Quadratic costs do not always work in MPC ⋆

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Abstract

We consider model predictive control (MPC) without terminal costs and constraints. Firstly, we rigorously show that MPC based on quadratic stage costs may fail, i.e., there does not exist a prediction horizon length such that a (controlled) equilibrium is asymptotically stable for the MPC closed loop although the system is, e.g., finite time controllable. Hence, stability properties of the infinite horizon optimal control problem are, in general, *not* preserved in MPC as long as purely quadratic costs are employed. This shows the necessity of using the stage cost as a design parameter to achieve asymptotic stability. Furthermore, we relax the standard controllability assumption employed in MPC without terminal costs and constraints to alleviate its verification.

Key words: Model predictive control, nonlinear systems, mobile robots, asymptotic stabilization, quadratic costs.

1 Introduction

Model predictive control (MPC) is nowadays a well-established control methodology — both from a more theoretical point of view [15,21,12] and in many different fields of application [19,4,24]. One of the key drivers for its success story is the simplicity of the basic idea: measure the current state, solve a finite-horizon optimal control problem online, and implement the first portion of the computed control strategy. Then, this loop is iteratively repeated ad infinitum to generate an input signal on the infinite time horizon. MPC is particularly attractive due to its capability to deal with constrained multi-input multi-output systems. However, its stability analysis is far from being trivial and closed-loop stability is not necessarily guaranteed if the MPC controller is not designed appropriately, see, e.g., [20]. In order to establish asymptotic stability of the MPC closed loop, there are two main approaches available in the literature. The first is to impose suitable additional terminal constraints and terminal costs in the repeatedly solved optimal control problem, see, e.g., [5,17] or the textbook [21]. Alternatively, a certain *controllability assumption* is required [18,10,11] in order to avoid the necessity to use such (artificial) terminal ingredients.

The goal of this paper is to shed light on various aspects and assumptions of MPC without terminal cost and terminal constraints. Within this setting, typically a certain controllability assumption is used, which is formulated in terms of an upper bound on the optimal value function, see, e.g., [27,14,13,23]. While such a controllability condition can be used to establish asymptotic stability of the MPC closed loop, its verification is in general a difficult task, see, e.g., [30] for a non-trivial example. As a first main contribution, we weaken this controllability condition, which might help to alleviate this difficulty. In doing so, we also show that this new relaxed condition is *sharp*.

Typically, MPC is applied to solve set point stabilization (tracking) problems. To this end, stage (running) costs are constructed such that the deviation from the desired state and the control effort are penalized. A typical choice in industrial practice is to use quadratic cost functions, i.e., the distance from the set point is weighted quadratically. As a second main contribution, we illustrate via a simple example (the nonholonomic integrator/robot) that when using such a quadratic stage cost, MPC might in general *not* be stabilizing — independent of the length of the prediction horizon. This means that

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stability properties of the infinite horizon optimal control problem are not necessarily preserved — even for a very large prediction horizon; rather the stage cost has to be suitably chosen to ensure asymptotic stability.

Finally, we discuss sufficient conditions under which quadratic stage cost functions can be used, i.e., under which the above described situation that quadratic stage cost functions might fail cannot occur. Besides the well known case where the linearization is stabilizable [5], we present a (nonlinear) local controllability condition which is suitable to this end.

The remainder of this paper is structured as follows. In Section 2, we introduce the considered problem setup and briefly recall stability results in MPC without terminal constraints and costs. In Section 3, we show how the standard controllability condition used in this setting can be relaxed and discuss implications of this relaxation for closed-loop performance statements. Section 4 shows that MPC with a quadratic stage cost might *not* be stabilizing independent of the length of the prediction horizon, before a sufficient condition is derived under which quadratic cost functions *work*. Finally, Section 5 concludes the paper.

Notation: $\mathbb{N}$ and $\mathbb{R}_{\geq 0}$ denote the natural and the non-negative real numbers. $\mathcal{B}_r(x)$ denotes the ball $\{y \in \mathbb{R}^n | \|y - x\| \leq r\}$ of radius $r \in \mathbb{R}_{>0}$ centered around $x \in \mathbb{R}^n$, where $\|\cdot\|$ is the Euclidean norm of the vector x.

2 Problem Formulation

In this work, nonlinear systems governed by ordinary differential equations of the form

$$\dot{x}(t) = f(x(t), u(t)) \tag{1}$$

with continuous vector field $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ are considered. The state and the control input at time $t \in \mathbb{R}_{\geq 0}$ are denoted by $x(t) \in \mathbb{R}^n$ and $u(t) \in \mathbb{R}^m$, respectively. In addition, the inputs of system (1) are subject to pointwise in time input constraints, i.e. $u(t) \in U \subseteq \mathbb{R}^m$, where U is supposed to be closed. For the sake of completeness, the control functions $u : \mathbb{R}_{\geq 0} \to \mathbb{R}^m$ are assumed to be measurable and locally (Lebesgue-)integrable, i.e. $u \in \mathcal{L}^1_{\text{loc}}([0, \infty), \mathbb{R}^m)$. Moreover, the vector field f is supposed to be continuous and locally Lipschitz with respect to its first argument such that existence and uniqueness of the solution $x(\cdot; x^0, u)$ of (1) for given control function u and initial state x^0 is at least locally ensured. To simplify the notation, the solution is denoted by $x(\cdot)$ if there is no ambiguity.

The control objective is to asymptotically stabilize a (controlled) equilibrium, which without loss of generality is assumed to be the origin, i.e. $f(0, 0) = 0$ and $0 \in U$. We want to fulfill this control task with model predictive control. To this end, the cost functional $J_T : \mathcal{L}^1([0, T), U) \times \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ is defined as

$$J_T(u, \hat{x}) := \int_0^T \ell(x(t; \hat{x}, u), u(t))\,\mathrm{d}t \tag{2}$$

based on the positive definite stage cost function $\ell : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}_{\geq 0}$. The corresponding (optimal) value function $V_T : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ is given by

$$V_T(\hat{x}) := \inf_{u \in \mathcal{L}^1([0,T),U)} J_T(u, \hat{x}). \tag{3}$$

Note that (2) is only well-defined if the solution trajectory $x(\cdot)$ exists on $[0, T]$. In the following, it is tacitly assumed that, for each state $\hat{x} \in \mathbb{R}^n$, there is at least one admissible control, i.e.

$$\{u \in \mathcal{L}^1([0, T), U) : x(\cdot; \hat{x}, u) \text{ exists on } [0, T]\} \neq \emptyset$$

holds for all $\hat{x} \in \mathbb{R}^n$. Moreover, let us suppose that the infimum of the right hand side in (3) is attained, i.e. existence of an admissible control $u^\star$ such that $V_T(\hat{x}) = J_T(u^\star, \hat{x})$ holds. Note that both the cost functional J_T and the value function V_T depend on the horizon length $T > 0$.

Using the introduced notation, the MPC scheme is as follows.

Algorithm 1 MPC Algorithm

Given: Prediction horizon length $T > 0$ and sampling period $\delta \in (0, T)$.
Set $t = 0$.
(1) Measure the current state $\hat{x} := x(t)$.
(2) Compute a minimizer $u^\star : [0, T) \to \mathbb{R}^m$ of (2).
(3) Implement $u^{MPC}(t + \tau) = u^\star(\tau)$ for $\tau \in [0, \delta)$, set $t = t + \delta$, and goto Step 1.

Algorithm 1 is an MPC scheme without terminal constraints and costs.

