Regulation of Differential Drive Robots using Continuous Time MPC without Stabilizing Constraints or Costs ⋆

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Abstract: In this paper, model predictive control (MPC) of differential drive robots is considered. Here, we solve the set point stabilization problem without incorporating stabilizing constraints and/or costs in the MPC scheme. In particular, we extend recent results obtained in a discrete time setting to the continuous time domain. To this end, so called swaps and replacements are introduced in order to validate a growth condition on the value function and, thus, to rigorously prove asymptotic stability of the MPC closed loop for nonholonomic robots.

Keywords: nonholonomic mobile robots, model predictive control, regulation, nonlinear control.

1. INTRODUCTION

Control problems associated with differential drive robots are of a particular interest due their nonholonomic constraints. For the set-point stabilization (regulation) problem of such robots a considerable number of controllers have been proposed in the literature, e.g. adaptive control, sliding mode control, backstepping control, fuzzy systems, and neural-networks-based intelligent control, cf. Yoo (2010) for a full review. However, the main drawbacks of the aforementioned controllers are the difficult handling of input constraints as well as the necessity to thoroughly tune their parameters, see, e.g. Michalek and Kozowski (2010) for details. Due to its capability to control constrained nonlinear systems, model predictive control (MPC) has been successfully applied to differential drive robots in many studies, see, e.g. Gu and Hu (2005, 2006); Xie and Fierro (2008).

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At each sampling instant, MPC solves a finite horizon optimal control problem based on a measurement of the current state to compute a control value. However, this does not directly imply stability of the resulting closed loop system. Conditions which guarantee stability have been extensively studied, cf. Grüne and Pannek (2011); Rawlings and Mayne (2009). For regulation of differential robots, MPC stability was proved using terminal region constraints and costs in Gu and Hu (2005) while a contraction constraint was used in Xie and Fierro (2008). MPC without stabilizing constraints but with terminal costs has been first studied for nonholonomic systems in Grimm et al. (2005). Most recently, MPC without stabilizing constraints or costs has been proposed for differential robots in a discrete time setting, cf. Worthmann et al. (2015), where it has been shown that stability can be guaranteed by an appropriate choice of the stage cost and prediction horizon length. This last approach has, among others, two main advantages: (a) it proposes an MPC scheme which is simpler to implement and tune, and (b) it provides a formal procedure to tune the stage cost.

In this paper, we extend the work of Worthmann et al. (2015) by applying the results developed in Reble and Allgöwer (2012); Worthmann et al. (2014) in order to analyze the system in a continuous time setting. While in the discrete time setting a simple reordering of a constructed coefficient sequence lead to bounds on the (optimal) value function, new techniques have to be introduced in the continuous time setting to make the approach not only theoretically sound but also applicable. Here, we introduce so called swaps and replacements. Moreover, since the problem is studied on a continuous domain also the intersampling behaviour of the MPC closed loop is analysed, which further extends the analysis presented in Worthmann et al. (2015).

The paper is organized as follows: in Section 2 a brief description of the considered problem and the proposed MPC scheme is given. In the subsequent Section 3, stability results from Reble and Allgöwer (2012) are recalled and swaps and replacements are incorporated in the respective setting. Then, in Section 4, growth bounds on the value function are derived based on feasible open-loop trajectories. These functions are later used in Section 5 in order to determine a minimal prediction horizon length such that asymptotic stability of the MPC closed loop is guaranteed. Conclusions are drawn in Section 6.

Notation: $\mathbb{R}$ and $\mathbb{N}$ denote real and natural numbers, respectively. $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$ represents the non-negative integers and $\mathbb{R}_{\geq 0}$ the non-negative real numbers. A continuous function $\eta : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is said to be of class $\mathcal{K}$ if it is zero at zero and strictly monotonically increasing. If it is, in addition, unbounded it is of class $\mathcal{K}_\infty$-function. A function $\beta : \mathbb{R}_{\geq 0} \times \mathbb{N}_0 \to \mathbb{R}_{\geq 0}$ is called a $\mathcal{KL}$-function if $\beta(\cdot, n) \in \mathcal{K}_\infty$ for all $n \in \mathbb{N}_0$ and $\beta(r, \cdot)$ is strictly monotonically decaying to zero for each $r > 0$. A function $c : I \to \mathbb{R}$ is said to be *piecewise continuous* — denoted by $\mathcal{PC}(I, \mathbb{R})$ — if, for every $a, b \in I$ with $a < b$, the interval $[a, b]$ admits a finite partition $a = t_1 < t_2 < \cdots < t_n = b$, $n \in \mathbb{N}$, such that c is continuous on every subinterval (t_i, t_{i+1}), $i \in \{1, \ldots, n-1\}$, has a right limit $\lim_{t \searrow t_1} c(t)$ at t_1, a left limit $\lim_{t \nearrow t_n} c(t)$ at t_n, and both left and right limit at every t_i, $i \in \{2, 3, \ldots, n-1\}$.

2. PROBLEM FORMULATION

In this section, we, first, present the continuous-time differential drive kinematic model as well as the regulation objective. Then, a continuous-time model predictive control scheme, without stabilizing constraints or costs, is proposed.

At time $t \in \mathbb{R}_{\geq 0}$ (seconds) the continuous time differential kinematic model is given by

$$\begin{pmatrix} \dot{x}(t) \\ \dot{y}(t) \\ \dot{\theta}(t) \end{pmatrix} = \dot{z}(t) = f(z(t), u(t)) = \begin{pmatrix} v(t)\cos(\theta(t)) \\ v(t)\sin(\theta(t)) \\ w(t) \end{pmatrix} \quad (1)$$

where $f : \mathbb{R}^3 \times \mathbb{R}^2 \to \mathbb{R}^3$ is an analytic vector field. The state $z = (x, y, \theta)^T \in Z \subsetneq \mathbb{R}^3$ consists of the (spatial) position $(x, y)^T$ (m,m) and the orientation angle θ (rad) of the robot. The control input is defined by $u = (v, w)^T \in U \subsetneq \mathbb{R}^2$, where v (m/s) and w (rad/s) are the linear and the angular speeds of the robot, respectively. State and control constraints are modelled by the sets Z and U, respectively. For simplicity of exposition, let Z and U be given by

$$Z = [\underline{x}, \bar{x}] \times [\underline{y}, \bar{y}] \times \mathbb{R} \quad \text{and} \quad U = [\underline{v}, \bar{v}] \times [\underline{w}, \bar{w}]$$

with $0 \times 0 \in \mathrm{int}(Z) \times \mathrm{int}(U)$.

