Variational integrators for nonsmooth systems

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Motivation: Examples of nonsmooth systems

Bouncing ball

Newton’s cradle

Docking maneuver of spacecraft

Electric circuits
Outline

- Overview of continuous model
- Overview of discrete model
- Lagrangian mechanics in nonsmooth setting
- Discrete Lagrangian mechanics in nonsmooth setting
- Variational collision integrators


- Numerical examples
  - Forced systems
  - Hybrid systems
  - Application to electric circuits

Overview of continuous model

Focus on elastic collision problems, however, extensions towards other scenarios possible

- Lagrangian \( L(q, \dot{q}) = \frac{1}{2} \dot{q}^T M \dot{q} - V(q) \), mass matrix \( M \), potential \( V \), \( q \in Q \subseteq \mathbb{R}^n \)

- subset \( C \subset Q \): admissible set, no contact

- boundary \( \partial C \): contact has just occurred but no interpenetration

- trajectory \( q : [0, T] \rightarrow Q \), \( q(t) \in C \) and \( q(t_i) \in \partial C \)

- \( q(t_i) \) is nonsmooth but continuous at contact time \( t_i \)

- apply Hamilton’s principle with variations both in the curve \( q(t) \) and the impact time \( t_i \)
Overview of continuous model

Hamilton’s principle

\[ \delta \int_0^T L(q(t), \dot{q}(t)) \, dt = \delta \left[ \int_0^{t_i} L(q, \dot{q}) \, dt + \int_{t_i}^T L(q, \dot{q}) \, dt \right] \]

\[ = \int_0^{t_i} \left[ \frac{\partial L}{\partial q} \cdot \delta q + \frac{\partial L}{\partial \dot{q}} \cdot \delta \dot{q} \right] \, dt + \int_{t_i}^T \left[ \frac{\partial L}{\partial q} \cdot \delta q + \frac{\partial L}{\partial \dot{q}} \cdot \delta \dot{q} \right] \, dt \]

\[ - \left[ L(q, \dot{q}) \cdot \delta t_i \right]_{t_i}^{t_i} \]

\[ = \int_0^{t_i} \left[ \frac{\partial L}{\partial q} - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}} \right] \cdot \delta q \, dt + \int_{t_i}^T \left[ \frac{\partial L}{\partial q} - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}} \right] \cdot \delta q \, dt \]

\[ - \left[ \frac{\partial L}{\partial \dot{q}} \cdot \delta q + L \cdot \delta t_i \right]_{t_i}^{t_i} \]

with \( \delta q(T) = \delta q(0) = 0 \) and \( q(t_i) \in \partial C \)
Overview of continuous model

Euler-Lagrange equations

\[
\frac{\partial L}{\partial q}(q, \dot{q}) - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}}(q, \dot{q}) = 0 \quad \Rightarrow \quad M\ddot{q} = -\nabla V(q)
\]

\[
\left[ \frac{\partial L}{\partial \dot{q}} \cdot \delta q + L \cdot \delta t_i \right]^{t_i^+}_{t_i^-} = 0
\]

It holds \( q(t_i) \in \partial C \Rightarrow \delta q(t_i) + \dot{q}(t_i) \cdot \delta t_i \in T\partial C \)

space of allowable \( \delta q(t_i) \) and \( \delta t_i \) is spanned by \( \delta q(t_i) \in T\partial C \)

with \( \delta t_i = 0 \) and \( \delta q(t_i) = -\dot{q}(t_i) \) with \( \delta t_i = 1 \)
Overview of continuous model

Euler-Lagrange equations

\[
\left[ \frac{\partial L}{\partial \dot{q}} \bigg|_{t_i^+} - \frac{\partial L}{\partial \dot{q}} \bigg|_{t_i^-} \right] \cdot \delta q(t_i) = 0 \quad \text{for all } \delta q(t_i) \in T \partial C
\]

\[
\left[ \frac{\partial L}{\partial \dot{q}} \cdot \dot{q} - L \right]_{t_i^+} - \left[ \frac{\partial L}{\partial \dot{q}} \cdot \dot{q} - L \right]_{t_i^-} = 0
\]

\[
\Rightarrow
\]

\[
\dot{q}(t_i^+) - \dot{q}(t_i^-) \in N_C(q_i(t)) \quad \text{jump orthogonal to boundary } \partial C
\]

\[
E_L(t_i^+) - E_L(t_i^-) = 0 \quad \text{energy conservation}
\]

with energy \( E_L(q, \dot{q}) = \frac{1}{2} \dot{q}^T M \dot{q} + V(q) \) and the normal cone \( N_C(q) \) to \( \partial C \) at \( q \)
Overview of discrete model

- \( L_d(q_0, q_1) = h \left[ \left( \frac{q_1-q_0}{h} \right)^T M \left( \frac{q_1-q_0}{h} \right) - V(q_0) \right] \)
- discrete curve \( \{q_k\}_{k=0}^N \) in \( C \) and times \( t_k = kh \)
- impact point \( \tilde{q} \in \partial C \) at impact time
  \( \tilde{t} = \alpha t_{i-1} + (1 - \alpha) t_i, \alpha \in [0, 1] \) s.t. \( \tilde{t} \in [t_{i-1}, t_i] \)
- discrete trajectory
  \( (q_0, t_0), \ldots, (q_{i-1}, t_{i-1}), (\tilde{q}, \tilde{t}), (q_i, t_i), \ldots, (q_N, t_N) \)
Overview of discrete model

