

A CALDERÓN TYPE INVERSE PROBLEM FOR QUANTUM TREES

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ABSTRACT. We solve the inverse problem of recovering a metric tree from the knowledge of the Dirichlet-to-Neumann matrix on the tree's boundary corresponding to the Laplacian with standard vertex conditions. This result can be viewed as a counterpart of the Calderón problem in the analysis of PDEs; in contrast to earlier results for quantum graphs, we only assume knowledge of the Dirichlet-to-Neumann matrix for a fixed energy, not of a whole matrix-valued function. The proof is based on tracing back the problem to an inverse problem for the Schur complement of the discrete Laplacian on an associated weighted tree. In addition, we provide examples which show that several possible generalizations of this result, e.g. to graphs with cycles, fail.

1. INTRODUCTION

Calderón's classical problem from Electrical Impedance Tomography consists in recovering, if possible, the isotropic conductivity of an inhomogeneous body uniquely from applying voltages to the surface of the body and measuring the corresponding current flux through the surface, see [7]. In mathematical terms the relation between voltage and current is given by the so-called Dirichlet-to-Neumann map on the boundary of a Euclidean domain Ω corresponding to the differential equation $\operatorname{div} \gamma \nabla u = 0$ on Ω . After a simple transformation (cf. [17]) the inverse problem of recovering the conductivity function γ from the knowledge of the Dirichlet-to-Neumann map is equivalent to recovering a positive electric potential q from the Dirichlet-to-Neumann map $M_{\Omega,q}$ defined by

$$M_{\Omega,q}u|_{\partial\Omega} = \partial_\nu u|_{\partial\Omega},$$

where u satisfies $-\Delta u + qu = 0$ in Ω and $u|_{\partial\Omega}$ and $\partial_\nu u|_{\partial\Omega}$ denote the trace and the derivative with respect to the unit normal, respectively, of u on the boundary. These equivalent problems were proven to be uniquely solvable under reasonable regularity assumptions; see, e.g., [3, 16, 17, 21]. Extensions to anisotropic conductivities and partial boundary data were made in, e.g., [2, 6, 9, 12, 18, 20].

Quantum graphs, i.e. differential operators on metric graphs, serve as approximations for partial differential operators on thin branching domains and have found a wide range of applications, see, e.g., the monograph [4]. A natural question analogous to Calderón's problem is whether or not a differential operator on a finite metric graph can be recovered from the knowledge of a corresponding Dirichlet-to-Neumann map on its boundary. Given a finite metric graph Γ and an electric potential q on Γ , define the Dirichlet-to-Neumann matrix $M_{\Gamma,q}(\lambda)$ for the differential equation $-f'' + qf = \lambda f$ on Γ for suitable $\lambda \in \mathbb{C}$ by the equation

$$M_{\Gamma,q}(\lambda)f_\lambda|_{\partial\Gamma} = \partial f_\lambda|_{\partial\Gamma},$$

where f_λ satisfies $-f_\lambda'' + qf_\lambda = \lambda f_\lambda$ inside the edges of Γ and standard (continuity-Kirchhoff) matching conditions on all vertices that do not belong to the "naive"

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boundary $\partial\Gamma$ consisting of all vertices of degree one; $f_\lambda|_{\partial\Gamma}$ and $\partial f_\lambda|_{\partial\Gamma}$ are the vectors of boundary evaluations of f_λ and its derivative, respectively. The function $\lambda \mapsto M_{\Gamma,q}(\lambda) \in \mathbb{C}^{\partial\Gamma \times \partial\Gamma}$ is a matrix-valued Herglotz–Nevanlinna function with a discrete set of poles on the real axis. In recent years the inverse problem of recovering the metric graph Γ and the potential q on it from the knowledge of the matrix function $\lambda \mapsto M_{\Gamma,q}(\lambda)$ has received a lot of attention. It was solved in [1, 5, 22] for the case that Γ is a tree, i.e. a graph without cycles; see also [8, 11, 23] for related results. On the other hand, it is not uniquely solvable for more general graphs if not further additional data is provided, see [10, 15].

To consider a problem that is entirely analogous to Calderón’s problem, one may ask if it is possible to recover Γ and q from $M_{\Gamma,q} := M_{\Gamma,q}(0)$ (or, more generally, from $M_{\Gamma,q}(\lambda_0)$ for a fixed λ_0) instead of requiring the knowledge of the whole matrix function $\lambda \mapsto M_{\Gamma,q}(\lambda)$. It is clear that at least recovering the coefficient function q is impossible in general as $M_{\Gamma,q}$ is just a finite matrix, whereas the Dirichlet-to-Neumann map in the above-mentioned PDE setting is an operator in an infinite-dimensional space of functions on the boundary. The aim of this paper is to show that, however, it is not entirely hopeless to recover information on a quantum graph from $M_{\Gamma,q}$. We consider the case $q = 0$ identically, so that the differential operator in question is actually the second derivative on each edge. The question we ask is whether or not the structure of the underlying metric graph is uniquely determined by the matrix $M_\Gamma := M_{\Gamma,0}(0)$. The main result of this paper is the following.

Theorem. *If Γ_1 and Γ_2 are two finite metric trees which have the same number of boundary vertices and satisfy*

$$M_{\Gamma_1} = M_{\Gamma_2},$$

then Γ_1 and Γ_2 coincide up to vertices of degree two.

We actually provide an explicit formula how the distances between each two boundary vertices of Γ can be obtained from M_Γ ; this leads to an explicit reconstruction algorithm for the tree Γ ; cf. Theorem 5.1 and Algorithm 1 below. In addition to this result we provide examples which show that this is not true as soon as Γ is allowed to have cycles and that in the general case not even topological properties as, e.g., the Betti number of Γ can be recovered from M_Γ .

The proof of our main result is based on a transformation into a problem for a corresponding weighted discrete graph and the corresponding discrete Laplacian. We identify the matrix M_Γ with the Schur complement of the discrete Laplacian with respect to a certain block matrix decomposition and solve the inverse problem of recovering a weighted discrete tree from that Schur complement of its discrete Laplacian. This is done with the help of properties of so-called reduced Laplacians established in [14]. We point out that the techniques used here are entirely different from the methods used for recovering information on a quantum graph from the matrix function $\lambda \mapsto M_{\Gamma,q}(\lambda)$.

The paper is structured as follows. In Section 2 we introduce our setting and provide a rigorous definition of the Dirichlet-to-Neumann matrix M_Γ . Afterwards, in Section 3 we verify the identification of M_Γ with the Schur complement of a discrete Laplacian. Section 4 contains the solution of the inverse problem for trees on the level of weighted discrete graphs, while Section 5 contains its translation into the original setting of metric graphs and thus the main result of this paper. Finally, in Section 6 we provide examples that rule out several naive generalizations of our main result.