Actuator dynamics can be modeled by first order low-pass filters as shown in (Backman et al., 2012, Subsection 2.1). Within this framework the theoretical developments of this paper can still be applied with minor adaptations. For given initial state z_0 and control function $u : \mathbb{R}_{\geq 0} \to U$, the trajectory governed by the dynamics (1) is denoted by $z(\cdot; z_0, u(\cdot))$. Furthermore, a control function $u(\cdot)$ is called admissible on the interval $[0, T)$, $T \in \mathbb{R}_{>0}$, if the conditions

$$u(t) \in U,\ t \in [0, T) \quad \text{and} \quad z_u(t; z_0) \in Z,\ t \in [0, T]$$

hold. The set of all admissible control functions $u(\cdot)$ on $[0, T)$ is denoted by $\mathcal{U}_T(z_0)$. Moreover, the set $\mathcal{U}_\infty(z_0)$ dentoes the set of all admissible control functions $u(\cdot) : \mathbb{R}_{\geq 0} \to \mathbb{R}^2$ satisfying $u(\cdot) \in \mathcal{U}_T(z_0)$ for all $T > 0$.

The control objective is to steer the robot to the origin — a (controlled) equilibrium, i.e. $f(0, 0) = 0$. Therefore, our goal is to design a feedback control law $\mu : \mathbb{R}^3 \to \mathbb{R}^2$ such that the resulting closed loop is asymptotically stable with respect to the origin $0_{\mathbb{R}^3}$, i.e. there exists a function $\beta \in \mathcal{KL}$ that satisfies

$$\|z_\mu(t; z_0)\| \leq \beta(\|z_0\|, t) \quad \forall\, t \geq 0$$

for all $z_0 \in Z$, where $z_\mu(t; z_0)$ is the closed loop trajectory induced by the feedback control $\mu(\cdot)$. To this end, we introduce continuous running (stage) costs $\ell : \mathbb{R}^3 \times \mathbb{R}^2 \to \mathbb{R}_{\geq 0}$ which serve as a performance criterion and satisfy

$$\ell(0, 0) = 0 \quad \text{and} \quad \inf_{u \in \mathbb{U}} \ell(z, u) > 0 \quad \forall\, z \in \mathbb{R}^3 \backslash \{0\}.$$

Analogously to Worthmann et al. (2015) the running costs $\ell(\cdot, \cdot)$ are chosen as

$$\ell(z, u) = q_1 x^4 + q_2 y^2 + q_3 \theta^4 + r_1 v^4 + r_2 w^4 \qquad (2)$$

with weighting parameters q_1, q_2, q_3, r_1, $r_2 \in \mathbb{R}_{>0}$. Based on the introduced running costs, the cost function and the corresponding (optimal) value function are defined as

$$J_T(z_0, u(\cdot)) := \int_0^T \ell(z_u(t; z_0), u(t))\, \mathrm{d}t \quad \text{and}$$

$$V_T(z_0) := \inf_{u(\cdot) \in \mathcal{U}_T(z_0)} J_T(z_0, u(\cdot))$$

for $T \in \mathbb{R}_{>0} \cup \{\infty\}$, respectively.

Ideally, an infinite-horizon optimal control problem minimizing $J_\infty(z_0, \cdot)$ is desirable. However, since solving this problem is computationally intractable, we utilize an MPC scheme in order to solve the problem on a finite optimization horizon $T \in \mathbb{R}_{>0}$. Algorithm 1 summarizes an MPC scheme without stabilizing constraints or costs tailored for this purpose.

Algorithm 1 MPC

Given: Prediction horizon T, control horizon $\delta \in (0, T)$, and the current state $\hat{z} \in Z$.

1: Compute an optimal control function $u^\star(\cdot) \in \mathcal{U}_T(\hat{z})$ such that $J_T(\hat{z}, u^\star(\cdot)) = V_T(\hat{z})$ holds.
2: Define the MPC feedback law $\mu_{T,\delta} : [0, \delta) \times Z \to U$ at $\hat{z}$ by $\mu_{T,\delta}(t; \hat{z}) := u^\star(t)$, i.e. the MPC feedback equals the first portion of the function $u^\star(t)$. Then, implement $\mu_{T,\delta}(t; \hat{z})|_{t \in [0,\delta)}$ at the plant. This yields
$$z_{\mu_{T,\delta}}(\delta; \hat{z}) = z(\delta; \hat{z}, u^\star(\cdot)).$$
3: Initialize the optimal control problem with $\hat{z} = z_{\mu_{T,\delta}}(\delta; \hat{z})$ and go to step 1.

The MPC closed loop control and trajectory resulting from applying Algorithm 1 are denoted by $\mu_{T,\delta}^{MPC}(\cdot; z_0)$ and $z_{T,\delta}^{MPC}(\cdot; z_0)$, respectively. Moreover, the resulting MPC closed loop cost is given by

$$J_\infty^{\mu_{T,\delta}}(z_0) := \int_0^\infty \ell(z_{T,\delta}^{MPC}(t; z_0), u_{T,\delta}^{MPC}(t; z_0))\, \mathrm{d}t$$

Since $0_{\mathbb{R}^2} \in U$ holds, recursive feasibility is ensured, which entails $\mathcal{U}_T(z_0) \neq \emptyset$. Hence, the existence of an admissible control function $u^\star(\cdot)$ minimizing $J_T(z_0, \cdot)$ is guaranteed in each MPC step. We assume that the solution of the MPC problem is obtained instantaneously. While for many cases of interest this is justified by recent algorithmic developments, see, e.g. Houska et al. (2011), this might not be true for more complex models. Future research will aim at investigating suboptimal MPC and the effect of time delays due to computation times.

3. STABILITY AND PERFORMANCE BOUNDS

In this section, first stability results presented in Reble and Allgöwer (2012) are recalled in a slighly reformulated version, see Braun et al. (2012). Second, we introduce the concept of admissible swaps and replacements. In Section 4, the combination of these two ingredients is exploited in order to rigorously prove asymptotic stability of the continuous time model of the mobile robot.

The following theorem (Reble and Allgöwer, 2012, Theorem 9) is used to determine an optimization horizon T such that asymptotic stability and a performance bound on the MPC closed loop are guaranteed.

Theorem 1. Assume existence of a monotonically increasing and bounded function $B : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ satisfying

$$V_t(z_0) \leq B(t) \cdot \ell^\star(z_0) =: B(t) \cdot \inf_{u \in \mathbb{U}} \ell(z_0, u) \quad \forall\, t \geq 0 \quad (3)$$

for all $z_0 \in Z$. Then, for given control horizon $\delta > 0$ and prediction horizon $T > \delta$ chosen so that $\alpha_{T,\delta} > 0$ holds for

$$\alpha_{T,\delta} := 1 - \frac{e^{-\int_\delta^T B(t)^{-1}\,\mathrm{d}t} \cdot e^{-\int_{T-\delta}^T B(t)^{-1}\,\mathrm{d}t}}{\left[1 - e^{-\int_\delta^T B(t)^{-1}\,\mathrm{d}t}\right]\left[1 - e^{-\int_{T-\delta}^T B(t)^{-1}\,\mathrm{d}t}\right]},$$

the relaxed Lyapunov inequality

$$V_T(z_{\mu_{T,\delta}}(\delta; z)) \leq V_T(z) - \alpha_{T,\delta} \int_0^\delta \ell(z_{\mu_{T,\delta}}(t; z), \mu_{T,\delta}(t, z))\,\mathrm{d}t$$

as well as the performance estimate $V_\infty^{\mu_{T,\delta}}(z) \leq \alpha_{T,\delta}^{-1} \cdot V_\infty(z)$ are satisfied for all $z \in Z$. If, in addition, there exists $\mathcal{K}_\infty$-functions $\underline{\eta}, \bar{\eta} : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ satisfying

$$\underline{\eta}(\|z\|) \leq \ell^\star(z) \leq \bar{\eta}(\|z\|) \qquad \forall\, z \in Z, \qquad (4)$$

the MPC closed loop is asymptotically stable.

