AN INFINITE DIMENSIONAL DESCRIPTOR SYSTEM MODEL FOR ELECTRICAL CIRCUITS WITH TRANSMISSION LINES

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Abstract. In this paper a model of linear electrical circuits with transmission lines is derived. The equations obtained by the modified nodal analysis (MNA) are boundary-coupled with the telegraph equations who describe the behavior of the transmission lines. The resulting system of equations turns out to be an abstract differential-algebraic system and it is formulated as a descriptor system whose (generalized) state space is an infinite dimensional Hilbert space.

Key words. Analytic circuit theory, partial differential-algebraic equations, infinite dimensional linear system theory

Introduction. Nowadays, electrical circuits consist of a very large number ($\approx 10^7$) of components like resistors, capacitors, inductors, free and controlled voltage and current sources. Additionally, these circuits are operated in some higher frequency domains. As a consequence, several longer connections between components cannot be modelled as a short circuit anymore but rather as a transmission line (see [8]) The model of such a transmission line is shown in Figure 0.1.

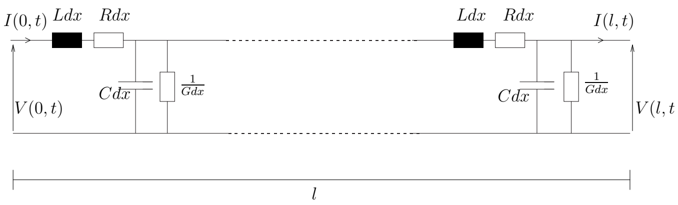

FIG. 0.1. *Transmission Line*

l is the length of the line and G, R, L, C are the conductivity, resistance, inductance and capacity per length unit. Due to the passivity of the line, G and R are assumed to be non-negative and L and C strictly positive. These components are continuously and homogeneously distributed along the line.
The current and the voltage along the transmission line fulfill the telegraph equations:

$$\frac{\partial}{\partial t}V(x,t) = -\frac{G}{C}V(x,t) - \frac{1}{C}\frac{\partial}{\partial x}I(x,t) \tag{0.1}$$

$$\frac{\partial}{\partial t}I(x,t) = -\frac{1}{L}\frac{\partial}{\partial x}V(x,t) - \frac{R}{L}I(x,t), \qquad x \in [0,l] \tag{0.2}$$

The aim of this work is to develop a model for electrical circuits containing several of these transmission lines. It is organized as follows: In the first section, the modified nodal analysis (MNA) for circuits with only lumped linear elements is presented as a descriptor system model. A state space model for the transmission lines is derived in the second section. The last section is about modelling linear circuits with lumped elements and transmission lines as a descriptor system while using the results of the first two sections.

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1. Circuits with Lumped Elements. The circuits are modelled as directed graph whose edges are weighted with the voltage-current relations of the electrical components (like e.g. Ohm's law). See e.g. [9], [7] or [4] for details.
Kirchhoff's current law (KCL) says that the net current outflow vanishes at any vertex of the graph, i.e.

$$A'i = 0,$$

where i is the vector containing the currents flowing through the circuit and A' is the reduced incidence matrix of the graph. According to *Kirchhoff's voltage law (KVL)*, every voltage can be derived from the node potential vector ϕ. In particular, we have

$$u = A'^T\phi.$$

u is the vector having the edge voltages as components.
One of the most powerful techniques in industrial circuit simulation is the so-called *modified nodal analysis (MNA)*. There, the relations of the electrical components are split into a part that can be solved for i and a part that can be solved for u. In the linear case, we have the form

$$\begin{pmatrix} I & P_1 \\ 0 & P_2 \end{pmatrix}\begin{pmatrix} i_1 \\ i_2 \end{pmatrix}\begin{pmatrix} Q_1 & 0 \\ Q_2 & I \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$$

for some coefficient matrices P_1, P_2, Q_1, Q_2. With the corresponding partition, we obtain from Kirchhoff's laws

$$A_1'i_1 + A_2'i_2 = 0 \quad \text{and} \quad \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} A_1'^T \\ A_2'^T \end{pmatrix}\phi.$$

