Digital redesign of nonlinear multi-input systems

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(joint work with Dragan Nešić, Karl Worthmann)

1. Introduction

At the Oberwolfach Control Theory Meeting 2005 I presented the following open problem:

Consider a single input control affine closed loop system

$$\dot{x}(t) = g_0(x(t)) + g_1(x(t))u(x(t)) \tag{1}$$

with $x \in \mathbb{R}^n$ and a smooth feedback controller $u : \mathbb{R}^n \to \mathbb{R}$ and the corresponding sampled–data system

$$\dot{x}_T(t) = g_0(x_T(t)) + g_1(x_T(t))u_T(x_T(iT))), \quad t \in [iT, (i+1)T),\ i = 0, 1, \ldots \tag{2}$$

with a family of sampled-data controllers $u_T : \mathbb{R}^n \to \mathbb{R}$ parameterized with the (sufficiently small) sampling rate $T > 0$ which are locally bounded uniformly in T but not necessarily continuous. We consider the mismatch after one time step given by

$$\Delta_T(x_0) := \|x(T, x_0, u) - x_T(T, x_0, u_T)\|,$$

with $x(t, x_0, u)$ and $x_T(t, x_0, u_T)$ denoting the solutions of (1) and (2), respectively, with initial value x_0 at time $t = 0$.

It is easy to prove that for $u_T \equiv u$ we obtain $\Delta_T = O(T^2)$[1] while for

$$u_T(x) = u(x) + \frac{T}{2}\frac{\partial u(x)}{\partial x}[g_0(x) + g_1(x)u(x)] \tag{3}$$

we obtain $\Delta_T = O(T^3)$ (this follows from [4, Theorem 4.11] setting $V(x) = x_i$ observing that positive definiteness of V is not needed). Remark 4.12 in [4] suggests that higher order cannot be obtained in general.

Problem: Find conditions on g_0, g_1, u under which $\Delta_T \le O(T^4)$ can be achieved.

In this report a solution to the problem and an extension to multi-input systems will be presented. In the talk, we will in addition discuss performance issues and present a novel numerical optimization approach based on these results.

2. Single-Input systems

We use the following notation: for two vector fields $f, g : \mathbb{R}^n \to \mathbb{R}^n$ we define the usual Lie bracket by $[f, g] = \frac{d}{dx}g \cdot f - \frac{d}{dx}f \cdot g$. Furthermore, for $k \in \mathbb{N}$ we define

$$u^k(x_0) := \left.\frac{d^k}{dt^k}\right|_{t=0} u(x(t, x_0, u)). \tag{4}$$

[1] $\Delta_T = O(T^m)$ means: for each compact $K \subset \mathbb{R}^n$ there is $C > 0$ with $\sup_{x\in K} \Delta_T(x) \le CT^m$

Note that with this notation (3) can be written as

$$u_T(x) = u(x) + \frac{T}{2}u^1(x).$$

Theorem 2.1: A feedback law u_T with $\Delta_T = O(T^4)$ exists if and only if there exists a bounded function $\alpha : \mathbb{R}^n \to \mathbb{R}$ satisfying

$$[g_0, g_1](x)u^1(x) = \alpha(x)g_1(x). \tag{5}$$

If this condition holds, then the feedback laws u_T are given by

$$u_T(x) = u(x) + \frac{T}{2}u^1(x) + \frac{T^2}{6}u^2(x) + \frac{T^2}{12}\alpha(x)$$

and these u_T are uniquely determined up to terms of order $O(T^3)$ for all x with $g_1(x) \neq 0$.

The proof of this theorem relies on comparing the Taylor expansion of $x(T, x_0, u)$ with the Fliess expansion of $x_T(T, x_0, u_T)$ in $T = 0$, see [1, Theorem 3.6] for details.

Remark 2.2: (i) Conditions for higher order $\Delta_T \leq O(T^5)$ can be stated similarly but become more and more involved. However, computer mathematics systems like, e.g., MAPLE can be used to check the conditions recursively and compute the corresponding u_T.

