Interaction of Open and Closed Loop Control in MPC ⋆

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Abstract

We present a novel MPC algorithm without terminal constraints and/or costs, for which the prediction horizon is reduced in comparison to standard MPC while asymptotic stability and inherent robustness properties are maintained. To this end, we derive simplified stability conditions and investigate the interplay of open and closed loop control in MPC. The insight gained from this analysis allows to close the control loop more often while keeping the computational complexity of basic MPC. Our findings are verified by numerical simulations.

Key words: Model predictive control, asymptotic stability, stability criterion, MPC algorithm, closed loop performance.

1 Introduction

Within the past two decades, model predictive control (MPC) has grown mature for both linear and nonlinear systems with and without stabilizing terminal constraints [8, 21]. Nowadays, MPC is applied in a variety of industrial scenarios [5, 19]. Besides its ability to deal with control and state constraints, the reason for its success is the simplicity of the basic idea: measure the current state of the system, optimize a cost over the control input using model based predictions of the system response, implement the first portion of the computed control, and repeat these steps ad infinitum.

Several attempts to reduce the computational complexity of MPC have been reported in the literature: in multiplexed MPC [12], the decision variables of multiple input multiple output (MIMO) systems are updated asynchronously, i.e. a single control input is updated at each time instant. Hence, the optimization problem is relaxed by dividing it into single decision variable problems. An alternative MPC design [13] splits up the cost function into two parts (control and state) and employs a smaller horizon for the control part whereas the remaining inputs are obtained via tailored linear control laws. Furthermore, MPC schemes utilizing variable control horizons, but with stabilizing costs and constraints, have also been reported in the literature, see, e.g. [23]. Here, we focus on MPC without terminal constraints and/or costs, see, e.g. [8, Section 7.4] and the references therein for a thorough discussion on MPC with or without additional stabilizing terminal ingredients.

Since the computational complexity grows rapidly with a prolonged prediction horizon, our goal is to reduce the horizon length while maintaining stability. In this respect, we perform a sensitivity analysis with respect to the prediction horizon and the time shift utilizing a relaxed Lyapunov inequality as stability criterion [7, 11]. Here, the time shift denotes the length of the control segment to be implemented in an MPC step, which should be kept small in order to make use of the inherent robustness of MPC, see, e.g. [14, 18]. The importance of these parameters was also emphasised in [1], where the linear setting with terminal costs was analysed in detail.

The contribution is both analytical and methodical. Firstly, we derive qualitative properties (symmetry and monotonicity) of the performance index $\alpha_{T,\delta}$ introduced in [22]. From these results, simplified stability conditions are inferred both for the exponentially stabilizable

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and the general case. In addition, asymptotic estimates for the growth of the *minimal stabilizing horizon* are deduced. In a second step, we incorporate this knowledge in a novel MPC algorithm. Without additional computational effort, complexity and robustness are balanced while stability is maintained. To this end, the proposed algorithm repeatedly checks whether the control loop can be closed prematurely in order to exploit the inherent robustness of MPC. Moreover, we demonstrate the effectiveness of our approach in a numerical case study.

The paper is organized as follows: In Section 2, the problem setup, basic definitions, and a basic MPC algorithm are described. Thereafter, results from [22] are briefly summarized before a new stability condition is derived. In Section 4, the interaction of the prediction horizon and the time shift is analyzed in detail. In the subsequent Section 5, the proposed algorithm is displayed before its applicability is demonstrated by means of a differential-drive robot. Conclusions are drawn in Section 7.

Notation: Throughout this work we denote the natural numbers including zero by $\mathbb{N}_0$ and the nonnegative reals by $\mathbb{R}_{\geq 0}$. The Euclidean norm is denoted by $\|\cdot\|$ while the open ball with center $x \in \mathbb{R}^n$ and radius r is defined as $\mathcal{B}_r(x) := \{y \in \mathbb{R}^n \mid \|y - x\| < r\}$. $\mathcal{L}^1([0,T), \mathbb{R}^m)$ represents the set of measurable and absolutely integrable functions $u : [0,T) \to \mathbb{R}^m$. Furthermore, we call a continuous function $\gamma : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ a class $\mathcal{K}_\infty$-function if it is zero at zero, strictly increasing, and unbounded. Similarly, a continuous function $\beta : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is said to be of class $\mathcal{KL}$ if $\beta(\cdot, t) \in \mathcal{K}_\infty$ holds for all $t \geq 0$ and, for each $r > 0$, β is strictly decreasing in its second argument with $\lim_{t\to\infty} \beta(r,t) = 0$.

2 Setup and Preliminaries

Let sets $\mathbb{X} \subseteq \mathbb{R}^n$ and $\mathbb{U} \subseteq \mathbb{R}^m$ as well as a continuous vector field $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ be given such that $f(x^*, u^*) = 0$ holds for a tuple $(x^*, u^*) \in \mathbb{X} \times \mathbb{U}$. If, in addition, f is locally Lipschitz w.r.t. its first argument, the continuous time system governed by the dynamics

$$\dot{x}(t) = f(x(t), u(t)) \tag{1}$$

can be introduced. We call $x(t) \in \mathbb{R}^n$ and $u(t) \in \mathbb{R}^m$ the state and control at time $t \in \mathbb{R}_{\geq 0}$. The sets $\mathbb{X}$ and $\mathbb{U}$ represent state and control constraints, respectively. A solution of (1) at time t for initial value $x(0) = x^0$ and control function $\mathbf{u} \in \mathcal{L}^1([0,t), \mathbb{R}^m)$ is abbreviated by $x(t; x^0, \mathbf{u})$.