Remark 1 *The above problem formulation can be extended to also include state constraints, i.e., $x(t) \in X \subseteq \mathbb{R}^n$ is required to hold for all $t \geq 0$. When considering MPC schemes without additional terminal constraints, the presence of such state constraints necessitates some additional assumptions or techniques in order to ensure recursive feasibility of the MPC algorithm, see, e.g., [1] or [12, Chapter 7]. The following results then also hold in such a setting including state constraints.*

Without terminal constraints and costs, the prediction horizon T has to be chosen large enough such that in combination with a certain controllability assumption, asymptotic stability can be concluded. This controllability assumption is typically stated as the existence of a suitable upper bound on the optimal value function, see, e.g. [27] and [23] for its counterpart in continuous time.

Assumption 2 (Growth Bound) *Let a continuous, monotonically increasing, and bounded function $B : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ be given such that, for each $x \in \mathbb{R}^n$ and each $t \in \mathbb{R}_{\geq 0}$, the following inequality holds for the optimal value function V_t as defined in* (3)*:*

$$V_t(x) \leq B(t) \cdot \inf_{u \in U} \ell(x, u). \tag{4}$$

Typically, $\inf_{u \in U} \ell(x, u) = \ell(x, 0)$ holds, e.g. if the control effort is penalized by an additive term in the stage cost ℓ. Using Assumption 2, one can obtain the following result, see [23].

Theorem 3 *Suppose that Assumption 2 is satisfied and that there exists a $\mathcal{K}_\infty$-function*[1] *η such that $\eta(\|x\|) \leq \inf_{u \in U} \ell(x, u)$ holds for all $x \in \mathbb{R}^n$. Then there exists $T > 0$ such that the MPC closed loop resulting from Algorithm 1 is (globally) asymptotically stable.*

In fact, the results in [13] (discrete time) and [23] (continuous time setting) also provide a technique to estimate the prediction horizon length T such that asymptotic stability holds (see [31] for the connection of both approaches). The basic idea is to interpret the value function V_T as a Lyapunov function. In addition, a performance estimate of the MPC closed loop compared to the infinite horizon optimal solution can be concluded (degree of suboptimality α).

On the one hand, our goal is to analyze and weaken Assumption 2 for the presented MPC scheme. On the other hand, we study the applicability of Algorithm 1 in Section 4; in particular, we show that quadratic stage costs, which are very often used in practice, are not always suitable. To this end, we consider a simple example to demonstrate that MPC might fail for quadratic stage cost ℓ in the sense that asymptotic stability cannot be achieved independently of the prediction horizon length T.

3 Asymptotic stability and performance with unbounded growth function B

In this section, we show that the results of Theorem 3 still hold if Assumption 2 is relaxed, in particular the boundedness of B, which – to the best of our knowledge – is always assumed in the literature. This relaxation might help to verify Assumption 2, which is quite difficult in general except for linear, stabilizable systems [1].

[1] A function $\eta : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is said to be of class $\mathcal{K}_\infty$ if it is continuous, zero at zero, strictly monotonically increasing, and unbounded.

We first discuss asymptotic stability in Subsection 3.1, before considering closed-loop performance (Subsection 3.2). Finally, in Subsection 3.3 we discuss various aspects resulting from the relaxed condition, including that it is *sharp* in the sense that the requirements on B cannot be further relaxed in order to still serve as a sufficient condition for asymptotic stability.

3.1 Asymptotic stability

Our first main result is as follows.

Theorem 4 *Suppose that Inequality* (4) *holds for all $t \in \mathbb{R}_{\geq 0}$ and each $x \in \mathbb{R}^n$ for a function $B : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ satisfying the asymptotic growth condition*

$$\limsup_{t \to \infty} \frac{B(t)}{t} =: \tilde{c} \in [0, 1). \tag{5}$$

In addition, assume existence of a $\mathcal{K}_\infty$-function η such that the inequality $\eta(\|x\|) \leq \inf_{u \in U} \ell(x, u)$ holds for all $x \in \mathbb{R}^n$. Then, for each $\delta > 0$, there exists a prediction horizon length $T^\star > \delta$ such that the origin is (globally) asymptotically stable w.r.t. the MPC closed-loop system for all prediction horizons T, $T \geq T^\star$.

Proof: Let $\delta > 0$ be given. We show existence of a prediction horizon length $T^\star$, $T^\star > \delta$, such that $\alpha = \alpha(T^\star) > 0$ holds with

$$\alpha(T) = 1 - \frac{\exp\left(-\int_\delta^T B(t)^{-1}\,\mathrm{d}t\right)\exp\left(-\int_{T-\delta}^T B(t)^{-1}\,\mathrm{d}t\right)}{\left[1 - \exp\left(-\int_\delta^T B(t)^{-1}\,\mathrm{d}t\right)\right]\left[1 - \exp\left(-\int_{T-\delta}^T B(t)^{-1}\,\mathrm{d}t\right)\right]}. \tag{6}$$

Then, [23, Theorem 9] and [2] guarantee asymptotic stability of the MPC closed loop and, thus, the assertion. Using Formula (6) for $\alpha(T)$, the condition $\alpha(T) > 0$ is equivalent to

$$1 - \exp\left(-\int_\delta^T B(t)^{-1}\,\mathrm{d}t\right) - \exp\left(-\int_{T-\delta}^T B(t)^{-1}\,\mathrm{d}t\right) > 0. \tag{7}$$

Assumption (5) implies the existence of a (time) instant $\bar{t} \geq 0$ and a constant c such that the function $\bar{B} : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ defined as

$$\bar{B}(t) := \begin{cases} c & t \leq \bar{t} \\ c + \tilde{c}(t - \bar{t}) & t > \bar{t} \end{cases}$$

satisfies the inequality $B(t) \leq \bar{B}(t)$ for all $t \geq 0$. Hence, using the monotonicity property [28], which can be directly transferred to the continuous time setting, and the definition of $\alpha(T)$ given by Formula (6), the condition

$$1-\exp\left(-\int_{\delta}^{T}\bar{B}(t)^{-1}\,\mathrm{d}t\right)-\exp\left(-\int_{T-\delta}^{T}\bar{B}(t)^{-1}\,\mathrm{d}t\right)>0 \tag{8}$$

is sufficient to conclude $\alpha(T)>0$ and, thus, the desired asymptotic stability of the MPC closed loop for $T-\delta>\max\{\bar{t},\delta\}$.[2] Hence, we get

$$\begin{aligned}
&1-e^{-\int_{\delta}^{T}\bar{B}(t)^{-1}\,\mathrm{d}t}-e^{-\int_{T-\delta}^{T}\bar{B}(t)^{-1}\,\mathrm{d}t}\\
=&1-e^{-\int_{T-\delta}^{T}\bar{B}(t)^{-1}\mathrm{d}t}\left(1+e^{-\int_{\delta}^{\bar{t}}\bar{B}(t)^{-1}\mathrm{d}t}e^{-\int_{\bar{t}}^{T-\delta}\bar{B}(t)^{-1}\mathrm{d}t}\right)\\
=&1-\sqrt[\tilde{c}]{\frac{c+\tilde{c}(T-\bar{t}-\delta)}{c+\tilde{c}(T-\bar{t})}}\left(1+\sqrt[\tilde{c}]{\frac{c}{c+\tilde{c}(T-\bar{t}-\delta)}}\cdot e^{\frac{\delta-\bar{t}}{c}}\right)\\
=&1-\sqrt[\tilde{c}]{\frac{1}{c+\tilde{c}(T-\bar{t})}}\left(\sqrt[\tilde{c}]{c+\tilde{c}(T-\bar{t}-\delta)}+\sqrt[\tilde{c}]{c}\cdot e^{\frac{\delta-\bar{t}}{c}}\right).
\end{aligned}$$

Consequently, the stability condition (8) holds if and only if the inequality

$$\sqrt[\tilde{c}]{c+\tilde{c}(T-\bar{t})}-\sqrt[\tilde{c}]{c+\tilde{c}(T-\bar{t}-\delta)}>\sqrt[\tilde{c}]{c}\cdot e^{\frac{\delta-\bar{t}}{c}} \tag{9}$$

is satisfied. Here, we point out that the right hand side is a (strictly positive) *constant* value for a given growth function $B:\mathbb{R}_{\geq 0}\to\mathbb{R}_{\geq 0}$ and fixed δ.

