Koopman operator in dynamics & control: From error bounds to closed-loop guarantees

Karl Worthmann, Optimization-based Control, Institute of Mathematics, TU Ilmenau

Joint work with L. Bold, F.M. Philipp (TU Ilmenau, Germany)
L. Magni, I. Schimperna (U Pavia, Italy), M. Schaller (TU Chemnitz, Germany)

24th European Control Conference (ECC), Reykjavík, Iceland

Outline

Koopman operator in systems & control:

From finite-data error bounds to closed-loop guarantees

Model Predictive Control (MPC): predict quantities of interest along the flow

- Recap including sufficient stability conditions
- Surrogate model in the optimization step

  $\rightsquigarrow$ **Proportional error bounds** to preserve controllability properties

  $\rightsquigarrow$ MPC with **closed-loop guarantees**

Verification of assumptions

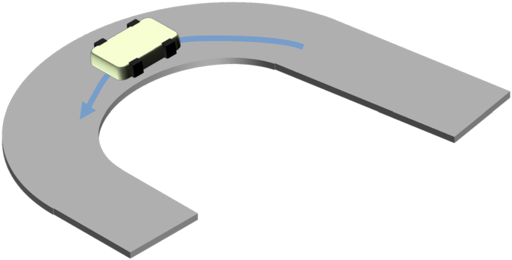
- **Koopman operator** and extended dynamic mode decompositon (EDMD)

  $\rightsquigarrow$ data-driven surrogate model
- Kernel EDMD $\rightsquigarrow$ L^∞**-bounds on the full approximation error**

Conclusions and outlook

R. Strässer, KW, I. Mezić, J. Berberich, M. Schaller, F. Allgöwer: *An overview of Koopman-based control: From error bounds to closed-loop guarantees*, **Annual Reviews in Control** 61:101035, 2026

Basic idea of model predictive control



Control based on repeated prediction and optimization

1. Obtain state measurement
2. Predict system and optimize input
3. Apply “optimal“ input value

Thanks to R. Findeisen, TU Darmstadt, Germany.

Model Predictive Control

Set-point stabilization: w.l.o.g. origin controlled equilibrium, i.e., $f(0,0) = 0$

- Control and state constraint sets $\mathbb{U} \subseteq \mathbb{R}^m$ and $\mathbb{X} \subseteq \mathbb{R}^n$

MPC scheme with prediction horizon $N \in \mathbb{N}_{\geq 2}$. Set time $n := 0$

1) Measure current state $\hat{x} := x(n)$
2) Minimize $J_N(\hat{x}, \mathbf{\bar{u}}) = \sum_{k=0}^{N-1} \|\bar{x}_{\mathbf{u}}(k;\hat{x})\|_Q^2 + \|\bar{u}(k)\|_R^2$ subject to
   - $\bar{x}_{\mathbf{u}}(k+1;\hat{x}) = f(\bar{x}_{\mathbf{u}}(k;\hat{x}), \bar{u}(k))$ with $\bar{x}_{\mathbf{u}}(0;\hat{x}) = \hat{x}$
   - $\bar{x}(k+1) \in \mathbb{X}$ and $\bar{u}(k) \in \mathbb{U}$ for $k \in \{0, 1, \ldots, N-1\}$

   $\rightsquigarrow$ optimal control sequence $\mathbf{\bar{u}}^\star = (\bar{u}^\star(k))_{k=0}^{N-1}$
3) Apply **first** input value $\bar{u}^\star(0)$

Static-state **feedback law** $\mu = \mu_N$ by $\mu_N(x(n)) := \bar{u}^\star(0)$

L. Grüne, J. Pannek: Nonlinear model predictive control: Theory and algorithms. Cham: Springer Int. Pub. 2016

J.B. Rawlings, D.Q. Mayne, M.M. Diehl: Model predictive control: theory, computation, and design, Nob Hill Pub. 2020

Stability of Model Predictive Control

Key stability criterion: relaxed Lyapunov inequality $\rightsquigarrow$ two sufficient conditions

- **Terminal conditions**, i.e., terminal region $\mathbb{X}_f \subseteq \mathbb{X}$ and cost $V_f : \mathbb{X}_f \to \mathbb{R}_{\geq 0}$, for which $\mu_f : \mathbb{X}_f \to \mathbb{U}$ satisfies *invariance* and *decrease* conditions

  $\rightsquigarrow$ I. Schimperna, L. Bold, J. Köhler, KW, L. Magni: **Wednesday (today) 5:10 pm, Uni 5**
  Stability of data-driven Koopman MPC with terminal conditions, arXiv:2511.21248

- **Cost controllability**, i.e., existence of sequence $(B_k)_{k=2}^{\infty}$ satisfying

$$V_k(\hat{x}) := \inf_{\mathbf{u} \in \mathcal{U}_k(\hat{x})} J_k(\hat{x}, \mathbf{u}) \leq B_k \|\hat{x}\|_Q^2 \qquad \forall\, k \in \{2, \ldots, N\}$$

with suboptimality degree $\alpha_N := 1 - \frac{(B_N - 1)\prod_{k=2} N(B_k - 1)}{\prod_{k=2}^{N} B_k - \prod_{k=2}^{N}(B_k - 1)} \in (0, 1]$

