Nonlinear MPC: the Impact of Sampling on Closed Loop Stability

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In this paper we consider model predictive control (MPC) schemes without stabilizing terminal constraints and/or costs for continuous time systems. While the estimates on the required prediction horizon length such that asymptotic stability of the MPC closed loop is guaranteed yield, in general, satisfactory results their applicability is limited due to the fact that the respective proofs assume that the input function can be switched arbitrarily often on compact time intervals. We present a technique which allows to determine a suitable discretization accuracy such that the obtained performance bound is preserved while the control signal is only switched at the sampling instants of the corresponding sampled data system.

1 Performance Estimates for Model Predictive Control Schemes

We consider continuous time systems governed by ordinary differential equations $\dot{z}(t) = f(z(t), u(t))$ subject to the control constraints $u(t) \in \mathbb{U}$. Here, $z(t) \in \mathbb{R}^n$ represents the state, $u(t) \in \mathbb{R}^m$ the control, and $\mathbb{U} \subseteq \mathbb{R}^m$ the set of admissible control values. For a given (initial) state $z_0 \in \mathbb{R}^n$ and control function $u : \mathbb{R}_{\geq 0} \to \mathbb{R}^m$ the solution is denoted by $z_u(\cdot; z_0)$. Our goal is to stabilize the origin $0_{\mathbb{R}^n}$ which is assumed to be a controlled equilibrium, i.e. $f(0,0) = 0$ and $0_{\mathbb{R}^m} \in \mathbb{U}$. Since we want to fulfil this control task in an optimal fashion, stage costs $\ell : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}_{\geq 0}$ satisfying the conditions $\ell(0,0) = 0$ and $\ell(z,u) > 0$ for all $z \neq 0$ are employed as a performance criterion. Then, a model predictive control algorithms is used:

- measure the current state $\hat{z} := z(t)$ at time t.
- minimize the cost functional $J_T(\hat{z}, \bar{u}) = \int_0^T \ell(\bar{z}_{\bar{u}}(s; \hat{z}), \bar{u}(s))\, ds$ subject to the dynamics $\dot{\bar{z}}_{\bar{u}}(s; \hat{z}) = f(\bar{z}_{\bar{u}}(s; \hat{z}), \bar{u}(s))$, $\bar{z}_{\bar{u}}(0; \hat{z}) = \hat{z}$, and the control constraints $\bar{u}(s) \in \mathbb{U}$, $s \in [0, T)$. We denote a corresponding optimal control by $\bar{u}^\star : [0, T) \to \mathbb{R}^m$.[1] Then, $V_T(\hat{z}) := \inf_{\bar{u}:[0,T)\to\mathbb{U}} = J_T(\hat{z}, \bar{u})$ holds.
- apply the first piece $\bar{u}^\star(t)|_{t\in[0,\delta)}$ of the computed control $\bar{u}^\star(\cdot)$ at the plant and increment t by δ time units.

This procedure allows to define a (static) state feedback $\mu_{T,\delta} : [0, \delta) \times \mathbb{R}^n \to \mathbb{R}^m$ as $\mu_{T,\delta}(s, \hat{z}) := \bar{u}^\star(s)$ depending on the prediction (optimization) horizon T and the control horizon δ, i.e. the control loop is closed every δ time units. The resulting MPC closed loop trajectory $z^{MPC}_{\mu_{T,\delta}}(\cdot; z_0)$ is generated iteteratively by $\dot{z}_{\mu_{T,\delta}}(t) = f(z_{\mu_{T,\delta}}(t), \mu_{T,\delta}(t - \lfloor t/\delta \rfloor \delta, z_{\mu_{T,\delta}}(\lfloor t/\delta \rfloor \delta)))$ with $z_{\mu_{T,\delta}}(0) = z_0$. The control function consisting of the concatenated input signals is denoted by $\mu^{MPC}_{T,\delta}(\cdot; z_0)$. Asymptotic stability of this MPC scheme can be ensured by the following theorem [5].