2. METRIC GRAPHS AND THE DIRICHLET-TO-NEUMANN MATRIX

Throughout this paper, \mathcal{G} denotes a finite graph consisting of a finite set $\mathcal{V} = \mathcal{V}(\mathcal{G})$ of vertices and a finite set $\mathcal{E} = \mathcal{E}(\mathcal{G})$ of edges. For each $v \in \mathcal{V}$ we denote by $\deg v$ its degree and by $\mathcal{E}(v)$ the set of all edges incident to v . Moreover, we call the set

$$\partial\mathcal{G} = \{v \in \mathcal{V} : \deg v = 1\}$$

of vertices of degree one the *boundary* of \mathcal{G} and the corresponding vertices *boundary vertices*. Each vertex which is not a boundary vertex is called *interior vertex*. In the following we assume for simplicity that \mathcal{G} has no multiple edges and no loops; those are not relevant to us since we will mostly deal with trees, i.e. graphs without cycles. We assume also that \mathcal{G} is *connected*, i.e. for any two vertices there exists a path connecting them.

For any given finite graph \mathcal{G} we denote by Γ the corresponding metric tree that is induced by a length function $L : \mathcal{E} \rightarrow (0, \infty)$; we interpret each edge e of Γ as interval $[0, L(e)]$ and obtain from this parametrization a natural metric d_Γ on Γ . For a function $f : \Gamma \rightarrow \mathbb{C}$ we denote by $f_e := f|_e$ its restriction to the edge e . We define the natural L^2 and Sobolev spaces on Γ by

$$L^2(\Gamma) := \bigoplus_{e \in \mathcal{E}} L^2(0, L(e)), \quad \tilde{H}^k(\Gamma) := \bigoplus_{e \in \mathcal{E}} H^k(0, L(e)), \quad k = 1, 2,$$

where $H^k(0, L(e))$, $k = 1, 2, \dots$, is the k -th order Sobolev space on the interval $(0, L(e))$, which consists of functions which are k -times differentiable almost everywhere with all derivatives in $L^2(0, L(e))$; these spaces are equipped with the usual inner products. Furthermore, we consider the space $H^1(\Gamma)$ which consists of all functions in $\tilde{H}^1(\Gamma)$ which are continuous at each vertex; note that for $f \in H^1(\Gamma)$ its evaluation $f(v)$ at a vertex v is well-defined. For $f \in H^1(\Gamma) \cap \tilde{H}^2(\Gamma)$ we define the normal derivative $\partial f(v)$ at the vertex v as

$$\partial f(v) := \sum_{e \in \mathcal{E}(v)} \frac{d}{dx} f_e(v),$$

where the derivatives are taken in the direction towards the vertex. Note that if v belongs to $\partial\mathcal{G}$ the sum in the definition of $\partial f(v)$ consists of only one summand.

We can now define one of the main players in our considerations.

Definition 2.1. Let Γ be a finite metric graph whose boundary consists of the vertices v_1, \dots, v_k . The *Dirichlet-to-Neumann matrix* corresponding to the Laplacian on Γ with standard vertex conditions is the $k \times k$ -matrix M_Γ that satisfies

$$M_\Gamma \begin{pmatrix} f(v_1) \\ \vdots \\ f(v_k) \end{pmatrix} = \begin{pmatrix} \partial f(v_1) \\ \vdots \\ \partial f(v_k) \end{pmatrix},$$

where $f \in H^1(\Gamma) \cap \tilde{H}^2(\Gamma)$ solves $f'' = 0$ inside every edge and $\partial f(v) = 0$ for all $v \in \mathcal{V} \setminus \partial\mathcal{G}$.

Some remarks are in order. First, any function f as in the definition of M_Γ is linear on every edge and the specified vertex conditions determine f uniquely for any given collection of boundary values $f(v_1), \dots, f(v_k)$; see, e.g., [19, Lemma 2.2]. Equivalently one may define M_Γ in the following way: If $f^{(l)}$ is the unique solution of $(f^{(l)})'' = 0$ inside every edge of Γ that satisfies $\partial f^{(l)}(v) = 0$ at each interior vertex v , $f^{(l)}(v_l) = 1$ and $f^{(l)}(v_j) = 0$ for $j = 1, \dots, k$, $j \neq l$, then

$$(M_\Gamma)_{l,m} = \partial f^{(l)}(v_m), \quad l, m = 1, \dots, k.$$

An easy integration-by-parts argument yields that the matrix M_Γ is hermitian and non-negative.

3. REDUCTION TO AN INVERSE PROBLEM FOR A WEIGHTED DISCRETE GRAPH

In this section we show that the Dirichlet-to-Neumann matrix M_Γ for any finite metric graph Γ is given by the Schur complement of the discrete Laplacian on the underlying weighted discrete graph \mathcal{G} with weights corresponding to the inverse edge lengths of the metric graph. As an immediate consequence, the inverse problem of determining Γ from the Dirichlet-to-Neumann matrix is equivalent to determining the weighted graph \mathcal{G} from the Schur complement of the corresponding discrete Laplacian; cf. Section 5.

Throughout this section we assume that \mathcal{G} is a finite discrete graph with nonempty boundary $\partial\mathcal{G}$. We denote the vertices of \mathcal{G} by v_1, \dots, v_n and assume that v_1, \dots, v_k are those vertices which belong to $\partial\mathcal{G}$, where $1 \leq k = |\partial\mathcal{G}| \leq n$. Let Γ be a metric graph associated with \mathcal{G} . We interpret the inverse edge lengths of the metric graph Γ as edge weights of the discrete graph \mathcal{G} by setting

$$w_e := \frac{1}{L(e)}, \quad e \in \mathcal{E}, \quad (3.1)$$

and consider the discrete Laplacian $L(\mathcal{G}) \in \mathbb{R}^{n \times n}$ on \mathcal{G} associated with the weights w_e , $e \in \mathcal{E}$, given by

$$(L(\mathcal{G}))_{i,j} = \begin{cases} -w_e & \text{if } e \text{ connects } v_i \text{ and } v_j, i \neq j, \\ 0 & \text{if } v_i, v_j \text{ are not adjacent,} \\ \sum_{e \in \mathcal{E}(v_i)} w_e & \text{if } i = j. \end{cases} \quad (3.2)$$

According to the division of the vertices into the boundary vertices v_1, \dots, v_k and the interior vertices v_{k+1}, \dots, v_n we can write the discrete Laplacian as a block matrix

$$L(\mathcal{G}) = \begin{pmatrix} \widehat{D} & -B^\top \\ -B & \widehat{L} \end{pmatrix}; \quad (3.3)$$

here $\widehat{D} \in \mathbb{R}^{k \times k}$ is the diagonal matrix whose i -th diagonal entry equals the weight w_e of the edge e connecting the boundary vertex v_i to an interior vertex in \mathcal{G} and B is such that every column contains exactly one nonzero entry; more precisely, the j -th column of B has the entry w_{e_j} at position l if the boundary vertex v_j is adjacent to the l -th interior vertex v_{k+l} and the edge connecting the two is e_j , and all further entries in this column are zero. Clearly, $\widehat{L} \in \mathbb{R}^{(n-k) \times (n-k)}$ is the matrix that is obtained from $L(\mathcal{G})$ after deletion of the rows and columns that correspond to boundary vertices.