Assumption: system is cost controllable

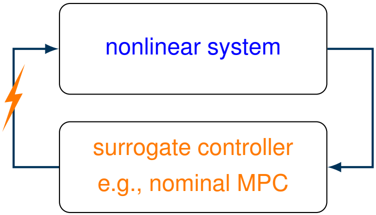
A. Boccia, L. Grüne, KW: *Stability and feasibility of state constrained MPC without stabilizing terminal constraints*, **Systems & Control Letters** 72:14-21, 2014

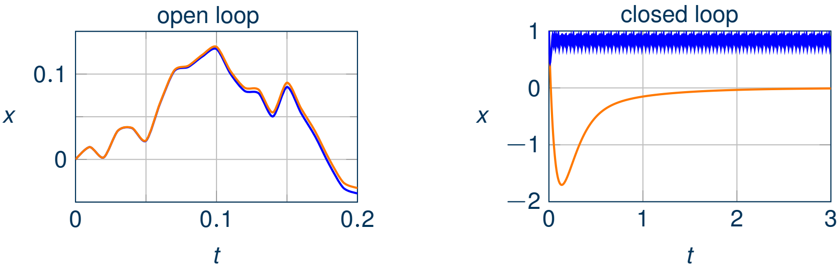
J.M. Coron, L. Grüne, KW: *Model predictive control, cost controllability, and homogeneity*, **SIAM J. Control and Optimization** 58(5):2979-2996

G. Grimm, M.J. Messina, S.E. Tuna, A.R. Teel: *Model predictive control: for want of a local control Lyapunov function, all is not lost*, **TACON** 50(5):546-558, 2005

L. Grüne, J. Pannek, M. Seehafer, KW: Analysis of unconstrained nonlinear MPC schemes with time varying control horizon, **SICON** 48(8):4938–4962, 2010

Challenge in Model Predictive Control





Thanks to R. Strässer, U Stuttgart, Germany.

Surrogate-model MPC: Asymptotic Stability

Suppose **cost controllability** on forward-invariant set S. Let surrogate f^ε, $\varepsilon \in (0, \bar{\varepsilon}]$, satisfy the following on a set Ω with $S \oplus \mathcal{B}_{\eta^\varepsilon \sum_{i=0}^{N-1} \bar{L}^i}(0) \subseteq \Omega$ for some $\bar{N}$:

- **proportional** and **uniform** error bound w.r.t. accuracy ε, i.e.,

$$\|f(x,u) - f^\varepsilon(x,u)\| \leq \min\{c_x^\varepsilon \|x\| + c_u^\varepsilon \|u\|, \eta^\varepsilon\} \qquad \forall\, x \in \Omega, u \in \mathbb{U}$$

  with $\lim_{\varepsilon \searrow 0} \max\{c_x^\varepsilon, c_u^\varepsilon, \eta^\varepsilon\} = 0$
- surrogate dynamics f^ε are L_{f^ε}-Lipschitz in x with $L_{f^\varepsilon} \leq \bar{L}$, uniform in u

Then, 1) $V_N^\varepsilon(\hat{x}) \leq B_N^\varepsilon \cdot \ell^\star(\hat{x})$ for all $\hat{x} \in S$ with $\lim_{\varepsilon \searrow 0} B_N^\varepsilon = B_N$ for $N \leq \bar{N}$

2) asymptotic stability w.r.t. MPC closed loop using surrogate f^ε, ε sufficiently small, in the optimization step + cost controllability w.r.t. f^ε instead of f

3) Results are symmetric w.r.t. f and f^ε

Verification of assumptions w.r.t. surrogate f^ε model using Koopman operator theory

- pointwise error bounds resulting from kernel-EDMD surrogates f^ε

L. Bold, L. Grüne, M. Schaller, KW: *Data-driven MPC with stability guarantees using extended dynamic mode decomposition*, **TACON** 70(1):534-541, 2025

I. Schimperna, KW, M. Schaller, L. Bold, L. Magni: *Data-driven model predictive control: Asymptotic stability despite approximation errors exemplified in the Koopman framework*, **Automatica**, to appear (arXiv:2505.05951)

Stability of MPC: some insight into the proofs

Uniform error bound, Lipschitz continuity, and relation between S and Ω

$\rightsquigarrow$ surrogate trajectories remain in the set Ω for admissible controls $\mathbf{u} \in \mathcal{U}_N(\hat{x})$

Lipschitz continuity, uniform in ε, and **proportional error bound**

$\rightsquigarrow$ growth bound is preserved for all $N \in \{1, \ldots, \bar{N}\}$