Discrete action principle with variations in \( q_k, \tilde{q} \) and \( \alpha \) (i.e. \( \tilde{\alpha} \))

\[
\delta \left[ \sum_{k=0}^{i-2} L_d(q_k, q_{k+1}, h) + L_d(q_{i-1}, \tilde{q}, \alpha h) + L_d(\tilde{q}, q_i, (1 - \alpha) h) + \sum_{k=i}^{N-1} L_d(q_k, q_{k+1}, h) \right]
\]

\[
= \sum_{k=1}^{i-2} \left[ D_2 L_d(q_{k-1}, q_k, h) + D_1 L_d(q_k, q_{k+1}, h) \right] \cdot \delta q_k \\
+ \sum_{k=i+1}^{N-1} \left[ D_2 L_d(q_{k-1}, q_k, h) + D_1 L_d(q_k, q_{k+1}, h) \right] \cdot \delta q_k \\
+ \left[ D_2 L_d(q_{i-2}, q_{i-1}, h) + D_1 L_d(q_{i-1}, \tilde{q}, \alpha h) \right] \cdot \delta q_{i-1} \\
+ \left[ D_2 L_d(q_{i-1}, \tilde{q}, \alpha h) + D_1 L_d(\tilde{q}, q_i, (1 - \alpha) h) \right] \cdot \delta \tilde{q} \\
+ \left[ D_2 L_d(\tilde{q}, q_i, (1 - \alpha) h) + D_1 L_d(q_i, q_{i+1}, h) \right] \cdot \delta q_i \\
+ \left[ D_3 L_d(q_{i-1}, \tilde{q}, \alpha h) - D_3 L_d(\tilde{q}, q_i, (1 - \alpha) h) \right] \cdot h \delta \alpha
\]

with \( \delta q_0 = \delta q_N = 0 \)
Overview of discrete model

Discrete Euler-Lagrange equations (away from impact)

\[ D_2 L_d(q_{k-1}, q_k, h) + D_1 L_d(q_k, q_{k+1}, h) = 0 \quad \forall k \text{ away from impact} \]

\[ \Rightarrow M \left( \frac{q_{k+1} - 2q_k + q_{k-1}}{h^2} \right) = -\nabla V(q_k) \]

\(n\) equations to solve for \(n\) unknowns \(q_{k+1}\)
Overview of discrete model

Discrete Euler-Lagrange equations (just before impact)

\[ D_2 L_d(q_{i-2}, q_{i-1}, h) + D_1 L_d(q_{i-1}, \ddot{q}, \alpha h) = 0 \]

\[ \Rightarrow M \left( \frac{\ddot{q} - q_{i-1}}{\alpha h} \right) - M \left( \frac{q_{i-1} - q_{i-2}}{h} \right) = -\alpha h \nabla V(q_{i-1}) \]

Together with \( \ddot{q} \in \partial C \) these are \( n + 1 \) equations to solve for \( n + 1 \) unknowns \( \ddot{q} \) and \( \alpha \)
Overview of discrete model

Discrete Euler-Lagrange equations (at impact)

\[
\left[ D_2 L_d(q_{i-1}, \tilde{q}, \alpha h) + D_1 L_d(\tilde{q}, q_i, (1 - \alpha) h) \right] \cdot \delta \tilde{q} = 0 \quad \forall \delta \tilde{q} \in T_{\partial C} \partial C
\]

\[
D_3 L_d(q_{i-1}, \tilde{q}, \alpha h) - D_3 L_d(\tilde{q}, q_i, (1 - \alpha) h) = 0
\]
Overview of discrete model

Discrete Euler-Lagrange equations (at impact)

\[
[D_2 L_d(q_{i-1}, \ddot{q}, \alpha h) + D_1 L_d(\ddot{q}, q_i, (1 - \alpha)h)] \cdot \delta \ddot{q} = 0 \quad \forall \delta \ddot{q} \in T \partial C
\]

\[
D_3 L_d(q_{i-1}, \ddot{q}, \alpha h) - D_3 L_d(\ddot{q}, q_i, (1 - \alpha)h) = 0
\]

\[
\Rightarrow M \left( \frac{q_i - \ddot{q}}{(1 - \alpha)h} \right) - M \left( \frac{\ddot{q} - q_{i-1}}{\alpha h} \right) + (1 - \alpha)h \nabla V(\dddot{q}) \in N_C(\dddot{q})
\]

\[
\left[ \frac{1}{2} \left( \frac{q_i - \ddot{q}}{(1 - \alpha)h} \right)^T M \left( \frac{q_i - \ddot{q}}{(1 - \alpha)h} \right) + V(\dddot{q}) \right]
\]

\[
- \left[ \frac{1}{2} \left( \frac{\ddot{q} - q_{i-1}}{\alpha h} \right)^T M \left( \frac{\ddot{q} - q_{i-1}}{\alpha h} \right) + V(q_{i-1}) \right] = 0
\]

\[
n \text{ equations to solve for } n \text{ unknowns } q_i
\]
Overview of discrete model

Discrete Euler-Lagrange equations (just after impact)

\[ D_2 L_d(\tilde{q}, q_i, (1 - \alpha)h) + D_1 L_d(q_i, q_{i+1}, h) = 0 \]

\[ \Rightarrow M \left( \frac{\tilde{q}_{i+1} - q_i}{h} \right) - M \left( \frac{q_i - \tilde{q}}{(1 - \alpha)h} \right) = -h \nabla V(q_i) \]

n equations to solve for n unknowns \( q_{i+1} \)
Variational mechanics in nonsmooth setting