The following proposition connects $L(\mathcal{G})$ to the Dirichlet-to-Neumann matrix; cf. also [13, Lemma 3.1].

Proposition 3.1. *Let Γ be a finite metric graph, let \mathcal{G} be the underlying discrete graph, equipped with the edge weights (3.1), and let $L(\mathcal{G})$ be the corresponding discrete Laplacian in (3.2). Then the matrix \widehat{L} in the decomposition (3.3) is invertible and we have*

$$M_\Gamma = \widehat{D} - B^\top \widehat{L}^{-1} B.$$

In other words, the Dirichlet-to-Neumann matrix coincides with the Schur complement of the discrete Laplacian $L(\mathcal{G})$.

Proof. Step 1. For any vector x in the kernel of \widehat{L} the augmented vector

$$\widetilde{x} := \begin{pmatrix} 0_{\mathbb{R}^k} \\ x \end{pmatrix}$$

satisfies $\widetilde{x}^\top L(\mathcal{G})\widetilde{x} = 0$. As $L(\mathcal{G})$ is nonnegative and the only eigenvectors corresponding to the eigenvalue 0 are non-zero multiples of the all-ones vector this implies $x = 0$, that is, \widehat{L} is invertible.

Step 2. We show that any function $f \in H^1(\Gamma) \cap \widetilde{H}^2(\Gamma)$ with $f_e'' = 0$ identically on each edge e satisfies

$$L(\mathcal{G}) \begin{pmatrix} f(v_1) \\ \vdots \\ f(v_n) \end{pmatrix} = \begin{pmatrix} \partial f(v_1) \\ \vdots \\ \partial f(v_n) \end{pmatrix}. \quad (3.4)$$

Indeed, for a fixed vertex v_j we assume without loss of generality that all edges are parametrized pointing towards v_j so that for any vertex v_i adjacent to v_j and the corresponding edge e connecting v_i to v_j we have

$$f_e(x) = \frac{x}{L(e)}f(v_j) + \frac{L(e) - x}{L(e)}f(v_i), \quad 0 \leq x \leq L(e),$$

and thus its derivative at the vertex v_j equals

$$f_e'(L(e)) = \frac{f(v_j)}{L(e)} - \frac{f(v_i)}{L(e)}.$$

Thus

$$\partial f(v_j) = \sum_{e \in \mathcal{E}(v_j)} f_e'(L(e)) = \sum_{e \in \mathcal{E}(v_j)} \frac{f(v_j)}{L(e)} - \sum_{e \in \mathcal{E}(v_j) \cap \mathcal{E}(v_i)} \frac{f(v_i)}{L(e)}.$$

It follows directly from the definition of $L(\mathcal{G})$ and (3.1) that the latter coincides with the j -th entry of $L(\mathcal{G})(f(v_1), \dots, f(v_n))^\top$, which proves (3.4).

Step 3. Let now f belong to $H^1(\Gamma) \cap \widetilde{H}^2(\Gamma)$ with $f_e'' = 0$ identically on every edge e and, additionally, $\partial f(v_{k+1}) = \dots = \partial f(v_n) = 0$. For such f we can rewrite (3.4)

$$\begin{pmatrix} \widehat{D} & -B^\top \\ -B & \widehat{L} \end{pmatrix} \begin{pmatrix} f(v_1) \\ \vdots \\ f(v_n) \end{pmatrix} = \begin{pmatrix} \partial f(v_1) \\ \vdots \\ \partial f(v_k) \\ 0 \\ \vdots \\ 0 \end{pmatrix},$$

and the last $n - k$ rows of this system of equations yield

$$\begin{pmatrix} f(v_{k+1}) \\ \vdots \\ f(v_n) \end{pmatrix} = \widehat{L}^{-1}B \begin{pmatrix} f(v_1) \\ \vdots \\ f(v_k) \end{pmatrix}.$$

Plugging this into the first k rows yields

$$\widehat{D} \begin{pmatrix} f(v_1) \\ \vdots \\ f(v_k) \end{pmatrix} - B^\top \widehat{L}^{-1}B \begin{pmatrix} f(v_1) \\ \vdots \\ f(v_k) \end{pmatrix} = \begin{pmatrix} \partial f(v_1) \\ \vdots \\ \partial f(v_k) \end{pmatrix},$$

which proves the desired expression for M_Γ . \square

We point out that in the special case that every vertex of \mathcal{G} is a boundary vertex (which is only possible if \mathcal{G} consists of two vertices and one edge connecting them) the block matrix decomposition (3.3) is trivial, i.e. it just consists of \widehat{D} , and the Schur complement in Proposition 3.1 then simply equals \widehat{D} or, equally, $L(\mathcal{G})$ itself.

Since the Dirichlet-to-Neumann matrix is invariant under adding or removing vertices of degree two in the quantum graph, also the Schur complement of the discrete Laplacian $L(\mathcal{G})$ is invariant under these operations. More specifically, if we replace an edge e by a path consisting of two edges e_1, e_2 such that $L(e_1) + L(e_2) = L(e)$ (and the weights in the discrete weighted graph satisfying the corresponding equation) then the Schur complement of the discrete Laplacian with respect to the boundary vertices does not change.

4. RECONSTRUCTION OF A WEIGHTED TREE FROM THE SCHUR COMPLEMENT OF THE DISCRETE LAPLACIAN

In this section we solve the inverse problem of recovering a weighted discrete tree from the Schur complement of the corresponding discrete Laplacian. Throughout this section all graphs are trees.

The proof of the main result of this section requires some preparation. One of its ingredients will be the following lemma, which can be found in [14, Proposition 1]. Here the *reduced Laplacian* $L_v(\mathcal{G})$ appears, which by definition is the matrix obtained from $L(\mathcal{G})$ by removing the line and column that correspond to the vertex v . We point out that the same argument as in Step 1 of the proof of Proposition 3.1 yields that the reduced Laplacian is invertible.

Lemma 4.1. *Let \mathcal{G} be a finite weighted tree. Then for any vertex $v \in \mathcal{V}$ the entry (i, j) of the inverse $L_v(\mathcal{G})^{-1}$ of the reduced Laplacian equals $\sum_{e \in P_{i,j}} \frac{1}{w_e}$, where $P_{i,j}$ is the set of edges that are on both the path from v_i to v and the path from v_j to v .*

We point out that in the particular situation discussed in Section 3, where the weights on the discrete tree \mathcal{G} are induced by the edge lengths of a corresponding metric tree Γ , we have $w_e = 1/L(e)$ for each edge e . Thus the previous lemma states that the (i, j) -th entry of $L_v(\mathcal{G})^{-1}$ equals the total length of $P_{i,j}$ according to the metric on Γ .