Determine prediction horizon $\bar{N}$ using cost controllability. Then, derive a relaxed Lyapunov inequality for V_N^ε along the MPC closed loop:

$$\begin{aligned} V_N^\varepsilon(f(\hat{x}, \mu_N^\varepsilon(\hat{x}))) &= V_N^\varepsilon(f(\hat{x}, \mu_N^\varepsilon(\hat{x}))) \pm V_N^\varepsilon(f^\varepsilon(\hat{x}, \mu_N^\varepsilon(\hat{x}))) \\ \leq \quad & V_N^\varepsilon(\hat{x}) - \alpha^\varepsilon \ell(\hat{x}, \mu_N^\varepsilon(\hat{x})) + V_N^\varepsilon(f(\hat{x}, \mu_N^\varepsilon(\hat{x}))) - V_N^\varepsilon(f^\varepsilon(\hat{x}, \mu_N^\varepsilon(\hat{x}))). \end{aligned}$$

Challenge: estimate the difference $V_N^\varepsilon(f(\cdot)) - V_N^\varepsilon(f^\varepsilon(\cdot))$ consistently w.r.t. the <u>quadratic state cost $\ell(x, u) = \|x\|$</u>$_Q^2 + \|u\|_R^2$

L. Bold, F.M. Philipp, M. Schaller, KW: Kernel-based Koopman approximants for control: Flexible sampling, error analysis, and stability, **SICON**, 2025

R.H. Moldenhauer, KW, R. Postoyan, D. Nešić, M. Granzotto: Discounted MPC and infinite-horizon optimal control under plant-model mismatch: Stability and suboptimality, arXiv:2604.08521

I. Schimperna, KW, M. Schaller, L. Bold, L. Magni: *Data-driven model predictive control: Asymptotic Stability despite Approximation Errors exemplified in the Koopman framework*, **Automatica**, to appear (arXiv:2505.05951)

Koopman-based prediction

Tutorial paper by I. Mezić, J. Cortés, KW, M. Lazar, A. Lederer:
Koopman operator theory: fundamentals, control, and applications, arXiv:2607.01819

Unknown nonlinear dynamics $x^+ = F(x)$ and observable $\varphi : \mathbb{R}^n \to \mathbb{R}$

$$\boxed{\text{initial state } x} \xrightarrow[\text{dynamics}]{} \boxed{F(x)} \xrightarrow[\text{evaluate}]{} \boxed{\varphi(F(x))}$$

Goal: Data-driven prediction of observable at successor state, i.e., $\varphi(F(x))$

- Linear Koopman operator $\mathcal{K}$ acting on (the space of) observables $\varphi : \mathbb{R}^n \to \mathbb{R}$ using observable data $(\varphi(x), \varphi(F(x)))$ only

$$\boxed{\text{observable } \varphi} \xrightarrow[\text{Koopman}]{} \boxed{\mathcal{K}\varphi} \xrightarrow[\text{evaluate}]{} \boxed{(\mathcal{K}\varphi)(x)}$$

Koopman identity: $(\mathcal{K}\varphi)(x) = \varphi(F(x))$ for all $x \in \mathbb{R}^n$

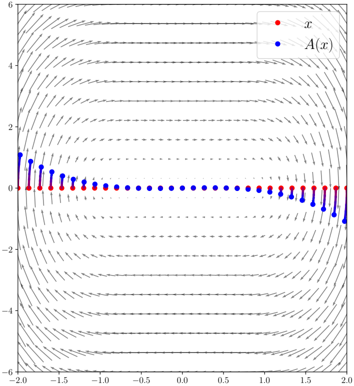
- Analogon for continuous-time systems: Koopman semigroup and generator

I. Mezić: Spectral properties of dynamical systems, model reduction and decompositions. **Nonlinear Dynamics** 41:309–325, 2005

S.L. Brunton, M. Budišić, E. Kaiser, J.N. Kutz: *Modern Koopman Theory for Dynamical Systems*, **SIAM Review** 64(2):229–340, 2022

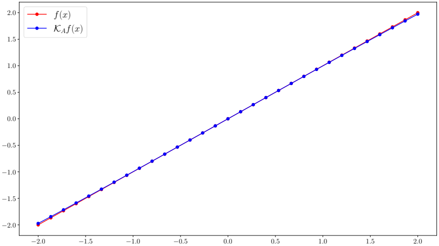
Example: Duffing Oscillator

Consider the ODE $\dot{x}(t) = \begin{pmatrix} x_2 \\ x_1 - 3x_1^3 \end{pmatrix}$. Let F be given by $\hat{x} \mapsto x(\Delta t, \hat{x})$.



On $X = [-2, 2]^2$, compute **observable** f of propagated initial values $\hat{x} = (\hat{x}_1, 0)$

- Example $f(x) = x_1$



Example: Duffing Oscillator

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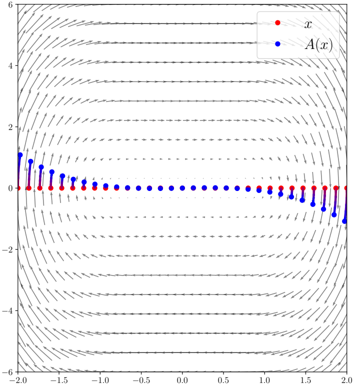
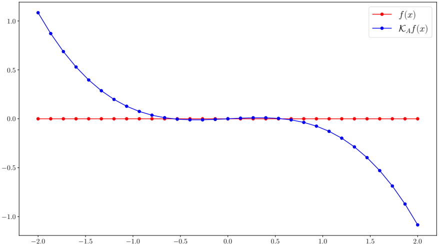


