1. Task

1.1. For two metal bars the angle of torsion is to be determined for different applied torques. The revealed relation is to be plotted and the shear modulus G is to be calculated from the slope of the linear fitting function.

1.2. The same metal bars as in 1.1 are used to form a torsion pendulum and the shear modulus is to be determined from its period of oscillation.

Literature:

Schenk, W.; Kremer, F. (Hrsg.) — Physikalisches Praktikum. Vieweg+Teubner Verlag | Springer Fachmedien Wiesbaden GmbH. 13., überarbeitete Auflage 2011, S. 66-68, S. 74-76

Walcher, W. — Praktikum der Physik. B. G. Teubner Stuttgart Leipzig Wiesbaden. 8. Auflage 2004, S. 64-66, S. 69-71

Stroppe, H. — Physik für Studenten der Natur- und Technikwissenschaften. Fachbuchverlag Leipzig im Carl Hanser Verlag. 11. Auflage 1999, S. 104-111

2. Background

Solid matter will be transformed under the influence of mechanical strains. If the strain is sufficiently small, the deformation is elastic, i.e. the object regains its original shape after relief of the strain. If a certain threshold strain is exceeded, plastic deformations take place and finally the material breaks or tears.

Within the elastic range of a body, there is a linear relationship between the applied stress and the resulting deformation which is called Hooke's law. The elastic behavior of homogenous, isotropic bodies is characterized by four material properties: the modulus of elasticity E, the shear modulus (also named torsional modulus) G, the modulus of compressibility K and the Poisson's ratio μ. Knowing two of these parameters (e.g. E and G), is sufficient to calculate the two others.

In the case of torsion, for an elastic deformation, the volume of the twisted object is maintained and parallel laminar layers slide past one another (see Fig. 1). The shear stress τ is thought of as a force $\vec{F}_S$ acting purely tangentially onto the cross-sectional area A (*generally infinitesimal*). One calculates τ formally as the norm of the shear force $\vec{F}_S$, related to the area increment A:

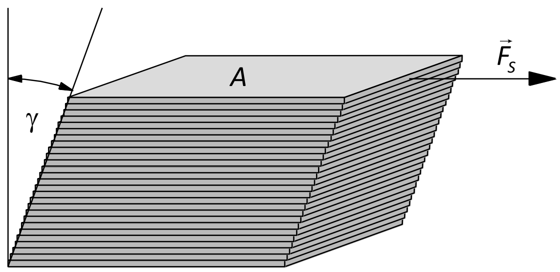

Fig. 1: Sketch for explanation of the shear modulus

$$\tau = \frac{\left|d\vec{F}_S\right|}{dA} . \tag{1}$$

Hooke's law, valid for small deformation angles γ, is in this case:

$$\tau = G\gamma \,. \qquad \textbf{(2)}$$

Characteristic values for the shear modulus G for metals are about several $10^{10}\, N/m^2$. For checking experimentally the relation in (2) and to measure G, the model described above will be applied to the **twisting of a bar** or a **„long wire"**, respectively (*i.e. in this case: the length l of the object under consideration is very large compared to its outer diameter 2R*).