While condition (4) holds trivially for the chosen running costs (2), it is crucial to verify the growth condition (3). The following lemma plays a vital role in constructing a function $B : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ such that all assumptions of Theorem 1 hold. The following definition is needed in order to formulate Lemma 3.

Definition 2. (Admissible Swap / Replacement). Let real numbers a, b, $0 \leq a < b$, and a function $c \in \mathcal{PC}(\mathbb{R}_{\geq 0}, \mathbb{R}_{\geq 0})$ be given. Then, for $\delta \in (0, b - a]$, the piecewise continuous function $\bar{c} : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ defined as

$$\bar{c}(t) := \begin{cases} c(b - a + t) & \text{for } t \in [a, a + \delta) \\ c(a - b + t) & \text{for } t \in [b, b + \delta) \\ c(t) & \text{otherwise} \end{cases}$$

is called an *admissible swap* if $\int_a^{a+t} c(s)\,\mathrm{d}s \leq \int_b^{b+t} c(s)\,\mathrm{d}s$ holds for all $t \in [0, \delta]$. Moreover, for $\delta > 0$ and a function $\tilde{c} \in \mathcal{PC}([a, \delta), \mathbb{R}_{\geq 0})$, $\bar{c} \in \mathcal{PC}(\mathbb{R}_{\geq 0}, \mathbb{R}_{\geq 0})$ given by

$$\bar{c}(t) := \begin{cases} \tilde{c}(t) & \text{for } t \in [a, a + \delta) \\ c(t) & \text{otherwise} \end{cases}$$

is said to be an *admissible replacement* if the condition $\int_a^{a+t} c(s)\,\mathrm{d}s \leq \int_a^{a+t} \tilde{c}(s)\,\mathrm{d}s$ holds for all $t \in [a, a + \delta]$.

Lemma 3. For initial value $z_0 \in Z$, let $u_{z_0}(\cdot) \in \mathcal{U}_\infty(z_0)$ and $c_{z_0} \in \mathcal{PC}(\mathbb{R}_{\geq 0}, \mathbb{R}_{\geq 0})$ satisfying $\int_0^\infty c_{z_0}(t)\,\mathrm{d}t < \infty$ be given such that the inequality

$$\ell(z(t; z_0, u_{z_0}(\cdot)), u_{z_0}(t)) \leq c_{z_0}(t) \cdot \ell^\star(z_0) \quad \forall\, t \geq 0 \quad (5)$$

holds (almost everywhere). Then, the monotonically increasing, bounded function $B_{z_0} : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ defined by $B_{z_0}(t) = \int_0^t c_{z_0}(s)\,\mathrm{d}s$, $t \in \mathbb{R}_{\geq 0}$, satisfies the inequality

$$V_t(z_0) \leq \int_0^t \ell(z(s; z_0, u_{z_0}(\cdot)), u_{z_0}(s))\,\mathrm{d}s \leq B_{z_0}(t) \ell^\star(z_0) \quad (6)$$

for all $t \geq 0$. Furthermore, Inequality (6), boundedness, and monotonicity are preserved for $\overline{B}_{z_0} : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ defined as

$$\overline{B}_{z_0}(t) := \int_0^t \bar{c}_{z_0}(s)\,\mathrm{d}s \quad (7)$$

if $\bar{c}_{z_0} \in \mathcal{PC}(\mathbb{R}_{\geq 0}, \mathbb{R}_{\geq 0})$ is composed of $c_{z_0}(\cdot)$ by finitely many admissible swaps and/or replacements. [1]

While the proof of Inequality (6) is straightforward, see Reble and Allgöwer (2011), the remaining part of Lemma 3 can be shown analogously to Worthmann (2012) observing that the function $\overline{B}_{z_0}$ resulting from admissible swaps and/or replacements is an upper bound of B_{z_0}, i.e. $\overline{B}_{z_0}(t) \geq B_{z_0}(t)$ holds for all $t \geq 0$.

Interchanging appropriate slices (or even doing that several times in order to realize a "squeeze" as demonstrated lateron) and/or replacing suitable intervals of a function $c_{z_0} \in \mathcal{PC}(\mathbb{R}_{\geq 0}, \mathbb{R}_{\geq 0})$ yields a function $\overline{B}_{z_0}(\cdot)$ which can be employed in order to cover Condition (3) not only for the particular initial value z_0 but on a whole set. Hence, admissible swaps and replacements turn out to be the essential tool in order to verify the main assumption of Theorem 1 for the differential drive robot in the subsequent section.

4. GROWTH BOUNDS ESTIMATE

In this section, a bounded function $B(t)_{t \in \mathbb{R}_{>0}}$ for which condition (3) holds is derived. To this end, the set Z is split up into two disjoint sets $\mathcal{M}_\rho$ and $\mathcal{M}_\rho^c := Z \backslash \mathcal{M}_\rho$, where $\mathcal{M}_\rho$ is an open set of initial conditions depending on a parameter $\rho \in [0, \infty)$ and defined by

$$\mathcal{M}_\rho := \left\{ z = \begin{pmatrix} x \\ y \\ \theta \end{pmatrix} \in Z : \ell^\star\left(\begin{pmatrix} x \\ y \\ 0 \end{pmatrix}\right) < \rho \right\}. \quad (8)$$

Then, using Lemma 3, bounded functions $B_{\mathcal{M}_\rho}, B_{\mathcal{M}_\rho^c} \in \mathcal{PC}(\mathbb{R}_{\geq 0}, \mathbb{R}_{\geq 0})$ are constructed such that Inequality (3) holds on $\mathcal{M}_\rho$ or $\mathcal{M}_\rho^c$, respectively. Finally, taking into account that the constant input function $u \equiv 0_{\mathbb{R}^2}$ is admissible on the infinite time horizon shows that $\overline{B}(t) = t$

[1] Indeed, Inequality (6) holds with $\overline{B}_{z_0} \in \mathcal{L}^1_{\mathrm{loc}}(\mathbb{R}_{\geq 0}, \mathbb{R}_{\geq 0})$ defined by $\overline{B}_{z_0}(t) := \sup_{I : |I| = t} \int_I c_{z_0}(s)\,\mathrm{d}s$ where I denotes arbitrary Lebesgue-measurable sets.