On $X = [-2, 2]^2$, compute **observable** f of propagated initial values $\hat{x} = (\hat{x}_1, 0)$

- Example $f(x) = x_2$



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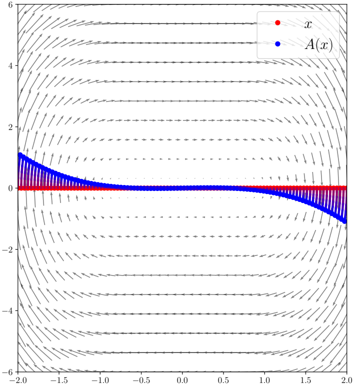
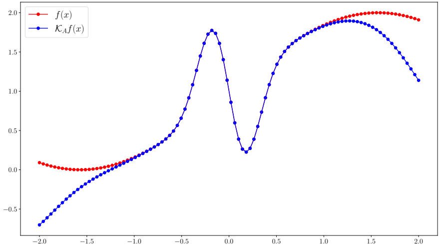
Example: Duffing Oscillator

Consider the ODE $\dot{x}(t) = \begin{pmatrix} x_2 \\ x_1 - 3x_1^3 \end{pmatrix}$. Let F be given by $\hat{x} \mapsto x(\Delta t, \hat{x})$.



On $X = [-2, 2]^2$, compute **observable** f of propagated initial values $\hat{x} = (\hat{x}_1, 0)$

- Example $f(x) = \ \ldots$



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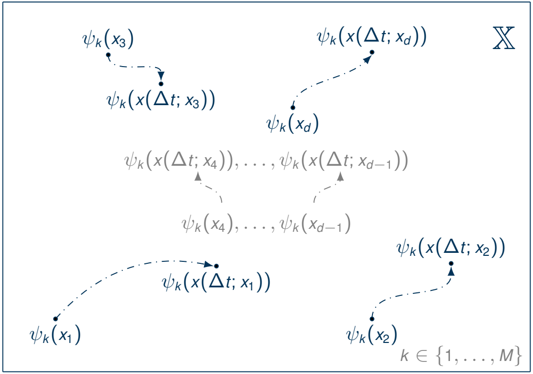
Extended Dynamic Mode Decomposition (EDMD)

Data matrices $X = (\Psi(x_1)|\cdots|\Psi(x_d))$, $Y = (\Psi(x(\Delta t, x_1))|\cdots|\Psi(x(\Delta t, x_d)))$



Observables generate

$$\mathbb{V} := \mathrm{span}(\{(\psi_k)_{k=1}^M\})$$

$$\Psi(x) = \begin{pmatrix} \psi_1(x) \\ \vdots \\ \psi_k(x) \\ \vdots \\ \psi_M(x) \end{pmatrix}$$

$\mathbb{X} \subset \mathbb{R}^n$, $\mathbb{X}$ compact

Estimation (regression problem): $\min_K \|KX - Y\|$ using data points $x_1, .., x_d$

- $\widehat{K}$ empirical estimator of the compression $P_{\mathbb{V}}\mathscr{K}^{\Delta t}|_{\mathbb{V}}$

M.O. Williams, I.G. Kevrekidis, C.W. Rowley: A data–driven approximation of the Koopman operator: EDMD. **J. Nonlinear Science** 25:1307-1346, 2015

M. Korda, I. Mezić: On convergence of extended dynamic mode decomposition to the Koopman operator. **J. Nonlinear Science** 28:687–710, 2018

EDMD: Probabilistic finite-data error bounds

Theorem (Estimation error). For system $\dot{x}(t) = f(x(t))$, observables ψ_k, $k \in \mathbb{Z}_{1:M}$,

- error bound $\varepsilon > 0$, and probabilistic tolerance $\delta \in (0, 1)$,

there is a number of data points $d_0 = \mathcal{O}\left(M^2/\delta\varepsilon^2\right)$ such that

$$\mathbb{P}\left(\|\mathscr{L}_{\mathbb{V}} - \tilde{\mathscr{L}}_d\|_F \leq \varepsilon\right) \geq 1 - \delta \qquad \forall\, d \geq d_0$$

for $\mathscr{L}_{\mathbb{V}} = P_{\mathbb{V}}\mathscr{L}|_{\mathbb{V}} := C^{-1}A$ with $C_{ij} = \langle \psi_i, \psi_j \rangle_{L^2_\mu(\mathbb{X})}$ and $A_{ij} = \langle \psi_i, \mathscr{L}\psi_j \rangle_{L^2_\mu(\mathbb{X})}$.

Analogous results for Koopman operator for a fixed time step Δt.

- I.i.d. and ergodic sampling, i.e., along trajectories for ergodic systems
- L^2-bound on the **projection error** using finite-elements dictionary

Choice of the dictionary $\rightsquigarrow$ eigenfunctions, deep learning

B. Lusch, J.N. Kutz, S.L. Brunton: *Deep learning for universal linear embeddings of nonlinear dynamics*, **Nature communications** 9:4950, 2018.