This yields the MNA equations

$$\begin{pmatrix} A_1'Q_1A_1'^T & A_1'P_1 - A_2' \\ Q_2A_1'^T + A_2' & P_2 \end{pmatrix}\begin{pmatrix} \phi \\ i_2 \end{pmatrix} = \begin{pmatrix} A_1'c_1 \\ c_2 \end{pmatrix}. \tag{1.1}$$

Let now $A' = (A_RA_CA_LA_IA_V)$ be the incidence matrix generated by the graph of the given circuit. A_R, A_C, A_L, A_I, A_V contain the resistive, capacitive, inductive branches and those of the current and voltage sources, respectively. Let i_R, i_C, i_L, i_I and i_V the corresponding current vectors. Using the partition

$$i_1 = \begin{pmatrix} i_R \\ i_C \end{pmatrix}, \quad i_2 = \begin{pmatrix} i_L \\ i_I \\ i_V \end{pmatrix}$$

this leads to the following system of equations:

$$\begin{aligned} A_CCA_C^T\frac{\mathrm{d}}{\mathrm{dt}}\phi + A_RR^{-1}A_R^T\phi + A_Li_L + A_Vi_V + A_Ii_I &= 0 \\ A_L^T\phi - L\frac{\mathrm{d}}{\mathrm{dt}}i_L &= 0 \\ A_V^T\phi - u_V &= 0. \end{aligned}$$

C, R and L are diagonal matrices whose entries are the capacities, resistances and inductances.
The voltage sources are divided into the free and controlled ones, i.e.

$$u_V = \nu_VA'^T\phi + \nu_CCA_C^T\frac{\mathrm{d}}{\mathrm{dt}}\phi + \nu_Li_L + \nu_{IV}i_V + \nu_fu_f, \tag{1.2}$$

where u_f represents the free voltages. ν_V, ν_C, ν_L, ν_{IV}, ν_f are matrices which represent the amplifying gains of the controlled sources whose controlling variables are voltages and capacitive currents, inductive currents and currents of voltage sources. Since resistive currents depends algebraically on their voltages, the resistive-current-controlled voltage sources can be seen as voltage-controlled voltage sources.
In the same way, we have the relation

$$A_I i_I = A_{IV}\mu_V A'^T\phi + A_{IC}\mu_C C A_C^T \frac{\mathrm{d}}{\mathrm{dt}}\phi + A_{IL}\mu_L i_L + A_{Ii_V}\mu_{Ii_V} i_V + A_f\mu_f i_f, \quad (1.3)$$

where $A_I = (A_{IV} A_{IC} A_{IL} A_{Ii_V} A_f)$ and some amplifying gain matrices $\mu_V, \mu_C, \mu_L, \mu_{Ii_V}, \mu_f$.
Using the equations (1.2) and (1.3), we obtain the system $E\dot{x} = Ax + Bu$ with

$$x = \begin{pmatrix} \phi \\ i_L \\ i_V \end{pmatrix}, \quad u = \begin{pmatrix} u_f \\ i_f \end{pmatrix},$$

$$E = \begin{pmatrix} A_C C A_C^T + A_{IC}\mu_C A_C^T & 0 & 0 \\ 0 & L & 0 \\ \nu_C C A_C^T & 0 & 0 \end{pmatrix},$$

$$A = \begin{pmatrix} -A_R R^{-1} A_R^T - A_{IV}\mu_V A'^T & -A_L - A_{IL}\mu_L & -A_V - A_{Ii_V}\mu_{Ii_V} \\ A_L^T & 0 & 0 \\ A_V^T - \nu_V A'^T & -\nu_L & -\nu_{IV} \end{pmatrix},$$

$$B = \begin{pmatrix} 0 & -A_f \\ 0 & 0 \\ -\nu_f & 0 \end{pmatrix}.$$

Remark 1.1. *Differential-algebraic control systems of the form $E\dot{x} = Ax + Bu$ are called descriptor systems. They are e.g. treated in [2].*