is an upper bound for $B(t)$ in Inequality (3). In conclusion, Inequality (3) holds for all $z_0 \in Z$ with

$$\overline{B}(t) := \min\{t, \max\{B_{\mathcal{M}_\rho}(t), B_{\mathcal{M}_\rho^c}(t)\}\}, \quad t \in \mathbb{R}_{\geq 0}. \quad (9)$$

The following approach is employed in order to construct $B_{\mathcal{M}_\rho}, B_{\mathcal{M}_\rho^c} \in \mathcal{PC}(\mathbb{R}_{\geq 0}, \mathbb{R}_{\geq 0})$ First, for $z_0 = (x_0, y_0, 0)^T \in \mathcal{M}_\rho$, an admissible control function $u_{z_0}(\cdot) \in \mathcal{U}[0,\infty)$ steering the robot to the origin in finite time is proposed. This function $u_{z_0}(\cdot)$ yields (suboptimal) running costs $\ell(z(t; z_0, u_{z_0}(\cdot)), u_{z_0}(t))$ such that, by definition of optimality, the following quotients can be estimated uniformly with respect to $z_0 = (x_0, y_0, 0)^T \in \mathcal{M}_\rho$ by

$$\ell(z(t; z_0, u_{z_0}(\cdot)), u_{z_0}(t)) \cdot \ell^\star(z_0)^{-1} \leq c^{\mathcal{M}_\rho}(t) \quad \forall\, t \in \mathbb{R}_{\geq 0} \quad (10)$$

with $c^{\mathcal{M}_\rho} \in \mathcal{PC}(\mathbb{R}_{>0}, \mathbb{R}_{\geq 0})$ such that Inequality (5) holds. Afterwards, the segments of the calculated function $c^{\mathcal{M}_\rho}$ are appropriately interchanged such that a function $\bar{c}^{\mathcal{M}_\rho} \in \mathcal{PC}(\mathbb{R}_{\geq 0}, \mathbb{R}_{\geq 0})$ according to Lemma 3 is obtained. Then, a bounded growth function $\overline{B}_{\mathcal{M}_\rho} : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is defined by $\overline{B}_{\mathcal{M}_\rho}(t) := \int_0^t \bar{c}^{\mathcal{M}_\rho}(s)\ \mathrm{d}s$, see Figure 2 for an illustration example. Finally, $\overline{B}_{\mathcal{M}_\rho}$ is used in order to ensure Condition (3) for all initial states contained in $\mathcal{M}_\rho$ with $\theta_0 \neq 0$. Then, an analogous construction is performed for $z_0 \in \mathcal{M}_\rho^c$. Due to symmetries (the robot can go back and forth), it is sufficient to consider initial positions with $(x_0, y_0)^T \geq 0_{\mathbb{R}^2}$.

4.1 Trajectory Generation for $z_0 \in \mathcal{M}_\rho^c$

For initial conditions $z_0 = (x_0, y_0, 0)^T \in \mathcal{M}_\rho^c$, the following simple manoeuvre is employed, see Figure 1 (left):

a) turn the robot to the angle $\phi := \arctan(y_0/x_0)$, $\phi \in [-\pi, \pi)$ such that the vehicle points towards (or in the opposite direction to) the origin $(0,0)^T \in \mathbb{R}^2$,
b) drive directly towards the origin,
c) turn the vehicle to the desired angle $\theta^\star = 0$.

The time needed in order to carry out this manoeuvre depends the constraints Z and U. We define the minimal time t_w required to turn the vehicle by 90 degrees as well as the minimal time t_v required to drive the vehicle from the farthest corner of the box $[\underline{x}, \bar{x}] \times [\underline{y}, \bar{y}]$ to the origin $0_{\mathbb{R}^2}$:

$$t_w := \frac{\pi/2}{\min\{-\underline{w}, \bar{w}, \pi/2\}},$$

$$t_v := \frac{\sqrt{\max\{-\underline{y}, \bar{y}\}^2 + \max\{-\underline{x}, \bar{x}\}^2}}{\min\{-\underline{v}, \bar{v}\}},$$

respectively. Here, we assume reasonable control constraints, i.e. $\min\{-\underline{w}, \bar{w}\} \leq \pi/2$. Additionally, to keep the presentation technically simple, the following inequality is imposed

$$r_2 \leq \frac{q_3}{2}. \quad (11)$$

Then, the proposed manoeuvre is the following. First, the vehicle stays at the initial position without moving for time $t_1 = t_w$, i.e. $u(t)|_{t \in [0,t_1)} = 0_{\mathbb{R}^2}$, which yields Inequality (10) with $c^{\mathcal{M}_\rho^c}(t)|_{t \in [0,t_1)} = 1$. This additional phase without any movement is introduced in order to facilitate handling initial positions with $\theta_0 \neq 0$.

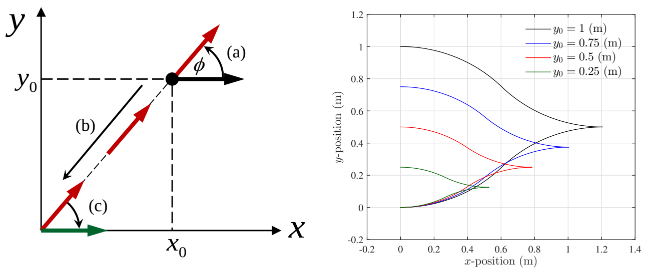

Fig. 1. Illustration of the manoeuvre for initial value $z_0 = (x_0, y_0, 0) \in \mathcal{M}_\rho^c$ (left) and the trajectories of parts (b) and (c) for $z_0 \in \mathcal{M}_\rho$ (right).

Next, the vehicle turns until time t_2 such that $\theta(t_2) = \phi$, where $t_2 - t_1 = t_w$. This is achieved by applying the input $u(t)|_{t \in [t_1, t_2)} = (0, \phi(t_2 - t_1)^{-1})^T \in U$ and yields the running costs

$$\ell(z(t; z_0, u(\cdot)), u(t)) = q_1 x_0^4 + q_2 y_0^2 + \frac{\phi^4\left(q_3(t - t_1)^4 + r_2\right)}{(t_2 - t_1)^4}.$$

Since $\phi \in [0, \pi/2)$, $\ell^\star(z_0) \geq \rho$, and Assumption (11) hold, Inequality (10) holds with

$$c^{\mathcal{M}_\rho^c}(t) := 1 + \frac{q_3 \pi^4}{16 t_w^4 \cdot \rho}\left((t - t_1)^4 + \frac{1}{2}\right) \quad \text{for } t \in [t_1, t_2).$$

Then, the vehicle drives towards the origin until time $t_3 = t_2 + t_v$, with constant speed $u(t) = (-\|(x_0, y_0)^T\| t_v^{-1}, 0)^T$, $t \in [t_2, t_3)$. This leads to $\ell(z(t; z_0, u(\cdot)), u(t))$ given by

$$\frac{(t_3 - t)^2}{t_v^2}\left[q_1\left(\frac{t_3 - t}{t_v}\right)^2 x_0^4 + q_2 y_0^2\right] + q_3 \phi^4 + \frac{r_1}{t_v^4}\left\|\begin{pmatrix} x_0 \\ y_0 \end{pmatrix}\right\|^4.$$