The next auxiliary observation deals with recovering a finite weighted tree from a set of distances. It requires the following definition.

Definition 4.2. Two weighted discrete trees $\mathcal{G}_1, \mathcal{G}_2$ are called *equal*, $\mathcal{G}_1 = \mathcal{G}_2$, if $|\mathcal{V}(\mathcal{G}_1)| = |\mathcal{V}(\mathcal{G}_2)| =: n$ and there exists a permutation matrix $P \in \mathbb{R}^{n \times n}$ with $L(\mathcal{G}_2) = P^\top L(\mathcal{G}_1)P$. Moreover, \mathcal{G}_1 and \mathcal{G}_2 are called *equal up to vertices of degree two* if they coincide after removing each vertex v of degree two and replacing the two edges e_1, e_2 incident to v by one edge e whose weight w_e satisfies

$$\frac{1}{w_e} = \frac{1}{w_{e_1}} + \frac{1}{w_{e_2}}.$$

In the following proposition we use distances of vertices in a weighted discrete tree \mathcal{G} . In fact, if v, w are two vertices in \mathcal{G} and the edges e_1, \dots, e_m form the unique path that connects v and w then we define

$$d_{\mathcal{G}}(v, w) = \sum_{i=1}^m \frac{1}{w_{e_i}}.$$

When identifying a weighted discrete tree with a metric tree Γ as above via $L(e) = 1/w_e$ for each edge e , the distances between vertices defined here coincide with the distances according to the metric on Γ . Hence two weighted discrete trees \mathcal{G}_1 and \mathcal{G}_2

are equal up to vertices of degree two if the associated quantum graphs are equal up to vertices of degree two.

Proposition 4.3. *Let $\mathcal{G}_1, \mathcal{G}_2$ be two finite, weighted trees with boundaries*

$$\partial\mathcal{G}_j = \{v_1^j, \dots, v_k^j\}, \quad j = 1, 2,$$

where $k = |\partial\mathcal{G}_1| = |\partial\mathcal{G}_2|$. Assume that the pairwise distances of boundary vertices in \mathcal{G}_1 and \mathcal{G}_2 coincide, i.e.

$$d_{\mathcal{G}_1}(v_i^1, v_m^1) = d_{\mathcal{G}_2}(v_i^2, v_m^2) \quad (4.1)$$

holds for $i, m = 1, \dots, k$. Then $\mathcal{G}_1 = \mathcal{G}_2$ up to vertices of degree two.

Proof. The proof makes use of the following two simple observations.

Observation 1: In every tree \mathcal{G} with at least three boundary vertices there exists a pair of boundary vertices that have a joint neighbor (i.e. are adjacent to a joint vertex, up to vertices of degree two).

Indeed, assume the converse and let v_1 be any interior vertex of \mathcal{G} ; without loss of generality we presume that \mathcal{G} does not contain vertices of degree two. Then v_1 has degree 3 or greater and there exists an edge e_{j_1} incident to v_1 which is not a boundary edge. Let v_2 be the other vertex to which e_{j_1} is incident. Then the degree of v_2 is at least 3 and by assumption only one edge incident to v_2 is a boundary edge. Thus there exists an edge $e_{j_2} \neq e_{j_1}$ incident to v_2 such that e_{j_2} is not a boundary edge. Following this procedure we obtain a chain of edges $e_{j_1}, e_{j_2}, e_{j_3}, \dots$ which form an infinite path through \mathcal{G} . Since \mathcal{G} does not contain cycles each edge of \mathcal{G} appears at most once in this path, and as \mathcal{G} is finite this is a contradiction.

Observation 2: If \mathcal{G} is a tree with $\partial\mathcal{G} = \{v_1, \dots, v_k\}$ then two boundary vertices v_{i_1}, v_{i_2} have a joint neighbor (up to vertices of degree two) if and only if there exists a constant $c \in \mathbb{R}$ with

$$d_{\mathcal{G}}(v_{i_1}, v_j) - d_{\mathcal{G}}(v_{i_2}, v_j) = c \quad \text{for all } j \in \{1, \dots, k\} \setminus \{i_1, i_2\}. \quad (4.2)$$

Indeed, if v_{i_1} and v_{i_2} have a joint neighbor vertex then (4.2) is clearly fulfilled. Assume conversely that (4.2) holds. Let w_j be the vertex of degree three or larger closest to v_{i_j} and let \hat{v}_j be a boundary vertex such that the paths from \hat{v}_j to v_{i_1} and from \hat{v}_j to v_{i_2} split from each other at w_j , $j = 1, 2$. Then (4.2) implies

$$\begin{aligned} c &= d_{\mathcal{G}}(v_{i_1}, \hat{v}_1) - d_{\mathcal{G}}(v_{i_2}, \hat{v}_1) \\ &= d_{\mathcal{G}}(v_{i_1}, w_1) + d_{\mathcal{G}}(w_1, \hat{v}_1) - (d_{\mathcal{G}}(v_{i_2}, w_2) + d_{\mathcal{G}}(w_1, w_2) + d_{\mathcal{G}}(w_1, \hat{v}_1)) \\ &= d_{\mathcal{G}}(v_{i_1}, w_1) - d_{\mathcal{G}}(v_{i_2}, w_2) - d_{\mathcal{G}}(w_1, w_2) \end{aligned} \quad (4.3)$$

and

$$\begin{aligned} c &= d_{\mathcal{G}}(v_{i_1}, \hat{v}_2) - d_{\mathcal{G}}(v_{i_2}, \hat{v}_2) \\ &= d_{\mathcal{G}}(v_{i_1}, w_1) + d_{\mathcal{G}}(w_1, w_2) + d_{\mathcal{G}}(w_2, \hat{v}_2) - (d_{\mathcal{G}}(v_{i_2}, w_2) + d_{\mathcal{G}}(w_2, \hat{v}_2)) \\ &= d_{\mathcal{G}}(v_{i_1}, w_1) + d_{\mathcal{G}}(w_1, w_2) - d_{\mathcal{G}}(v_{i_2}, w_2). \end{aligned} \quad (4.4)$$

Now we get from (4.3) and (4.4)

$$d_{\mathcal{G}}(v_{i_1}, w_1) - d_{\mathcal{G}}(v_{i_2}, w_2) - d_{\mathcal{G}}(w_1, w_2) = d_{\mathcal{G}}(v_{i_1}, w_1) + d_{\mathcal{G}}(w_1, w_2) - d_{\mathcal{G}}(v_{i_2}, w_2)$$

and therefore $d_{\mathcal{G}}(w_1, w_2) = 0$, that is v_{i_1} and v_{i_2} have the joint neighbor $w_1 = w_2$.