**Difficulty:** lack of smoothness of mappings prevents us from using differential calculus on the manifold of mappings

**Instead:** extend problem to nonautonomous case

- configuration variables and time are functions of a separate parameter $\tau$
- allows the impact to be fixed in $\tau$ space while remaining variable in both configuration and time spaces
- relevant space of configurations is a smooth manifold
Lagrangian mechanics in nonsmooth setting

- configuration manifold $Q$, submanifold $C \subset Q$ (admissible configurations), contact set $\partial C$, regular Lagrangian $L : TQ \to \mathbb{R}$

- path space $\mathcal{M} = \mathcal{T} \times Q([0, 1], \tau_i, \partial C, Q)$ with
  \[
  \mathcal{T} = \{ c_t \in C^\infty([0, 1], \mathbb{R}) \mid c'_t > 0 \text{ in } [0, 1] \}
  \]
  \[
  Q([0, 1], \tau_i, \partial C, Q) = \{ c_q : [0, 1] \to Q \mid c_q \text{ is a } C^0, \text{ piecewise } C^2, \quad c_q(\tau) \text{ has one singularity at } \tau_i, \quad c_q(\tau_i) \in \partial C \} \]

- path $c \in \mathcal{M}$ consists of $c = (c_t, c_q)$

- associated curve $q : [c_t(0), c_t(1)] \to Q$, $q(t) = c_q(c_t^{-1}(t))$

- denote the space of all these paths $q(t) \in Q$ as $C$

- moment of impact $\tau_i$ is fixed in $\tau$ space but allowed to vary in $t$ space as $t_i = c_t(\tau_i)$
Lagrangian mechanics in nonsmooth setting

- nonautonomous formulation of autonomous mechanical system leads to smoothness of the manifold of mappings

It holds

- $\mathcal{T}$ is a smooth manifold
- $\mathcal{Q}([0, 1], \tau_i, \partial C, Q)$ is a smooth manifold
- $\mathcal{M}$ is a smooth manifold

[Fetecau, Marsden, Ortiz, West 2003]
Lagrangian mechanics in nonsmooth setting

- Let’s denote $\mathcal{M}$ as $\tau$ space and $\mathcal{C}$ as $t$ space
- a variation in the path $c \in \mathcal{M}$ is given by the variations in $T$ and in $Q$ by $\delta c = (\delta c_t, \delta c_q) \in T\mathcal{M}$
- the tangent space to $\mathcal{M}$ is given as $T\mathcal{M} = TT \times TQ$
  with

$$TQ = \bigcup_{c_q \in Q} T_{c_q} Q$$

$$= \{ c_v : [0, 1] \rightarrow T_{c_q} Q \mid c_v \in C^1([0, \tau_i) \cup (\tau_i, 1], T_{c_q} Q), c_v(\tau_i) \in T_{c_q(\tau_i)} \partial \mathcal{C} \}$$

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Lagrangian mechanics in nonsmooth setting

- note that \( c_q(\tau) = q(c_t(\tau)) \Rightarrow c'_q(\tau) = \dot{q}(t) \cdot c'_t(\tau) \) where
  \( c' \) denotes the derivative w.r.t. \( \tau \) and \( \dot{q} \) w.r.t. \( t \)

- action map on \( t \) space with \( t = c_t(\tau) \)

\[
\mathcal{S}(q) = \int_{c_t(0)}^{c_t(1)} L(q(t), \dot{q}(t)) \, dt = \int_0^T L(q(t), \dot{q}(t)) \, dt
\]

- action map \( \mathcal{S} : M \rightarrow \mathbb{R} \) on \( \tau \) space

\[
\mathcal{S}(c_t, c_q) = \int_0^1 L\left( c_q(\tau), \frac{c'_q(\tau)}{c'_t(\tau)} \right) c'_t(\tau) \, d\tau
\]
Lagrangian mechanics in nonsmooth setting

Hamilton’s principle with variations $\delta c = (\delta c_t, \delta c_q) \in T_c \mathcal{M}$

$$d \mathcal{S}(c) \cdot \delta c = \int_0^1 \left[ \frac{\partial L}{\partial q} \delta c_q + \frac{\partial L}{\partial \dot{q}} \frac{c'_t}{c'_t} - \frac{\partial L}{\partial \dot{q}} \frac{c'_q}{(c'_t)^2} \right] c'_t + L \delta c'_t \, d\tau$$

$$= \int_0^1 \left[ \frac{\partial L}{\partial q} \delta c_q + \frac{\partial L}{\partial \dot{q}} \left( \frac{\delta c'_q}{c'_t} - \frac{c'_q \delta c'_t}{(c'_t)^2} \right) \right] c'_t \, d\tau + \int_0^1 L \delta c'_t \, d\tau$$

$$= \int_0^{\tau_i} \left[ \frac{\partial L}{\partial q} c'_t - \frac{d}{d\tau} \frac{\partial L}{\partial \dot{q}} \right] \delta c_q \, d\tau + \frac{\partial L}{\partial \dot{q}} \delta c_q \bigg|_0^{\tau_i}$$

$$+ \int_{\tau_i}^1 \left[ \frac{\partial L}{\partial q} c'_t - \frac{d}{d\tau} \frac{\partial L}{\partial \dot{q}} \right] \delta c_q \, d\tau + \frac{\partial L}{\partial \dot{q}} \delta c_q \bigg|_{\tau_i}^1$$

