathcal{J}(x,u) \;\middle|\; (x,u) \in \mathcal{B}_{[E,A,B]},\ Ex(0) = Ex_0 \text{ and } \lim_{t\to\infty} Ex(t) = 0 \right\}. \tag{5.2}$$

Note that one could analogously consider a cost functional for the negative time axis (cf. [Voi15, Sec. 3.8] ). We write $V^+$ to emphasize that only the values $(x(t), u(t))$ for $t \geq 0$ influence the cost functional in (5.1).
To guarantee

$$\left\{ (x,u) \in \mathcal{B}_{[E,A,B]} \;\middle|\; Ex(0) = Ex_0 \text{ and } \lim_{t\to\infty} Ex(t) = 0 \right\} \neq \emptyset$$

for a given initial value $x_0 \in \mathbb{K}^n$, we assume the system $[E, A, B]$ to be behaviourally stabilizable. Thus, we ensure that we do not infimize on the empty set and hence $V^+(x_0) < \infty$ holds for all $x_0 \in \mathcal{V}_{\mathrm{diff}}$. If in addition $V^+(x_0) > -\infty$ holds for all $x_0 \in \mathcal{V}_{\mathrm{diff}}$, we call the OCP (5.1) *feasible* for the system $[E, A, B]$.

## 5.1 Solution via Lur'e equations

To state a lower bound for the optimal value function $V^+$, we use a solution of the KYP inequality (3.2). The relationship between the solvability of the (3.2) and $V^+$ is content of Proposition 5.2, which is proved with help of the following lemma.

**Lemma 5.1.** *Consider $[E, A, B] \in \Sigma_{n,m}(\mathbb{K})$ with $\mathcal{V}_{\text{diff}} \subseteq \mathbb{K}^n$. Assume $P \in \mathbb{K}^{n\times n}$ is a solution of the KYP inequality (3.2). Then for all $(x, u) \in \mathcal{B}_{[E,A,B]}$ and $0 \le t_1 \le t_2$ it holds*

$$x(t_1)^* E^* P E x(t_1) - x(t_2)^* E^* P E x(t_2) \le \int_{t_1}^{t_2} \begin{pmatrix} x(\tau) \\ u(\tau) \end{pmatrix}^* \begin{bmatrix} Q & S \\ S^* & R \end{bmatrix} \begin{pmatrix} x(\tau) \\ u(\tau) \end{pmatrix} \mathrm{d}\tau.$$

**Proof:** Let $(x, u) \in \mathcal{B}_{[E,A,B]}$ and $0 \le t_1 \le t_2$. Since by definition of $\mathcal{V}_{\text{sys}}$ it holds that $\begin{pmatrix} x(t) \\ u(t) \end{pmatrix} \in \mathcal{V}_{\text{sys}}$ for almost all $t \in \mathbb{R}$, we obtain

$$\begin{aligned} & x(t_2)^* E^* P E x(t_2) - x(t_1)^* E^* P E x(t_1) \\ = \; & \int_{t_1}^{t_2} \tfrac{\mathrm{d}}{\mathrm{d}\tau}\, x(\tau)^* E^* P E x(\tau)\, \mathrm{d}\tau \\ = \; & \int_{t_1}^{t_2} x(\tau)^* E^* P \left(\tfrac{\mathrm{d}}{\mathrm{d}\tau}\, E x(\tau)\right)(\tau) + \left(\tfrac{\mathrm{d}}{\mathrm{d}\tau}\, E x(\tau)\right)^* P E x(\tau)\, \mathrm{d}\tau \\ = \; & \int_{t_1}^{t_2} x(\tau)^* E^* P [A x(\tau) + B u(\tau)] + [A x(\tau) + B u(\tau)]^* P E x(\tau)\, \mathrm{d}\tau \\ = \; & \int_{t_1}^{t_2} \begin{pmatrix} x(\tau) \\ u(\tau) \end{pmatrix}^* \begin{bmatrix} A^* P E + E^* P A & E^* P B \\ B^* P E & 0 \end{bmatrix} \begin{pmatrix} x(\tau) \\ u(\tau) \end{pmatrix} \mathrm{d}\tau \\ \overset{(3.2)}{\ge} & -\int_{t_1}^{t_2} \begin{pmatrix} x(\tau) \\ u(\tau) \end{pmatrix}^* \begin{bmatrix} Q & S \\ S^* & R \end{bmatrix} \begin{pmatrix} x(\tau) \\ u(\tau) \end{pmatrix} \mathrm{d}\tau. \end{aligned}$$

□

Thus, we achieve infiniteness of the optimal value function.