I. Mezić: *On numerical approximations of the Koopman operator*, **Mathematics** 10:1180, 2022

F. Nüske, S. Peitz, F.M. Philipp, M. Schaller, KW: *Finite-data error bounds for Koopman-based prediction and control*, **J. Nonlinear Science** 33:14, 2023

F.M. Philipp et al.: *Variance representations and convergence rates for data-driven approximations of Koopman operators*, **Physica D** 492:135223, 2026

C. Zhang, E. Zuazua: *A quantitative analysis of Koopman operator methods for system identification and predictions*, **Comptes Rendus. Mécanique** '23

Discrete- and continuous-time: extensions

Time-homogeneous discrete-time **Markov process** including **stochastic systems**

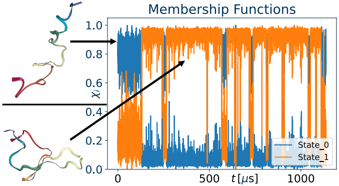
- Stochastic differential equations (SDE): $\mathrm{d}X_t = f(X_t)\,\mathrm{d}t + \sigma(X_t)\,\mathrm{d}W_t$, e.g., folding kinetics of a 35-amino acid protein (Molecular Dynamics)

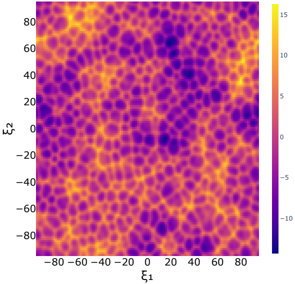
$$\left(\mathcal{K}^t \varphi\right)(\hat{x}) = \mathbb{E}\big(\varphi(X_t) \mid X_0 = \hat{x}\big)$$

- Discrete-time dynamical systems with noise: $x^+ = T(x) + \varepsilon$

and **deterministic systems**, including discrete-time ones,

- nonlinear PDEs, e.g., the Kuramoto-Sivashinsky eq. modelling the propagation of a chaotic flame front





F.M. Philipp, ..., KW: Variance representations and convergence rates [...] approximations [...], **Physica D: Nonlinear Phenomena** 492:135223, 2026

Consistency and reprojection

Koopman operator satisfies identity $(\mathcal{K}^t \circ \Psi)(\hat{x}) = \Psi(x(t;\hat{x}))$, $\hat{x} \in \mathbb{X}$, $t \geq 0$.
But, in general, $\nexists\, \bar{x} \in \mathbb{R}^n : \hat{K}\Psi(\hat{x}) = \Psi(\bar{x})$ for the compression $\hat{K} = P_{\mathbb{V}}\mathcal{K}^{\Delta t}|_{\mathbb{V}}$

P. van Goor, R. Mahony, M. Schaller, KW: *Reprojection methods for Koopman-based modelling and prediction*, 62**nd CDC 315-321, 2023**

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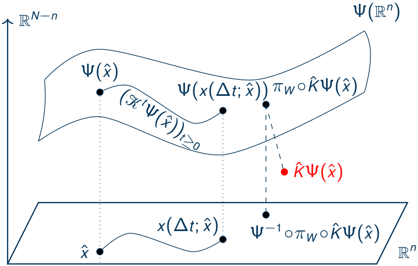
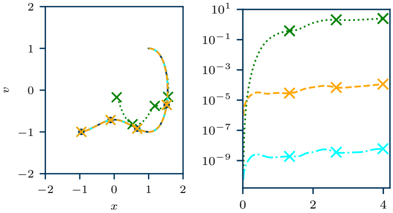




$$\ddot{x}(t) = x(t) - x(t)^3$$

Compression with $\mathbb{V}_5$ (· · ·), **coordinate reprojection** with $\mathbb{V}_3$ (- - -) and $\mathbb{V}_5$ (-·-·-)

- More elaborated reprojections based on differential-geometric arguments
- Deep neural network to learn invariant Koopman representations

P. van Goor, R. Mahony, M. Schaller, KW: *Reprojection methods for Koopman-based modelling and prediction*, 62**nd CDC 315-321, 2023**

Maximum-likelihood reprojections for reliable Koopman-based predictions and bifurcation analysis of parametric dynamical systems, arXiv:2506.17817

E. Yeung, S. Kundu, N. Hodas: *Learning deep neural network representations for Koopman operators of nonlinear dynamical systems*, **ACC** 2019

kEDMD approximants & uniform error bounds

RKHS: Hilbert space $\mathbb{H}$ such that the **reproducing property** $f(x) = \langle f, k(x, \cdot) \rangle$ holds for all $f \in \mathbb{H}$ with symmetric strictly positive definite kernel k on $\mathbb{X}$.