$$+ \int_0^{\tau_i} \frac{d}{d\tau} \left[ \frac{\partial L}{\partial \dot{q}} c'_q \frac{c'_t}{c'_t} - L \right] \delta c_t \, d\tau - \left[ \frac{\partial L}{\partial \dot{q}} c'_q \frac{c'_t}{c'_t} - L \right] \delta c_t \bigg|_0^{\tau_i}$$

$$+ \int_{\tau_i}^1 \frac{d}{d\tau} \left[ \frac{\partial L}{\partial \dot{q}} c'_q \frac{c'_t}{c'_t} - L \right] \delta c_t \, d\tau - \left[ \frac{\partial L}{\partial \dot{q}} c'_q \frac{c'_t}{c'_t} - L \right] \delta c_t \bigg|_{\tau_i}^1$$
Lagrangian mechanics in nonsmooth setting

with $\delta c(0) = \delta c(1) = 0$

$$= \int_0^{\tau_i} \left[ \frac{\partial L}{\partial q} c'_t - \frac{d}{d\tau} \frac{\partial L}{\partial \dot{q}} \right] \delta c_q + \frac{d}{d\tau} \left[ \frac{\partial L}{\partial \dot{q}} c'_t - L \right] \delta c_t d\tau + \int_{\tau_i}^1 \cdots d\tau$$

$$= \text{EL}_{c_q}$$

$$= \text{EL}_{c_t}$$

$$+ \frac{\partial L}{\partial \dot{q}} \delta c_q - \left[ \frac{\partial L}{\partial q} c'_t - L \right] \delta c_t \bigg|_{\tau_i^-}^{\tau_i^+} = 0 \quad \forall \delta c \ in \ T_c \mathcal{M}$$

with the Euler-Lagrange equations $\text{EL}_{c_q}, \text{EL}_{c_t}$ and the

Lagrangian one-form $\Theta_L = \frac{\partial L}{\partial \dot{q}} \ dc_q - \left[ \frac{\partial L}{\partial q} c'_t - L \right] dc_t$

$\Rightarrow$ $c$ is solution iff EL equations are satisfied on smooth parts and Lagrangian one-form as a zero jump at $\tau_i$
Lagrangian mechanics in nonsmooth setting

- this means in $t$ space

$$\frac{\partial L}{\partial q} - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}} = 0 \quad \text{in} \quad [0, t_i) \cup (t_i, T]$$

$$\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}} \dot{q} - L \right) = 0 \quad \text{in} \quad [0, t_i) \cup (t_i, T]$$

with $0 = c_t(0), \ T = c_t(1)$ and $t_i = c_t(\tau_i)$

- note that

$$\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}} \dot{q} - L \right) = \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}} \right) \dot{q} + \frac{\partial L}{\partial \dot{q}} \ddot{q} - \frac{dL}{dt}$$

$$= \left[ \frac{\partial L}{\partial q} \dot{q} + \frac{\partial L}{\partial \dot{q}} \ddot{q} \right] - \frac{dL}{dt} = 0$$

$\Rightarrow$ preservation of energy $E : TQ \to \mathbb{R}, \ E = \frac{\partial L}{\partial \dot{q}} \dot{q} - L$ on smooth parts (redundant to first EL equations)
Lagrangian mechanics in nonsmooth setting

- with energy term the Lagrangian one-form is written as

$$\Theta_L = \frac{\partial L}{\partial \dot{q}} dc_q - Edc_t$$

- the condition $\Theta_L|_{\tau_i^-} = \Theta_L|_{\tau_i^+}$ leads to

$$\left. \frac{\partial L}{\partial \dot{q}} \right|_{t=t_i^-} \cdot \delta c_q = \left. \frac{\partial L}{\partial \dot{q}} \right|_{t=t_i^+} \cdot \delta c_q$$

for any $\delta c_q \in T_{c_q(\tau_i)} \partial C$

$$E(q(t_i^-), \dot{q}(t_i^-)) = E(q(t_i^+), \dot{q}(t_i^+))$$

Weierstrass-Erdmann-type conditions for impact

(i) conservation of linear momentum in the tangent direction to $\partial C$

(ii) conservation of energy during elastic impact
Lagrangian mechanics in nonsmooth setting

Example: elastic impact of point mass

▶ admissible set \( q \in C = \{(x, y) \in \mathbb{R}^2 \mid y \geq 0\} \)

▶ \( L(q, \dot{q}) = \frac{1}{2} \dot{q}^T M \dot{q} - V(q) \), \( M = \text{diag}(m, m) \) with \( m \in \mathbb{R} \)

▶ \( \partial C = \{(x, y) \in \mathbb{R}^2 \mid y = 0\} \)

▶ \( T_q \partial C = \{(\dot{x}, \dot{y}) \in \mathbb{R}^2 \mid (0 \ 1) \begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \dot{y} = 0\} \)