2. The Transmission Lines. In this section a state space model for transmission lines is derived. Let the equations (0.1) and (0.2) be given. We always assume without loss of generality that the length of the line is normalized, i.e. $l = 1$. This can be done by a transformation of the parameters. For convenience, we assume that the parameters C, L, G and R are constant along the line.
As input, we choose $\begin{pmatrix} u_{T0}(t) \\ i_{T1}(t) \end{pmatrix} := \begin{pmatrix} V(0,t) \\ I(1,t) \end{pmatrix}$ and output of the system is supposed to be $\begin{pmatrix} u_{T1}(t) \\ i_{T0}(t) \end{pmatrix} := \begin{pmatrix} V(1,t) \\ I(0,t) \end{pmatrix}$.
The partial differential equations (0.1) and (0.2) can be formulated as an abstract ordinary differential equation in the function space $\mathcal{L}_2^2[0,1] = \left\{ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} : x_1, x_2 \in \mathcal{L}_2[0,1] \right\}$:

$$\frac{\mathrm{d}}{\mathrm{dt}} \begin{pmatrix} V(x,t) \\ I(x,t) \end{pmatrix} = \underbrace{\begin{pmatrix} -\frac{G}{C} & -\frac{1}{C}\frac{\partial}{\partial x} \\ -\frac{1}{L}\frac{\partial}{\partial x} & -\frac{R}{L} \end{pmatrix}}_{\mathfrak{A}} \begin{pmatrix} V(x,t) \\ I(x,t) \end{pmatrix} \quad (2.1)$$

with initial data

$$\begin{pmatrix} V(\cdot,0) \\ I(\cdot,0) \end{pmatrix} = \begin{pmatrix} V_0 \\ I_0 \end{pmatrix} \in \mathcal{L}_2^2[0,1]. \quad (2.2)$$

and boundary conditions

$$\begin{pmatrix} V(0,t) \\ I(1,t) \end{pmatrix} = \begin{pmatrix} u_{T0}(t) \\ i_{T1}(t) \end{pmatrix}. \tag{2.3}$$

The differential operator

$$\begin{aligned} \mathfrak{U} : D(\mathfrak{U}) \subset \mathcal{L}_2^2[0,1] &\to \mathcal{L}_2^2[0,1] \\ \begin{pmatrix} V \\ I \end{pmatrix} &\mapsto \begin{pmatrix} -\frac{G}{C}V - \frac{1}{C}\frac{\partial}{\partial x}I \\ -\frac{1}{L}\frac{\partial}{\partial x}V - \frac{R}{L}I \end{pmatrix} \end{aligned}$$

is unbounded and its domain is

$$D(\mathfrak{U}) = \left\{ \begin{pmatrix} V \\ I \end{pmatrix} \in \mathcal{L}_2^2[0,1] \text{ with } V, I \text{ are absolutely continuous, and } \frac{\partial}{\partial x}V, \frac{\partial}{\partial x}I \in \mathcal{L}_2[0,1] \right\},$$

which is dense in $\mathcal{L}_2^2[0,1]$.

Altogether the equations (2.1), (2.2) and (2.3) can be reformulated as an *abstract boundary control problem*(see [1], p.121 ff.)

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{dt}} \begin{pmatrix} V \\ I \end{pmatrix} &= \mathfrak{U} \begin{pmatrix} V \\ I \end{pmatrix} \\ \mathfrak{P}z(t) &= u(t), \end{aligned} \tag{2.4}$$

where $\mathfrak{P} : D(\mathfrak{P}) \to \mathbb{C}^2$, $\begin{pmatrix} V \\ I \end{pmatrix} \mapsto \begin{pmatrix} V(0) \\ I(1) \end{pmatrix}$ is the so-called *boundary operator* and it is defined on

$$D(\mathfrak{P}) = \left\{ \begin{pmatrix} V \\ I \end{pmatrix} \in \mathcal{L}_2^2[0,1] : V \text{ is continuous in } x = 0 \text{ and } I \text{ is continuous in } x = 1 \right\} \subset D(\mathfrak{U})$$

and has the kernel

$$\ker(\mathfrak{P}) = \left\{ \begin{pmatrix} V \\ I \end{pmatrix} \in D(\mathfrak{P}) : V(0) = I(1) = 0 \right\}$$

THEOREM 2.1. *The control system (2.4) is a boundary control system, i.e. the following assertions hold.*

1. *The operator $A : D(A) \to \mathcal{L}_2[0,1]^2$ with $D(A) = D(\mathfrak{P}) \cap D(\mathfrak{U})$ and*

$$Az = \mathfrak{U}z \quad \textit{for } z \in D(A) \tag{2.5}$$

is an infinitesimal generator of a C_0 semigroup on $\mathcal{L}_2^2[0,1]$.