Definition 1 *For initial value $x^0 \in \mathbb{X}$ and $T > 0$, a control function $\mathbf{u} \in \mathcal{L}^1([0,T), \mathbb{U})$ is called admissible if $x(t; x^0, \mathbf{u}) \in \mathbb{X}$ holds for all $t \in (0,T]$. The set of such functions is denoted by $\mathcal{U}_T(x^0)$. The set $\mathcal{U}_\infty(x^0)$ consists of all functions $\mathbf{u} : [0,\infty) \to \mathbb{U}$ where restrictions $\mathbf{u}|_{[0,T)}$ satisfy $\mathbf{u}|_{[0,T)} \in \mathcal{U}_T(x^0)$ for all $T > 0$.*

We want to apply model predictive control (MPC). To this end, we first construct the following infinite horizon optimal control problem: for a given $x^0 \in \mathbb{X}$, minimize

$$J_\infty(x^0, \mathbf{u}) := \int_0^\infty \ell(x(t; x^0, \mathbf{u}), u(t)) \, \mathrm{d}t$$

with respect to $\mathbf{u} \in \mathcal{U}^\infty(x^0)$ with continuous stage costs $\ell : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}_{\geq 0}$ satisfying

$$\ell(x^*, u^*) = 0 \quad \text{and} \quad \inf_{u \in \mathbb{U}} \ell(x, u) > 0 \quad \forall\, x \in \mathbb{X} \backslash \{x^*\}.$$

The stage costs link the performance measure $J_\infty(x^0, \cdot)$ to a stabilization task. However, the curse of dimensionality renders the infinite horizon problem to be computationally hard, see, e.g., [3]. Hence, the MPC cost functional is retrieved by truncating the infinite horizon via

$$J_T(x, \mathbf{u}) := \int_0^T \ell(x(t; x, \mathbf{u}), u(t)) \, \mathrm{d}t, \tag{2}$$

and the solution of the infinite horizon problem is approximated by a series of finite horizon problems. These are tractable using, for instance, discretization methods and nonlinear optimization, cf. [8, Chapter 12] and the references therein. For a detailed analysis of the repercussions of using zero order hold implementations instead of control functions, see, e.g. [28, 29].

For $T \in \mathbb{R}_{\geq 0} \cup \{\infty\}$, the optimal value function is defined as $V_T(x) := \inf_{\mathbf{u} \in \mathcal{U}_T(x)} J_T(x, \mathbf{u})$. Existence of a minimizer $\mathbf{u}^\star \in \mathcal{U}_T(x)$, i.e. $V_T(x) = J_T(x, \mathbf{u}^\star)$, is assumed in order to avoid technical difficulties. The computed control function $\mathbf{u}^\star$ is applied on the time interval $[0, \delta)$. Then, the finite horizon optimal control problem is shifted forward in time. The parameter $\delta > 0$ denotes the time shift in MPC. As a result, we obtain Algorithm 1 which summarizes the described method and induces the MPC control law

$$u^{\mathrm{MPC}}_{T,\delta}(t) := \mathbf{u}^\star(t - k\delta; x(k\delta; x^0, \mathbf{u}^{\mathrm{MPC}}_{T,\delta})$$

for all $t \in [k\delta, (k+1)\delta)$ and all $k \in \mathbb{N}_0$.

Algorithm 1 MPC Algorithm

Given: $T > \delta > 0$. Set $t = 0$ and $x(0) = x^0$.
(1) Measure the current state $\hat{x} := x(t)$.
(2) Compute a minimizer $\mathbf{u}^\star = \mathbf{u}^\star(\cdot; \hat{x}) \in \mathcal{U}_T(\hat{x})$ of (2).
(3) Implement $u^\star(s; \hat{x})$, $s \in [0, \delta)$, shift the horizon forward in time by δ time units, and goto Step 1.

Note that neither measurement errors nor disturbances, which may also represent modeling error, are taken into account. In the defective case, Algorithm 1 counteracts flaws by closing the control loop each δ time units. Our goal is that the (controlled) equilibrium x^* is asymptotically stable for the MPC closed loop meaning that a $\mathcal{KL}$-function β exists such that

$$\|x(t; x^0, \mathbf{u}_{T,\delta}^{\mathrm{MPC}}) - x^*\| \leq \beta(\|x^0 - x^*\|, t) \qquad (3)$$

holds for all $t \geq 0$ and $x^0 \in \mathbb{X}$. However, the respective stability analysis is far from being trivial, see, e.g. [20].

3 Stability of the MPC Closed Loop

In general, the value function $V_T : \mathbb{X} \to \mathbb{R}_{\geq 0}$ is not a Lyapunov function. Yet, if the control task is well-posed, the value function V_T *decreases* under reasonable conditions meaning that the relaxed Lyapunov inequality

$$V_T(\hat{x}) \geq V_T(x_{\mathbf{u}^\star}(\delta)) + \alpha \int_0^\delta \ell(x_{\mathbf{u}^\star}(s), u^\star(s; \hat{x}))\, \mathrm{d}s \quad (4)$$

holds with $x_{\mathbf{u}^\star}(s) := x(s; \hat{x}, \mathbf{u}^\star(\cdot; \hat{x}))$ and $\alpha \in (0, 1]$. This is the key ingredient to prove stability and performance properties in MPC without terminal constraints [7].