▶ momentum condition

\[
\left( \frac{\partial L}{\partial \dot{q}} \bigg|_{t=t_i^+} - \frac{\partial L}{\partial \dot{q}} \bigg|_{t=t_i^-} \right) \cdot \delta q = 0 \quad \text{for any } \delta q \in T_q \partial C
\]

\[
\Rightarrow M \left( \begin{pmatrix} \dot{x}^+ \\ \dot{y}^+ \end{pmatrix} - \begin{pmatrix} \dot{x}^- \\ \dot{y}^- \end{pmatrix} \right) \perp T_q \partial C
\]

\[
\Rightarrow M \left( \begin{pmatrix} \dot{x}^+ \\ \dot{y}^+ \end{pmatrix} - \begin{pmatrix} \dot{x}^- \\ \dot{y}^- \end{pmatrix} \right) \in N_C(q)
\]

\[
\Rightarrow M \left( \begin{pmatrix} \dot{x}^+ \\ \dot{y}^+ \end{pmatrix} - \begin{pmatrix} \dot{x}^- \\ \dot{y}^- \end{pmatrix} \right) = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \lambda \Rightarrow \dot{x}^+ = \dot{x}^-, \dot{y}^+ = \dot{y}^- + \lambda
\]
Lagrangian mechanics in nonsmooth setting

Example: elastic impact of point mass

- discrete energy preservation

\[ E(q(t_i^-), \dot{q}(t_i^-)) = E(q(t_i^+), \dot{q}(t_i^+)) \]

\[ \Rightarrow \frac{1}{2} m ((\dot{x}^-)^2 + (\dot{y}^-)^2) + V(x^-, y^-) = \]

\[ \frac{1}{2} m ((\dot{x}^-)^2 + (\dot{y}^- + \lambda)^2) + V(x^+, y^+) \]

\[ \Rightarrow q^- = q^+ \quad \lambda = -2\dot{y}^- \]

- substitute in momentum equation

\[ \dot{y}^+ = \dot{y}^- + \lambda = -\dot{y}^- \]

\[ \Rightarrow \text{elastic impact: sign change in normal direction of velocity} \]
Lagrangian mechanics in nonsmooth setting

**Symplecticity**
- it holds \((F^t)^*\Omega_L = \Omega_L\) with flow \(F^t\) and the extended symplectic form

\[
\Omega_L = \omega_L + dE \wedge dt
\]

with the canonical symplectic form \(\omega_L = d\theta_L\) and the canonical Lagrangian one-form \(\theta_L = \frac{\partial L}{\partial \dot{q}} dq\)

**Noether theorem**
- If the Lagrangian \(L : TQ \rightarrow \mathbb{R}\) is invariant under the group action \(\Phi : G \times Q \rightarrow Q\) and the group action leaves \(\partial C\) invariant, then the corresponding Lagrangian momentum map \(J_L : TQ \rightarrow \mathfrak{g}^*\) is a conserved quantity of the flow so that

\[
J_L \circ F^t = J_L \quad \text{for all times } t.
\]
Lagrangian mechanics in nonsmooth setting

Example: elastic impact of point mass in plane

- Lagrangian \( L(x, y, \dot{x}, \dot{y}) = \frac{1}{2} m (\ddot{x}^2 + \ddot{y}^2) \) is invariant under translations \( \Phi_g : (x, y) \mapsto (x + u, y + v) \)

- \( (x, y) \in \partial C = \{(x, y) \in \mathbb{R}^2 \mid y = 0\} \) is not invariant under \( \Phi_g \)

- but is invariant under \( \Phi^x_g : (x, y) \mapsto (x + u, y) \)

\[ \Rightarrow \text{Noether: only linear momentum in } x\text{-direction is conserved, } p_x = \text{const}, \text{ but linear momentum in } y\text{-direction, } p_y, \text{ is not conserved at impact} \]
Nonsmooth discrete Lagrangian mechanics

- discrete Lagrangian \( L_d(q_0, q_1, h) \approx \int_0^h L(q, \dot{q}) \, dt \)
- \( t_k = k h, \ k = 0, \ldots, N \), fix \( \tilde{\alpha} \in [0, 1] \), fixed impact time \( \tilde{\tau} = t_{i-1} + \tilde{\alpha} h \), actual impact time \( \tilde{t} = t_{i-1} + \alpha h \)
- \( \alpha = t_d(\tilde{\alpha}) \) with strictly increasing function \( t_d : [0, 1] \to [0, 1] \)
- that means we know the step at which impact occurs but not the impact time \( \tilde{t} \) which is allowed to vary according to variations in \( \alpha \)
Nonsmooth discrete Lagrangian mechanics

- discrete path space \( \mathcal{M}_d = \mathcal{T}_d \times \mathcal{Q}_d(\tilde{\alpha}, \partial C, Q) \) with

\[
\mathcal{T}_d = \{ t_d(\tilde{\alpha}) \mid t_d \in C^\infty([0, 1], [0, 1]), t_d \text{ onto}, t_d' > 0 \text{ in } [0, 1] \}
\]

\[
\mathcal{Q}_d(\tilde{\alpha}, \partial C, Q) = \{ q_d : \{ t_0, \ldots, t_{i-1}, \tilde{\tau}, t_i, \ldots, t_N \} \rightarrow Q, q_d(\tilde{\tau}) \in \partial C \}
\]

- discrete trajectory

\[
(\alpha, q_d) = (\alpha, \{ q_0, \ldots, q_{i-1}, \tilde{q}, q_i, \ldots, q_N \})
\]

with \( q_k = q_d(t_k) \) for \( k = \{0, \ldots, N\} \), \( \tilde{q} = q_d(\tilde{\tau}) \) and \( \alpha = t_d(\tilde{\alpha}) \)
Nonsmooth discrete Lagrangian mechanics

- discrete action map

\[ S_d(\alpha, q) = \sum_{k=0}^{i-2} L_d(q_k, q_{k+1}, h) + \sum_{k=i}^{N-1} L_d(q_k, q_{k+1}, h) + L_d(q_{i-1}, \tilde{q}, \alpha h) + L_d(\tilde{q}, q_i, (1 - \alpha)h) \]