Let us now come to the assertion of the proposition, which we prove by induction over k . Clearly, if $k = 2$ then both \mathcal{G}_1 and \mathcal{G}_2 consist (up to vertices of degree two) of one edge of weight $d_{\mathcal{G}}(v_1, v_2)^{-1}$ and, thus, are equal.

Assume now that the assertion is true whenever each of the two trees has $k - 1$ boundary vertices and let $\mathcal{G}_1, \mathcal{G}_2$ satisfy $|\partial\mathcal{G}_1| = |\partial\mathcal{G}_2| = k$ and (4.1). By Observation 1, \mathcal{G}_1 contains two boundary vertices that have a joint neighbor, without loss of generality v_1^1 and v_2^1 . As this property is determined by the distances between

boundary edges only, see Observation 2, and these distances coincide for \mathcal{G}_1 and \mathcal{G}_2 by (4.1), it follows that also v_1^2 and v_2^2 have a joint neighbor in \mathcal{G}_2 . Let \mathcal{G}'_j be the tree obtained from \mathcal{G}_j by removing the edge e_1^j incident to v_1^j , $j = 1, 2$, (ignoring vertices of degree two). Then we have

$$\partial\mathcal{G}'_j = \{v_2^j, v_3^j, \dots, v_k^j\}, \quad j = 1, 2,$$

and by the induction assumption and (4.1) (for $i, m = 2, \dots, k$) it follows $\mathcal{G}'_1 = \mathcal{G}'_2$ up to vertices of degree two.

It remains to show that e_1^1 and e_1^2 have the same weight and are attached to the same interior point x of $\mathcal{G}'_1 = \mathcal{G}'_2$. In fact, let x^j be the point on $\mathcal{G}'_1 = \mathcal{G}'_2$ to which e_1^j is attached, $j = 1, 2$. The numbers

$$d_1^j = d_{\mathcal{G}_j}(v_1^j, x^j) \quad \text{and} \quad d_2^j = d_{\mathcal{G}_j}(v_2^j, x^j), \quad j = 1, 2,$$

are uniquely determined by the system of linear equations

$$\begin{aligned} d_1^j + d_2^j &= d_{\mathcal{G}_j}(v_1^j, v_2^j), \\ d_1^j - d_2^j &= d_{\mathcal{G}_j}(v_1^j, v_i^j) - d_{\mathcal{G}_j}(v_2^j, v_i^j), \quad i \neq 1, 2, \end{aligned} \quad (4.5)$$

$j = 1, 2$, and the right-hand sides are independent of j due to (4.1). Hence $d_1^1 = d_1^2$ and $d_2^1 = d_2^2$, that is,

$$\frac{1}{w_{e_1^1}} = d_1^1 = d_1^2 = \frac{1}{w_{e_1^2}}$$

and

$$d_{\mathcal{G}_1}(v_2^1, x^1) = d_2^1 = d_2^2 = d_{\mathcal{G}_2}(v_2^2, x^2),$$

and as v_1^j and v_2^j have a joint neighbor (up to vertices of degree two), this implies $x^1 = x^2$. It follows $\mathcal{G}_1 = \mathcal{G}_2$ up to vertices of degree two. \square

We are now in the position to prove the main result of this section.

Theorem 4.4. *Let \mathcal{G} be a finite weighted tree with boundary $\partial\mathcal{G} = \{v_1, \dots, v_k\}$ and let*

$$L(\mathcal{G}) = \begin{pmatrix} \widehat{D} & -B^\top \\ -B & \widehat{L} \end{pmatrix} \in \mathbb{R}^{n \times n} \quad (4.6)$$

be the corresponding discrete Laplacian, written in block-matrix form as in (3.3). Moreover, let

$$S := \widehat{D} - B^\top \widehat{L}^{-1} B \in \mathbb{R}^{k \times k}$$

denote the Schur complement of $L(\mathcal{G})$ and let $S_{i_0} \in \mathbb{R}^{k \times k}$ be the matrix obtained from S by replacing the i_0 -th diagonal entry by one and all other entries in the i_0 -th row and column by zero. Then for each $j \neq i_0$

$$\langle S_{i_0}^{-1} \mathbf{e}_j, \mathbf{e}_j \rangle = d_{\mathcal{G}}(v_j, v_{i_0}), \quad (4.7)$$

i.e. the j -th diagonal entry of $S_{i_0}^{-1}$ equals the distance between the boundary vertices v_{i_0} and v_j .

In particular, if \mathcal{G}_1 and \mathcal{G}_2 are two finite weighted trees such that the Schur complements of $L(\mathcal{G}_1)$ and $L(\mathcal{G}_2)$ coincide, then $\mathcal{G}_1 = \mathcal{G}_2$ up to vertices of degree two.

Proof. Step 1: In this step we calculate an explicit expression for $S_{i_0}^{-1}$. Denote by U the matrix obtained from B by replacing its i_0 -th column by zero. Moreover, let $\dot{\mathcal{G}}$ denote the weighted tree obtained from \mathcal{G} by removing all boundary vertices

(and their incident edges) and by $L(\dot{\mathcal{G}})$ the corresponding discrete Laplacian on $\dot{\mathcal{G}}$. Note that for $l = 1, \dots, n - k$

$$B^\top \mathbf{e}_l = \sum w_{e_j} \mathbf{e}_j, \quad (4.8)$$

where the sum is taken over all j such that the boundary vertex v_j is adjacent to the l -th interior vertex v_{k+l} , and e_j is the edge incident to v_j . As a consequence, the matrix $B\widehat{D}^{-1}B^\top \in \mathbb{R}^{(n-k) \times (n-k)}$ is diagonal and its l -th diagonal entry equals

$$\langle \widehat{D}^{-1}B^\top \mathbf{e}_l, B^\top \mathbf{e}_l \rangle = \sum w_{e_j},$$

with the sum being taken over the same j as for (4.8). Therefore the matrix \widehat{L} in the decomposition (4.6) can be written

$$\widehat{L} = L(\dot{\mathcal{G}}) + B\widehat{D}^{-1}B^\top = L(\dot{\mathcal{G}}) + w_{e_{i_0}} \mathbf{e}_{l_0} \mathbf{e}_{l_0}^\top + U\widehat{D}^{-1}U^\top, \quad (4.9)$$

where e_{i_0} is the edge incident to the boundary vertex v_{i_0} and l_0 is such that v_{k+l_0} is the vertex at the other end of e_{i_0} .