**Proposition 5.2.** *Consider $[E, A, B] \in \Sigma_{n,m}(\mathbb{K})$. If $P \in \mathbb{K}^{n\times n}$ is a solution of the KYP inequality (3.2), then the functional $V^+$ as in (5.2) fulfils*

$$x_0^* E^* P E x_0 \le V^+(x_0). \tag{5.3}$$

**Proof:** Let $P \in \mathbb{K}^{n\times n}$ be a solution of the KYP inequality (3.2) and $x_0 \in \mathcal{V}_{\text{diff}}$ and $(x, u) \in \mathcal{B}_{[E,A,B]}$ fulfil $Ex(0) = Ex_0$ and $\lim_{t\to\infty} Ex(t) = 0$. Thus, Lemma 5.1 provides

$$x_0^* E^* P E x_0 \le \int_0^{\infty} \begin{pmatrix} x(\tau) \\ u(\tau) \end{pmatrix}^* \begin{bmatrix} Q & S \\ S^* & R \end{bmatrix} \begin{pmatrix} x(\tau) \\ u(\tau) \end{pmatrix} \mathrm{d}\tau = J(x, u).$$

Due to the definition of $V^+$ we obtain (5.3). □

As solutions of the Lur'e equation (4.1) yield particular solutions of the KYP inequality (3.2), it seems obvious that they provide stronger properties in view of the OCP (5.1). Theorem 5.3 confirms this hypothesis. While a solution of the KYP inequality (3.2) only helps to estimate a lower bound for the optimal value function, a Lur'e solution provides an explicit characterization.

**Theorem 5.3.** *Consider* $[E, A, B] \in \Sigma_{n,m}(\mathbb{K})$. *Assume* $(X, K, L)$ *to be a solution of the Lur'e equation* (4.1). *Then for all* $x_0 \in \mathcal{V}_{\text{diff}}$ *and* $(x, u) \in \mathcal{B}_{[E,A,B]} \cap \mathcal{L}^2(\mathbb{R}, \mathbb{K}^{n+m})$ *with* $Ex(0) = Ex_0$ *and* $\lim_{t\to\infty} Ex(t) = 0$ *it holds that*

$$x_0^* E^* X E x_0 + \|Kx + Lu\|^2_{\mathcal{L}^2(\mathbb{R},\mathbb{K}^{n+m})} = \mathcal{J}(x, u). \tag{5.4}$$

**Proof:** Let $x_0 \in \mathcal{V}_{\text{diff}}$ and $(x, u) \in \mathcal{B}_{[E,A,B]} \cap \mathcal{L}^2(\mathbb{R}, \mathbb{K}^{n+m})$ with $Ex(0) = Ex_0$ and $\lim_{t\to\infty} Ex(t) = 0$. Similar to the calculations in the proof of Lemma 5.1 we get

$$\begin{aligned}
&x(t_2)^* E^* X E x(t_2) - x(t_1)^* E^* X E x(t_1) \\
=\;& \int_{t_1}^{t_2} \begin{pmatrix} x(\tau) \\ u(\tau) \end{pmatrix}^* \begin{bmatrix} A^*XE + E^*XA & E^*XB \\ B^*XE & 0 \end{bmatrix} \begin{pmatrix} x(\tau) \\ u(\tau) \end{pmatrix} \mathrm{d}\tau \\
\overset{(4.1a)}{=}& -\int_{t_1}^{t_2} \begin{pmatrix} x(\tau) \\ u(\tau) \end{pmatrix}^* \begin{bmatrix} Q & S \\ S^* & R \end{bmatrix} \begin{pmatrix} x(\tau) \\ u(\tau) \end{pmatrix} \mathrm{d}\tau + \int_{t_1}^{t_2} \begin{pmatrix} x(\tau) \\ u(\tau) \end{pmatrix}^* \begin{bmatrix} K^* \\ L^* \end{bmatrix} \begin{bmatrix} K & L \end{bmatrix} \begin{pmatrix} x(\tau) \\ u(\tau) \end{pmatrix} \mathrm{d}\tau.
\end{aligned}$$