2. *There exists a $B \in \mathcal{L}(\mathbb{C}^2, \mathcal{L}_2^2[0,1])$ such that for all $u \in \mathbb{C}^2$, $Bu \in D(\mathfrak{U})$, the operator $\mathfrak{U}B$ is an element of $\mathcal{L}(\mathbb{C}^2, \mathcal{L}_2^2[0,1])$ and*

$$\mathfrak{P}Bu = u, \quad u \in \mathbb{C}^2 \tag{2.6}$$

Proof.

1. The fact that A generates a C_0-semigroup on $\mathcal{L}_2^2[0,1]$ can be proven using the *Lumer-Philips-Theorem* [6].

2. A right inverse $B : \mathbb{C}^2 \to \mathcal{L}_2^2[0,1]$ of $\mathfrak{P}$ is e.g. given by

$$B\begin{pmatrix} V_1 \\ I_2 \end{pmatrix} = \begin{pmatrix} V_1 \cdot 1_{[0,1]} \\ I_2 \cdot 1_{[0,1]} \end{pmatrix},$$

whereas $1_{[0,1]}$ is the constant function mapping to 1.

□

As state of the system we take $z(t) = \begin{pmatrix} V(\cdot,t) \\ I(\cdot,t) \end{pmatrix} - Bu(t)$. For weakly differentiable inputs u, it satisfies the following abstract differential equation in $\mathcal{L}_2^2[0,1]$ (see [1]).

$$\begin{aligned} \dot{z}(t) &= Az(t) - B\dot{u}(t) + \mathfrak{U}Bu(t) \\ z_0 &= \begin{pmatrix} V(0,\cdot) \\ I(0,\cdot) \end{pmatrix} - Bu(0). \end{aligned} \tag{2.7}$$

Since A generates a C_0-semigroup, the unique solvability of the the initial value problem (2.7) is guaranteed.
The output y is then determined by

$$y(t) = Cz(t) + Du(t) \tag{2.8}$$

with $C\begin{pmatrix} z_1(\cdot) \\ z_2(\cdot) \end{pmatrix} = \begin{pmatrix} z_1(1) \\ z_2(0) \end{pmatrix}$ and $D = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

Remark 2.1. *$C : \mathcal{L}_2^2[0,1] \to \mathbb{C}^2$ is the output operator and not a capacity. In the rest of this paper it will be always clear from the context which of both is meant.*
The equations 2.7 and 2.8 can also be rewritten in the following way:

$$\begin{aligned} \dot{z} &= Az - \tfrac{\mathrm{d}}{\mathrm{dt}} B_{T10} u_{T0} + B_{T11} u_{T0} - \tfrac{\mathrm{d}}{\mathrm{dt}} B_{T21} i_{T1} + B_{T21} I_{T1} \\ u_{T1} &= C_{T1} z + u_{T0} \\ i_{T0} &= C_{T0} z + i_{T1}, \end{aligned} \tag{2.9}$$

with $B_{T10} = \begin{pmatrix} 1_{[0,1]} \\ 0 \end{pmatrix}$, $B_{T20} = \begin{pmatrix} 0 \\ 1_{[0,1]} \end{pmatrix}$, $B_{T11} = \begin{pmatrix} -\frac{G}{C} 1_{[0,1]} \\ 0 \end{pmatrix}$, $B_{T21} = \begin{pmatrix} 0 \\ -\frac{R}{L} 1_{[0,1]} \end{pmatrix}$, $C_{T1}\begin{pmatrix} V \\ I \end{pmatrix} = V(1)$ and $C_{T0}\begin{pmatrix} V \\ I \end{pmatrix} = I(0)$.