The following growth condition, which was introduced in [24], suffices to determine a prediction horizon T such that Inequality (4) holds.

Assumption 2 *Let a monotonically increasing bounded function $B : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ be given such that*

$$V_t(x) \leq B(t) \cdot \ell^\star(x) \qquad \forall\, t \geq 0$$

holds for all $x \in \mathbb{X}$ with $\ell^\star(x) := \inf_{u \in \mathbb{U}} \ell(x, u)$.

Assumption 2 links an assessment of the current state in terms of the stage costs to the growth of the value function w.r.t. the optimization horizon t. B is typically defined as $B(t) := \int_0^t \rho(s)\, \mathrm{d}s$ for an integrable function $\rho : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$, see [26] for a non-trivial example. The particular choice $\rho(t) = Ce^{-\mu t}$ (overshoot $C \geq 1$ and decay rate $\mu > 0$) corresponds to systems, which are exponentially stabilizable w.r.t. their *stage costs*, see, e.g. [4] for linear or [9] for nonlinear systems.

The following theorem can be found in [22] (modified by some straightforward changes analogous to [9] to take the imposed state constraints into account); a detailed analysis of recursive feasibility can be found in [4].

Theorem 3 *Let Assumption 2 hold and the degree of suboptimality $\alpha_{T,\delta}$ be defined as*

$$1 - \frac{\exp\left(-\int_\delta^T B(t)^{-1}\, \mathrm{d}t\right) \cdot \exp\left(-\int_{T-\delta}^T B(t)^{-1}\, \mathrm{d}t\right)}{\left[1 - \exp(-\int\limits_\delta^T B(t)^{-1}\, \mathrm{d}t)\right]\left[1 - \exp(-\int\limits_{T-\delta}^T B(t)^{-1}\, \mathrm{d}t)\right]}. \quad (5)$$

Then, if the prediction horizon T and the time shift $\delta \in (0, T)$ are so that $\alpha_{T,\delta} > 0$ holds, the relaxed Lyapunov Inequality (4) and the performance estimate

$$\int_0^\infty \ell(x(s; x^0, \mathbf{u}_{T,\delta}^{MPC}), u_{T,\delta}^{MPC}(s; x^0))\, \mathrm{d}s \leq \frac{V_\infty(x^0)}{\alpha_{T,\delta}} \quad (6)$$

hold for all $x^0 \in \mathbb{X}$. If, in addition, $\mathcal{K}_\infty$-functions η, ζ exist such that the running costs ℓ satisfy

$$\eta(\|x - x^\star\|) \leq \inf_{u \in \mathbb{U}} \ell(x, u) \quad \text{and} \quad V_T(x) \leq \zeta(\|x - x^\star\|) \quad (7)$$

for all $x \in \mathbb{X}$, then the equilibrium x^* is asymptotically stable for the MPC closed-loop.

The following proposition helps to identify the contribution of the prediction horizon T to $\alpha_{T,\delta}$.

Proposition 4 *Suppose that the function B of Assumption 2 is constant, i.e. $B(t) = \gamma \in \mathbb{R}_{>0}$ for all $t \geq 0$. Then, the stability condition $\alpha_{T,\delta} > 0$ simplifies to*

$$T > \delta - \gamma \ln(1 - e^{-\delta/\gamma}). \qquad (8)$$

Moreover, the minimal stabilizing horizon $\widehat{T}$ defined as

$$\widehat{T} := \inf\{T \mid \alpha_{\tilde{T},\delta} > 0 \quad \forall\, \tilde{T} > T\} \qquad (9)$$

grows asymptotically like $\gamma \ln \gamma$.

Proof: Let δ be arbitrary but fixed. Since $B \equiv \gamma$ holds, the integrals in (5) can be computed. Then, the stability condition $\alpha_{T,\delta} > 0$ reads $1 - e^{-\delta/\gamma} > e^{-(T-\delta)/\gamma}$ or, equivalently, $\ln(1 - e^{-\delta/\gamma}) > \delta/\gamma - T/\gamma$. This proves Inequality (8).
The assertion on the asymptotic growth of $\widehat{T}$ can be expressed as

$$\lim_{\gamma \to \infty} \frac{\delta - \gamma \ln(1 - e^{-\delta/\gamma})}{\gamma \ln \gamma} = 1. \qquad (10)$$

Hence, showing $\lim_{\gamma \to \infty} -\ln(1 - e^{-\delta/\gamma}) / \ln \gamma = 1$ completes the proof. To this end, we apply l'Hôpital's rule:

$$\begin{aligned} \lim_{\gamma \to \infty} \frac{-\ln(1 - e^{-\delta/\gamma})}{\ln \gamma} &= \lim_{\gamma \to \infty} \frac{\delta \gamma^{-1} e^{-\delta/\gamma}}{1 - e^{-\delta/\gamma}} \\ &= \lim_{\gamma \to \infty} (1 - \delta \gamma^{-1}) = 1. \end{aligned}$$

Since the last limit exists, the two prior equality signs are justified. Hence, Equality (10) holds. □

Corollary 5 reveals the importance of Proposition 4 by showing that Condition (8) is a sufficient stability criterion. Note that this stability condition is always satisfied for a sufficiently long prediction horizon $T \in (\delta, \infty)$.