Let us write $L := L(\dot{\mathcal{G}}) + w_{e_{i_0}} \mathbf{e}_{l_0} \mathbf{e}_{l_0}^\top$, so that (4.9) can be rewritten

$$\widehat{L} = L + U\widehat{D}^{-1}U^\top. \quad (4.10)$$

The matrix L is invertible: indeed, $Lx = 0$ implies

$$0 \leq \langle L(\dot{\mathcal{G}})x, x \rangle = -w_{e_{i_0}} x_{l_0}^2 \leq 0 \quad (4.11)$$

and, hence, $\langle L(\dot{\mathcal{G}})x, x \rangle = 0$, which implies $x \in \ker L(\dot{\mathcal{G}})$. But then x is a multiple of the all-ones vector and (4.11) implies $x = 0$. It follows that in fact L is a positive matrix, and since $U\widehat{D}^{-1}U^\top$ is nonnegative, also $L + U\widehat{D}^{-1}U^\top$ is positive and, in particular, invertible. Therefore (4.10) together with the Sherman-Morrison-Woodbury formula implies

$$\widehat{L}^{-1} = L^{-1} - L^{-1}U(\widehat{D} + U^\top L^{-1}U)^{-1}U^\top L^{-1}, \quad (4.12)$$

note that the matrix $U^\top L^{-1}U$ is nonnegative and hence $\widehat{D} + U^\top L^{-1}U$ is invertible. Multiplying (4.12) from left and right with U^\top and U , respectively, we find

$$\begin{aligned} U^\top \widehat{L}^{-1}U &= U^\top L^{-1}U - U^\top L^{-1}U(\widehat{D} + U^\top L^{-1}U)^{-1}U^\top L^{-1}U \\ &= (I + \widehat{D}(\widehat{D} + U^\top L^{-1}U)^{-1} - I)U^\top L^{-1}U \\ &= \widehat{D} - \widehat{D}(\widehat{D} + U^\top L^{-1}U)^{-1}\widehat{D}, \end{aligned}$$

which yields

$$(\widehat{D} - U^\top \widehat{L}^{-1}U)^{-1} = \widehat{D}^{-1}(\widehat{D} + U^\top L^{-1}U)\widehat{D}^{-1}.$$

Observe that the matrix $\widehat{D} - U^\top \widehat{L}^{-1}U$ can be obtained from S by setting its i_0 -th diagonal entry equal to $w_{e_{i_0}}$ and all further entries in the i_0 -th row and i_0 -th column to zero. Due to this particular structure of the i_0 -th row and column of this matrix it follows that

$$S_{i_0}^{-1} = \left[\widehat{D}^{-1}(\widehat{D} + U^\top L^{-1}U)\widehat{D}^{-1} \right]_{i_0}, \quad (4.13)$$

where the index i_0 again indicates that the i_0 -th diagonal entry is reset to one and all other entries in the i_0 -th row and column are reset to zero.

Step 2: Here we show that the j -th diagonal entry of (4.13) equals $d_{\mathcal{G}}(v_j, v_{i_0})$, the distance of the j -th boundary vertex of \mathcal{G} to v_{i_0} . Note that for $j \neq i_0$

$$\begin{aligned} \langle S_{i_0}^{-1} \mathbf{e}_j, \mathbf{e}_j \rangle &= \langle \widehat{D}^{-1}(\widehat{D} + U^\top L^{-1}U)\widehat{D}^{-1} \mathbf{e}_j, \mathbf{e}_j \rangle \\ &= \langle \widehat{D}^{-1} \mathbf{e}_j, \mathbf{e}_j \rangle + \langle L^{-1}U\widehat{D}^{-1} \mathbf{e}_j, U\widehat{D}^{-1} \mathbf{e}_j \rangle, \end{aligned} \quad (4.14)$$

that $\widehat{D}^{-1} \mathbf{e}_j = \frac{1}{w_{e_j}} \mathbf{e}_j$, where e_j is the edge incident to v_j , and that, hence,

$$U \widehat{D}^{-1} \mathbf{e}_j = \frac{1}{w_{e_j}} U \mathbf{e}_j = \frac{1}{w_{e_j}} w_{e_j} \mathbf{e}_l$$

if the l -th interior vertex v_{k+l} is the one that is adjacent to v_j . From this, (4.14) and Lemma 4.1 we get

$$\langle S_{i_0}^{-1} \mathbf{e}_j, \mathbf{e}_j \rangle = \frac{1}{w_{e_j}} + \langle L^{-1} \mathbf{e}_l, \mathbf{e}_l \rangle = \frac{1}{w_{e_j}} + d_{\mathcal{G}}(v_{k+l}, v_{i_0}) = d_{\mathcal{G}}(v_j, v_{i_0}),$$

where we have used that L is the result of reducing the Laplacian on the subtree spanned by the vertex set $(\mathcal{V} \setminus \partial \mathcal{G}) \cup \{v_{i_0}\}$ with respect to v_{i_0} . This proves the first assertion of the theorem.

Step 3: We derive the second assertion of the theorem. Indeed, if $\mathcal{G}_1, \mathcal{G}_2$ are two weighted graphs such that $L(\mathcal{G}_1)$ and $L(\mathcal{G}_2)$ have the same Schur complement then the first assertion of this theorem implies that \mathcal{G}_1 and \mathcal{G}_2 have the same set of distances between its boundary vertices. Hence Proposition 4.3 implies $\mathcal{G}_1 = \mathcal{G}_2$ up to vertices of degree two. \square

Remark 4.5. Theorem 4.4 states that the Schur complement of the discrete Laplacian $L(\mathcal{G})$ determines \mathcal{G} and, thus, $L(\mathcal{G})$ itself uniquely if \mathcal{G} is a tree. It is easy to see that in general the Schur complement of a matrix with respect to a block decomposition does not determine the original matrix uniquely. For instance the Schur complements of the block matrices

$$\begin{pmatrix} A & B \\ B^\top & C \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} A & UB \\ (UB)^\top & UCU^\top \end{pmatrix},$$

equal $A - B^\top C^{-1} B$ and $A - B^\top U^\top U C^{-1} U^\top U B$, respectively, and therefore they coincide for any orthogonal matrix U .

5. RECONSTRUCTION OF A METRIC TREE FROM THE DIRICHLET-TO-NEUMANN MATRIX

In this section we provide the main result of this paper. Its proof is based on Theorem 4.4 above.

Theorem 5.1. *Assume that Γ_1, Γ_2 are finite, compact metric trees and that the corresponding Dirichlet-to-Neumann matrices M_{Γ_1} and M_{Γ_2} in Definition 2.1 satisfy*

$$M_{\Gamma_1} = M_{\Gamma_2}. \tag{5.1}$$

Then Γ_1 and Γ_2 coincide up to vertices of degree two.

Moreover, if Γ is a finite, compact metric tree with Dirichlet-to-Neumann matrix M_Γ and M_{Γ, i_0} denotes the matrix obtained from M_Γ by replacing the i_0 -th diagonal entry by one and all further entries in the i_0 -th row and column by zero then

$$\langle M_{\Gamma, i_0}^{-1} \mathbf{e}_j, \mathbf{e}_j \rangle = d_\Gamma(v_j, v_{i_0}), \tag{5.2}$$

i.e. the j -th diagonal entry of M_{Γ, i_0}^{-1} coincides with the distance between the boundary vertices v_{i_0} and v_j in Γ .