Since $\phi \leq \pi/2$ and the control effort is smaller than $\min\{-\underline{v}, \bar{v}\}$, Inequality (5) holds with

$$c^{\mathcal{M}_\rho^c}(t) := \left(\frac{t_3 - t}{t_v}\right)^2 + \frac{q_3(\pi/2)^4 + r_1 \min\{-\underline{v}, \bar{v}\}^4}{\rho} \quad (12)$$

for $t \in [t_2, t_3)$. Finally, the vehicle turns until time $t_4 = t_3 + t_w$ such that $\theta(t_4) = 0$ using the input $u(t) = (0, -\phi t_w^{-1})^T$ for $t \in [t_3, t_4)$. Thus, the corresponding running costs are

$$\ell(z(t; z_0, u(\cdot)), u(t)) = (\phi \cdot t_w^{-1})^4 \cdot \left[q_3(t_4 - t)^4 + r_2\right]. \quad (13)$$

Then, invoking Condition (11) ensures Inequality (5) with

$$c^{\mathcal{M}_\rho^c}(t) := \frac{q_3 \pi^4}{16 t_w^4 \rho}\left[(t_4 - t)^4 + \frac{1}{2}\right] \quad \text{for } t \in [t_3, t_4) \quad (14)$$

and $c^{\mathcal{M}_\rho^c} \equiv 0$ on $[t_4, \infty)$. In conclusion, the constructed function $c^{\mathcal{M}_\rho^c}(\cdot)$ is of class $\mathcal{PC}(\mathbb{R}_{\geq 0}, \mathbb{R}_{\geq 0})$, bounded, and integrable on $[0, \infty)$. Hence, it can be used to define $B_{\mathcal{M}_\rho^c} : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ by $B_{\mathcal{M}_\rho^c}(t) := \int_0^t c^{\mathcal{M}_\rho^c}(s)\ \mathrm{d}s$ such that Inequality (6) holds for all $z_0 \in \mathcal{M}_\rho^c$ with $\theta_0 = 0$.

Initial Conditions inside $\mathcal{M}_\rho^c$ with $\theta_0 \neq 0$: We show that Condition (3) holds for all $z_0 \in \mathcal{M}_\rho^c$ with growth bound (7) for $\bar{c}(\cdot)$ obtained by swapping and flipping $c(\cdot)$. To this end, initial values $z_0 \in \mathcal{M}_\rho^c$ with $\theta_0 \in [-\pi, 0) \cup (0, \pi)$ must be considered. Here, we distinguish four intervals for the angle θ_0 similar to Worthmann et al. (2015).

Case 1: let θ_0 be contained in the interval $(0, \phi]$. Since $\theta_0 \in (0, \phi)$ holds, there uniquely exists a time $t^\star \in (t_w, 2t_w]$ such that $\theta_0 = \theta(t^\star; (x_0\ y_0\ 0)^T, u(\cdot))$ holds. Hence, using the control function defined by

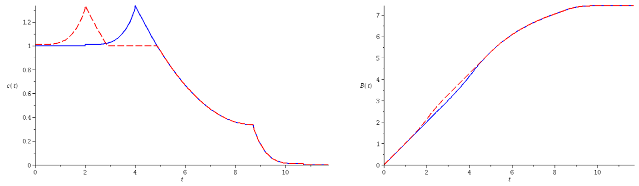

Fig. 2. Illustration of the functions $c^{\mathcal{M}_\rho^c}$ (blue solid line) and $\bar{c}^{\mathcal{M}_\rho^c}$ (red dashed line) and the resulting accumulated bounds $B_{\mathcal{M}_\rho^c}$ and $\overline{B}_{\mathcal{M}_\rho^c}$ based on the parameters specified in Section 5 with $q_2 = 5$ and $\rho = 2$.

$$\tilde{u}(t) = \begin{cases} 0_{\mathbb{R}_2} & \text{for } t \in [0, t^\star), \\ u(t) & \text{for } t \geq t^\star \end{cases}$$

steers the robot to the origin analogously to the proposed manoeuvre for $\theta_0 = 0$ only skipping a (small) portion of $u(\cdot)$. For $t \in [0, t^\star)$, $\ell(t; z_0, \tilde{u}(\cdot)) = \ell^\star(z_0)$ holds, which implies

$$\ell(t; z_0, \tilde{u}(\cdot), \tilde{u}(t)) \leq c^{\mathcal{M}_\rho^c}(t)\ell^\star(z_0) \qquad (15)$$

since $c^{\mathcal{M}_\rho^c}(t) \geq 1$ holds for all $t \in [0, t^\star)$. In addition, $\ell(t; z_0, \tilde{u}(\cdot), \tilde{u}(t)) = \ell(t; z_0, u(\cdot), u(t))$ holds while the quantity $\ell^\star(z_0)$ on the right hand side of the desired inequality has become larger ($\theta_0 > 0$), which again implies (15) for $t \geq t^\star$. Hence, Condition (3) is shown for the constructed function $B(\cdot)$.

Case 2: let $\theta_0 \in (\phi, \pi]$ hold. The first part of the manoeuvre is performed by turning the robot until time $t_2 = 2t_w$ such that $\theta(t_2) = \phi$ is achieved using the input $\tilde{u}(t) = (0, -\Delta\theta \cdot t_w^{-1})$, $t \in [0, t_2)$, $\Delta\theta = (\theta_0 - \phi)/2$. This yields the running costs $\ell(z(t; z_0, \tilde{u}(\cdot)), \tilde{u}(t))$ given by

$$q_1 x_0^4 + q_2 y_0^2 + q_3 \left(\theta_0 - \frac{t\Delta\theta}{t_w}\right)^4 + r_2 \left(\frac{\Delta\theta}{t_w}\right)^4 \qquad (16)$$

for $t \in [0, t_2)$. Using $\theta_0 \geq \delta := t\Delta\theta t_w^{-1} > 0$ and the estimate

$$\begin{aligned}\left(\theta_0 - t\Delta\theta t_w^{-1}\right)^4 &= \theta_0^4 - \delta\theta_0^3 - 3\delta\theta_0(\theta_0 - \delta)^2 - \delta^3(\theta_0 - \delta) \\ &\leq \theta_0^4 - t\Delta\theta t_w^{-1}\theta_0^3\end{aligned}$$

and, thus, invoking Assumption (11) leads to

$$\ell(z(t; z_0, \tilde{u}(\cdot)), \tilde{u}(t)) \leq \ell^\star(z_0) - \left[\frac{q_3\Delta\theta}{2t_w}\right]\left[2t\theta_0^3 - \left(\frac{\Delta\theta}{t_w}\right)^3\right] \qquad (17)$$

for $t \in [0, 2t_w)$. Next, swap the function $c^{\mathcal{M}_\rho^c}(\cdot)$ with the parameters $(a, b, \delta) = (0, t_w, t_w)$ and denote the outcome by $\bar{c}^{\mathcal{M}_\rho^c}(\cdot)$ in order to construct a new bound $\overline{B} : \mathbb{R}_{\geq}0 \to \mathbb{R}_{\geq 0}$, which still satisfies Inequality (6) for all initial values considered so far. Then,

$$\ell(z(t; z_0, \tilde{u}(\cdot)), \tilde{u}(t)) \leq \bar{c}^{\mathcal{M}_\rho^c}(t)\ell^\star(z_0) \qquad (18)$$

holds for $t \in [0, t_w)$ according to (16) and for $t \in [t_w, 2t_w)$ since the right hand side of (17) is less than $\ell^\star(z_0)$ on this interval. Finally, the remaining parts of the manoeuvre can be dealt with analogously to Case 1 showing that Condition (3) is ensured with the accumulated bound $\overline{B}(\cdot)$.