Next, we define the function $g:(\delta+\max\{\bar{t},\delta\},\infty)\to\mathbb{R}$,

$$T\mapsto\sqrt[\tilde{c}]{c+\tilde{c}(T-\bar{t}-\delta)}.$$

Using this definition, the left hand side of (9) can be written as the difference $g(T+\delta)-g(T)$. Since g is a $\mathcal{C}^\infty$-function on its domain, using Taylor series expansion theory implies the existence of $\xi\in(T,T+\delta)$ such that

$$g(T+\delta)-g(T)=\delta g'(T)+\frac{\delta^2}{2}g''(\xi)$$

holds. Thus, the left hand side of (9) equals

$$\delta(c+\tilde{c}(T-\bar{t}-\delta))^{\frac{1-\tilde{c}}{\tilde{c}}}+\frac{\delta^2(1-\tilde{c})}{2}(c+\tilde{c}(\xi-\bar{t}-\delta))^{\frac{1-2\tilde{c}}{\tilde{c}}}.$$

Here, both summands are positive. Hence, the unboundedness of the first summand ($\tilde{c}^{-1}-1>0$) implies that the left hand side of (9) grows unboundedly for T tending to infinity. This shows the existence of a prediction horizon length $T^\star$ such that inequality (9) is satisfied for all $T\geq T^\star$. Thus, also the stability condition (8) holds for all $T\geq T^\star$, which allows to conclude the assertion. □

[2] This restriction is only made to simplify the upcoming calculations.

3.2 Recovery of Optimal Performance

In the previous section we derived asymptotic stability of the MPC closed loop assuming an asymptotic growth bound on the value function. To this end, the main idea was to exploit the degree of suboptimality $\alpha=\alpha(T)$ as introduced in [13] and [23]. Besides guaranteeing asymptotic stability if $\alpha(T)>0$, this performance index links the MPC closed-loop cost $J_\infty(u^{\mathrm{MPC}},\hat{x})$ with the infinite horizon optimal cost $V_\infty(\hat{x})$, i.e.

$$J_\infty(\hat{x},u^{\mathrm{MPC}})\leq\alpha(T)^{-1}V_T(\hat{x})\leq\alpha(T)^{-1}V_\infty(\hat{x})$$

holds, see [11,23] for details. This means that α specifies how suboptimal the closed loop resulting from application of MPC is compared to the infinite horizon optimal performance. When using Assumption 2, i.e., when using a bounded function B, it is well known that MPC yields approximately optimal performance for a sufficiently large prediction horizon since $\lim_{T\to\infty}\alpha(T)=1$ holds independently of the parameter δ, see, e.g., [13] and [22, Proposition 3.17]. The following lemma shows that this property cannot be directly concluded based on Formula (6) for unbounded growth functions B as in Inequality (5).

Lemma 5 *Let $B:\mathbb{R}_{\geq 0}\to\mathbb{R}_{\geq 0}$ be a monotonically increasing function satisfying*

$$B(t)=B(\bar{t})+\tilde{c}(t-\bar{t})\qquad\forall\, t\geq\bar{t}.$$

with $\tilde{c}\in(0,1)$. Then, for $\delta=\xi T$, $\xi\in(0,1)$, the degree of suboptimality $\alpha(T)$ given by Formula (6) converges to

$$1-\frac{(\xi-\xi^2)^{1/\tilde{c}}}{\left(1-\xi^{1/\tilde{c}}\right)\left(1-(1-\xi)^{1/\tilde{c}}\right)}\in(0,1)$$

for prediction horizon T tending to infinity.

Proof: Let the prediction horizon T be chosen such that $\min\{\xi,1-\xi\}T\geq\bar{t}$ holds. Then,

$$\begin{aligned}
e^{-\int_{\rho T}^{T}B(t)^{-1}\,\mathrm{d}t} &= \exp\left(-\int_{\rho T}^{T}\frac{1}{B(\bar{t})+\tilde{c}(t-\bar{t})}\,\mathrm{d}t\right)\\
&= \exp\left(-\tilde{c}^{-1}\ln\left(\frac{B(\bar{t})+\tilde{c}(T-\bar{t})}{B(\bar{t})+\tilde{c}(\rho T-\bar{t})}\right)\right)\\
&\overset{T\to\infty}{\longrightarrow}\rho^{1/\tilde{c}}
\end{aligned}$$

holds for $\rho\in\{\xi,1-\xi\}$. Consequently,

$$\alpha(T)\overset{T\to\infty}{\longrightarrow}1-\frac{\xi^{1/\tilde{c}}(1-\xi)^{1/\tilde{c}}}{\left(1-\xi^{1/\tilde{c}}\right)\left(1-(1-\xi)^{1/\tilde{c}}\right)}$$

holds. Next, we show that this limit is always strictly positive for $\tilde{c}\in(0,1)$ (and strictly negative for $\tilde{c}>1$).

This assertion is equivalent to

$$1 - \xi^{1/\tilde{c}} - (1-\xi)^{1/\tilde{c}} > 0.$$

But, since $\xi \in (0,1)$ and $1 = \xi + (1-\xi)$ hold, the claimed inequality holds for $\tilde{c} \in (0,1)$, i.e. $\lim_{T\to\infty} \alpha(T) > 0$. Hence, taking $\xi \in (0,1)$ into account shows $\lim_{T\to\infty} \alpha(T) \in (0,1)$, which completes the proof. □

Although the function B satisfies the asymptotic growth condition (5), Lemma 5 shows that, even for $T \to \infty$, optimal performance cannot be concluded from (6). The following proposition is the key in order to resolve this apparent contradiction. Here, we show that boundedness of B can indeed be assumed in Assumption 2 without loss of generality. In particular, we provide a guideline on the construction of such a bounded growth function $\widetilde{B}$ once an unbounded one is found.