- take variations

\[ (\delta \alpha, \delta q_d) = (\delta \alpha, \{\delta q_0, \ldots, \delta q_{i-1}, \delta \tilde{q}, \delta q_i, \ldots, \delta q_N\}) \]

in the tangent space \( TM_d = T\mathcal{T}_d \times TQ_d \)
Nonsmooth discrete Lagrangian mechanics

same discrete Euler-Lagrange equations as before

► away from impact (solve for $q_{k+1}$)

\[ D_2 L_d(q_{k-1}, q_k, h) + D_1 L_d(q_k, q_{k+1}, h) = 0 \quad \forall k \text{ away from impact} \]

► just before impact (solve for $\tilde{q}$ and $\alpha$)

\[ D_2 L_d(q_{i-2}, q_{i-1}, h) + D_1 L_d(q_{i-1}, \tilde{q}, \alpha h) = 0 \]

\[ \tilde{q} \in \partial C \]

► at impact (solve for $q_i$)

\[ [D_2 L_d(q_{i-1}, \tilde{q}, \alpha h) + D_1 L_d(\tilde{q}, q_i, (1 - \alpha) h)] \cdot \delta \tilde{q} = 0 \quad \forall \delta \tilde{q} \in T \partial C \]

\[ D_3 L_d(q_{i-1}, \tilde{q}, \alpha h) - D_3 L_d(\tilde{q}, q_i, (1 - \alpha) h) = 0 \]

► just after impact (solve for $q_{i+1}$)

\[ D_2 L_d(\tilde{q}, q_i, (1 - \alpha) h) + D_1 L_d(q_i, q_{i+1}, h) = 0 \]
Nonsmooth discrete Lagrangian mechanics

▶ discrete Lagrangian one form

\[ \Theta^+_{L_d}(q_k, q_{k+1}) = D_2 L_d(q_k, q_{k+1})dq_{k+1} \]
\[ \Theta^-_{L_d}(q_k, q_{k+1}) = -D_1 L_d(q_k, q_{k+1})dq_k \]

▶ discrete energy \[ E_d(q_k, q_{k+1}, h) = -D_3 L_d(q_k, q_{k+1}, h) \]

▶ discrete Euler-Lagrange equations at impact

\[ \Theta^+_{L_d}(q_{i-1}, \ddot{q}, \alpha h) \cdot \delta \ddot{q} = \Theta^-_{L_d}(\ddot{q}, q_i, (1 - \alpha)h) \cdot \delta \ddot{q} \quad \forall \delta \ddot{q} \in T \partial C \]
\[ E_d(q_{i-1}, \ddot{q}, \alpha h) = E_d(\ddot{q}, q_i, (1 - \alpha)h) \]

discrete version of Weierstrass-Erdmann-type conditions

(i) conservation of discrete linear momentum in the tangent direction to \( \partial C \)

(ii) conservation of discrete energy during elastic impact
Nonsmooth discrete Lagrangian mechanics

Symplecticity

- discrete Lagrangian symplectic form \( \Omega_{L_d} = d\Theta^+_L = d\Theta^-_L \)
  with coordinate expression

\[
\Omega_{L_d}(q_0, q_1) = \frac{\partial^2 L_d}{\partial q_0^i \partial q_1^j} dq_0^i \wedge dq_1^j
\]

is preserved under the discrete flow: \((F_{L_d}^N)^*(\Omega_{L_d}) = \Omega_{L_d}\)

Noether theorem

- If the discrete Lagrangian \( L_d : Q \times Q \to \mathbb{R} \) is invariant under the group action \( \Phi : G \times Q \to Q \) and the group action leaves \( \partial C \) invariant, then the corresponding discrete Lagrangian momentum map \( J_{L_d} : Q \times Q \to g^* \) is a conserved quantity of the discrete flow so that

\[
J_{L_d} \circ F_{L_d} = J_{L_d}.
\]
Implementation

- assume $\partial C = \{ q \mid g(q) = 0 \}$ with $g : Q \to \mathbb{R}$
- tangent space $T\partial C = \{ (q, \nu) \mid g(q) = 0, \nabla g(q) \cdot \nu = 0 \}$
- if $\delta \tilde{q} \in T\partial C$ it holds $\delta \tilde{q} \in \text{null}(\nabla g(\tilde{q}))$
- for $w \cdot \delta \tilde{q} = 0 \ \forall \delta \tilde{q} \in T\partial C$ it must hold $w \in \text{im}(\nabla g^T(\tilde{q}))$
- **DEL just before impact** (solve for $\tilde{q}$ and $\alpha$)
  \[
  D_2 L_d(q_{i-2}, q_{i-1}, h) + D_1 L_d(q_{i-1}, \tilde{q}, \alpha h) = 0
  \]
  \[
  g(\tilde{q}) = 0
  \]
  which are $n + 1$ equations for the $n + 1$ unknowns $\tilde{q}$ and $\alpha$
- **DEL at impact** (Lagrange multiplier approach)
  \[
  D_2 L_d(q_{i-1}, \tilde{q}, \alpha h) + D_1 L_d(\tilde{q}, q_i, (1 - \alpha)h) = \nabla g^T(\tilde{q}) \cdot \lambda
  \]
  \[
  D_3 L_d(q_{i-1}, \tilde{q}, \alpha h) - D_3 L_d(\tilde{q}, q_i, (1 - \alpha)h) = 0
  \]
  which are $n + 1$ equations for the $n + 1$ unknowns $q_i$ and Lagrange multiplier $\lambda \in \mathbb{R}$
Numerical examples