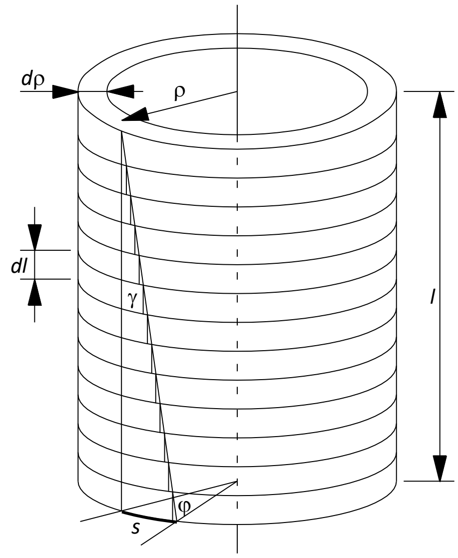

Fig. 2a: Stacking of rings

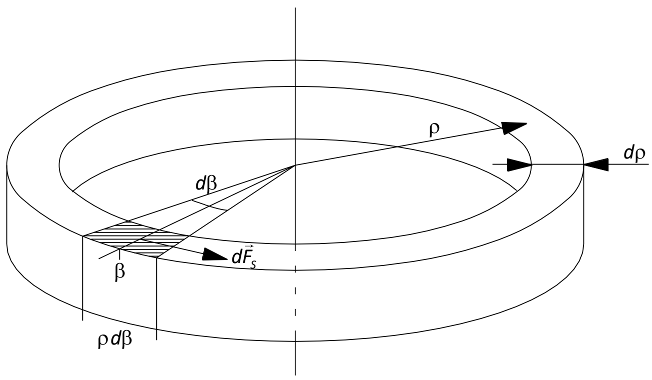

Fig. 2b: Single ring with infinitesimal section area

1. The bar is considered to consist of thin slices of infinitesimal height dl.
2. Each disc consists of an infinite number of rings spanning from the outside edge to the center of the plane. Each ring is characterized by its mean radius ρ and the infinitesimal thickness $d\rho$ (Fig. 2a).
3. Each ring consists of radial segments with a central angle β and the infinitesimal width $\rho d\beta$. One gets for the infinitesimal area increments (see Fig. 2b)

   $$dA = \rho\, d\beta\, d\rho \,. \qquad \textbf{(3)}$$

   Only area increments next to each other will be moved tangentially under the applied force $d\vec{F}_s$. The rings within a slice do not move relative to each other!
4. For a particular segment, characterized by radius ρ and angle β, an infinitesimal torque

   $$dM_{\rho\beta} = \rho\, dF_s \qquad \textbf{(4)}$$

   is acting with respect to the axis of symmetry.
5. In the case of no shear strain, all area increments lying on-top of each other have a particular envelope line. Under application of shear strain this line will be inclined by the angle γ.

6. According Fig. 2a the following relation applies

$$l\,\gamma = s = \rho\,\varphi\,. \tag{5}$$

From (1), (2) and (5) one obtains

$$dF_S = \frac{G\,\varphi}{l}\rho\,dA\,. \tag{6}$$

Insertion with (3) in (4) yields

$$dM_{\rho\beta} = \frac{G\,\varphi}{l}\rho^2\,\rho\,d\beta\,d\rho\,. \tag{7}$$

The infinitesimal torque of a whole ring is obtained by summing up (integration) over β from $0 \rightarrow 2\pi$, i.e.

$$dM_\rho = 2\pi\frac{G\,\varphi}{l}\rho^3\,d\rho\,. \tag{8}$$

After all, the following integration over ρ from $0 \rightarrow R$ yields the experimentally verifyable expression:

$$M = \frac{\pi\,G\,R^4}{2l}\,\varphi\,. \tag{9}$$

In the first part of the experiment, a <u>static approach</u> will be used to determine the shear modulus G of the metal bars. According to eq. (9) the linear dependence $M(\varphi)$ will be recorded, plotted and fitted with a linear function. The slope of the expected line is:

$$S = \frac{\pi\,G\,R^4}{2l} = \frac{\pi\,G\,d^4}{32l}\,. \tag{10}$$

The quantities to be measured are:
the suspended masses m, the radius R_S of the disc with the thread laid around to induce the static torque M, the rotation angle φ, as well as the length l and the diameter $d = 2R$ of the corresponding rod.

In the second part of the experiment, a <u>dynamic approach</u> will be applied to determine the shear modulus G of the same bars. Here, a mass fixed at the end of the bar will be rotated by the starting angle φ_0 and leads to a twisting of the bar. The resulting restoring dynamic torque M will lead to a periodic mechanic oscillation. Equation (9) is interpreted in the sense of the rotation of a solid body:

$$J_1\,\ddot{\varphi} = -M = -\frac{\pi\,G\,R^4}{2l}\,\varphi = -D\,\varphi\,. \tag{11}$$