Proposition 6 *Suppose that the conditions of Theorem 4 are satisfied, i.e., Inequality* (4) *holds for all* $t \geq 0$ *and each* $x \in \mathbb{R}^n$ *for a function* $B : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ *satisfying the asymptotic growth condition* (5)*. Then, there exists a bounded function* $\tilde{B} : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ *also satisfying Inequality* (4)*.*

Proof: Applying Theorem 4 (for some fixed δ) allows to conclude existence of a large enough prediction horizon T such that $\alpha(T) > 0$. As discussed above, a positive value $\alpha(T)$ in turn implies the performance estimate $J_\infty(x, u^{MPC}) \leq \alpha(T)^{-1} V_T(x)$ for all $x \in \mathbb{R}^n$, see [11,23]. Since the definition of the optimal value function implies $V_\infty(\hat{x}) \leq J_\infty(u_{MPC}, \hat{x})$, this allows to conclude

$$V_\infty(x) \leq \alpha(T)^{-1} V_T(x) \leq \alpha(T)^{-1} B(T) \inf_{u\in U} \ell(x,u).$$

Since $V_t(x) \leq V_\infty(x)$ holds for all $t \geq 0$, it follows that

$$V_t(x) \leq \alpha(T)^{-1} B(T) \inf_{u\in U} \ell(x,u)$$

holds for all $t \geq 0$. Hence, Inequality (4) can also be satisfied by the bounded function $\widetilde{B}$ defined as

$$\widetilde{B}(t) := \min\{B(t), \alpha(T)^{-1} B(T)\}. \qquad (10)$$

□

The following theorem is essentially a corollary of Theorem 4 and Proposition 6. It shows that one of the basic properties of MPC also holds true for unbounded functions B satisfying condition (5), namely that MPC yields approximately optimal performance for prediction horizon T approaching infinity.

Theorem 7 (Optimal Performance) *Let the conditions of Theorem 4 be satisfied and a (desired) performance index* $\overline{\alpha} < 1$ *be given. Then, for sampling period* $\delta > 0$ *or sampling period* $\delta = \gamma \cdot T$ *with* $\gamma \in (0,1)$*, there exists a prediction horizon* $T^\star$ *such that* $\alpha(T) > \overline{\alpha}$ *holds for all* $T \geq T^\star$*.*

Proof: Proposition 6 shows that Condition (5) implies that Assumption 2 also holds with a bounded growth function $\widetilde{B} : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$. Since $\widetilde{B}$ is bounded, there exists a constant $\eta > 0$ such that $B(t)^{-1} \geq \eta$ holds for all $t \geq 0$. Hence, the inequality

$$\int_\delta^T B(t)^{-1}\,\mathrm{d}t \geq (T-\delta)\eta$$

holds. The condition on δ implies that $T - \delta \to \infty$ as $T \to \infty$. This implies

$$\exp\Big(-\int_\delta^T B(t)^{-1}\,\mathrm{d}t\Big) \leq \exp(-(T-\delta)\eta) \stackrel{T\to\infty}{\longrightarrow} 0.$$

In conclusion $\alpha(T) \to 1$ for $T \to \infty$ in view of Formula (6), which completes the proof. □

3.3 Discussion

We now discuss several aspects of the previous results. First, we note that in MPC without terminal costs and constraints, the main difficulty is the verification of Assumption 2. Our results in Section 3.1 show that it is sufficient to devise a possibly unbounded growth function B as long as it satisfies Condition (5). While Proposition 6 ensures that in this case, in fact also a bounded function B exists, verifying Inequality (4) with an unbounded function B for a given system might in general be easier. Hence, our results contribute to alleviate the verification process needed to establish Assumption 2.

Moreover, Proposition 6 can be used to derive better estimates for a stabilizing prediction horizon T as shown in the following academic example.

Example 8 *Let us consider* $B(t) := 2 + 4t/5$*, i.e.* $\tilde{c} = 4/5$ *in* (5)*. For sampling period* $\delta = 1$*, a horizon length* $T = 213$ *is necessary for* $\alpha(T) > 0$*. On the other hand, when the sampling period* δ *is chosen proportionally to the prediction horizon* T *in Formula* (6)*,* $\alpha(T) > 0$ *holds for much shorter* T*, see Figure 1. For example, choosing* $\delta = T/2$ *yields a positive suboptimality degree* $\alpha(T)$ *for* $T > 5 \cdot \sum_{i=1}^4 \sqrt[5]{2^{-i}} \approx 14.31$*. However,*

$$\lim_{T\to\infty} \alpha(T) \stackrel{Lemma\ 5}{=} 1 - 2^{-5/2}/(1-2^{-5/4})^2 \approx 0.474$$

holds, i.e. we cannot conclude (near) optimal performance — even for a very large prediction horizon T*.*

We make use of Proposition 6 to obtain shorter stabilizing prediction horizons also for fixed δ*. For* $T \approx 28.69486$

and $\delta = T/2$, $B(T)\alpha(T)^{-1} \leq 97.21$ holds.[3] *Hence, constructing a growth function via* (10) *yields smaller bounds for* $t \gtrsim 119.01$. *This reduces the required prediction horizon length*

$$\widehat{T} := \min\{k\delta : \alpha(k\delta) > 0 \text{ and } k \in \mathbb{N}_{\geq 2}\}$$

for $\delta = 1$ *from* 213 *to* 133, *i.e. by more than* 37.5%. *Moreover, (near) optimal performance is recovered for sufficiently large prediction horizon* T *as shown in Figure 1.*

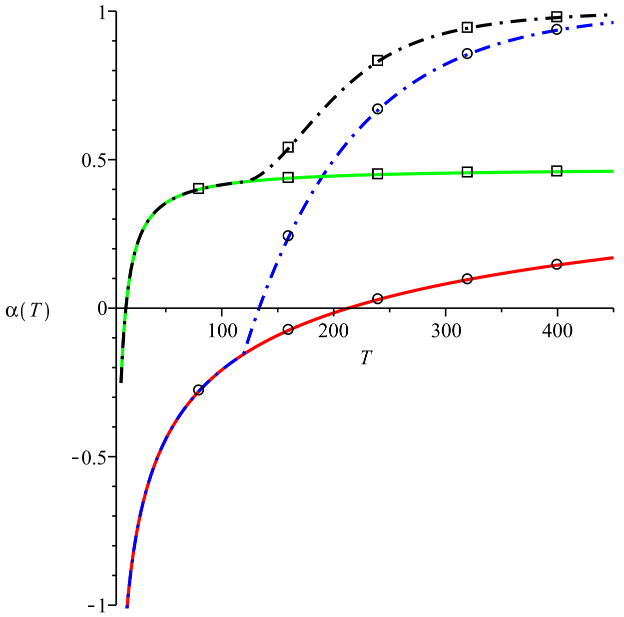

Fig. 1. Visualization of the performance index $\alpha(T)$ determined by (6) for growth bound $B(t) := 2 + 4t/5$ with $\delta = 1$ (solid red line; ∘) and $\delta = T/2$ (solid green line; □). The performance index $\alpha(T)$ is also drawn based on the growth function $\widetilde{B}(t) := \min\{B(t), 97.21\}$ for $\delta = 1$ (dashed-dotted blue line; ∘) and $\delta = T/2$ (dashed-dotted black line; □).

While the actual numbers may seem arbitrary, the idea to make use of estimates based on $\delta = T/2$ and to exploit these for determining a stabilizing horizon $\widehat{T}$ for a desired sampling period δ may be helpful and comes without any additional costs during runtime of the MPC algorithm. Note that $\alpha(T)$ is maximized at $\delta = T/2$ in the exponentially stabilizable case [13] and can, thus, be used as a good rule of thumb.

Finally, we briefly discuss how precise or conservative the sufficient condition (5) is. Clearly, it is not necessary for asymptotic stability, which can be seen by the following simple example.

[3] For the considered function B it can be numerically shown that there exists a minimizer $T^\sharp$ of $\alpha(T)^{-1}B(T)$.