3. The Setup of the Network Equations.

3. The Setup of the Network Equations. Let $A' = (A_R A_C A_L A_V A_{T0} A_{T1} A_I)$ be the reduced incidence matrix of the circuit. A_{T0} and A_{T1} include the branches of the left and right boundaries of the transmission lines and i_{T0}, i_{T1} are the corresponding current vectors.
The MNA equations read then

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{dt}} A_C C A_C^T \phi + A_R R^{-1} A_R^T \phi & \\ + A_L i_L + A_V i_V + A_{T0} i_{T0} + A_{T1} i_{T1} + A_I i_I &= 0 \end{aligned} \tag{3.1}$$

$$A_L^T \phi - L \frac{\mathrm{d}}{\mathrm{dt}} i_L = 0 \tag{3.2}$$

$$A_V^T \phi - u_V = 0. \tag{3.3}$$

Now let the circuit have n_T transmission lines which are modelled as in the previous section with

$$\begin{array}{rcl} \dot{z}_i &=& A_i z - \frac{\mathrm{d}}{\mathrm{dt}} B_{T10i} u_{T0i} + B_{T11i} u_{T0i} - \frac{\mathrm{d}}{\mathrm{dt}} B_{T21i} i_{T1i} + B_{T21i} i_{T1i} \\ u_{T1i} &=& C_{T1i} z + u_{T0i} \\ i_{T0i} &=& C_{T0i} z + i_{T1i} \qquad \text{for } i = 1 \dots n. \end{array}$$

We define

$$\begin{aligned} A_T &:= \mathrm{diag}(A_1, \dots, A_{n_T}), \\ B_{T10} &:= \mathrm{diag}(B_{101}, \dots, B_{10n_T}), \\ B_{T11} &:= \mathrm{diag}(B_{111}, \dots, B_{11n_T}), \\ B_{T20} &:= \mathrm{diag}(B_{201}, \dots, B_{20n_T}), \\ B_{T21} &:= \mathrm{diag}(B_{211}, \dots, B_{21n_T}), \\ C_{T0} &:= \mathrm{diag}(C_{T01}, \dots, C_{T0n_T}), \\ C_{T1} &:= \mathrm{diag}(C_{T11}, \dots, C_{T1n_T}) \text{ and} \\ z &:= \begin{pmatrix} z_1 \\ \vdots \\ z_{n_T} \end{pmatrix}. \end{aligned}$$

Then we have

$$\frac{\mathrm{d}}{\mathrm{dt}} z - A_T z - \frac{\mathrm{d}}{\mathrm{dt}} B_{T10} A_{T0}^T \phi + B_{T20} A_{T0}^T \phi - \frac{\mathrm{d}}{\mathrm{dt}} B_{T11} i_{T1} + B_{T21} i_{T1} = 0 \qquad (3.4)$$

$$C_{T1} z + u_{T0} - u_{T1} = 0 \qquad (3.5)$$

$$C_{T0} z + i_{T1} - i_{T0} = 0. \qquad (3.6)$$

The equations for the controlled sources are

$$\begin{aligned} u_V &= \nu_V A'^T \phi + \nu_C C A_C^T \frac{\mathrm{d}}{\mathrm{dt}} \phi + \nu_L i_L + \nu_{IV} i_V + \nu_{T0} i_{T0} + \nu_{T1} i_{T1} + \nu_f u_f, \\ A_I i_I &= A_{IV} \mu_V A'^T \phi + A_{IC} \mu_C C A_C^T \frac{\mathrm{d}}{\mathrm{dt}} \phi + A_{IL} \mu_L i_L + A_{Ii_V} \mu_{i_V} i_V \\ &\quad + A_{IT0} \mu_{T0} i_{T0} + A_{IT1} \mu_{T1} i_{T1} + A_{If} \mu_f i_f. \end{aligned}$$

Plugging these two equations and the relation (3.6) into the equations (3.1) and (3.3), we obtain the following differential-algebraic system:

$$E\dot{x} = Ax + Bu,$$

with

$$x = \begin{pmatrix} \phi \\ i_L \\ i_V \\ i_{T1} \\ z \end{pmatrix}, \quad u = \begin{pmatrix} u_f \\ i_f \end{pmatrix}.$$

The generalized state space of this system is then $X = \mathbb{R}^{n_\phi + n_L + n_V + n_T} \times \mathcal{L}_2[0,1]^{2n_T}$, where n_ϕ, n_L, n_V, n_T are the the numbers of nodes, inductances, voltage sources and transmission lines.