Proof. Let Γ_1, Γ_2 be two finite, compact metric graphs that satisfy (5.1). Moreover, let $\mathcal{G}_1, \mathcal{G}_2$ be the corresponding weighted discrete trees with weights obtained through the identification (3.1). As the number of rows (or columns) of the Dirichlet-to-Neumann matrices is equal to the number of boundary vertices,

both graphs \mathcal{G}_1 and \mathcal{G}_2 have the same number of boundary vertices. Now we employ Proposition 3.1 and (5.1) to obtain

$$\widehat{D}_1 - B_1^\top \widehat{L}_1^{-1} B_1 = M_{\Gamma_1} = M_{\Gamma_2} = \widehat{D}_2 - B_2^\top \widehat{L}_2^{-1} B_2,$$

where we use the decompositions

$$L(\mathcal{G}_1) = \begin{pmatrix} \widehat{D}_1 & -B_1^\top \\ -B_1 & \widehat{L}_1 \end{pmatrix} \quad \text{and} \quad L(\mathcal{G}_2) = \begin{pmatrix} \widehat{D}_2 & -B_2^\top \\ -B_2 & \widehat{L}_2 \end{pmatrix}$$

for the discrete Laplacians of the weighted trees \mathcal{G}_1 and \mathcal{G}_2 . By Theorem 4.4, the trees \mathcal{G}_1 and \mathcal{G}_2 coincide up to vertices of degree two and therefore also $\Gamma_1 = \Gamma_2$ up to vertices of degree two.

If Γ is any finite, compact metric tree then the assertion (5.2) follows immediately from the identity

$$M_\Gamma = \widehat{D} - B^\top \widehat{L}^{-1} B$$

according to the decomposition (3.3) of the discrete Laplacian on the corresponding weighted tree and (4.7) in Theorem 4.4. \square

Remark 5.2. As uniqueness of metric trees is determined up to permutations of vertices, the statement of Theorem 5.1 remains valid if there exists a permutation matrix $P \in \mathbb{R}^{k \times k}$ such that $P^\top M_{\Gamma_1} P = M_{\Gamma_2}$.

From the identity (5.2) in Theorem 5.1 and the proof of Proposition 4.3 we obtain Algorithm 1 for the reconstruction of a metric tree from the Dirichlet-to-Neumann matrix M_Γ . The output of this algorithm is the adjacency matrix $A \in \mathbb{R}^{n \times n}$ of the metric tree Γ defined by

$$A_{i,j} = \begin{cases} L(e_{i,j}) & \text{if } e_{i,j} \text{ connects } v_i \text{ and } v_j, \\ 0 & \text{else.} \end{cases}$$

This matrix determines the metric graph uniquely.

6. EXAMPLES

In this section we provide examples which show that Theorem 5.1 does not extend, e.g., to graphs with cycles, to the Dirichlet-to-Neumann matrix on only a part of the boundary or to the so-called Weyl vector.

We start with an example showing that one cannot recover graphs with cycles uniquely from the Dirichlet-to-Neumann matrix.

Example 6.1. Let Γ_1 be an equilateral graph with edge lengths one that consists of a cycle with three pending edges attached to different points on the cycle, see the left-hand side of Figure 6.1. The discrete Laplacian of the underlying weighted

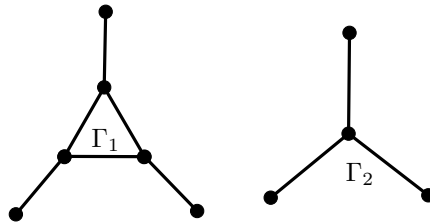


FIGURE 6.1. The equilateral graphs from Example 6.1. If Γ_1 has edge lengths 1 and Γ_2 has edge lengths $4/3$ then the two graphs have the same Dirichlet-to-Neumann-matrix.

Algorithm 5.1: Recover a metric tree from the Dirichlet-to-Neumann matrix.

1 **Input:** Dirichlet-to-Neumann matrix $M_\Gamma \in \mathbb{R}^{k \times k}$;
2 Compute $D_0 := (d_\Gamma(v_i, v_j))_{i,j=1}^k$ via (5.2) and set $I := \{1, \dots, k\}$;
3 **if** $k = 2$ **then**
4 let $A := \begin{pmatrix} 0 & d_\Gamma(v_1, v_2) \\ d_\Gamma(v_1, v_2) & 0 \end{pmatrix}$;
5 **else**
6 Set $A_0 = 0 \in \mathbb{R}^{k \times k}$ and $l = 1$;
7 **repeat**
8 By means of (4.2) choose a maximal set of indices $\{i_1, \dots, i_{p_l}\} \subset I$
9 such that the corresponding vertices $v_{i_1}, \dots, v_{i_{p_l}}$ are incident to the
10 same interior vertex;
11 Set $I := (I \setminus \{i_1, \dots, i_{p_l}\}) \cup \{k+l\}$;
12 Use D_{l-1} and (4.5) to compute the distances $d_\Gamma(v_{i_j}, v_{k+l})$,
13 $j = 1, \dots, p_l$;
14 Enlarge the adjacency matrix $A_l := \begin{pmatrix} A_{l-1} & a_l \\ a_l^\top & 0 \end{pmatrix} \in \mathbb{R}^{(k+l) \times (k+l)}$ with
15
$$a_{l,i} = \begin{cases} d_\Gamma(v_{k+l}, v_i), & \text{if } i \in \{i_1, \dots, i_{p_l}\}, \\ 0, & \text{else,} \end{cases}$$

16 $1 \leq i \leq k+l-1$;
17 Enlarge the distance matrix $D_l := \begin{pmatrix} D_{l-1} & d_l \\ d_l^\top & 0 \end{pmatrix} \in \mathbb{R}^{(k+l) \times (k+l)}$ with
18
$$d_{l,i} := \begin{cases} d_\Gamma(v_{k+l}, v_i), & \text{if } i \in \{i_1, \dots, i_{p_l}\}, \\ d_\Gamma(v_{i_1}, v_i) - d_\Gamma(v_{i_1}, v_{k+l}), & \text{if } i \in I \setminus \{k+l\}, \\ 0, & \text{else,} \end{cases}$$

19 $1 \leq i \leq k+l-1$;
20 $l = l + 1$;
21 **until** $|I| = 1$;
22 Set $A := A_{l-1}$.

graph \mathcal{G}_1 is then given by

$$L(\mathcal{G}_1) = \begin{pmatrix} 1 & 0 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 \\ -1 & 0 & 0 & 3 & -1 & -1 \\ 0 & -1 & 0 & -1 & 3 & -1 \\ 0 & 0 & -1 & -1 & -1 & 3 \end{pmatrix}.$$