Example 9 *Consider the system*

$$\dot{x} = f(x,u) \quad \text{with} \quad f(x,u) = \begin{cases} -x^2 & \text{if } x \geq 0 \\ x^2 & \text{if } x < 0 \end{cases}.$$

For an initial condition $x^0 \neq 0$, the solution is given by $x(t; x^0, u) = x^0/(1 + |x^0|t)$. Choosing $\ell(x,u) = x^2 + u^2$, it trivially follows that the optimal input is $u \equiv 0$, and the optimal value function for horizon T is given by

$$V_T(x^0) = \int_0^T \left(\frac{x^0}{1+|x^0|t}\right)^2 \mathrm{d}t = |x^0|\left(1 - \frac{1}{1+|x^0|T}\right).$$

Hence, we obtain $\sup_{x^0 \in \mathbb{R}\setminus\{0\}} V_T(x^0)/\ell(x^0,0) = T$, i.e., $B(t) = t$. Yet, the system is trivially asymptotically stable independent of the prediction horizon T.

On the other hand, Condition (5) is *sharp* in the sense that asymptotic stability cannot be guaranteed in general anymore when it is relaxed, i.e., if $\limsup_{t\to\infty} B(t)t^{-1} \geq 1$ holds. This can, e.g., be seen by the nonholonomic robot as considered in Subsection 4.1. Here, by applying the zero input, the robot stays at its initial position and hence one immediately obtains satisfaction of Assumption 2 with $B(t) = t$ for all stage cost functions satisfying $\inf_{u\in U} \ell(x,u) = \ell(x,0)$ (which is, e.g., the case if $\ell(x,u) = x^T Q x + u^T R u$ with Q, R symmetric and positive definite). However, when using such a quadratic stage cost, there does not exist a prediction horizon $T > 0$ such that asymptotic stability for the resulting MPC closed-loop system can be guaranteed, as shown below in Section 4. Hence, Condition (5) (and, thus, also Assumption 2) cannot be satisfied for this example with quadratic stage cost.

4 Quadratic stage costs do not always work

In this section, we show that quadratic stage cost functions do not always result in an asymptotically stable closed-loop system independently of the prediction horizon length T, which will be done using the nonholonomic robot example (see Subsection 4.1). Afterwards, in Subsection 4.2, we discuss sufficient conditions under which quadratic cost function can be employed.

4.1 Nonholonomic Robot Example

The system dynamics of the nonholonomic robot are modeled as

$$\dot{x}(t) = \frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} x_1(t) \\ x_2(t) \\ \theta(t) \end{pmatrix} = \begin{pmatrix} \cos(\theta(t)) & 0 \\ \sin(\theta(t)) & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} u_1(t) \\ u_2(t) \end{pmatrix} \quad (11)$$

where the state vector $x \in \mathbb{R}^n$, $n = 3$, consists of the position in the plane $(x_1, x_2)^T$ and the orientation θ of the robot, while the control is given by $u = (u_1, u_2)^T \in \mathbb{R}^m$, $m = 2$. Obviously, for the control $u = (0, 0)^T$, every state $x \in \mathbb{R}^n$ is a (controlled) equilibrium. As shown, e.g., in [16, pp. 83-89], the nonholonomic robot (unicycle) is (locally) equivalent to Brockett's nonholonomic integrator (see [3]). Hence, all structural assertions can be directly transferred to this example and vice versa.

It is well known that there does neither exist a continuous (static) state feedback law $\mu : \mathbb{R}^n \to \mathbb{R}^m$ that asymptotically stabilizes this system [3] nor a continuously differentiable control Lyapunov function, see, e.g., [6]. This makes MPC design with terminal region and cost very difficult, see, e.g. [8,9] where a non-quadratic terminal cost is employed and the origin is not located in the interior of the terminal region. Moreover, whenever Assumption 2 (or variants of it) were verified for this system in the context of MPC without terminal constraints and costs, non-quadratic stage costs ℓ were used, see [10,22] (exponential controllability of the non-holonomic integrator with respect to its stage costs) and [29] for a more elaborated growth bound. In the following, we use Pontryagin's maximum principle to show that for each prediction horizon $T > 0$, there exists an initial value $\hat{x}$ such that the optimal solution is $u \equiv 0$, i.e., the robot stays at its initial value. This means that no prediction horizon exists which asymptotically stabilizes the closed-loop system when using a quadratic stage cost function.

Consider the finite horizon optimal control problem

$$\min_{u \in \mathcal{L}^1([0,T),U)} \int_0^T x(t)^T Q x(t) + u(t)^T R u(t) \,\mathrm{d}t \qquad (12)$$

with $x(t) = x(t; x^0, u)$ subject to the system dynamics (11) and the initial condition $x(0; x^0, u) = x^0$. Before we begin with our analysis, let us recall some notations and Pontryagin's maximum principle, see, e.g., [25, Subsection 2.2.3]. The *Hamiltonian* is defined as

$$\mathcal{H}(x, u, \lambda) = -x^T Q x - u^T R u + \lambda^T f(x, u).$$

Then, since no terminal costs are present, every optimal control function $u^\star \in \mathcal{L}^1([0,T), U)$ satisfies the following necessary conditions

$$\begin{aligned}
&\dot{x}^\star(t) = f(x^\star(t), u^\star(t)), \quad x^\star(0) = x^0, \\
&\dot{\lambda}(t) = -\mathcal{H}_x(x^\star(t), u^\star(t), \lambda(t)), \quad \lambda(T) = 0, \\
&\mathcal{H}(x^\star(t), u^\star(t), \lambda(t)) \geq \mathcal{H}(x^\star(t), u, \lambda(t)) \quad \forall\, u \in U
\end{aligned}$$

for almost all $t \in [0, T]$. [4] Hence, the adjoint $\lambda : [0, T] \to \mathbb{R}^3$ is governed by the linear differential equation

$$\dot{\lambda}(t) = 2Qx^\star(t) - \lambda(t)^T \begin{pmatrix} 0 & 0 & -\sin(\theta^\star(t)) u_1^\star(t) \\ 0 & 0 & \cos(\theta^\star(t)) u_1^\star(t) \\ 0 & 0 & 0 \end{pmatrix}.$$

Based on these definitions, we can formulate the following lemma.

Lemma 10 *Let the prediction horizon T, $T \in (0, \infty)$, be given. Then, if the initial condition $x^0 \in \mathbb{R}^n \setminus \{0\}$ is chosen such that*

$$(Qx^0)_3 = 0 = (Qx^0)_1 \cos(\theta^0) + (Qx^0)_2 \sin(\theta^0) \qquad (13)$$

hold, the control function $u^\star \equiv 0$ satisfies the necessary optimality conditions for the minimization problem (12) *with symmetric and positive definite matrices Q and R.*

Proof: See Appendix A.

Next, we show that in every neighborhood of the origin there exists an initial value x^0 satisfying Condition (13).

Lemma 11 *Let $\varepsilon > 0$ be given. Moreover, let the matrix Q be symmetric and positive definite. Then, there exists an initial value $x^0 \in \mathcal{B}_\varepsilon(0) := \{x \in \mathbb{R}^n \mid \|x\| \leq \varepsilon\}$ with $x^0 \neq 0$ satisfying the conditions* (13)*.*

Proof: See Appendix B.

Finally, we use Lemmata 10 and 11 to show that, for a given prediction horizon T, $T \in (0, \infty)$, there always exists an initial condition x^0 satisfying (13) such that $u^\star$ is the *global* optimum of the optimal control problem (12).