Corollary 5 *Let $\delta > 0$ be given and Assumption 2 hold. Suppose that class $\mathcal{K}_\infty$-functions η, ψ exist satisfying*

$$\eta(\|x-x^\star\|) \leq \inf_{u\in\mathbb{U}} \ell(x,u) \leq \psi(\|x-x^\star\|) \quad \forall\, x \in \mathbb{X}. \quad (11)$$

Then, if the prediction horizon $T \in (\delta, \infty)$ is chosen such that Inequality (8) holds with $\gamma = \lim_{t\to\infty} B(t)$, the MPC closed loop is asymptotically stable.

Proof: Since the growth bound B of Assumption 2 is monotonically increasing and bounded, its limit $\gamma := \lim_{t\to\infty} B(t) \in (0,\infty)$ exists. Consequently, $B(t) \leq \gamma$ holds for all $t \geq 0$. Hence, Assumption 2 also holds with $\widetilde{B} : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$, $t \mapsto \gamma$, instead of B. Moreover, the inequality $\widetilde{\alpha}_{T,\delta} \leq \alpha_{T,\delta}$ follows for the resulting $\widetilde{\alpha}_{T,\delta}$-value as observed in [25] — $\alpha_{T,\delta}$ results from the solution of a minimization problem where the function B pops up in several (infinitely many) inequality constraints. However, since $B(t) \leq \widetilde{B}(t)$ holds for every t, the constraints are relaxed and, thus, the feasible region is enlarged, which explains why $\widetilde{\alpha}_{T,\delta}$ is a lower bound for the original minimization problem. Hence, (8) implies $\alpha_{T,\delta} \geq \widetilde{\alpha}_{T,\delta} > 0$.
Furthermore, combining the inequality

$$V_T(x) \leq B(T) \cdot \ell^\star(x) = B(T) \inf_{u\in\mathbb{U}} \ell(x,u)$$

of Assumption 2 with (11) yields the validity of (7) with $\mathcal{K}_\infty$-function $\zeta(r) := B(T)\psi(r) \in \mathcal{K}_\infty$. In conclusion, all assumptions of Theorem 3 are satisfied, which yields the assertion. $\square$

The following (adademic) example shows that the performance index $\alpha_{T,\delta}$ becomes significantly better if the precise shape of the growth function B is known. However, it also points out why the seemingly cumbersome definition (9) is needed to avoid possible pitfalls.

Example 6 *Let the growth bound $B : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ be*

$$B(t) := \begin{cases} 0.85 \cdot \sqrt{t} & t \in [0,2) \\ 0.85 \cdot \sqrt{2} + 0.5((t-1)^2 - 1) & t \in [2,3) \\ 0.85 \cdot \sqrt{2} + 11/6 - t^{-1} & t \in [3,\infty) \end{cases}.$$

with upper bound $\gamma := 0.85 \cdot \sqrt{2} + 11/6$. Here, Formula (8) with $\delta = 0.1$ yields $\delta - \gamma \cdot \ln(1 - e^{-\delta/\gamma}) \lesssim 10.51$ meaning $\alpha_{T,\delta} > 0$ holds for all $T \geq 10.51$. However, if the precise shape of B is taken into account, $\alpha_{T,\delta}$ yields the values depicted in Figure 1 according to (5). In particular, $T = 1.9 = 19 \cdot \delta$ yields a strictly positive α-value. However, this is not the minimal stabilizing horizon $\widehat{T}$ according to (9) since $\alpha_{T,\delta} < 0$ holds for all $T \in [2.613, 3.495]$.

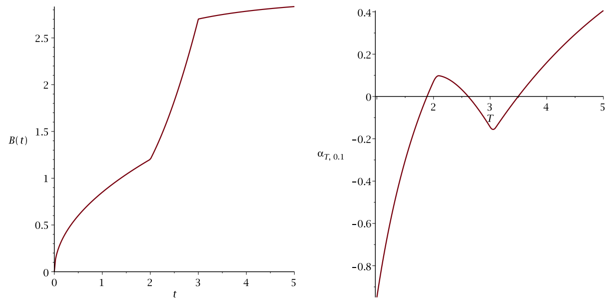

Figure 1. Visualization of the growth bound B of Example 6 and the resulting suboptimality index $\alpha_{T,\delta}$, $\delta = 0.1$.

4 The Time Shift in MPC

At the center of our analysis is the stability criterion $\alpha_{T,\delta} > 0$, which offers two degrees of freedom: the optimization horizon T and the time shift δ. The following consideration motivates to investigate the impact of the time shift on the suboptimality degree $\alpha_{T,\delta}$ in detail.

In the proof of Theorem 3 the relaxed Lyapunov Inequality (4) is summed up along the MPC closed loop. Considering two consecutive time steps we obtain

$$V_T(x) \geq V_T(\bar{x}(\delta)) + \alpha \int_0^\delta \ell(\bar{x}(s), u^{\mathrm{MPC}}_{T,\delta}(s;x))\,\mathrm{d}s,$$

$$V_T(\bar{x}(\delta)) \geq V_T(\bar{x}(2\delta)) + \alpha \int_\delta^{2\delta} \ell(\bar{x}(s), u^{\mathrm{MPC}}_{T,\delta}(s;x))\,\mathrm{d}s$$

where we used the abbreviation $\bar{x}(t) := x(t; x, \mathbf{u}^{\mathrm{MPC}}_{T,\delta})$. Summation of these terms reveals

$$V_T(x) \geq V_T(\bar{x}(2\delta)) + \alpha \int_0^{2\delta} \ell(\bar{x}(s), u^{\mathrm{MPC}}_{T,\delta}(s;x))\,\mathrm{d}s$$

or, written down for m steps

$$V_T(x) \geq V_T(\bar{x}(m\delta)) + \alpha \int_0^{m\delta} \ell(\bar{x}(s), u^{\mathrm{MPC}}_{T,\delta}(s;x))\,\mathrm{d}s. \quad (12)$$

We like to stress that this implication does not hold vice versa. In fact, Inequality (12) even allows an intermediate increase of the value function along the MPC closed-loop trajectory provided a sufficient decrease after $m\delta$ time units (m steps). Hence, the m-step Lyapunov Inequality (12) is a relaxation of its counterpart (4).