For $\mathcal{X} = \{x_1, \ldots, x_M\} \subset \mathbb{X}$, the feature space $V_{\mathcal{X}}$ is $\operatorname{span}(\{k_X(\cdot, x_i), x_i \in \mathcal{X}\})$

$\rightsquigarrow$ **kernel EDMD**: EDMD with data-informed dictionary: $\psi_k = k_X(\cdot, x_k)$

Uniform bounds on the approximation error by RKHS interpolation:

$$\begin{aligned}
\|\widehat{\mathcal{K}} - \mathcal{K}\|_{\mathcal{N}(Y) \to L^\infty(X)} &= \|P_{\mathcal{X}} \mathcal{K}(P_{\mathcal{Y}} - I) + (P_{\mathcal{X}} - I)\mathcal{K}\|_{\mathcal{N}(Y) \to L^\infty(X)} \\
&\leq C_1 h_{\mathcal{X}}^{k+1/2} + C_2 h_{\mathcal{Y}}^{k+1/2}
\end{aligned}$$

using orthogonal projections $P_{\mathcal{X}}$ and $P_{\mathcal{Y}}$ & Wendland kernels

- $\|P_{\mathcal{X}}\|_{L^\infty(X) \to L^\infty(X)} = 1$ (projection) and $\|\mathcal{K}\|_{L^\infty(X) \to L^\infty(X)} = 1$
- $\|P_{\mathcal{X}} f - f\|_{\mathcal{N}(X) \to L^\infty(X)} \to 0$ for orthog. proj. $P_{\mathcal{X}}$ to $V_{\mathcal{X}}$: approximation theory
- $\|\mathcal{K}\|_{\mathcal{N}(Y) \to \mathcal{N}(X)} < \infty$ results from inclusion $\mathcal{K}(\mathcal{N}(Y)) \subseteq \mathcal{N}(X) = H^{\sigma_{d,k}}(X)$

Proportional bounds $|(\widehat{\mathcal{K}} f)(x) - (\mathcal{K} f)(x)| \leq C h_{\mathcal{X}}^{k-1/2} \operatorname{dist}(x, \mathcal{X}) \|f\|_{\mathcal{N}_{\Phi_{n,k}}(\Omega)}$

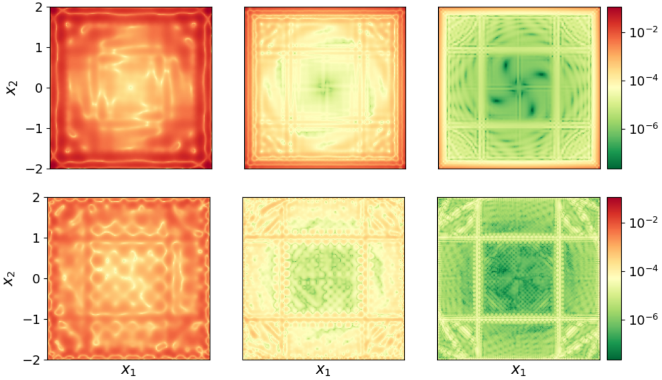
L. Bold, F.M. Philipp, M. Schaller, KW: *Kernel-based Koopman approximants for control*: [...], **SIAM J. Control and Optimization** 63(6):4044-407, 2025

F. Köhne, F.M. Philipp, M. Schaller, A. Schiela, KW: *L^∞-error bounds for approximations of the Koopman operator by kEDMD*, **SIADS** 24(1):501–529, 2025

R. Yadav, A. Mauroy: Approximation of the Koopman operator via Bernstein polynomials, **Communications in Nonlinear Science ...**, 147:108819, 2025

Kernel EDMD: simulation results

Prediction error of the kEDMD surrogate $\widehat{F} = \Upsilon\left(\Psi_{F(\mathcal{X})}^{\top}(\mathbf{K}_{\mathcal{X}} + \lambda I)^{-1}\mathbf{k}_{\mathcal{X}}(x)\right)$ for $x^{+} = F(x) := \frac{1}{8}[\mathrm{diag}(\|x\|^2 - 1) + \text{anti-diag}(-1\ 1)]x$ on a uniform/Padua grid



Comparable error bounds and simulation results for stochastic dynamics

M. Hertel, F.M. Philipp, M. Schaller, KW: *Koopman operator for stochastic dynamics: kernel approximants with uniform error bounds*, **SIADS**, arxiv:2512.20247

On Koopman operators for control systems

Thought experiment: Koopman-invariant subspace w.r.t. drift $\dot{x}(t) = f(x(t))$, i.e.,

$$A\Psi(x) := (\mathscr{L}\Psi)(x) = \dot{\Psi}(x(t)) = \left[\frac{\partial\Psi(x(t))}{\partial x}\right] f(x(t)) \in \operatorname{span}\{\Psi\}.$$

We consider nonlinear control system $\tilde{f} : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ given by

$$\dot{x} = \tilde{f}(x,u) = \tilde{f}(x,0) + [\tilde{f}(x,u) - \tilde{f}(x,0)] =: f(x) + G(x,u)$$

with $G(x,0) = 0$. Then, for $\Psi \in \mathscr{C}^1$, we have

$$\dot{\Psi}(x(t)) = A\Psi(x(t)) + \left[\frac{\partial\Psi(x(t))}{\partial x}\right] G(x(t),u(t)) =: A\Psi(x(t)) + \mathscr{B}(x(t),u(t)).$$

Rewritting $\mathscr{B}$ as $\int_0^1 \frac{\partial\mathscr{B}}{\partial u}(x(t),\lambda u(t))\,\mathrm{d}\lambda\, u(t) =: B(x(t),u(t))u(t)$ yields **LPV form** $B_z(p(t))u(t)$. Then, assuming **control affinity**, i.e., $G(x,u) = \sum_{i=1}^m g_i(x)u_i$, the state-dependent scheduling variable $p(t) = \mu(\Phi(x(t)))$ is independent of $u(t)$.