Case 3: let $\theta_0 \in (-\pi + \phi, 0)$ hold. First, the robot uses the control input $\tilde{u}(t) = (0, \Delta\theta t_w^{-1})^T$, $t \in [0, t_w)$, with $\Delta\theta$ defined as $\max\{0, \phi - \pi/2 - \theta_0\}$ in order to achieve $\psi := \phi - \theta(t_w; z_0, \tilde{u}(\cdot)) \leq \pi/2$. Then, the control signal $\tilde{u}(t) = (0, \psi t_w^{-1})^T$, $t \in [t_w, 2t_w)$, is employed, which yields $\theta(2t_w; z_0, \tilde{u}(\cdot)) = \phi$. Here, we have to show that the inequality

$$\int_0^t \ell(x(s; z_0, \tilde{u}(\cdot)), \tilde{u}(s))\, \mathrm{d}s \leq \int_0^t \bar{c}^{\mathcal{M}_\rho^c}(s)\, \mathrm{d}s \cdot \ell^\star(z_0) \quad (19)$$

holds for all $t \in [0, 2t_w)$. Proceeding analogously to Case 2 leads to Estimate (17) with θ_0 replaced by $-\theta_0$ for $s \in [0, t_w)$. Hence, $\ell(x(t; z_0, \tilde{u}(\cdot)), \tilde{u}(t)) \leq \bar{c}^{\mathcal{M}_\rho^c}(t)\ell^\star(z_0)$, $t \in [0, t_w)$, and $\int_0^{t_w} \ell(x(s; z_0, \tilde{u}(\cdot)), \tilde{u}(s))\, \mathrm{d}s \leq t_w\ell^\star(z_0)$ hold. Using the second of these inequalities and [2]

$$\ell(z(t; z_0, \tilde{u}(\cdot)), \tilde{u}(t)) \leq \bar{c}^{\mathcal{M}_\rho^c}(t - t_w)\ell^\star(z_0), \qquad t \in [t_w, 2t_w),$$

shows the desired Inequality (19).

Case 4: let $\theta_0 \in (-\pi, -\pi + \phi)$ hold. First, the robot is turned until time t_w such that $\theta(t_w; z_0, \tilde{u}(\cdot)) = -\pi + \phi$ holds using the input $\tilde{u}(t) = (0, \Delta\theta t_w^{-1})^T$ for $t \in [0, t_w)$ with $\Delta\theta = |\theta_0| - \pi + \phi$. During the second part of the manoeuvre the robot is driven to the origin until time $t_w + t_v$. Then, the robot is turned until time $2t_w + t_v$ such that $\theta(2t_w + t_v; z_0, \tilde{u}(\cdot)) = -\pi/2$ holds using $\tilde{u}(t) = (0, \Delta\theta t_w^{-1})^T$ with $\Delta\theta = \pi/2 - \phi$ for $t \in [2t_w, 2t_w + t_v)$. Hence, the running costs of the first phase are given by (16) while those of the third part are given by

$$q_3 \left[\phi - \pi + \frac{(t - t_w - t_v)\Delta\theta}{t_w}\right]^4 + r_2 \left(\frac{\Delta\theta}{t_w}\right)^4.$$

for $t \in [t_w + t_v, 2t_w + t_v)$. Next, note that $\bar{c}^{\mathcal{M}_\rho^c}(\cdot)$ is strictly monotonically decreasing on the time interval $[2t_w, 2t_w + t_v)$, cp. (12). If $\lim_{t \nearrow 2t_w + t_v} \bar{c}^{\mathcal{M}_\rho^c}(t) \geq 1$ holds, set $t^\star := 2t_w + t_v$. Otherwise let $t^\star \in (2t_w, 2t_w + t_v)$ be the unique time satisfying $\bar{c}^{\mathcal{M}_\rho^c}(t^\star) = 1$. Then, cut $\bar{c}^{\mathcal{M}_\rho^c}(t)|_{[2t_w, t^\star)}$, squeeze it in at time t_w and denote the outcome with a slight abuse of notation again with $\bar{c}^{\mathcal{M}_\rho^c}$. If $t^\star - 2t_w \leq t_w$ one swap with parameters $(a, b, \delta) = (t_w, 2t_w, t^\star - 2t_w)$ suffices, otherwise several swaps are necessary in order to perform this action.

Clearly, Inequality (18) holds on the intervals $[0, t_w/2)$ and $[t_w, t_w + t_v)$. Moreover, the inequality

$$\ell(z(t_w + t_v + t; z_0, \tilde{u}(\cdot)), \tilde{u}(t_w + t_v + t)) \leq \bar{c}^{\mathcal{M}_\rho^c}(t_w/2 + t)\ell^\star(z_0)$$

is satisfied for $t \in [0, t_w/2)$. Furthermore, the inequality $\ell(z(t; z_0, u(\cdot)), u(t)) \leq \ell^\star(z_0)$ holds on the intervals $[t_w/2, t_w)$ and $[3t_w/2 + t_v, 2t_w + t_v)$ by proceeding analogously to Case 2 (Taylor expansion). In conclusion, using the accumulated bound $\overline{B}_{\mathcal{M}_\rho^c}(t) := \int_0^t \bar{c}^{\mathcal{M}_\rho^c}(s)\, \mathrm{d}s$ yields the desired Inequality (3).

4.2 Trajectory Generation for $z_0 \in \mathcal{M}_\rho$

For initial conditions $z_0 \in \mathcal{M}_\rho$, the following manoeuvre is used in order to derive a growth bound $\overline{B}_{\mathcal{M}_\rho} : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ satisfying Inequality (3). Again, we begin with z_0 whose angular deviation is equal to zero, i.e. $\theta_0 = 0$:

a) drive towards the y-axis until $(0, y_0, 0)^T$ is reached.
b) drive forward while slightly steering in order to reduce the y-component to $y_0/2$ — a position $(\tilde{x}, y_0/2, 0)^T$ for some $\tilde{x} > 0$ is reached.
c) carry out a symmetric manoeuvre while driving backward such that the origin $0_{\mathbb{R}^3}$ is attained.