Here, J_1 is the unknown moment of inertia of the mass fixed to the bar and D is the so-called torsion constant of the rod. It corresponds to the slope S from the first experimental part. Without consideration of damping, a reasonable assumption in this case, the period of oscillation of this torsion pendulum is:

$$T_1 = 2\pi\sqrt{\frac{J_1}{D}}\,. \qquad \textbf{(12)}$$

If an additional mass of known moment of inertia J^* attached is to the bar, it is $J_2 = J_1 + J^*$ and the period of oscillation is

$$T_2 = 2\pi\sqrt{\frac{J_2}{D}}\,. \qquad \textbf{(13)}$$

From (12) and (13) the unknown J_1 can be eliminated by substitution and one gets for D:

$$D = 4\pi^2 \frac{J^*}{T_2^2 - T_1^2} = \frac{\pi G R^4}{2l}\,. \qquad \textbf{(14)}$$

From this equation G can be derived to:

$$G = 128\pi \frac{l}{d^4} \cdot \frac{J^*}{T_2^2 - T_1^2}\,. \qquad \textbf{(15)}$$

One sees that the shear modulus (torsional modulus) G depends on the measurable quantities J^*, l, d, T_1 and T_2.

3. Experimental Procedure and Data Analysis

3.1. Static Method

Around the circumference of a disc with the radius R_S a thread is laid. Masses suspended at one end of the thread lead to the tangential force $F_G = m_{ges}\, g$ and therefore induce the torque $M = m_{ges}\, g\, R_S$. The angle of torsion φ can be read at the angular indexing of the disc in degrees. The length l and the diameter $d = 2R$ of the bar will measured with a ruler and a micrometer screw gauge. The diameter needs to be measured at minimum in 10 different positions of the bar. For calculation, the arithmetic average will be used.

φ will be read for 5 to 10 different masses m_{ges} in steps of $\Delta m = 100g$. There, φ_{max} must not exceed angles of 45°-50° due to the risk of permanent deformations of the bar. The following quantities are already known:

radius of the disc guiding the thread:	$R_S = (88{,}0 \pm 0{,}1)mm$
mass of the hook for suspension of masses:	$m = 100g$
smaller weights:	$m = 100g$
larger weights:	$m = 200g$

The given weight of the pieces of mass should be considered as nominal values and need to be specified by weighing!

In a prepared table one fills in the measured values m_{ges} and φ, as well as the calculated torque M and the angle of torsion expressed in radians. The graphical representation $M(\varphi)$ yields data points, which can be fitted by a linear function (not through origin) and from its slope S the shear modulus can be calculated according to eq. (10).

The length l and the diameter $d = 2R$ of the bar have uncertainties of Δl and Δd, which need to be determined. The uncertainty ΔS of the slope S of the regression line is indicated by the analysis software. From these, the combined uncertainty of the shear modulus needs to be calculated.

3.2. Dynamic Method

To determine the period T_1 one turns the disc by max. 45° and measures at least two times the overall time of 50 oscillations. If there are no large differences observed between the two values (given the expected error of measurement), their average $\overline{T}_1$ will be used for the following calculations.

The determination of period T_2 is done in the same way, but the additional mass in the shape of a flat hollow cylinder needs to be put onto the disc. For calculation of the moment of inertia, one uses

$$J^* = \frac{m_Z}{8}\left(d_a{}^2 + d_i{}^2\right). \qquad \textbf{(16)}$$

Here, the mass m_Z, the outer diameter d_a and the inner diameter d_i of the hollow cylinder need to be measured with a caliper. Finally, together with (15) one gets:

$$G = 16\pi \frac{m_Z\, l}{d^4} \cdot \frac{d_a{}^2 + d_i{}^2}{T_2^2 - T_1^2}. \qquad \textbf{(17)}$$

For calculation of the combined uncertainty, it is recommended to start with the relative uncertainties of the measured parameter. In particular, it is $\Delta d_a = \Delta d_i = \Delta d$ and $\Delta T_1 = \Delta T_2 = \Delta T$.
Hint: The calculation is much easier if one substitutes $d_a{}^2 + d_i^2 = u$ and $T_2^2 - T_1^2 = v$.

The determined values for the shear modulus G need to be compared regarding the different experimental approaches (static vs. dynamic) and also compared to the expected values for the anticipated metals.