We use Proposition 3.1 to calculate the Dirichlet-to-Neumann matrix for Γ_1 and obtain

$$M_{\Gamma_1} = I_3 - I_3 \begin{pmatrix} 3 & -1 & -1 \\ -1 & 3 & -1 \\ -1 & -1 & 3 \end{pmatrix}^{-1} I_3 = \frac{1}{4} \begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix}.$$

Let now Γ_2 be an equilateral 3-star with edge lengths $4/3$, see the right-hand side of Figure 6.1. Then the discrete Laplacian of the associated weighted graph \mathcal{G}_2 equals

$$L(\mathcal{G}_2) = \frac{3}{4} \begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \\ -1 & -1 & -1 & 3 \end{pmatrix}$$

and, hence, the Dirichlet-to-Neumann matrix on Γ_2 is given by

$$M_{\Gamma_2} = \frac{3}{4}I_3 - \frac{1}{4} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \end{pmatrix} = M_{\Gamma_1}.$$

Therefore the Dirichlet-to-Neumann matrix alone cannot determine a metric graph uniquely if cycles are allowed.

Another consequence of the previous example is that it is not possible either to recover the Betti number (i.e. the number of independent cycles) from the knowledge of M_Γ only.

The next example shows that the Dirichlet-to-Neumann matrix for a proper subset of the boundary vertices (i.e. the Schur complement of the discrete Laplacian with respect to this subset) does not determine a metric tree uniquely.

Example 6.2. Consider an equilateral 3-star Γ_1 with edge lengths one and a path graph Γ_2 consisting of a single edge of length 2; cf. Figure 6.2. The discrete

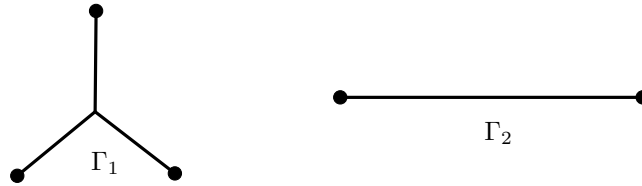


FIGURE 6.2. The equilateral graphs from Example 6.2. If Γ_1 has edge lengths one and Γ_2 has length 2 then the Dirichlet-to-Neumann matrix of Γ_1 with respect to only two boundary vertices coincides with M_{Γ_2} .

Laplacians of the corresponding weighted trees \mathcal{G}_1 and \mathcal{G}_2 are

$$L(\mathcal{G}_1) = \begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \\ -1 & -1 & -1 & 3 \end{pmatrix} \quad \text{and} \quad L(\mathcal{G}_2) = \begin{pmatrix} 1/2 & -1/2 \\ -1/2 & 1/2 \end{pmatrix} = M_{\Gamma_2}.$$

The Dirichlet-to-Neumann matrix of Γ_1 with respect to only two boundary vertices (i.e. the 2×2 -matrix defined as in (3.4) on the first two boundary vertices, thereby imposing a Neumann boundary condition on the solution f at the remaining boundary vertex) is given— analogously to the considerations in the proof of Proposition 3.1—by the Schur complement of $L(\mathcal{G}_1)$ with respect to the first two boundary vertices of the graph,

$$S_{\Gamma_1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ -1 & 3 \end{pmatrix}^{-1} \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 1/2 & -1/2 \\ -1/2 & 1/2 \end{pmatrix},$$

which coincides with M_{Γ_2} .

In some places in the literature inverse problems for quantum trees were solved under the assumption that not the whole λ -dependent Dirichlet-to-Neumann matrix but only its diagonal, the so-called *Weyl vector* is available, see, e.g. [11, 22]. The following example shows that the knowledge of the diagonal of M_Γ is not sufficient to recover a metric tree.

Example 6.3. Consider, on the one hand, the equilateral star Γ_1 consisting of four edges of length 1 each, see the left-hand side of Figure 6.3. Then the corresponding

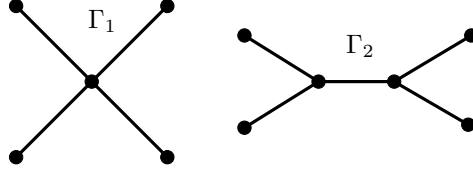


FIGURE 6.3. The two equilateral graphs from Example 6.3. If Γ_1 has edge lengths one and Γ_2 has edge lengths $5/6$ then M_{Γ_1} and M_{Γ_2} have the same diagonal.

weighted star graph \mathcal{G}_1 has edge weights $w_e = 1$ for each edge e and its discrete Laplacian is given by

$$L(\mathcal{G}_S) = \begin{pmatrix} 1 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & -1 \\ -1 & -1 & -1 & -1 & 4 \end{pmatrix}.$$

Using Proposition 3.1 we get the corresponding Dirichlet-to-Neumann matrix

$$M_{\Gamma_S} = I_4 - \frac{1}{4} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix} = \frac{1}{4} \begin{pmatrix} 3 & -1 & -1 & -1 \\ -1 & 3 & -1 & -1 \\ -1 & -1 & 3 & -1 \\ -1 & -1 & -1 & 3 \end{pmatrix}.$$

In particular, each diagonal entry of M_{Γ_S} equals $3/4$.

On the other hand, for the symmetric “double star” graph Γ_2 on the right-hand side of Figure 6.3 with four boundary edges of length $1/a$ and one interior edge of length $1/b$ the discrete Laplacian of the corresponding weighted tree \mathcal{G}_2 equals

$$L(\mathcal{G}_2) = \begin{pmatrix} a & 0 & 0 & 0 & -a & 0 \\ 0 & a & 0 & 0 & -a & 0 \\ 0 & 0 & a & 0 & 0 & -a \\ 0 & 0 & 0 & a & 0 & -a \\ -a & -a & 0 & 0 & 2a+b & -b \\ 0 & 0 & -a & -a & -b & 2a+b \end{pmatrix}$$

and, thus, by Proposition 3.1,

$$M_{\Gamma_2} = aI_4 - \frac{1}{4a(a+b)} \begin{pmatrix} a^2(2a+b) & a^2(2a+b) & a^2b & a^2b \\ a^2(2a+b) & a^2(2a+b) & a^2b & a^2b \\ a^2b & a^2b & a^2(2a+b) & a^2(2a+b) \\ a^2b & a^2b & a^2(2a+b) & a^2(2a+b) \end{pmatrix}.$$

In particular, each diagonal entry equals $a - \frac{a(2a+b)}{4(a+b)}$. Setting, e.g., $a = b = 6/5$ (that is, the corresponding metric graph is equilateral and each edge has length

5/6) we get all diagonal entries equal to $3/4$ and thus the diagonals of M_{Γ_1} and M_{Γ_2} coincide in this case while Γ_1 and Γ_2 differ from each other.

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