While analysing (12) is difficult, computing the performance index $\alpha_{T,m\delta}$ and, thus, establishing the inequality

$$V_T(x) \geq V_T(x_{\mathbf{u}^\star}(m\delta)) + \alpha \int_0^{m\delta} \ell(x_{\mathbf{u}^\star}(s), u^\star(s;x))\,\mathrm{d}s \quad (13)$$

with abbreviation $x_{\mathbf{u}^\star}(t) := x(t; x, \mathbf{u}^\star)$ is straightforward using Theorem 3. We emphasize the difference in the loop structure between (12) and (13): The former is obtained by reoptimizing every δ time units. In contrast to that, the computed control function is applied for $m\delta$ time units without closing the control loop in (13).

Hence, since large values of T lead, in general, to long computing times, our goal is to make use of the parameter δ in order to derive weaker conditions to ensure asymptotic stability. To this end, we first conduct a sensitivity analysis of the suboptimality degree $\alpha_{T,\delta}$ w.r.t. the time shift δ and, then, show that suitably choosing δ significantly reduces the *minimal stabilizing horizon* $\widehat{T}$ as defined in (9). A methodology to mitigate possibly occurring robustness issues resulting from longer *open-loop* intervals (larger δ-values) is presented in Section 5.

Next, we present our main result.

Theorem 7 *Suppose that Assumption 2 holds with growth function $B : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ given by*

1) *$B(t) = \gamma$ for all $t \geq 0$ or*
2) *$B(t) = \int_0^t Ce^{-\mu s}\,\mathrm{d}s$ with $C \geq 1$, $\mu > 0$.*

Let $T \in \mathbb{R}_{>0}$ be given and $\alpha_{T,\delta}$, $\delta \in (0,T)$, be given by Formula (5). Then, $\alpha_{T,\delta} = \alpha_{T,T-\delta}$ holds for all $\delta \in (0,T)$ (symmetry w.r.t. δ) and $\alpha_{T,\delta}$ is monotonically increasing with respect to δ on $(0,T/2]$ (monotonicity). In particular, the following properties hold:

- *Maximum at $\delta = T/2$: $\alpha(T,T/2) = \max_{\delta \in (0,T)} \alpha_{T,\delta}$*
- *$\alpha_{T,\delta} > 0$ implies $\alpha_{T,\hat{\delta}} > 0$ for all $\hat{\delta} \in [\delta, T-\delta]$*

Furthermore, the stability condition $\alpha_{T,T/2} > 0$ holds if and only if ($B \equiv \gamma$ or $B(t) = \int_0^t Ce^{-\mu s}\,\mathrm{d}s$)

$$T > 2\ln 2 \cdot \gamma \quad \text{or} \quad T > 2\mu^{-1}\ln(2^C - 1).$$

Moreover, the minimal stabilizing horizon $\widehat{T}$ defined in Proposition 4 grows asymptotically like $2\kappa \cdot \nu$ with constant $\kappa \in (0, \ln 2]$ and $\nu := \gamma$ or $\nu := C\mu^{-1}$, respectively. For dissipative exponentially stabilizable systems, i.e. $C = 1$, any prediction horizon $T > 0$ suffices, which corresponds to the special case $\kappa = 0$.

Proof: The claimed symmetry property follows directly from Formula (5) - even without the stipulations on B formulated by 1) or 2).
Plugging $B(t) = \gamma$ in Formula (5), solving the integrals and combining the terms in the nominator yields

$$\alpha_{T,\delta} = 1 - \frac{\exp(-T/\gamma)}{(1-\exp((\delta - T)/\gamma))(1-\exp(-\delta/\gamma))}.$$

This expression certainly diverges to $-\infty$ for both $\delta \to 0$ and $\delta \to T$. Next, we compute the derivative w.r.t. δ:

$$\frac{\partial \alpha_{T,\delta}}{\partial \delta} = \frac{\exp(-T/\gamma)\cdot(\exp(-\delta/\gamma) - \exp((\delta-T)/\gamma))}{\gamma(1-\exp((\delta-T)/\gamma))^2(1-\exp(-\delta/\gamma))^2}.$$

Hence, the derivative equals zero if and only if $\exp(-\delta/\gamma) = \exp((\delta - T)/\gamma)$, which, in turn, is equivalent to $\delta = T/2$. Hence, $\alpha_{T,\delta}$ is strictly monotonically increasing on $(0, T/2]$ (decreasing on $[T/2, T)$), which shows the monotonicity for B defined in 1). The monotonicity property for B given by 2) is proven in our conference paper [10]. The other properties (maximum and *stability range*) are straightforward consequences.