The operators E, A and B are given by

$$E = \begin{pmatrix} A_C C A_C^T + A_{IC}\mu_C C A_C^T & 0 & 0 & 0 & 0 \\ 0 & L & 0 & 0 & 0 \\ \nu_C C A_C^T & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ B_{T10} A_{T0}^T & 0 & 0 & B_{T11} & I \end{pmatrix}, \quad B = \begin{pmatrix} 0 & -A_{I_f}\mu_f \\ 0 & 0 \\ -\nu_f & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$$A = \begin{pmatrix} -A_R R^{-1} A_R^T - A_{IV}\mu_V A'^T & -A_L - A_L\mu_L & -A_V - A_{Ii_V}\mu_{Ii_V} & -A_{T0} - A_{T1} - A_{IT0}\mu_{IT0} - A_{IT1}\mu_{IT1} & -A_{IT1}\mu_{IT1} C_{T0} - A_{T0} C_{T0} \\ A_L^T & 0 & 0 & 0 & 0 \\ A_V^T - \nu_V A'^T & -\nu_L & -\nu_{IV} & -\nu_{T0} - \nu_{T1} & \nu_{T0} C_{T0} \\ A_{T0}^T - A_{T1}^T & 0 & 0 & 0 & C_{T1} \\ B_{T20} A_{T0}^T & 0 & 0 & B_{T21} & A_T \end{pmatrix}$$

with $E : X \to X$, $A : D(A) = \mathbb{R}^{n_\phi + n_L + n_V + n_T} \times D(A_1)^{n_T} \subset X \to X$, $B : \mathbb{R}^{n_{V_f} + n_{I_f}} \to X$, where n_{V_f} and n_{I_f} are the numbers of free voltage and current sources.

Conclusion. In this work, a generalized state space model for circuits with transmission lines and lumped linear elements was derived. The state space turned out to be an infinite dimensional Hilbert space. Such systems are also known as abstract differential-algebraic systems (ADAS) and are treated e.g. in [5]. There typical problems of differential algebraic equations like index concepts and consistent initialization are generalized to the infinite dimensional case and can be applied to the system presented in this paper.

Acknowledgment. This work was supported by the DFG-Gaduiertenkolleg Mathematik und Praxis, Universität Kaiserslautern.

REFERENCES

[1] Curtain, R.F., Zwart, H.J., *An Introduction to Infinite-Dimensional Linear Systems Theory*, Springer (1995)

[2] Dai, , *Singular Control Systems*, Springer (1988)

[3] Estévez Schwarz, D., Tischendorf, C., *Structural analysis of electric circuits and consequences for MNA*, Int. J. Circuit Theory Appl. 28, No.2, 131-162 (2000)

[4] Hennig, E., Sommer, R., *Rechnergestützte Berechnung elektrischer Schaltungen* Vorlesungsmanuskript Universität Kaiserslautern, Fachbereich Elektrotechnik, (1998)

[5] Lamour, R., Maerz, R., Tischendorf, C., *PDAEs and Further Mixed Systems as Abstract Differential Algebraic Systems* Preprint, Humboldt Universit"at zu Berlin (2001)

[6] Renardy, M., Rogers R.C., *An Introduction to Partial Differential Equations, Springer*,(1992)

[7] Reis, T., *Model Reduction of Multidimensional Systems- With Application to Electrical Networks*, Diploma Thesis, University of Kaiserslautern (2002)

[8] Unger, H.G.., *Theorie der Leitungen*, Vieweg (1967)

[9] Zerz, E., *Topics in Multidimensional Linear Systems Theory*, Springer (2000)