L.C. Iacob, R. Tóth, M. Schoukens: *Koopman form of nonlinear systems with inputs*, **Automatica** 162:111525, 2024

Data-based modeling for control-affine systems

Conclusion: **Bilinear surrogate** using control affinity of the Koopman generator

$$\dot{\psi} = \mathscr{L}^{u(t)}\psi \quad \text{with} \quad \mathscr{L}^{u(t)} = \mathscr{L}^0 + \sum\nolimits_{i=1}^{m} u_i(t)(\mathscr{L}^{e_i} - \mathscr{L}^0)$$

Linear surrogates (**EDMDc**) $\dot{\psi} = \mathscr{L}\psi + \mathscr{B}u$ have been successfully applied, e.g., LQR control. But have their defiencies in view of the previous derivation since the function B is, in general, state dependent, even for $\dot{x}(t) = f(x(t)) + Bu(t)$.

LPV form: choice of scheduling variable is key (poly. optimization, LFRs etc.)

Controlled **Koopman invariance** of the dictionary [Goswami, Paley '21]. Remedies:

- Reprojection based on measurements to enhance reliability (data consistency)
- Control-coherent Koopman modeling
- Kernel EDMD for control with Koopman-invariant RKHS and L^∞-bounds

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M. Korda, I. Mezić: *Linear predictors for nonlinear dynamical systems: Koopman operator meets model predictive control*, **Automatica** 93:149-160, 2018

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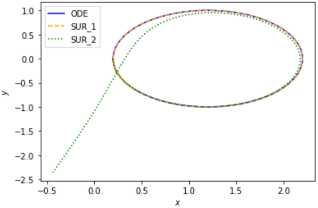
Example: non-holonomic robot

First-principles: kinematic model of the differential-drive robot

$$\dot{x}(t) = \frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} x_1(t) \\ x_2(t) \\ \theta(t) \end{pmatrix} = \begin{pmatrix} \cos(\theta(t)) \\ \sin(\theta(t)) \\ 0 \end{pmatrix} u_1(t) + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} u_2(t).$$

Bilinear surrogate model with reprojection



L. Bold, M. Rosenfelder, H. Eschmann, H. Ebel, KW: *On Koopman-based surrogate models for non-holonomic robots*, **IFAC-PapersOnLine**, 2024

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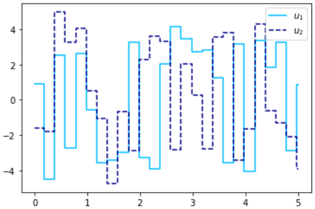
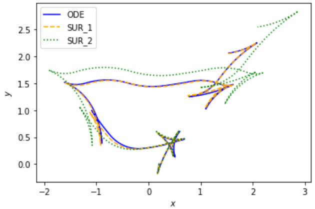




Experimental validation on a custom-build mobile robot ✓

L. Bold, M. Rosenfelder, H. Eschmann, H. Ebel, KW: *On Koopman-based surrogate models for non-holonomic robots*, **IFAC-PapersOnLine**, 2024

M. Rosenfelder, L. Bold, H. Eschmann, P. Eberhard, KW, H. Ebel: *Data-driven predictive control of nonholonomic robots based on a bilinear Koopman realization: Data does not replace geometry*, **Robotics and Autonomous Systems** 105156, 2025

Bilinear Koopman approximants for control

Discrete-time control-affine system $f(x,u) = g_0(x) + G(x)u$

- $\mathcal{O}(\Delta t^2)$-error if discretization of continuous-time system with zero-order hold

Flexible sampling: Clustering at *virtual observation points* x_j, $j \in \mathbb{Z}_{1:d}$, using **cluster size** $r_{\mathfrak{X}}$ to approximate $g_0(x_j), G(x_j)$

- $\bar{d}_j$ data points $(x_{jk}, u_{jk}, x_{jk}^+)$, $x_{jk} = f(x_{jk}, u_{jk})$, with $x_{jk} \in \mathcal{B}_{r_{\mathfrak{X}}}$, $k \in \mathbb{Z}_{1:\bar{d}_j}$
- *Exciting* inputs $u_0, u_1, \ldots, u_m, \ldots, u_{\bar{d}_j}$: NCO, scaling, subspace angles

Theorem (Proportional error bound): The kernel EDMD surrogate $\widehat{f}$ satisfies

$$\|f(x,u) - \widehat{f}(x,u)\|_\infty \leq c_x^\varepsilon \|x\| + c_u^\varepsilon \|u\|$$

with $c_u^\varepsilon = C_1 h_{\mathfrak{X}}^{k-1/2} \operatorname{dist}(x, \mathfrak{X}) + C_2 \|K_{\mathfrak{X}}^{-1}\| r_{\mathfrak{X}}$ and c_x^ε given by

$$c_x^\varepsilon := C_3 h_{\mathfrak{X}}^{k-1/2} + C_4 \left(\max_{j \in \mathbb{Z}_{1:\bar{d}_j}} \sqrt{2\bar{d}_j} \|V_j^\dagger\| \right) C(\mathfrak{X}) r_{\mathfrak{X}}$$