Theorem 1.1 *If $\alpha \in (0, 1]$ exists such that, for each $z \in \mathbb{R}^n$, the* relaxed Lyapunov inequality

$$V_T(z_{\mu_{T,\delta}}(\delta; z)) \leq V_T(z) - \alpha \int_0^\delta \ell(z_{\mu_{T,\delta}}(s; z), \mu_{T,\delta}(s, z))\, ds \tag{1}$$

holds. Then, the suboptimality estimate $J_\infty(z_0, \mu_{T,\delta}) := \int_0^\infty \ell(z^{MPC}_{\mu_{T,\delta}}(t; z_0), \mu^{MPC}_{T,\delta}(t; z_0))\, dt \leq V_\infty(z_0)/\alpha$ holds. If, in addition, the (technical) conditions $\underline{\eta}(\|z\|) \leq \inf_{u\in\mathbb{U}} \ell(x, u)$ and $V_T(z) \leq \overline{\eta}(\|z\|)$ hold for $\mathcal{K}_\infty$-functions $\underline{\eta}$, $\overline{\eta}$, asymptotic stability of the receding horizon closed loop can be concluded.

If constants $C \geq 1$, $\mu > 0$ can be determined such that, for each $z_0 \in \mathbb{R}^n$, an open loop control function $u_{z_0} : \mathbb{R}_{\geq 0} \to \mathbb{U}$ exists satisfying

$$\ell(z_{u_{z_0}}(t; z_0), u_{z_0}(t)) \leq Ce^{-\mu t} \min_{u\in\mathbb{U}} \ell(z_0, u) \quad \text{for all} \quad t \geq 0, \tag{2}$$

i.e. a controllability condition, then the assumptions of Theorem 1.1 are satisfied with $\alpha_{T,\delta} := 1 - \frac{\zeta(\delta)}{\zeta(T)-\zeta(\delta)} \cdot \frac{\zeta(T-\delta)}{\zeta(T)-\zeta(T-\delta)}$ where $\zeta(s)$ is defined as $\sqrt[C]{e^{\mu s} - 1}$. In particular, for given control horizon δ and a desired performance bound $\overline{\alpha} \in (0, 1)$,

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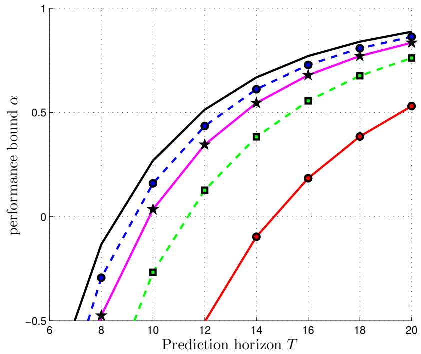
[1] Existence of a minimizer is assumed in order to avoid technical difficulties, see [2] for details and further explanations on nonlinear MPC algorithms.

there exists T such that $\alpha_{T,\delta} \geq \overline{\alpha}$, i.e. the required prediction horizon length in order to guarantee asymptotic stability of the MPC closed loop (or a desired suboptimality index $\overline{\alpha}$) can be calculated.

Example 1.2 (Non-holonomic integrator) We consider the differential equation given by $f : \mathbb{R}^3 \times \mathbb{R}^2 \to \mathbb{R}^3$, $(z, u) \mapsto (u_1, u_2, z_1 u_2 - z_2 u_1)^T$ with stage costs $\ell(x, u) = z_1^2 + z_2^2 + \nu|x_3| + u_1^2 + u_2^2$, $\nu = \pi$. Using a particular control function depending on the initial state z_0 allows us to determine an overshoot bound $C = 3$ and a decay rate $\mu = 0.5$ such that the controllability condition (2) holds. Then, for a fixed control horizon $\delta = 2$, the performance bound $\alpha_{T,\delta}$ can be evaluated according to the stated formula, see [6, Section 2.2] for details. Hence, asymptotic stability of the MPC closed loop is, e.g. ensured for prediction horizon $T \gtrsim 8.7$ as it can be seen in the figure on the right (solid line without markers).