Theorem 12 *Let us consider the minimization problem* (12) *with symmetric and positive definite matrices Q and R. Furthermore, let the prediction horizon T, $T \in (0, \infty)$, be given. Then, for every $\varepsilon > 0$ there exists an initial value $x^0 \in \mathcal{B}_\varepsilon(0) \setminus \{0\}$ such that $u^\star \equiv 0$ is the unique optimal control.*

Proof: We want to apply [25, Theorem 2.1] to verify a sufficiency condition. In combination with the already derived preparatory lemmata, this will allow us to choose a suitable initial condition, for which optimality of $u^\star$ can be concluded. To this end, let us define the *derived Hamiltonian* $\mathcal{H}^0 : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R} \cup \{\infty\}$ by

$$\mathcal{H}^0(x, \lambda) = \max_{u \in \mathbb{R}^m} \mathcal{H}(x, u, \lambda).$$

As a preliminary step, we compute an optimizer. Note that we already plug in the adjoint λ resulting from Pontryagin's maximum principle, which was calculated in

[4] Note that the transversality condition $\lambda_i(T) = 0$, $i \in \{1, 2, \ldots, n\}$ immediately implies $\lambda_0 \neq 0$ for the abnormal multiplier λ_0. Hence, it does not occur in our formulation of the maximum principle.

the proof of Lemma 10. The first derivative of the Hamiltonian $\mathcal{H}$ with respect to u is

$$\frac{\partial \mathcal{H}}{\partial u} = -2Ru + \lambda^T \begin{pmatrix} \cos(\theta) & 0 \\ \sin(\theta) & 0 \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} \lambda_1 \cos(\theta) + \lambda_2 \sin(\theta) - 2(r_{11}u_1 + r_{12}u_2) \\ \lambda_3 - 2(r_{21}u_1 + r_{22}u_2) \end{pmatrix}.$$

Using the necessary optimality condition $\lambda_3 \equiv 0$, yields $r_{21}u_1 + r_{22}u_2 = 0$ for $(\partial\mathcal{H}/\partial u)_2 = 0$, i.e.

$$u_2 = -\frac{r_{21}u_1}{r_{22}}.$$

Hence, $(\partial\mathcal{H}/\partial u)_1 = 0$ is equivalent to

$$u_1 = r_{22} \cdot \frac{\lambda_1 \cos(\theta) + \lambda_2 \sin(\theta)}{2\det(R)}.$$

Since $\mathcal{H}$ is strictly concave with respect to u, $(u_1\, u_2)^T$ is the unique maximizer of $\mathcal{H}$ with respect to u for given x and λ with $\lambda_3 \equiv 0$. Having an analytic expression allows for rewriting $\mathcal{H}^0(x, \lambda)$: Plugging u_1 and u_2 in $u^T Ru$ yields $r_{22}^{-1} \det(R)u_1^2$. Furthermore, we get

$$(\cos(\theta)\lambda_1 + \sin(\theta)\lambda_2)u_1 = 2r_{22}^{-1} \det(R)u_1^2$$

for $\lambda^T f(x, u)$. Hence

$$\mathcal{H}^0(x, \lambda) = -x^T Qx + r_{22}^{-1} \det(R)u_1^2. \tag{14}$$

In order to show that $u^\star \equiv 0$ is an optimal control, we verify the sufficiency condition [25, Theorem 2.1], i.e., we show that $\mathcal{H}^0(x, \lambda(t))$ is concave in x for each $t \in [0, T]$. To this end, we show that the second derivative of $\mathcal{H}^0$ with respect to x is negative definite. Using (14), we see that this second derivative is essentially $-Q$. Only the entry $-q_{33}$ is perturbed by the second derivative of

$$r_{22}^{-1} \det(R)u_1^2 = \frac{r_{22}}{4\det(R)} \cdot (\lambda_1 \cos(\theta) + \lambda_2 \sin(\theta))^2$$

with respect to θ. However, Lemma 11 implies that for each $T > 0$, $\lambda_1(t)$ and $\lambda_2(t)$ can be made arbitrarily small for all $t \in [0, T]$ by choosing a sufficiently small initial value x^0. Hence, the perturbation can be made small enough such that the entry in the matrix of the second derivative is dominated by $-q_{33}$. Since the determinant of a matrix is continuous with respect to its entries, we have $\operatorname{sign}(-det(Q)) = \operatorname{sign}(\frac{\partial^2 \mathcal{H}^0}{\partial \theta^2})$. Hence, negative definiteness of the second derivative of $\mathcal{H}^0$ with respect to x holds.

Uniqueness: In fact, the above arguments show that for each $T > 0$, there exists an initial value $x^0 \in \mathbb{R}^n \backslash \{0\}$ such that $u^\star \equiv 0$ is an optimal solution for all prediction horizons $\widetilde{T} \in [T, T + \Delta T]$ for some $\Delta T > 0$. Let us assume that there exists another optimal control $\tilde{u} : [0, T) \to \mathbb{R}^m$. Then,

$$J_T(u^\star, \hat{x}) = J_T(\tilde{u}, \hat{x}) \tag{15}$$

holds. Since $\tilde{u} \neq u^\star$ (in the $\mathcal{L}^1$-sense) $\int_0^T \tilde{u}(t)^T R\tilde{u}(t)\,\mathrm{d}t > 0$ must hold. Hence, there must exist a time instant $\tilde{t}$ such that $x(\tilde{t})^T Qx(\tilde{t}) < (x^0)^T Qx^0$ holds in view of Equality (15). Then, on $[T, T + \Delta T]$ we construct the control function

$$\hat{u}(t) = \begin{cases} \hat{u}(t) = \tilde{u}(t) & \text{for } t \in [0, \tilde{t}) \\ \hat{u}(t) = 0 & \text{for } t \in [\tilde{t}, \tilde{t} + \Delta T) \\ \hat{u}(t) = \tilde{u}(t - \Delta T) & \text{for } t \in [\tilde{t} + \Delta T, T + \Delta T) \end{cases}.$$

Hence, the overall costs associated with $\hat{u}$ are

$$\begin{aligned} J_{T+\Delta T}(\hat{u}, x^0) &= J_T(\tilde{u}, x^0) + \int_{\tilde{t}}^{\tilde{t}+\Delta T} x(\tilde{t})^T Qx(\tilde{t})\,\mathrm{d}t \\ &\overset{(15)}{=} J_T(u^\star, x^0) + \int_{\tilde{t}}^{\tilde{t}+\Delta T} x(\tilde{t})^T Qx(\tilde{t})\,\mathrm{d}t \\ &< \int_0^{T+\Delta T} (x^0)^T Qx^0\,\mathrm{d}t. \end{aligned}$$

However, the latter are the costs associated with $u^\star(t) = 0$, $t \in [0, T + \Delta T)$, which contradicts the optimality of this control function on $[0, T + \Delta T)$. □

We note that the control function $u^* \equiv 0$ is admissible (and hence optimal) even in the setting with control *and* state constraints as long as the initial state x^0 is feasible. In conclusion, we have shown that asymptotic stability for the MPC closed-loop system is *not* guaranteed if a quadratic stage cost is used. This means that in general, MPC might not work for quadratic stage costs — independently of the horizon length $T \in (0, \infty)$ and the particular choice of the weighting matrices Q and R. In particular, it follows that it is crucial to use the stage cost as a design parameter and choose it suitably such that asymptotic stability can be guaranteed. To this end, the results of Section 3, in particular Theorem 4, can be used. For the nonholonomic robot, certain non-quadratic stage cost functions can be employed to this end, see [10,22,29,30].