Considering $t_1 = 0$ and $t_2 \to \infty$ we arrive at (5.4). □

As we want to minimize the value of the cost functional $\mathcal{J}$ for a given $x_0 \in \mathcal{V}_{\text{diff}}$, it suffices to minimize the second summand on the left hand side of (5.4). In [IR17, Theorem 6.6 (a)] it is shown that there exists a sequence of solution trajectories $(x, u)$ infimizing $\|Kx + Lu\|^2_{\mathcal{L}^2(\mathbb{R},\mathbb{K}^{n+m})}$.

**Theorem 5.4.** *Consider the behaviourally stabilizable system* $[E, A, B] \in \Sigma_{n,m}(\mathbb{K})$ *and a stabilizing solution* $(X, K, L)$ *of the Lur'e equation* (4.1). *Then for all* $\varepsilon > 0$ *and* $x_0 \in \mathcal{V}_{\text{diff}}$ *there exists some* $(x, u) \in \mathcal{B}_{[E,A,B]}$ *such that*

$$u \in \mathcal{L}^2(\mathbb{R}_{\geq 0}, \mathbb{K}^m), \quad \lim_{t\to\infty} Ex(t) = 0, \quad \textit{and} \quad \|Kx + Lu\|_{\mathcal{L}^2(\mathbb{R}_{\geq 0},\mathbb{K}^m)} < \varepsilon.$$

Hence, the optimal value function can be calculated.

**Theorem 5.5.** *Consider the behaviourally stabilizable system* $[E, A, B] \in \Sigma_{n,m}(\mathbb{K})$. *Let* $(X, K, L)$ *be a stabilizing solution of the Lur'e equation* (4.1). *Then the optimal value function* (5.2) *fulfils*

$$V^+(x_0) = x_0^* E^* X E x_0 \quad \forall\, x_0 \in \mathcal{V}_{\text{diff}}.$$

**Proof:** This statement is a direct consequence of Theorem 5.3 and Theorem 5.4. □

Theorem 5.5 provides the minimal costs for each $x_0 \in \mathcal{V}_{\text{diff}}$. Together with Theorem 4.20 and Theorem 5.4 the optimal solution $(x, u)$ of the OCP (5.1) can be characterized.

**Corollary 5.6.** *Let $[E, A, B] \in \Sigma_{n,m}(\mathbb{K})$ behaviourally stabilizable and weighting matrices $Q = Q^* \in \mathbb{K}^{n\times n}$, $S \in \mathbb{K}^{n\times m}$ and $R = R^* \in \mathbb{K}^{m\times m}$ be given. Further, let $(X, K, L) \in \mathbb{K}^{n\times n} \times \mathbb{K}^{q\times n} \times \mathbb{K}^{q\times m}$ be a stabilizing solution of the Lur'e equation* (4.1) *and the projector* $\Pi$ *defined as in* (4.5). *Then $(x, u) \in \mathcal{L}^2_{\mathrm{loc}}(\mathbb{R}, \mathbb{K}^n) \times \mathcal{L}^2_{\mathrm{loc}}(\mathbb{R}, \mathbb{K}^m)$ is a solution of the OCP* (5.1) *if, and only if, $(x, u)$ solves the differential-algebraic boundary value problem*

$$\begin{aligned} \tfrac{\mathrm{d}}{\mathrm{d}t}\left(\Pi E x(t)\right) &= Ax(t) + Bu(t) \qquad\qquad Ex(0) = Ex_0 \quad \lim_{t\to\infty} Ex(t) = 0 \\ 0 &= Kx(t) + Lu(t). \end{aligned}$$

## 5.2 Example

To illustrate our results of Section 5.1 we apply them to an example. Consider the cost functional $J$ as in (5.1) with weighting matrices

$$Q = I_4, \quad S = 0_{4\times 2}, \quad R = I_2,$$

the DAE system (2.11) with system matrices

$$E = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & -1 \\ 0 & 0 \end{bmatrix},$$

and the initial value $x_0 = (1, 2, 3, 4)^\top$, where $n = 4$ and $m = 2$. Note that the system is given in FEF (2.7), where $n_1 = 2$, $n_2 = 2$, and $n_3 = 0$, i.e. $W = T = I_4$ and $F = 0_{2\times 4}$.