(ii) The condition (5) is rather restrictive. Hence, Theorem 2.1 shows that a mismatch $\Delta_T \leq O(T^4)$ can hardly be expected in general, regardless of how u_T is chosen. In particular, the seemingly "natural" Taylor-like choice

$$u_T(x) = u(x) + \frac{T}{2}u^1(x) + \frac{T^2}{6}u^2(x)$$

only works if $\alpha \equiv 0$. A sufficient condition for $\alpha \equiv 0$ is $[g_0, g_1] \equiv 0$, i.e., the vector fields commute.

(iii) A sufficient condition for (5) is $[g_0, g_1] \in \operatorname{span}\langle g_1 \rangle$. In [3] it was shown that this condition is necessary and sufficient for the fact that for each smooth controller $u : \mathbb{R}^n \to \mathbb{R}$ there exists u_T satisfying $\Delta_T \leq O(T^k)$ for arbitrary $k \in \mathbb{N}$.

3. Multi-Input systems

We now extend our result to multi-input control affine systems of the form

$$\dot{x}(t) = g_0(x(t)) + \sum_{i=1}^{m} g_i(x(t))u_i(x(t)) \tag{6}$$

with vector fields $g_i = (g_{i,1}, \ldots, g_{i,n})^T$, $i = 1, \ldots, m$, $m \in \mathbb{N}$, $m \leq n$, and controller $u = (u_1, \ldots, u_m)^T$. We write the right hand side of the system briefly as

$$g_0(x) + G(x)u(x) \quad \text{with} \quad G(x) = \begin{pmatrix} g_{1,1}(x) & \cdots & g_{m,1}(x) \\ \vdots & \ddots & \vdots \\ g_{1,n}(x) & \cdots & g_{m,n}(x) \end{pmatrix}. \tag{7}$$

and use definition (4) also for these vector valued feedback laws.

As in the single input case for $u_T \equiv u$ we get $\Delta_T = O(T^2)$ sets while for $u_T(x) = u(x) + \frac{T}{2}u^1(x)]$ we obtain $\Delta_T = O(T^3)$, cf. [2, Theorem 4.1 (i)-(ii)]. For $\Delta_T \leq O(T^4)$, Theorem 2.1 generalizes as follows, see [2, Theorem 4.1 (iii)]. Again, the proof relies on Taylor and Fliess expansions of the solution.

Theorem 3.1: For the multi-input system (6), a feedback law u_T with $\Delta_T \leq O(T^4)$ exists if there exists a bounded function $\alpha : \mathbb{R}^n \to \mathbb{R}^m$ satisfying

$$\sum_{i=1}^{m} \left[[g_0, g_i](x) + \sum_{\substack{j=1 \\ j \neq i}}^{m} [g_j, g_i](x) u_{0,j}(x) \right] u_i^1(x) = \sum_{i=1}^{m} \alpha_i(x) g_i(x). \tag{8}$$

If this condition holds, then the feedback laws u_T are given by

$$u_T(x) = u(x) + \frac{T}{2}u^1(x) + \frac{T^2}{6}u^2(x) + \frac{T^2}{12}\alpha(x)$$

and these u_T are uniquely determined up to terms of order $O(T^3)$ for all x for which $G(x)$ has full column rank. For these x condition (8) is also necessary.

As in the case of Theorem 2.1, the results can be extended to higher orders which is most conveniently done recursively using a computer mathematics system such as MAPLE. This recursive design procedure leads to a feedback of the form

$$u_T(x) = u(x) + \frac{T}{2}\tilde{u}^1(x) + \frac{T^2}{6}\tilde{u}^2(x) + \ldots$$

in which each $\tilde{u}^k$ is the solution of a least squares problem of the form $G(x)\tilde{u}^k(x) = b^k(x)$. If this problem is solvable with residual 0 for $k = 1, \ldots, m$, then u_T is a sampled-data feedback yielding $\Delta_T \leq O(T^{m+2})$. In particular, this shows that

(i) the problem is solvable for arbitrary order $O(T^k)$, $k \in \mathbb{N}$, if $G(x)$ is square and invertible for all $x \in \mathbb{R}^n$

(ii) the problem is in general not solvable for $\Delta_T \leq O(T^4)$ if $G(x)$ is not square, i.e., when $m < n$.

References

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