4.2 A Sufficient Condition

In this section, we present sufficient conditions under which the situation discussed in Subsection 4.1 cannot occur and, thus, the MPC controlled system is asymptotically stable while solely quadratic stage cost functions ℓ

are used. In particular, we present sufficient conditions such that Assumption 2 can be satisfied at least locally around the origin with a bounded function B when using quadratic stage costs. Together with the results of [1], this implies that the closed loop is asymptotically stable on arbitrary sublevel sets of the infinite horizon value function V_∞.

In the remainder of this section, we assume that $0 \in \operatorname{int}(U)$ holds. We first recall the following definition of local controllability (compare [26, Section 3.7]).

Definition 13 *The system $\dot{x}(t) = f(x(t), u(t))$ is said to be locally controllable at $x = 0$ in time $\tau > 0$ if for each $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon) > 0$ such that for each $y_1, y_2 \in B_\delta(0)$ there exists an input $u \in \mathcal{L}^\infty([0,\tau), \mathbb{R}^m)$ such that*

$$x(\tau; y_1, u) = y_2 \quad \text{and} \quad \|(x(t; y_1, u)^T\, u(t)^T)\| \leq \varepsilon$$

hold for almost all $t \in [0, \tau)$.

Local controllability is in general not sufficient for satisfaction of Assumption 2 when using quadratic stage cost functions. This can, e.g., be seen by considering the nonholonomic integrator, which is locally controllable at the origin but Assumption 2 cannot be satisfied (since otherwise, by Theorem 4, the closed loop would by asymptotically stable, which is not the case as shown above).

Our next result shows that a slightly strengthened local controllability condition is sufficient for (local) satisfaction of Assumption 2.

Proposition 14 *Let parameters r, τ, $\bar{\varepsilon} > 0$ be given such that the system is locally controllable at $x = 0$ in time τ with $\delta(\varepsilon) \geq r\varepsilon$ for all $\varepsilon \in (0, \bar{\varepsilon}]$. If the stage cost ℓ is quadratic, i.e., $\ell(x, u) := x^T Q x + u^T R u$ with symmetric and positive definite matrices Q, R, then Inequality (4) is satisfied for all $t \in \mathbb{R}_{\geq 0}$ and all $x^0 \in B_{r\bar{\varepsilon}}(0)$ with bounded function B.*

Proof: By assumption, for each $x^0 \in B_{r\bar{\varepsilon}}(0)$, there exists an input signal[5] $u \in \mathcal{L}^1([0,\tau), U)$ such that $x(\tau; x^0, u) = 0$ and $\|(x(t; x^0, u)^T\, u(t)^T)\| \leq \|x^0\|/r$ for almost all $t \in [0, \tau)$. Using this input signal results in

$$\begin{aligned} V_T(x^0) \leq J_{\max\{T,\tau\}}(u; x^0) &= \int_0^\tau \ell(x(t; x^0, u), u(t))\,\mathrm{d}t \\ &\leq \tau \frac{\lambda_{\max}(Q) + \lambda_{\max}(R)}{r^2} \|x^0\|^2 \\ &\leq \tau \frac{\lambda_{\max}(Q) + \lambda_{\max}(R)}{\lambda_{\min}(Q) r^2} \cdot \inf_{u \in U} \ell(x^0, u) \end{aligned} \tag{16}$$

for all $T \geq 0$. This proves the assertion with $B(t) \equiv \tau(\lambda_{\max}(Q) + \lambda_{\max}(R))/(\lambda_{\min}(Q) r^2)$. $\square$

Next, we show that controllability of the linearization at the origin is a sufficient condition for local controllability as required in Proposition 14. This can be shown by a suitable extension of the proof of [26, Theorem 7], where controllability of the linearization is shown to be sufficient for local controllability (but not with $\delta(\varepsilon) \geq r\varepsilon$).

Proposition 15 *Let the linearization of the system at the origin be controllable. Then, for each $\tau > 0$, there exists $\bar{\varepsilon} = \bar{\varepsilon}(\tau) > 0$ such that the system is locally controllable at $x = 0$ in time τ with $\delta(\varepsilon) \geq r\varepsilon$ for all $\varepsilon \in (0, \bar{\varepsilon}]$.*

Proof: Let $\tau > 0$ be given. Then, the map

$$\Psi : \mathbb{R}^n \times \mathcal{L}^1([0,\tau), \mathbb{R}^m) \to \mathcal{AC}([0,\tau], \mathbb{R}^n)$$

is defined by $\Psi(x^0, u) = x(\cdot; x^0, u)|_{[0,\tau]}$, i.e., Ψ maps an initial value and a control function to a state trajectory on the time interval $[0, \tau]$. Here, $\mathcal{AC}([0,\tau], \mathbb{R}^n)$ denotes the space of absolutely continuous functions on $[0, \tau]$. Since Ψ is C^1 [26, Theorem 1,p. 57], a Taylor series expansion at $(0, 0)$, i.e., for initial condition $x^0 = 0 \in \mathbb{R}^n$ and the constant control $u \equiv 0$, yields

$$\Psi(x^0, u) = \Psi_*[0,0](x^0, u) + o(\|x^0\|^2 + \|u\|^2_{\mathcal{L}^1}), \tag{17}$$

compare [7, Theorem 5.1]. Here, $\Psi_*[0,0]$ is the (Frechet) differential of Ψ at $(0, 0)$. Furthermore, using the assumed controllability of the linearization, the implicit function theorem allows to conclude existence of some $\varepsilon_1 > 0$ and a C^1-mapping $j : B_{\varepsilon_1}(0) \times B_{\varepsilon_1}(0) \subset \mathbb{R}^n \times \mathbb{R}^n \to \mathcal{L}^\infty([0,\tau), \mathbb{R}^m)$ such that

$$x(\tau; y_1, j(y_1, y_2)) = y_2 \quad \forall\, y_1, y_2 \in B_{\varepsilon_1}(0)$$

holds with $j(0, 0) = 0$ [26, Proof of Theorem 7]. Again, since j is C^1, Taylor series expansion at $(0, 0)$ yields

$$j(y_1, y_2) = j_*[0,0](y_1, y_2) + o(\|(y_1^T\, y_2^T)\|^2). \tag{18}$$

From (18), it follows that there exist constants $\varepsilon_2 \in (0, \varepsilon_1]$ and $r_1 > 0$ such that $\|j(y_1, y_2)\|_{\mathcal{L}_\infty} \leq r_1 \|(y_1^T\, y_2^T)\|$ for all $y_1, y_2 \in B_{\varepsilon_2}(0)$. Together with (17), this implies that there exist constants $\varepsilon_3 \in (0, \varepsilon_2]$ and $r_2 > 0$ such that $\|\Psi(y_1, j(y_1, y_2))\|_{\mathcal{L}_\infty} \leq r_2 \|(y_1^T\, y_2^T)\|$ for all $y_1, y_2 \in B_{\varepsilon_3}(0)$. Let us consider states y_1, $y_2 \in B_{\varepsilon_3}(0)$. Then, using the notation $u := j(y_1, y_2)$, the last inequality implies

$$\|(x(t; y_1, u)^T\, u(t)^T)\| \leq \sqrt{r_1^2 + r_2^2}\, \|(y_1^T\, y_2^T)\|$$

for almost all $t \in [0, \tau)$. Hence, the assertion follows using the definitions $r = 1/\sqrt{2(r_1^2 + r_2^2)}$ and $\bar{\varepsilon} = \varepsilon_3 / r$. $\square$

[5] Since $0 \in \operatorname{int}(U)$, we can choose $\bar{\varepsilon}$ small enough such that $u \in \mathcal{L}^1([0,\tau), U)$ and not only $u \in \mathcal{L}^1([0,\tau), \mathbb{R}^m)$.