At first, we show that the optimal value function $V^+$ is finite with help of the KYP inequality. Then, we determine a solution of Lur'e equation (4.1) and thereby calculate $V^+(x_0)$.

### Finiteness of the optimal value function

Choose $S_\infty = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}^\top$. Obviously, $\operatorname{im} S_\infty = \ker E$ holds true. Further, we have

$$\operatorname{rk}\begin{bmatrix} E & AS_\infty & B \end{bmatrix} = \operatorname{rk}\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix} = 4 = n,$$

i.e. the system $[E, A, B]$ is impulse controllable. Therefore, $x_0 \in \mathbb{R}^4 = \mathcal{V}_{\text{diff}}$ holds. Corollary 2.44 (d) yields

$$\mathcal{V}_{\text{sys}} = \begin{bmatrix} A & B \end{bmatrix}^{-1} E\,\mathbb{R}^4 = \left\{ \begin{pmatrix} x \\ u \end{pmatrix} \in \mathbb{K}^6 \;\middle|\; x_3 = u_2, x_4 = 0 \text{ and } x_1, x_2, u_1, u_2 \in \mathbb{R} \right\}.$$

Since

$$\operatorname{rk} \begin{bmatrix} \lambda E - A & B \end{bmatrix} = \operatorname{rk} \begin{bmatrix} \lambda & -1 & 0 & 0 & 1 & 0 \\ -1 & \lambda & 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 & 0 & -1 \\ 0 & 0 & 0 & -1 & 0 & 0 \end{bmatrix} = 4 = n$$

holds for all $\lambda \in \mathbb{C}$, the system $[E, A, B]$ is behaviourally controllable and thus behaviourally stabilizable. Further, the Popov function $\Phi$ is given by

$$\Phi(s) = \begin{bmatrix} 1 - \frac{1}{s^2-1} & 1 \\ 1 & 2 - \frac{1}{s^2-1} \end{bmatrix}.$$

Hence, $\Phi(\mathrm{i}\omega)$ is Hermitian for all $\omega \in \mathbb{R}$. Due to the minor criterion and

$$1 + \frac{1}{\omega^2+1} \geq 0 \quad \text{and} \quad \det(\Phi(\mathrm{i}\omega)) = \left(1 + \frac{1}{\omega^2+1}\right)\left(2 + \frac{1}{\omega^2+1}\right) - 1 \geq 0$$

the Popov function fulfils $\Phi(\mathrm{i}\omega) \geq 0$ for all $\omega \in \mathbb{R}$ with $\det(\mathrm{i}\omega E - A) \neq 0$. According to Theorem 3.2 (b2) there exists some $P = P^* \in \mathbb{R}^{4\times 4}$ such that the KYP inequality (3.2) holds. Hence, Proposition 5.2 yields that the cost functional $V^+$ fulfils

$$V^+(x_0) \geq x_0^* E^* P E x_0 \quad \forall\, x_0 \in \mathcal{V}_{\text{diff}}.$$

**Solution of the Lur'e equation**

The previous findings, i.e. behavioural stabilizability and positive semi-definiteness of the Popov function on the imaginary axis, together with Theorem 4.14 (b) provide the existence of a stabilizing solution of the Lur'e equation (4.1). To calculate a solution $(X, K, L)$ of (4.1) at first we determine a solution of the associated ODE Lur'e equation, which can be transformed into an algebraic Riccati equation.

Consider the system and weighting matrices partitioned according to the FEF (2.7). Then the ODE Lur'e equation

$$\begin{bmatrix} A_{11}^* X_{11} + X_{11} A_{11} + Q_{11} & X_{11} B_1 + S_1 - Q_{12} B_2 \\ B_1^* X_{11} + S_1^* - B_2^* Q_{12}^* & B_2^* Q_{22} B_2 - B_2^* S_2 - S_2^* B_2 + R \end{bmatrix} = \begin{bmatrix} K_1^* \\ L_1^* \end{bmatrix} \begin{bmatrix} K_1 & L_1 \end{bmatrix} \tag{5.5}$$

is equivalent to the algebraic Riccati equation

$$A_{11}^* X_{11} + X_{11} A_{11} - (X_{11} B_1 + \tilde{S})\, \tilde{R}^{-1} (X_{11} B_1 + \tilde{S})^* + Q_{11} = 0, \tag{5.6}$$

where

$$\tilde{S} = S_1 - Q_{12}B_2 = 0 \quad \text{and} \quad \tilde{R} = B_2^* Q_{22} B_2 - B_2^* S_2 - S_2^* B_2 + R = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}.$$