Let $\delta = T/2$ be given. For $B \equiv \gamma$ the stability criterion (8) can be rewritten as $1 > 2e^{-T/(2\gamma)}$, which is equivalent to $T > 2\ln 2 \cdot \gamma$. In the exponentially stabilizable case, the criterion $\alpha_{T,\delta} > 0$ is equivalent to

$$\left(e^{\mu T} - 1\right)^{\frac{1}{C}} \geq \left[\left(e^{\mu\delta} - 1\right)^{\frac{1}{C}} + \left(e^{\mu(T-\delta)} - 1\right)^{\frac{1}{C}}\right].$$

Utilizing $\delta = T/2$, the right hand side of this inequality simplifies to $2(e^{\mu T/2} - 1)^{\frac{1}{C}}$. Hence, the inequality is equivalent to $e^{\mu T} - 1 \geq 2^C(e^{\mu T/2} - 1)$ and can be rewritten as

$$\left(e^{\mu T/2} - \frac{2^C}{2}\right)^2 \geq \left(1 - \frac{2^C}{2}\right)^2.$$

In summary, $T \geq 2\mu^{-1}\ln(2^C - 1)$ holds if and only if $\alpha_{T,\delta} > 0$ is satisfied implying the second simplified stability condition.

For $B \equiv \gamma$ we directly obtain $\kappa = \ln(2)$ for the claimed asymptotic behavior. In the exponentially controllable case, $B(t)$ converges to $C\mu^{-1}$ for t approaching infinity. As already used in the proof of Corollary 5, $\alpha_{T,\delta}$ is monotonically decreasing with increasing accumulated bound B. Hence, the *worst case* is that also the overshoot C grows unboundedly for ν tending to infinity. Then, $\ln(2^C - 1) = \ln(\exp(\mathrm{C})^{\ln(2)} - 1)$ converges to $\ln(2) \cdot C$ which yields the assertion. If C is bounded by a constant ξ while $\nu = C\mu^{-1}$ grows unboundedly, κ is given by $\ln(2^\xi - 1)\xi \in (0, \ln(2))$. The assertion w.r.t. $C = 1$ follows directly from the stability condition. □

Theorem 7 reveals two facts about $\alpha_{T,\delta}$: For one, if the overall cost $C\mu^{-1}$ can be kept constant, reducing the overshoot C is more important than faster decay. [1] And secondly, comparing the derived estimate with its counterpart from Proposition 4 shows that using the control horizon $\delta = T/2$ yields an improved suboptimality estimate since the growth rate is linear instead of $\gamma \cdot \ln(\gamma)$.

[1] This can also be seen from the proof of Proposition 4 since the corresponding growth functions satisfy a strict inequality condition for every $t \geq 0$.

Theorem 7 has also remarkable consequences for algorithm design (as shown in the subsequent secion) and in the context of networked control systems, see, e.g. [10].

5 Algorithm Design and Robustness

The proposed stability condition is weakened by checking it after, e.g. $m\delta$ time units instead of δ. However, this implies that the system stays longer in open loop, which is not desirable from a robustness point of view. Hence, there is a conflict between relaxing the stability criterion and closed-loop robustness.

Our goal is to develop an algorithm which reduces the time period during which the system operates in open loop while maintaining asymptotic stability without prolonging the horizon T. The key idea is to exploit the weakened stability condition (V_T has only to decrease after $m\delta$ time units) but close the control loop whenever possible, see Step (2b) of Algorithm 2.

Algorithm 2 Loop Closing MPC Algorithm

Given: desired performance bound $\alpha > 0$, time shift $m\delta$, $m \in \mathbb{N}$, and prediction horizon $T = N\delta$ with $N = 2m$.
Set $t := 0$, $k := 0$, $\bar{c} := 0$, and $\bar{V}_T := \infty$.
(1) Measure the current state $\bar{x} := x(t)$.
(2) Compute a minimizer $\bar{\mathbf{u}}^\star \in \mathcal{U}_T(\bar{x})$ of (2).
  (a) If $k = m$ holds, set $\bar{V}_T := \infty$.
  (b) If $J_T(\bar{x}, \bar{\mathbf{u}}^\star) \leq \bar{V}_T - \alpha\bar{c}$ holds, set $k := 0$, $\bar{c} := 0$, $\hat{x} := \bar{x}$, $\hat{\mathbf{u}}^\star := \bar{\mathbf{u}}^\star$, and $\bar{V}_T := J_T(\bar{x}, \bar{\mathbf{u}}^\star)$.
(3) Implement $\hat{u}^\star(t)$, $t \in [k\delta, (k+1)\delta)$, and augment $\bar{c}$ by $\int_{k\tau}^{(k+1)\tau} \ell(x(s; \hat{x}, \hat{\mathbf{u}}^\star), \hat{u}^\star(s))\ \mathrm{d}s$. Increment k, augment t by δ, and goto Step (1).

Algorithm 2 does not cause any additional computational effort in comparison to Algorithm 1. The resulting MPC control $\mathbf{u}_T^{\mathrm{MPC}}$ is still the piecewise concatenation of the control signals $\hat{u}^\star(t)$, $t \in [k\delta, (k+1)\delta)$. However, it is decided at runtime in Step (2b) when to switch to a newly computed control function. Hence, the time varying version of Theorem 3, which can be found in [10], is required in order to conclude asymptotic stability of the MPC closed loop.