Instead of the (nonlinear) local controllability condition of Proposition 14, also stabilizability of the linearization at the origin can be used as a sufficient condition to show that when using a quadratic stage cost, Inequality (4) is satisfied for all $t \in \mathbb{R}_{\geq 0}$ and all x in a neighborhood of the origin with bounded B. Namely, in [5] it was shown that in this case, there exists some $P > 0$ such that for all x in a neighborhood of the origin and all $t \in \mathbb{R}_{\geq 0}$, $V_t(x) \leq x^T P x$. Hence (4) is satisfied with $B(t) \equiv \lambda_{\max}(P)/\lambda_{\min}(Q)$.

In conclusion, we have shown that quadratic stage cost functions are a suitable choice for stabilizing MPC if the system satisfies a certain local controllability condition (see Proposition 14) or if the linearization at the origin is stabilizable. Regarding the nonholonomic robot: while this system is locally controllable at the origin, the stronger local controllability property as required in Proposition 14 is not satisfied. In addition, the linearization at the origin is not stabilizable. Moreover, we have proven that also Assumption 2 cannot be satisfied based on a quadratic stage cost function. And indeed, MPC does not work with quadratic stage cost for this example.

Remark 16 *When using MPC schemes with terminal cost and constraints, it is well known how these additional ingredients can be designed when using quadratic state cost functions under the condition that the linearization is stabilizable (see, e.g. [5]). On the other hand, if this is not the case, the design of a suitable terminal cost and terminal region might be very difficult. For example, as discussed above for the nonholonomic robot, in [8,9] a non-quadratic terminal cost is employed and the origin is not located in the interior of the terminal region. For the setting with terminal cost and constraints, rigorously studying the potential failure of MPC using quadratic (stage and terminal) cost functions is an interesting subject for future work.*

5 Conclusions

In this paper, we have shown that the typical choice of quadratic stage cost functions is not always suitable for designing stabilizing MPC controllers. In particular, there might *not* exist a (sufficiently large) prediction horizon such that the resulting closed-loop system is asymptotically stable. This shows the necessity of using the stage cost as a design parameter which has to be suitably chosen.

Furthermore, we reconsidered the standard controllability assumption used as a sufficient condition in order to establish closed-loop asymptotic stability in MPC without terminal constraints and costs. We showed that this assumption can be relaxed, which might alleviate its verification. Finally, we discussed sufficient conditions under which this controllability assumption can be satisfied when using quadratic stage cost functions. In particular, we proposed a nonlinear (uniform) local controllability condition. If these conditions are satisfied, it follows that quadratic stage cost functions are a suitable choice in MPC.

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A Proof of Lemma 10

The corresponding state trajectory is given by $x^\star(t) \equiv x^0$ and evaluating the adjoint yields $\dot{\lambda}(t) = -2Qx^0$, $\lambda(T) = 0$. Note that Condition (13) implies $\lambda_3(t) \equiv 0$. Consequently, the third condition of Pontryagin's maximum principle reads

$$0 \geq -u^T R u + \lambda(t)^T \begin{pmatrix} \cos(\theta^0)u_1 \\ \sin(\theta^0)u_1 \\ u_2 \end{pmatrix}.$$

This inequality is equivalent to

$$\begin{aligned} u^T R u &\geq \lambda_1(t)\cos(\theta^0)u_1 + \lambda_2(t)\sin(\theta^0)u_1 \\ &= 2(t-T)\left((Qx^0)_1\cos(\theta^0) + (Qx^0)_2\sin(\theta^0)\right)u_1 \\ &\overset{(13)}{=} 0. \end{aligned}$$

Hence, using the positive definiteness of the matrix R shows the assertion. □

B Proof of Lemma 11

We rewrite the first condition of (13) as

$$q_{31}x_1^0 = -q_{32}x_2^0 - q_{33}\theta^0 \tag{B.1}$$

and distinguish the following four cases. For each case, we provide analytic formulas for initial values $x^0 \in \mathbb{R}^n \backslash \{0\}$ satisfying Condition (13). Then, for each case, it becomes clear that x^0 can be scaled such that $\|x^0\| \leq \varepsilon$ holds.

Case 1 ($q_{31} = 0 = q_{32}$): Equation (B.1) implies $\theta^0 = 0$. Then, for every $x_2^0 \neq 0$, choosing $x_1^0 = -q_{11}^{-1}q_{12}x_2^0$ ensures also the second equation of (13).

Case 2 ($q_{31} = 0 \neq q_{32}$): Let $\theta^0 \neq 0$ be sufficiently small such that the inequality

$$c := q_{11}\cos(\theta^0) + q_{12}\sin(\theta^0) > 0$$

holds. Next, set $x_2^0 = -q_{32}^{-1} q_{33} \theta^0$. Then, (B.1) holds. As a consequence, (13) is satisfied for

$$\begin{aligned} cx_1 &= \cos(\theta^0)(q_{32}^{-1} q_{12} q_{33})\theta^0 \\ &\quad + \sin(\theta^0)(q_{32}^{-1} q_{22} q_{33} - q_{32})\theta^0. \end{aligned}$$

Here, we point out that c does not converge to zero for $\theta^0 \to 0$. Hence, x^0 can be chosen arbitrarily small.

Case 3 ($q_{32} = 0 \neq q_{31}$): This case can be treated analogously to Case 2 if $q_{12} \neq 0$ holds. Otherwise ($q_{12} = 0$), $\theta^0 = x_1^0 = 0$ solves (13) — independently of x_2^0.

Case 4 ($q_{32} \neq 0 \neq q_{31}$): we set

$$x_2^0 = -\frac{q_{31} x_1^0 + q_{33} \theta^0}{q_{32}} \tag{B.2}$$

in order to ensure that (B.1) holds. Plugging this into the second equation of (13) yields

$$cx_1^0 = \left(\left(q_{31} - \frac{q_{12} q_{33}}{q_{32}} \right) \cos(\theta^0) + \left(q_{32} - \frac{q_{22} q_{33}}{q_{32}} \right) \sin(\theta^0) \right) \theta^0$$

with $c = c(\theta^0)$ given by

$$-\left(q_{11} - \frac{q_{12} q_{31}}{q_{32}} \right) \cos(\theta^0) - \left(q_{12} - \frac{q_{22} q_{31}}{q_{32}} \right) \sin(\theta^0).$$

Hence, the construction of x^0 is analogously to Case 2 if $q_{11} - q_{12} q_{31} q_{32}^{-1} \neq 0$ holds.

Otherwise ($q_{11} q_{32} = q_{12} q_{31}$) we know that $q_{12} \neq 0$. Here, we set $\theta^0 = 0$. Then, the second equation of (13) simplifies to $q_{11} x_1^0 + q_{12} x_2^0 = 0$. Plugging our assumption $q_{11} q_{32} = q_{12} q_{31}$ into this equation and multiplying by $q_{32}/q_{12} \neq 0$ leads to

$$q_{31} x_1^0 + q_{32} x_2^0 = 0.$$

But this is precisely Equation (B.1) meaning that the two conditions of (13) coincide. Since this condition is solved by each element of the respective linear subspace, a suitable initial condition x^0 clearly exists.

We like to stress once more that by providing explicit formulas for our desired initial values x^0, $x^0 \neq 0$, it is clear that this can always be chosen such that it is contained in an ε-neighborhood of the origin. This afterthought completes the proof. □

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