Particle colliding with rigid surface

- particle moving under gravity in \((x, y)\)-plane
  \[ L(q, \dot{q}) = \frac{1}{2} \dot{q}^T I \dot{q} - mgy, \quad q = (x, y) \in \mathbb{R}^2 \]
- colliding and bouncing on a horizontal rigid floor located at \(y = 0\) \(\Rightarrow\) \(\partial C = \{(x, y) \in \mathbb{R}^2 \mid y = 0\}\)
- admissible set \(y \geq 0\)

- midpoint rule
- time step \(h = 0.1\)
- 1000 impacts
  
  \[ q_0 = (0, 1) \]
  \[ \dot{q}_0 = (-2, 0) \]
Numerical examples

Rotating nonconvex rigid body colliding with rigid surface

- $Q = SE(2)$, $q = (x, y, \theta)$
- nonpenetration condition $y = \frac{L}{2}(|\sin \theta| + |\cos \theta|)$
- $M = \text{diag}(m, m, I)$ with inertia $I = \frac{29}{192} mL^2$
- $q_0 = (0, 3.5, 0)$, $\dot{q}_0 = (-2, 0, 5)$

- midpoint rule with $h = 0.005$ and 1500 impacts
Numerical examples

Rotating nonconvex rigid body colliding with rigid surface

- almost preserved energy
- second order accuracy (as for smooth systems)
Numerical examples

Mass-spring system

- Lagrangian $L = \frac{1}{2} m \ddot{q}^2 - \frac{1}{2} c q^2, \ q \in \mathbb{R}$
- admissible set $q \leq 0$
- $\partial C = \{ q \in \mathbb{R} \mid q = 0 \}$
- $m = 1, c = 5, q_0 = -1, \dot{q}_0 = -2.05$

Different integrators

- collision VI, Lagrangian (backward Euler) **EBD Fetecau**
- collision VI, Hamilton-Pontryagin (backward Euler) **EBD**
- collision VI, Hamilton-Pontryagin (forward Euler) **EFD**
- collision VI, Hamilton-Pontryagin (midpoint rule) **MPR**
Numerical examples

Mass-spring system

- step size $h = 0.01$

- different collision times (first versus second order)
- different amplitudes (back-/forward Euler vs. midpoint)
Numerical examples

Mass-spring system

- step size $h = 0.01$

almost energy preservation

shifted average value for first order methods
Numerical examples

Mass-spring system: consider 2 variants of EBD integrator

- neglect $\delta \ddot{q} \in T\partial C$ and discrete energy preservation
- instead switch sign of $y$-momentum $\tilde{p}_x^+ = \tilde{p}_x^-, \tilde{p}_y^+ = -\tilde{p}_y^-

**var 1** keep time grid as before

$$-D_1 L_d(\ddot{q}, q_i, (1 - \alpha)h) = D_2 L_d(q_{i-1}, \ddot{q}, \alpha h)$$

**var 2** reset time grid after switch

$$-D_1 L_d(\ddot{q}, q_i, h) = D_2 L_d(q_{i-1}, \ddot{q}, \alpha h)$$

energy

energy loss for variants of EBD
Numerical examples

Mass-spring system: influence of different step sizes

▶ energy

▶ configuration

$h = 0.02$

$h = 0.04$

$h = 0.06$
Forced systems: continuous model

- Lagrange-d’Alembert principle

\[ \delta \int_{0}^{T} L(q, \dot{q}) \, dt + \int_{0}^{T} f(q, \dot{q}, u) \delta q \, dt = 0 \]

- Forced EL equations

\[ \frac{\partial L}{\partial q} - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}} = f(q, \dot{q}, u) \text{ in } [0, t_i) \cup (t_i, T] \]

- EL equations at impact

\[ \left. \frac{\partial L}{\partial \dot{q}} \right|_{t=t_i^-} \cdot \delta c_q = \left. \frac{\partial L}{\partial \dot{q}} \right|_{t=t_i^+} \cdot \delta c_q \text{ for any } \delta c_q \in T_{c_q(t_i)} \partial C \]

\[ E(q(t_i^-), \dot{q}(t_i^-)) = E(q(t_i^+), \dot{q}(t_i^+)) \]
Forced systems: discrete model

▶ forced DEL away from impact (solve for $q_{k+1}$)
\[ D_2 L_d(q_{k-1}, q_k, h) + D_1 L_d(q_k, q_{k+1}, h) + f_d^+(q_{k-1}, q_k, h) + f_d^-(q_k, q_{k+1}, h) = 0 \]

▶ forced DEL just before impact (solve for $\tilde{q}$ and $\alpha$)
\[ D_2 L_d(q_{i-2}, q_{i-1}, h) + D_1 L_d(q_{i-1}, \tilde{q}, \alpha h) + f_d^+(q_{i-2}, q_{i-1}, h) + f_d^-(q_{i-1}, q_i, \alpha h) = 0 \]