If the horizon T is determined such that the relaxed Lyapunov inequality holds with time shift $m\delta$, stability is ensured by Theorem 3. Hence, the *exit strategy* of Step (2a) is not needed. The main difference in contrast to [17] is that no additional computational effort is caused in comparison to Algorithm 1. Only the relaxed Lyapunov inequality is evaluated to decide whether an update is carried out and the control loop is closed while maintaining stability. The potential of Algorithm 2 to work despite a shortened prediction horizon T and, thus, to (drastically) reduce the computation time is illustrated by the example displayed in Figure 2.

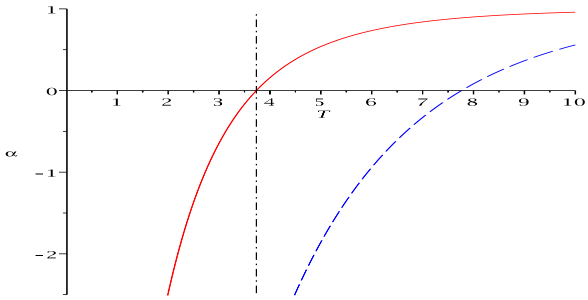

Figure 2. Performacne index $\alpha_{T,0.1}$ (dashed blue) and $\alpha_{T,T/2}$ (solid red) for $B(t) = \int_0^t Ce^{-\mu s}\ \mathrm{d}s$ ($C = 3.5$ and $\mu = 1.25$).

Algorithm 2 can be extended such that the control loop is closed even more frequently. To this end, a so called overall performance (stability) check is introduced which is based on the observation that it suffices to ensure

$$V_T(\hat{x}(t)) \leq V_T(x^0) - \alpha \int_0^t \ell(\hat{x}(s), u_T^{\mathrm{MPC}}(s))\ \mathrm{d}s \quad (14)$$

at each update time instant where the abbreviation $\hat{x}(s) := x(s; x^0, \mathbf{u}_T^{\mathrm{MPC}})$ is used. This relaxed update criterion can be easily incorporated in Step 2(b) of Algorithm 2 and reflects the accumulated slack up to time t. The idea to make use of the accumulated slack was introduced in [6]. In addition, an overall (online) performance measure at time t is given by (see also [17])

$$\alpha(t) := \frac{V_T(x_0) - V_T(\hat{x}(t))}{\int_0^t \ell(\hat{x}(s), u_T^{\mathrm{MPC}}(s))\ \mathrm{d}s}.$$

6 Numerical Case Study: Nonholonomic Robot

In this section, we apply Algorithm 2 to a nonholonomic 4-wheel skid steering mobile robot (SSMR). Within the respective model, we suppose that no longitudinal slip of the wheels occurs. Furthermore, tire deformation is neglected, i.e. the contact surface between the wheels of the robot and the motion plane are points, and the instantaneous center of rotation (ICR) lies symmetrically on the local forward motion axis X_R, see [15] for details. Hence, the motion of the robot is described by the kinematic model

$$\begin{pmatrix} \dot{x}(t) \\ \dot{y}(t) \\ \dot{\theta}(t) \end{pmatrix} = \begin{pmatrix} \cos(\theta(t)) & x_{ICR}\sin(\theta(t)) \\ \sin(\theta(t)) & -x_{ICR}\cos(\theta(t)) \\ 0 & 1 \end{pmatrix} \begin{pmatrix} v(t) \\ w(t) \end{pmatrix}.$$

We denote the generalized coordinates of the robot by $(x, y, \theta)^\top$ (m, m, rad), where x and y specify the Cartesian position of the center of mass of the robot while θ determines its orientation with respect to the inertial frame of reference (X_I, Y_I). The control input is denoted by $(v, w)^\top$ (m/s, rad/s), where v and w are the linear and the angular speeds of the robot, respectively. Last, $x_{ICR} = 8 \cdot 10^{-3}$ (m) is the coordinate of ICR along the X_R axis. The actuator dynamics are approximated by a first order low-pass filter

$$\begin{pmatrix} \dot{v}(t) \\ \dot{w}(t) \end{pmatrix} = \begin{pmatrix} k_v(v_d(t) - v(t)) \\ k_w(w_d(t) - w(t)) \end{pmatrix}, \qquad (15)$$

where $k_v = 3.8$ (s^{-1}) and $k_w = 4$ (s^{-1}) are the dynamic coefficient parameters while v_d and w_d are the desired control values fed to the robot actuators, cf. [2]. The parameters x_{ICR}, k_v and k_w were identified experimentally. In summary, the state vector $\mathbf{x}$ is augmented to $(x, y, \theta, v, w)^\top$ and the control vector is given by $\mathbf{u} = (v_d, w_d)^\top$. State and control constraints are modeled via $\mathbb{X} = [-2, 2]^2 \times \mathbb{R} \times [-0.25, 0.6] \times [-0.7, 0.7]$ and $\mathbb{U} = [-0.25, 0.6] \times [-0.7, 0.7]$, respectively.

We numerically investigate Algorithm 2 with $\delta = 0.1$(s) and a reference performance index $\bar{\alpha} = 0.55$. The control objective is to steer the mobile robot to the origin $0_{\mathbb{R}^5}$ starting from initial condition $\mathbf{x}^0 = (0, 1, 0, 0, 0)^\top \in \mathbb{X}$: a control problem known as parallel parking. Based on our recent study [26, 27], we make use of the running costs $\ell(\mathbf{x}, \mathbf{u})$ defined as

$$q_1 x^4 + q_2 y^2 + q_3 \theta^4 + q_4 v^4 + q_5 w^4 + r_1 v_d^4 + r_2 w_d^4$$

with weighting parameters $q = (1, 5, 0.1, 0.5, 0.05)^\top$ and $r = (0.1, 0.01)^\top$, where units are adjusted such that ℓ is dimensionless. We point out that MPC based on purely quadratic costs is not stabilizing without terminal conditions — even for the kinematic model without the actuator dynamics (15) as rigorously proved in [16].