[2] The running costs $\ell(z(t; z_0, \tilde{u}(\cdot)), \tilde{u}(t))$, $t \in [t_w, 2t_w)$, are given by $q_1 x_0^4 + q_2 y_0^2 + q_3(\theta(t_w; z_0, \tilde{u}(\cdot)) + \psi(t - t_w)t_w^{-1})^4 + r_2\psi^4 t_w^{-4}$.

For different trajectories for parts (b) and (c) of the above manoeuvre, see Figure 1 (right). The time needed in order to perform this manoeuvre depends on the constraints Z and U as in Subsection 4.1. To this end, we define

$$t_w := \frac{\pi}{\min\{-\underline{w}, \bar{w}, \pi\}} \quad \text{and} \quad t_v := \frac{\sqrt[4]{\rho/q_1}}{\min\{-\underline{v}, \bar{v}, \sqrt[4]{\rho/q_1}\}}$$

where the vehicle can turn by 180 degrees in time t_w and drive to the y-axis in time t_v, respectively. In addition to Inequality (11), the condition

$$r_1 \leq \frac{q_1}{2} \tag{20}$$

is imposed to keep the presentation technically simple.

We begin with initial conditions $z_0 = (x_0, y_0, 0)^T \geq 0$. First, the vehicle doesn't move for $t_1 = t_w$ time units. Hence, Inequality (10) holds with $c^{\mathcal{M}_\rho}(t) := 1$, $t \in [0, t_w)$. Then, the robot drives towards the y-axis until time $t_2 = t_w + t_v$ using $u(t) = (-x_0 t_v^{-1}, 0)^T \in U$, $t \in [t_w, t_w + t_v)$, which allows to derive the estimate

$$\begin{aligned} \ell(z(t; z_0, u(\cdot)), u(t)) &\overset{(20)}{\leq} q_1 x_0^4 \left[1 - \frac{t - t_w}{t_v}\right]^4 + q_2 y_0^2 + \frac{q_1 x_0^4}{2t_v^4} \\ &\leq \ell^\star(z_0) - q_1 x_0^4 t_v^{-1} \left[(t - t_w) - \frac{1}{2t_v^3}\right] \end{aligned}$$

for $t \in [t_w, t_w + t_v)$; note that $\int_{t_w}^{t_w+t_v} (t - t_w) - 0.5t_v^{-3} \, \mathrm{d}t > 0$ holds while the integrand itself is (strictly) monotonically increasing but starting with the negative value $0.5t_v^{-3}$. Hence, there exists a time $t^\star \in (t_w, t_w + t_v)$ satisfying $\int_{t_w}^{t^\star} (t - t_w) - 0.5t_v^{-3} \, \mathrm{d}t = 0$. Then, using the function $c^{\mathcal{M}_\rho}(t) = 1 - t_v^{-1}(t - t_w - 0.5t_v^{-3})$, $t \in [t_w, t^\star)$, and $c^{\mathcal{M}_\rho}(t) \equiv 1$ on $[t^\star, t_w + t_v)$ ensures

$$\int_{t_w}^{t} \ell(x(s, z_0, u(\cdot)), u(s)) \, \mathrm{d}s \leq \int_{t_w}^{t} c^{\mathcal{M}_\rho}(s) \, \mathrm{d}s \cdot \ell^\star(z_0)$$

for all $t \in [t_w, t_w + t_v)$. The next part of the manoeuvre is performed in four seconds with constant control effort, i.e. $\|u(\cdot)\|$ is constant on $[t_v + t_w, t_v + t_w + 4)$. During the first two seconds the angle is first decreased to $-\arctan(\sqrt{y_0})$ by using

$$v(t) = -\frac{y_0 \arctan(\sqrt{y_0})}{\frac{4}{\sqrt{y_0+1}} - 4} \quad \text{and} \quad w(t) = -\arctan(\sqrt{y_0})$$

and then put back to zero ($u(t + 1) = (v(t), -w(t))^T$, $t \in [t_w + t_v, t_w + t_v + 1)$) such that $y(t_v + t_w + 2; z_0, u(\cdot)) = y_0/2$ is attained. Afterwards, these two moves are carried out backwards in order to reach the origin ($u(t + 2) = (-v(t), -w(t))^T$ and $u(t + 3) = (-v(t), w(t))^T$). Note that this strategy ensures not to move when starting at the origin. Using the estimate $\sin(s \cdot \arctan \sqrt{y_0}) \leq s\sqrt{y_0}$, the x-component of the state trajectory exhibits the upper bound $(t - t_w - t_v)\sqrt{y_0} \cdot y_0\sqrt{y_0 + 1}/(4(\sqrt{1 + y_0} - 1))$ on the time interval $[t_w + t_v, t_w + t_v + 2)$ while the y-component drops to $y(t_w + t_v + n) = (1 - n/4)y_0$ for $n \in \{0, 1, 2, 3, 4\}$. The manoeuvre has to be suitably adapted if either control constraints enforce $v(\cdot)$ or $w(\cdot)$ to be smaller or the state constraints are violated. However, since this manoeuvre is constructed for small y_0, constraints can be neglected.

Next, we evaluate the running costs and determine $c^{\mathcal{M}_\rho}(\cdot)$ on $[t_v + t_w, t_v + t_w + 4)$ such that Inequality (10) holds. Here, the estimate $|v(t)|^4 \leq (1.5 + \sqrt{\rho/q_2})^4$ is employed (Worthmann et al., 2015, Section IV.B). Analogously to this estimate, the inequality

$$x(t)^4 \leq (2 - |t - t_w - t_v - 2|)^4 (\sqrt{\rho/q_2} + 1.5)^4 \cdot y_0^2/64 =: \zeta_x(t) y_0^2$$

is obtained based on the above bound on the x-component of the state trajectory. Furthermore, let us define $\zeta_y(t) := (1 - \lfloor t - t_w - tv \rfloor/4)^2$ and $\zeta_\theta(t) := (1 - |t - t_w - t_v - 1|)^4$, $t \in [t_v + t_w, t_v + t_w + 2)$, and $\zeta_\theta(t) := (1 - |t - t_w - t_v - 3|)^4$, $t \in [t_v + t_w + 2, t_v + t_w + 4)$. In conclusion, the running costs on the considered time interval can be estimated by $c^{\mathcal{M}_\rho}(t) \cdot \ell^\star(z_0)$ with $c^{\mathcal{M}_\rho}(t)$ defined as

$$\left(\zeta_y(t) + \left[\frac{q_1 \zeta_x(t) + q_3 \zeta_\theta(t)}{q_2} + \frac{r_1(1.5 + \sqrt{\rho/q_2})^4}{64 q_2} + \frac{r_2}{q_2}\right]\right)$$

using $q_2 y_0^2 \leq \ell^\star(z_0)$. Following analogously to the conclusion of Subsection 4.1, the function $B_{\mathcal{M}_\rho} : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ satisfying (6) for all $z_0 \in \mathcal{M}_\rho$ with $\theta_0 = 0$ is defined.