2 Sampling and Performance

Although the proposed approach works well, in the proof of Theorem 1.1 it is assumed that the control signal can be switched infinitely many times on the compact optimization interval $[0, T)$ — independently of the (possibly piecewise constant) functions used in order to verify Inequality (2). For implementational reasons it is, however, desirable to obtain stability certificates for control inputs which are constant on each sampling interval. This can be achieved by considering the exact discrete time representation $z^+ = f_\tau(z, w)$ of the sampled data system, cf. [4], where τ denotes the length of the sampling intervals. Doing so leads — based on results similar to Theorem 1.1 and Condition 2 for discrete time systems [1, 3] — to more conservative performance estimates, see [5] and the above figure (solid line, ∘). This leads to the question whether structural properties of control functions used in order to verify Inequality (2) can be preserved while maintaining the superior performance bounds of the continuous time approach.

A key idea in order to solve this problem is to decouple the control horizon δ and the discretization parameter τ. To be more precise, the interval $[0, \delta)$ is subdivided into $k \in \mathbb{N}$ (equidistant) sampling intervals ($k\tau = \delta$). Then, the MPC algorithm is adapted such that the first k control values are implemented on each interval $[n\delta, (n+1)\delta)$ in the discrete setting, so called multistep MPC. Hence, the relation between $\delta = k\tau$ and T is invariant with respect to the discretization accuracy τ. This allows us to apply the results obtained in [1, 3] and, thus, to preserve structural properties of the control functions since only finitely many switches are required in the respective proofs. In [6] we proved that the discrete time performance estimates converge for discretization parameter τ tending to zero ($k \to \infty$) to their upper bound given by $\alpha_{T,\delta}$. For iteratively refined discretizations, this convergence was shown to be monotone.

Example 2.1 (Non-holonomic integrator - revisited) The performance estimates corresponding to $\tau = \delta/k$ for $k = 1$ (solid line,∘), $k = 2$ (dashed line, □), $k = 4$ (solid line, ⋆), and $k = 8$ (dashed line, ∘) are illustrated in the above figure: finer discretizations imply improved suboptimality bounds. Indeed, monotone convergence to the continuous time values $\alpha_{T,\delta}$ can be observed.

In conclusion, firstly, the controllability condition (2) has to be verified for an overshoot bound C and a decay rate μ. Secondly, the continuous time performance estimate $\alpha_{T,\delta}$ is computed in order to determine a prediction horizon length for which asymptotic stability of the MPC closed loop is guaranteed. Finally, a sampling rate is determined in the discrete setting such that satisfactory performance (arbitrarily close to the upper bound $\alpha_{T,\delta}$ of the continuous time approach) is achieved. This allows to rigorously prove stability while structural properties of the class of admissible control functions are preserved.

References

[1] L. Grüne, Analysis and design of unconstrained nonlinear MPC schemes for finite and infinite dimensional systems. SIAM J. Control Optim., Vol. **48**, pp. 1206-1228 (2009)

[2] L. Grüne and J. Pannek, Nonlinear Model Predictive Control: Theory and Algorithms. Springer, London (2011)

[3] L. Grüne, J. Pannek, M. Seehafer, and K. Worthmann, Analysis of unconstrained nonlinear MPC schemes with varying control horizon. SIAM J. Control Optim., Vol. **48**(8), pp. 4938-4962 (2010)

[4] D. Nešić and A. Teel, A framework for stabilization of nonlinear sampled-data systems based on their approximate discrete-time models. IEEE Transactions on Automatic Control, Vol. **49**, pp. 1103-1122 (2004)

[5] M. Reble and F. Allgöwer, Unconstrained Model Predictive Control and Suboptimality Estimates for Nonlinear Continuous-Time Systems. Automatica, Vol. **48**, pp. 1812-1817 (2012)

[6] K. Worthmann, M. Reble, L. Grüne, and F. Allgöwer, The Role of Sampling for Stability and Performance in Unconstrained Nonlinear Model Predictive Control. SIAM J. Control Optim., Vol. **52**(1), pp. 581-605 (2014).