It can be easily verified that $X_{11} = \begin{bmatrix} 2.3479 & 2.2601 \\ 2.2601 & 2.9021 \end{bmatrix}$ solves the algebraic Riccati equation (5.6). Further, from the condition $L_1^* L_1 = \tilde{R}$ we infer

$$L_1 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \quad \text{and thus} \quad K_1 = \begin{bmatrix} 2.3495 & 2.2345 \\ 0 & 0.7260 \end{bmatrix}.$$

Since

$$\operatorname{rk} \begin{bmatrix} -\lambda I_{n_1} + A_{11} & B_1 \\ K_1 & L_1 \end{bmatrix} = \operatorname{rk} \begin{bmatrix} -\lambda & 1 & 1 & 0 \\ 1 & -\lambda & 0 & 1 \\ 2.3495 & 2.2345 & 1 & 1 \\ 0 & 0.7260 & 0 & 1 \end{bmatrix} = 4 = n_1 + q$$

holds for all $\lambda \in \overline{\mathbb{C}_+}$, the triple $(X_{11}, K_1, L_1)$ is a stabilizing solution of the ODE Lur'e equation (5.5). Therefore, Lemma 4.15 (b) yields that $(X, K, L)$, where

$$X = W^* \begin{bmatrix} X_{11} & 0 \\ 0 & 0 \end{bmatrix} W, \quad K = \begin{bmatrix} K_1 & 0 \end{bmatrix} T^{-1} - LF \quad \text{and} \quad L = L_1,$$

is a stabilizing solution of the Lur'e equation (4.1). Due to $W = T = I_4$ and $F = 0_{2\times 4}$ we get

$$X = \begin{bmatrix} 2.3479 & 2.2601 & 0 & 0 \\ 2.2601 & 2.9021 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad K = \begin{bmatrix} 2.3495 & 2.2345 & 0 & 0 \\ 0 & 0.7260 & 0 & 0 \end{bmatrix} \quad \text{and} \quad L = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}. \tag{5.7}$$

**Remark 5.7.** Note that regularity of $\tilde{R}$ is necessary to apply the transformation into the Riccati equation (5.6). Hence, this approach does not work in general. ◊

### Optimal costs and boundary value problem

The triple $(X, K, L)$ as in (5.7) is a stabilizing solution of the Lur'e equation (4.1). Hence, Theorem 5.5 yields that the optimal cost functional fulfils

$$V^*(x_0) = x_0^* E^* X E x_0 = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}^* \begin{bmatrix} 2.3479 & 2.2601 & 0 & 0 \\ 2.2601 & 2.9021 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix} = 22.9966.$$

Furthermore, Corollary 5.6 yields that $(x,u) \in \mathcal{L}^2_{\mathrm{loc}}(\mathbb{R},\mathbb{K}^4) \times \mathcal{L}^2_{\mathrm{loc}}(\mathbb{R},\mathbb{K}^2)$ is a solution of the OCP (5.1) if, and only if, $(x,u)$ solves the boundary value problem

$$\begin{aligned}
\dot{x}_1(t) &= x_2(t) + u_1(t) & x_1(0) &= 1, \quad \lim_{t\to\infty} x_1(t) = 0\\
\dot{x}_2(t) &= x_1(t) + u_2(t) & x_2(0) &= 2, \quad \lim_{t\to\infty} x_2(t) = 0\\
0 &= x_3(t) - u_2(t) & &\\
0 &= x_4(t) & &\\
0 &= 2.3495\, x_1(t) + 2.2345\, x_2(t) + u_1(t) + u_2(t) & &\\
0 &= 0.7260\, x_2(t) + u_2(t). & &
\end{aligned}$$

# 6 Summary and Outlook

In this thesis we elaborated the results of [RRV15]. A differential-algebraic version of the Kalman-Yakubovich-Popov (KYP) lemma was proved. It states the solvability of the matrix inequality as a sufficient criterion for positive semi-definiteness of the Popov function. If the differential-algebraic system fulfils certain system properties, necessity holds true.