We determined the shortest prediction horizon $T = 1.8$ (s) such that the value function evaluated along the MPC closed loop $V_T(\mathbf{x}(n\delta; \mathbf{x}^0, \mathbf{u}^{\mathrm{MPC}}_{T,\delta}))$, $n \in \mathbb{N}_0$, generated by Algorithm 1 satisfies the relaxed Lyapunov inequality (5). Then, we employed Algorithm 2 for the shorter prediction horizon $T = 1.6$ (s). Here, the evolution of the value function satisfies the suboptimality estimate (12) with $\alpha = \bar{\alpha}$ and $m\delta = 0.8$ (s). The closed-loop trajectories are considered until the condition $\|\mathbf{x}(n\delta; \mathbf{x}^0, \mathbf{u})\| \leq 0.05$ is met for $n \in \mathbb{N}_0$ with $\mathbf{u} = \mathbf{u}^{\mathrm{MPC}}_{T,\delta}$ (Algorithm 1) and $\mathbf{u} = \mathbf{u}^{\mathrm{MPC}}_{T}$ (Algorithm 2), respectively. As displayed in Figure 3, the robot is steered to the target state.

In Figure 4 (left), the evolution of the performance index evaluated along the MPC closed-loop trajectories is displayed. For $T = 1.6$ (s), the desired performance bound $\bar{\alpha} = 0.55$ is not met for Algorithm 2 at each time instant. However, as it can be inferred from Figure 4 (right) – where the evolution of the index k of Algorithm 2 is presented – the performance index is ensured

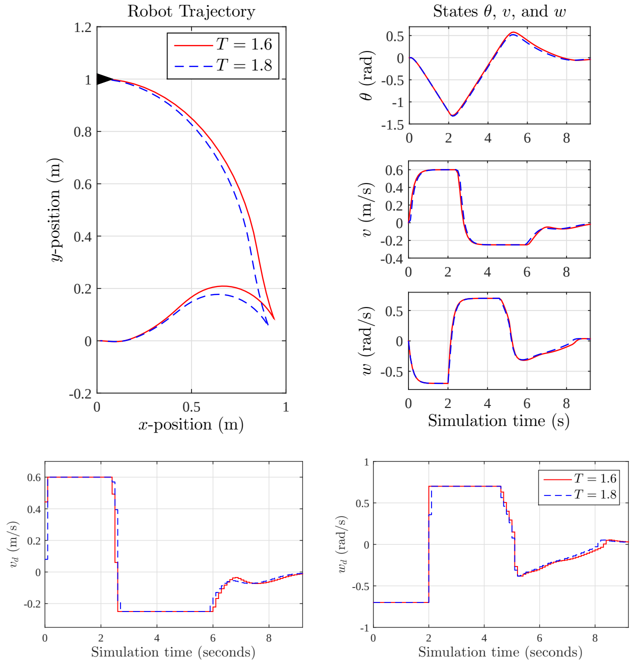

Figure 3. MPC closed-loop trajectories and controls for horizons $T = 1.6$ (s) (red; Algorithm 2) and $T = 1.8$ (s) (blue; Algorithm 1) with $\mathbf{x}^0 = (0, 1, 0, 0, 0)^\top$ and $\tau = 0.1$ (s).

at each *update* time instant: $k > 1$ is employed during two time intervals. Moreover, the index k, which signifies the number of applied open-loop control actions, satisfies $k \leq \lfloor T/(2\tau) \rfloor$. Hence, using Algorithm 2 allows for shortening the prediction horizon T while still being able to close the control loop most of the time without generating additional numerical effort and retaining the desired performance bound $\bar{\alpha}$.

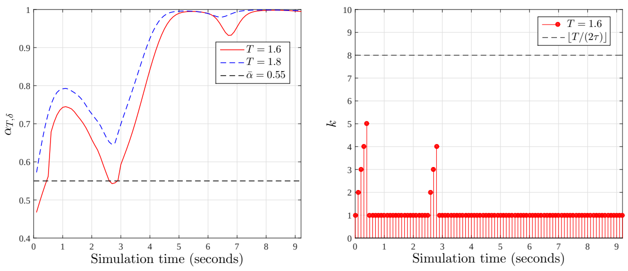

Figure 4. Evolution of the performance index along the MPC closed loop for $\mathbf{x}^0 = (0, 1, 0, 0, 0)^\top$, $\delta = 0.1$ (s), and $T = 1.6$ (s) (solid red line; Algorithm 2) and $T = 1.8$ (s) (blue dashed line; Algorithm 1). In addition, the evolution of the index k used in Algorithm 2 is shown (right).

7 Conclusion

Based on simplified stability conditions and a detailed analysis of the interplay of open and closed loop control in MPC, we developed a novel algorithm without stabilizing constraints and/or costs. Since this allows to employ a shorter prediction horizon, the computational complexity of each MPC step is reduced while the inherent robustness of MPC is maintained by incorporating conditions to close the control loop whenever possible.

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