Initial Conditions inside $\mathcal{M}_\rho$ with $\theta_0 \neq 0$: Next, the final $\overline{B}_{\mathcal{M}_\rho}(t)$, $t \in \mathbb{R}_{>0}$, is constructed such that the growth bound (3) also holds for z_0 with $\theta_0 \in [-\pi, 0) \cup (0, \pi)$ and, thus, for all initial conditions $z_0 \in \mathcal{M}_\rho$.

For $\theta_0 \neq 0$, the robot first turns till time t_w using $\tilde{u}(t) = (0, -\theta_0/t_w)^T$, $t \in [0, t_w)$ such that $\theta(t_w; z_0, \tilde{u}(\cdot)) = 0$ is attained. This yields the running costs $\ell(z(t; z_0, \tilde{u}(\cdot)), \tilde{u}(t))$, $t \in [0, t_w)$, given by

$$q_1 x_0^4 + q_2 y_0^2 + q_3 \theta_0^4 \left[1 - \left(\frac{t}{t_w}\right)\right]^4 + r_2 \left(\frac{\theta_0}{t_w}\right)^4.$$

Using $1 - (t/t_w) \in [0, 1]$ and Assumption (11), i.e. $r_2 \leq \frac{q_3}{2}$, leads to the estimate

$$\ell(z(t; z_0, \tilde{u}(\cdot)), \tilde{u}(t)) \leq \ell^\star(z_0) - q_3 \left(\frac{\theta_0^4}{t_w}\right) \left[t - \frac{1}{2t_w^3}\right]$$

for $t \in [0, t_w)$. Hence, there uniquely exists a time $t^\star \in (0, t_w)$ such that $\int_0^{t^\star} t - (2t_w^3)^{-1} \, \mathrm{d}t = 0$ holds. Then, the function $\tilde{c} : [0, t^\star) \to \mathbb{R}_{\geq 0}$ defined as $\tilde{c}(t) := 1 - t_w^{-1}(t - (2t_w^3)^{-1})$ satisfies the condition $\int_0^t \tilde{c}(s) - c(s) \, \mathrm{d}s \geq 0$ for all $t \in [0, t^\star)$ and is, thus, an admissible replacement. Since the remaining parts of the manoeuvre are performed as before, the resulting function $\bar{c}(\cdot)$, see Definition 2, yields the desired growth bound $\overline{B}_{\mathcal{M}_\rho}(t) : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$.

5. NUMERICAL RESULTS

The derived growth function $\overline{B}(\cdot)$, which is defined as in (9) and satisfies Inequality (6), is now employed in order to determine a prediction horizon T such that the resulting MPC closed loop is asymptotically stable. Let the control and the state constraints be given by $U = [-0.6, 0.6] \times [-\pi/4, \pi/4]$ and $Z = [-2, 2]^2 \times \mathbb{R}$, respectively. Additionally, we define the weighting parameters of the running costs $\ell(\cdot, \cdot)$ as $q_1 = 1$, $q_3 = 0.1$, $r_1 = q_1/2$, and $r_2 = q_3/2$ while q_2 is restricted to the set $\{2, 5\}$. Then, for a fixed control horizon δ, a stabilizing horizon $\hat{T}$ can be computed via Algorithm 2. Note that the sampling period corresponds to the control horizon δ in Algorithm 2.

Figure 3 shows the results of Algorithm 2. As can be noticed, a slight improvement in the prediction horizon length is observed in comparison to the discrete time results presented in Worthmann et al. (2015) due to the less conservative growth bound based on swaps and

Algorithm 2 Calculation of a stabilizing horizon $\hat{T}$

Given: Control bound $\underline{v}, \bar{v}, \underline{w}, \bar{w}$; box constraint $\underline{x}, \bar{x}, \underline{y}, \bar{y}$; weighting parameters q_1, q_2, q_3, r_1, r_2; a control horizon δ.
Initialization: Set prediction horizon $T = \delta$ and $\bar{\alpha} = 0$.

while $\bar{\alpha} = 0$ **do**
    Increment T by δ.
    Compute $\rho \in \mathbb{R}_{\geq 0}$ such that the maximum of $\int_0^T \bar{c}^{\mathcal{M}_\rho^c}(s)\,\mathrm{d}s$ and $\int_0^T \bar{c}^{\mathcal{M}_\rho}(s)\,\mathrm{d}s$ is minimized.
    For each $t \in [T-\delta, T)$, define $\overline{B}(t)$ as $\min\{t, \max\{\int_0^t \bar{c}^{\mathcal{M}_\rho^c}(s)\,\mathrm{d}s, \int_0^t \bar{c}^{\mathcal{M}_\rho}(s)\,\mathrm{d}s\}\}$
    Set $\bar{\alpha} = 1 - \dfrac{e^{-\int_\delta^T \overline{B}(t)^{-1}\mathrm{d}t}\, e^{-\int_{T-\delta}^T \overline{B}(t)^{-1}\mathrm{d}t}}{\left[1-e^{-\int_\delta^T \overline{B}(t)^{-1}\mathrm{d}t}\right]\left[1-e^{-\int_{T-\delta}^T \overline{B}(t)^{-1}\mathrm{d}t}\right]}$.
end while

Output Stabilizing horizon length $\hat{T}$.

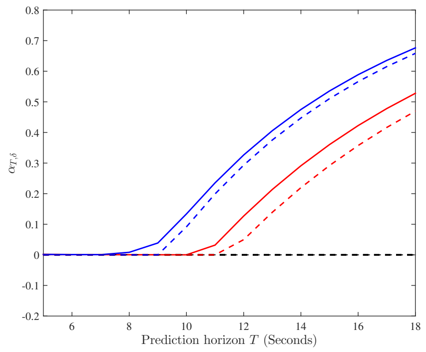

Fig. 3. Performance index $\alpha_{T,\delta}$, $\delta = 1$, in dependence on the prediction horizon T (solid) in comparison to its discrete time counterpart (dashed) based on the parameters of Section 5 with $q_2 \in \{2, 5\}$ (red/blue).

replacements. Indeed, $\hat{T}$ is the minimal stabilizing horizon, which yields $\bar{\alpha} > 0$ while being a multiple of the sampling period δ.

6. CONCLUSIONS

In this paper, we extended results obtained in a discrete time setting Worthmann et al. (2015) to the continuous time domain. The main tool needed in order to fulfil this task are the newly introduced swaps and replacements, which allow us to exploit particular open loop trajectories to deduce growth bounds for sets of initial conditions. Then, asymptotic stability of the MPC closed loop can be rigorously proven for nonholonomic mobile robots without using stabilizing constraints and/or costs. In particular, the intersampling behaviour is covered in contrast to the discrete time framework. Furthermore, the estimates on the required prediction horizon length are slightly improved in comparison to our previous work in the discrete time setting, see also Worthmann et al. (2014).

For the sake of simplicity, in this paper we considered a simple vehicle model. More complex models, which account for skidding and slipping of the robot wheels have been investigated in the literature, see, e.g. Yoo (2010). Future research will aim at extending the proposed framework to more advanced vehicle models.

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