Further, we studied the Lur'e equation. Solutions of this matrix equation are rank-minimizing solutions of the KYP inequality. Existence of a solution of the Lur'e equation is ensured if the considered system has no uncontrollable modes on the imaginary axis. Actually, under the condition that the system is behaviourally stabilizable, there even exist stabilizing solutions. If a solution of the Lur'e equation is stabilizable, this solution is an extremal solution of the KYP inequality. Furthermore, we related the existence of a solution of the Lur'e equation to properties of associated even matrix pencils and deflating subspaces. The proof of this theorem enabled us to explain, how deflating subspaces and even matrix pencils can be used to determine a solution of the Lur'e equation. The problem in implementing the given guideline is to find the required matrices fulfilling several rank conditions and deflating subspace properties. In [Rei11, Section 4] the author shows that such matrices can be constructed via the *even Kronecker canonical form* in the ODE case. It seems worth trying to transfer this approach to the DAE case.

Using solutions of the Lur'e equation and KYP inequality, we elaborated the results of [RRV15] on solving the linear-quadratic optimal control problem. We proved that solvability of the KYP inequality provides finiteness of the optimal value function. For an arbitrary system we obtained that the optimal cost functional can be rewritten by using a solution of the Lur'e equation. If further the solution is stabilizing, the minimal costs can be calculated without knowing the optimal solution. The optimal solution can be characterized via a differential boundary value problem. In Chapter 5 we only stated the existence of an infimizing sequence of solution trajectories. Hence, it is not clear, whether an optimal solution exists. This remains to be shown. [IR17] solved this problem for ODE systems. The authors believe that their approach to solve the ODE problem is appropriate to solve the optimal control problem for DAE systems.

Besides the presented approach [RRV15] there are further methods to handle the linear quadratic optimal control problem. For instance we quote [LMT13], where the considerations are based on projector analysis, and [Meh91], where the focus is laid on systems with index at most one. The advantage of the approach in [RRV15] is that neither impulse controllability has to be assumed nor do we need to make any assumptions on the index of the pencil. This benefit finds its application in solving the

Lur'e equation instead of the *generalized algebraic Riccati equation* (see [KTK99]). For the existence of a stabilizing solution of the generalized Riccati equation impulse controllability and a pencil with index at most one are required. To solve the optimal control problem Lur'e equations admit weaker solvability conditions: neither do the weighting matrices need to be positive semi-definite (see [Kur02]), nor does the system need to be impulse controllable (see [KTK99]).

# Nomenclature

Throughout the whole thesis we use the standard notation i, $\bar{\lambda}$, $A^*$, $A^{-*}$, $I_n$, $0_{m\times n}$, $\operatorname{rk} A$, $\ker A$, $\operatorname{im} A$, $\overline{S}$, $\emptyset$, $\frac{\mathrm{d}}{\mathrm{d}\,t}\,x(t)$ or equivalently $\dot{x}(t)$ and $x^{(k)}(t)$ for the imaginary unit, the complex conjugate of $\lambda \in \mathbb{C}$, the conjugate transpose of a complex matrix and its inverse, the identity matrix of size $n \times n$ and the zero matrix of size $m \times n$ (subscripts are omitted if clear from context), the rank of a matrix $A$, the kernel and the image of a linear map $A$, the closure of a set $S$, the empty set, the derivative of a differentiable function $x : \mathbb{R} \to \mathbb{K}^n$ at $t$ and the $k$th derivative of a $k$ times differentiable function $x : \mathbb{R} \to \mathbb{K}^n$. Further, the following abbreviations are used:

| | |
|---|---|
| $A^{-\mathbb{1}}$ ............... | the preimage of a linear map $A$, page 23 |
| $A^{-1}$ ................ | the inverse of a matrix $A$, page 23 |
| $A^{+}$ ................ | the right inverse of a matrix $A$, page 11 |
| $\mathcal{AC}_{\mathrm{loc}}(\mathbb{R},\mathbb{K}^n)$ ....... | the set of locally absolutely continuous functions $f : \mathbb{R} \to \mathbb{K}^n$, page 6 |
| $\mathcal{B}_{[E,A,B]}$ ............ | the solution behaviour of the system $[E, A, B]$, page 6 |
| $\mathbb{C}_+, \mathbb{C}_-$ ............ | the open sets of complex numbers with positive and negative real part, resp., page 15 |
| $\mathbb{C}_{>\alpha}$ ............... | the set of complex numbers $s$ with $\mathrm{Re}(s) > \alpha$ for $\alpha \in \mathbb{R}$, page 7 |
| $\mathcal{C}^1(\mathbb{R},\mathbb{K}^n)$ .......... | the space of continuously differentiable functions $f : \mathbb{R} \to \mathbb{K}^n$, page 6 |
| $\mathcal{D}_\alpha(\mathbb{R}_{\geq 0},\mathbb{K}^n)$ ....... | the set of all locally integrable functions $f : \mathbb{R}_{\geq 0} \to \mathbb{K}^n$ such that $\mathrm{e}^{-\alpha\cdot} f(\cdot)$ is (globally) integrable for $\alpha \in \mathbb{R}$, page 7 |
| $\operatorname{ess\,sup}_{t\in\mathbb{R}} \lVert f(t)\rVert$ ... | the essential supremum of $f : \mathbb{R} \to \mathbb{K}^{n+m}$ measurable, page 15 |
| $\mathrm{Gl}_k(\mathbb{K})$ ............ | the linear group of invertible $k \times k$ matrices with coefficients in $\mathbb{K}$, page 8 |
| $\operatorname{ind} sE - A$ ........ | the index of a regular matrix pencil $sE - A$, page 9 |
| $\mathcal{J}(x, u)$ ............ | the cost associated with a solution $(x, u) \in \mathcal{B}_{[E,A,B]}$, page 81 |
| $\mathbb{K}$ .................. | either the field $\mathbb{R}$ of real numbers or $\mathbb{C}$ of complex numbers, page 5 |
| $\mathbb{K}[s]$ ............... | the polynomial ring with coefficients in $\mathbb{K}$, page 7 |
| $\mathbb{K}[s]^{\ell\times n}$ ............ | the ring of matrices with coefficients in $\mathbb{K}[s]$, page 7 |
| $\mathbb{K}(s)$ ............... | the field of rational functions with coefficients in $\mathbb{K}$, page 7 |
| $\mathcal{L}(f)(s)$ ............ | the Laplace transformation of a function $f \in \mathcal{D}_\alpha(\mathbb{K}^n)$ at $s$, page 7 |
| $\mathcal{L}^2_{(\mathrm{loc})}(\mathbb{R},\mathbb{K}^n)$ ....... | the set of measurable and (locally) square integrable functions $f : \mathbb{R} \to \mathbb{K}^n$, resp., page 6 |

| | |
|---|---|
| $\mathbb{N}, \mathbb{N}_0$ | the set of natural numbers and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, resp., page 9 |
| $\operatorname{nil\,ind} M$ | the nilpotency index of a nilpotent matrix $M$, page 9 |
| $\Phi : \mathbb{C} \to \mathbb{K}(s)^{m\times m}$ | the Popov function, page 31 |
| $\mathcal{R}^{m\times n}$ | the set of $m \times n$ matrices with entries in a ring $\mathcal{R}$, page 5 |
| $\operatorname{Re}(s), \operatorname{Im}(s)$ | the real/imaginary part of a complex number $s$, resp., page 7 |
| $\operatorname{rk}_{\mathbb{K}(s)}$ | the rank of a matrix with respect to the field $\mathbb{K}(s)$, page 7 |
| $\mathbb{R}_{\geq 0}$ | the set of real numbers greater or equal 0, page 6 |
| $\Sigma_{n,m}(\mathbb{K})$ | the set of systems $[E, A, B]$ such that the pencil $sE - A$ is regular, page 14 |
| $\sigma(A)$ | the spectrum of a matrix $A$, page 53 |
| $\mathcal{V}_{\text{diff}}$ | the space of consistent initial differential variables, page 6 |
| $\mathcal{V}_{\text{sys}}$ | the system space, page 22 |
| $V^+(x_0)$ | the optimal value of the OCP (5.1) for a given initial value $x_0$, page 81 |
| $\mathcal{V}^\perp$ | the orthogonal complement of a linear sub,space $\mathcal{V}$, page 35 |
| $\mathcal{X} \oplus \mathcal{Y}$ | the direct sum of two linear subspaces $\mathcal{X}$ and $\mathcal{Y}$, page 35 |
| $\overset{W,T,F}{\simeq}_{fe}$ | feedback equivalence, page 10 |
| $\overset{W,T}{\simeq}_{se}$ | system equivalence, page 10 |
| $\simeq, \overset{W,T}{\simeq}$ | pencil equivalence, page 8 |
| $x \perp y$ | $x$ orthogonal $y$, page 35 |
| $=_\mathcal{V}, \geq_\mathcal{V}$ | relation on a subspace $\mathcal{V}$: $M =_\mathcal{V} N :\Leftrightarrow x^*Mx = x^*Nx \quad \forall\, x \in \mathcal{V}$, $M \geq_\mathcal{V} N :\Leftrightarrow x^*Mx \geq x^*Nx \quad \forall\, x \in \mathcal{V}$, resp., page 31 |
| $\overset{\text{a.e.}}{=}$ | equality almost everywhere, page 12 |