Conclusion: Data-driven surrogate $f^\varepsilon = \widehat{f}$ with adjustable *accuracy*

L. Bold, F.M. Philipp, M. Schaller, KW: Kernel-based Koopman approximants for control: Flexible sampling, error analysis ..., **SIAM J. Control Optim.**

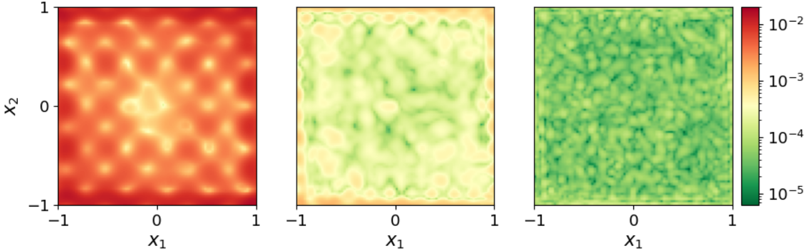
M. Schmitz, L. Bold, F.M. Philipp, M. Rosenfelder, P. Eberhard, H. Ebel, KW: On excitation of control-affine systems and its use for data-driven Koopman approximants, arXiv:2511.03734

Kernel EDMD for control: simulation results

Forward Euler with $\Delta t = 0.05$ for the duffing oscillator, i.e.,

$$x^+ = f(x,u) = x + \Delta t \begin{bmatrix} x_2 \\ x_1 - 3x_1^3 u \end{bmatrix} = \underbrace{\begin{bmatrix} x_1 + \Delta t x_2 \\ x_2 + \Delta t x_1 \end{bmatrix}}_{:=g_0(x)} + \underbrace{\begin{bmatrix} 0 \\ -\Delta t 3 x_1^3 \end{bmatrix}}_{:=G(x)} u.$$

Approximation error $\max_{j\in[1:20]} \|f(x,u_j) - \widehat{f}(x,u_j)\|$ for the kEDMD surrogate on $\Omega = [-2,2]^2$ with $d \in \{441, 1681, 6561\}$ cluster points using a Padua grid



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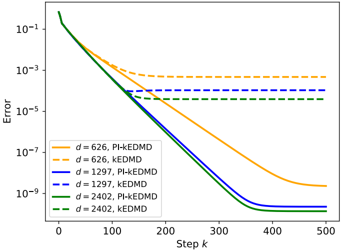
Numerical simulations: four-tank process

Euler discretization with $\Delta t = 10s$ of the ODE

$$\begin{pmatrix} \dot h_1 \\ \dot h_2 \\ \dot h_3 \\ \dot h_4 \end{pmatrix} = -\frac{\sqrt{2g}}{S} \begin{pmatrix} a_1\sqrt{h_1} + a_3\sqrt{h_3} \\ a_2\sqrt{h_2} + a_4\sqrt{h_4} \\ a_3\sqrt{h_3} \\ a_4\sqrt{h_4} \end{pmatrix} + \begin{bmatrix} \frac{\gamma_a}{S} & 0 \\ 0 & \frac{\gamma_b}{S} \\ 0 & \frac{1-\gamma_b}{S} \\ \frac{1-\gamma_a}{S} & 0 \end{bmatrix} \begin{pmatrix} q_a \\ q_b \end{pmatrix}.$$

Evolution of the MPC closed-loop error $\|x(k) - x^*\|$, $k \in \Delta t\mathbb{N}_0$, for the (shifted) controlled equilibrium (x^*, u^*), where kEDMD is leveraged in the optimization step.

- prediction horizon $N = 10$
- $S = [0.2, 1.36]^2 \times [0.2, 1.30]^2 m$
- $\mathbb{U} = [0, 3.26] \times [0, 4] m^3/h$
- weighting $Q = \mathrm{Id}$, $R = 10^{-4}\,\mathrm{Id}$
- uniform grid: $\sqrt[4]{d-1} \in \{5, 6, 7\}$



PI-kEDMD exploits $f^{\varepsilon}(x^*, u^*) = 0$.

I. Schimperna, KW, M. Schaller, L. Bold, L. Magni: *Data-driven MPC: Asymptotic Stability despite Approximation Errors ...*, **Automatica**, arXiv:2505.05951

Conclusion and outlook

Data-driven MPC with stability guarantees

- Finite-data error bounds **proportional** to *state* $\|x\|$ and *control* $\|u\|$

  $\rightsquigarrow$ data-based model predictive control with guarantees using terminal conditions or cost controllability

  $\rightsquigarrow$ data-driven controller design with certified region of attraction

Verification of all assumptions in the Koopman framework

- Kernel EDMD: L^∞**-bounds on full approximation error**
- Controller design using kernel EDMD with flexible sampling $\rightsquigarrow$ input-output data

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