▶ forced DEL at impact (solve for $q_i$)
\[
\begin{align*}
&\left[ D_2 L_d(q_{i-1}, \tilde{q}, \alpha h) + f_d^+(q_{i-1}, \tilde{q}, \alpha h) \\
&+ D_1 L_d(\tilde{q}, q_i, (1 - \alpha)h) + f_d^-(\tilde{q}, q_i, (1 - \alpha)h) \right] \cdot \delta \tilde{q} = 0 \quad \forall \delta \tilde{q} \in T \partial C
\end{align*}
\]
\[ D_3 L_d(q_{i-1}, \tilde{q}, \alpha h) + \alpha h f_d^+(q_{i-1}, \tilde{q}, \alpha h) \frac{\tilde{q} - q_{i-1}}{\alpha h} \\
- D_3 L_d(\tilde{q}, q_i, (1 - \alpha)h) - (1 - \alpha) h f_d^-(\tilde{q}, q_{i-1}, (1 - \alpha)h) \frac{q_i - \tilde{q}}{(1 - \alpha)h} = 0 \]
Numerical examples

Damped mass-spring system

- Lagrangian $L = \frac{1}{2}m\dot{q}^2 - \frac{1}{2}cq^2$, $q \in \mathbb{R}$
- damping force $f = -d \cdot \dot{q}$
- $\partial C = \{ q \in \mathbb{R} \mid q = 0 \}$
- $m = 1$, $c = 5$, $d = 0.05$

configuration with and without collision  
energy with and without collision
Hybrid systems

A hybrid dynamical system is represented by a hybrid automaton, which is described by
\( \{(X(j), G(j), g^{(j,i)}, r^{(j,i)}) \mid j, i = 1, \ldots, p\} \).

- \( X(j) \) is the state space for the discrete state \( j \), with the according system dynamics \( \dot{x} = f^{(j)}(x), \ x \in X(j) \).

- \( G(j) \subset X(j) \) is the guard set of state \( j \). It is divided into the subguards \( g^{(j,i)} \subset G(j) \).

- \( r^{(j,i)} : g^{(j,i)} \rightarrow X(i) \) is a reset map.

- If the dynamic state reaches a subguard of the discrete state \( x(t) \in g^{(j,i)} \), the system switches instantaneously to the discrete state \( i \).

- The new dynamic state is calculated by the reset map \( x(t^+) = r^{(j,i)}(x(t^-)) \).
Elastic impact system as hybrid system

- dynamics $\dot{x} = f^{(j)}$ are given by Euler-Lagrange equations with state space $X^{(j)} = TQ$ with

$$q \in Q = \{(x, y) \in \mathbb{R}^2 | y \geq 0\} \text{ for all } j$$

- guard (only defined on $Q$)

$$g^{(j,i)} = \{(x, y) \in \mathbb{R}^2 | y = 0\} = \partial C$$

- reset map $r^{(j,i)} : (q^-, \dot{q}^-) \mapsto (q^+, \dot{q}^+)$ with

$$(x^+, y^+, \dot{x}^+, \dot{y}^+) = (x^-, y^-, \dot{x}^-, -\dot{y}^-)$$

In the variational approach the reset map is derived from the variational principle and given by the momentum jump condition and the energy preservation equation.
Electric circuit as hybrid system

- state space $X^{(j)}$ given by Pontryagin bundle $TQ \oplus T^*Q$ with $(q, v, p) \in TQ \oplus T^*Q$ for all $j$
- restrictions due to KCL and KVL defined as distribution $\Delta_Q \subset TQ$ and annihilator $\Delta_Q^0 \subset T^*Q$

\[
\Rightarrow X^{(j)} = \Delta_Q \oplus \Delta_Q^0 =: \mathcal{D}
\]

- dynamics are given by the Lagrange-d’Alembert-Pontryagin principle
- What is the hybrid part?
Electric circuit as hybrid system

- **Diode model**
  
  **state 1:** \( u_T \geq -V_0 \cdot R_D \),
  \[
  v_{\text{diode}}^{(1)}(u_T) = \frac{1}{R_D} \cdot u_T
  \]
  
  **state 2:** \( u_T < -V_0 \cdot R_D \),
  \[
  v_{\text{diode}}^{(2)}(u_T) = -V_0,
  \]
  
  \( u_T \): terminal voltage

- **different forces for states**
  
  \[
  f_{\text{diode}}^{(1)} = -R_D v_{\text{diode}}
  \]
  
  \[
  f_{\text{diode}}^{(2)} = -1 \cdot v_{\text{diode}} - U_0 - u_T
  \]

- **guard** \( G = \{(q, v, p) \in \mathcal{D} \mid u_T = -V_0 \cdot R_D = u_{T,\text{crit}}\} \)

- **reset map** \( r = id \) on \( \mathcal{D} \)
Linear circuit with diode

$L = 3$, $C = 2$, $R_1 = 5$, $R_2 = 50$

$u_{S_1}(t) = \sin(t)$, $u_{S_2}(t) = 2 \cdot \cos(t)$

$R_D = 10^{-2}$, $U_0 = 10^{-4}$

$u_T = -U_0 \cdot R_D = -10^{-6}$

currents (EFD, $h = 0.08$)

energy (EFD, MPR)
Conclusions

- **(Discrete) Lagrangian mechanics** to model nonsmooth systems (e.g. elastic impact, non-elastic impacts etc.)
- **Extended phase space** allows to take the coupling of variations in configuration and switching time into account
- **Variational collision integrator** includes discrete energy balance that provides a good long-time energy behaviour
- **Symplecticity** and momentum preserving (Noether theorem)
- Generalization to **hybrid system** (modelled as hybrid automaton)
- Application to **hybrid electric circuits** including diodes
Literature overview

Literature used in this lecture


Variational collision integrators for constrained systems


Optimal control of nonsmooth systems using Lagrangian mechanics