Moreover, the blockdiagonal matrix composed of $A_i \in \mathbb{K}^{m_i \times n_i}$ with $m_i$, $n_i \in \mathbb{N}_0$ for $i = 1, \dots, k$ is denoted by $\operatorname{diag}(A_1, \dots, A_k) \in \mathbb{K}^{m\times n}$, where $m = m_1 + \dots + m_k$ and $n = n_1 + \dots + n_k$.

# List of Acronyms

**DAE** .............. differential-algebraic equation

**FEF** .............. feedback equivalence form

**KYP** .............. Kalman-Yakubovich-Popov

**OCP** .............. optimal control problem

**ODE** .............. ordinary differential equation

**QWF** ............. quasi Weierstraß form

**SEF** ............... system equivalence form

# List of Figures

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# Co-Autor Statement & Eigenständigkeitserklärung

Die vorliegende Arbeit ist in Gruppenarbeit von Karen Höhler und Philipp Sauerteig angefertig worden.

Im Rahmen dieser Gemeinschaftsarbeit war **Karen Höhler** federführend für die Ausarbeitung des Kapitels 2, mit Ausnahme des Abschnitts 2.4, verantwortlich. Darüber hinaus stammt im Kapitel 4 die ausgearbeitete Erstfassung der Beweise von Theorem 4.14 und Lemma 4.15 von ihr. Einen wesentlichen Anteil trug sie zu den Beweisen von Theorem 3.2, Lemma 3.12, Proposition 3.4, Theorem 4.4, Theorem 4.20 bei, indem sie den jeweiligen Erstentwurf überarbeitete, ergänzte und strukturierte. Zudem gestaltete sie die in der Arbeit enthaltenen Grafiken.

**Philipp Sauerteig** übernahm maßgebend die Ausarbeitung von Kapitel 3. Dazu lieferte er insbesondere Erstfassungen zu den Beweisen von Theorem 3.2, Proposition 3.4 und Lemma 3.12. Zudem füllte er den Abschnitt 2.4 mit Inhalten. Zu Kapitel 4 trug er durch die Erarbeitung von Erstversionen der Beweise von Proposition 4.5, Theorem 4.18 und Theorem 4.4 wesentlich bei.

Die theoretische Ausarbeitung von Abschnitt 4.6 sowie die von Kapitel 5 erfolgte in sehr enger Zusammenarbeit. Ein passendes Beispiel für den Abschnitt 4.6 erarbeitete Karen Höhler, während das Beispiel im Abschnitt 5.2 von Philpp Sauerteig ausgearbeitet wurde. Alle nicht näher zugeordneten Inhalte sind in Zusammenarbeit entstanden und können keiner Einzelperson explizit zugewiesen werden.

Abgesehen von den oben genannten Gruppenmitgliedern haben keine weiteren Personen beim Verfassen dieser Arbeit mitgewirkt. Es wurden nur die angegebenen Quellen und Hilfsmittel verwendet. Alle von uns aus anderen Veröffentlichungen übernommenen Passagen sind als solche gekennzeichnet. Ferner versichern wir, dass die eingereichte Arbeit weder vollständig noch in wesentlichen Teilen Gegenstand eines anderen Prüfungsverfahrens gewesen ist.

Ilmenau, den 11. September 2017

Karen Höhler

Philipp Sauerteig