One-dimensional perturbations of singular matrix pencils

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joint work with

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Kronecker form under perturbations

We consider pencils $sE - A$ with $E, A \in \mathbb{C}^{n \times m}$ with Kronecker canonical form

$$\begin{bmatrix} sI_{n_0} - J & 0 & 0 & 0 \\ 0 & sN_\alpha - I_{|\alpha|} & 0 & 0 \\ 0 & 0 & sK_\beta - L_\beta & 0 \\ 0 & 0 & 0 & sK_\gamma^\top - L_\gamma^\top \end{bmatrix}$$

Question: How do the lengths of the Jordan chains and of the row and column minimal indices, i.e. the entries of β and γ change under perturbations like

- rank one $s(E + wv^\top) - (A + wu^\top)$, $w \in \mathbb{C}^n$, $u, v \in \mathbb{C}^m$
- rank one $s(E + vw^\top) - (A + uw^\top)$, $w \in \mathbb{C}^m$, $u, v \in \mathbb{C}^n$
- adding and erasing columns in the pencil

One-dimensional perturbations

- We say that two subspaces $L, M \subseteq \mathbb{C}^n$ are one-dimensional perturbations iff

$$\max\left\{\dim \frac{L}{L \cap M}, \dim \frac{M}{L \cap M}\right\} \leq 1.$$

- We associate with the matrix pencil $sE - A$ the subspaces

$$AE^{-1} = \mathrm{ran} \begin{bmatrix} E \\ A \end{bmatrix}, \quad E^{-1}A = \ker[A, -E]$$

with the inverse $E^{-1} = \{(Ex, x) : x \in \mathbb{C}^n\}$ and $A \equiv \mathit{graph}\, A$.

- We say that the pencils $sE - A$ and $s\hat{E} - \hat{A}$ are one-dimensional perturbations if

$$L = E^{-1}A,\ M = \hat{E}^{-1}\hat{A} \quad \text{or} \quad L = AE^{-1},\ M = \hat{A}\hat{E}^{-1}.$$

Examples of 1−dim perturbations

- Erasing or adding a row/column in the pencil is 1-dim perturbation in kernel/range representation.
- What is the relation to the widely considered rank one perturbations of the form

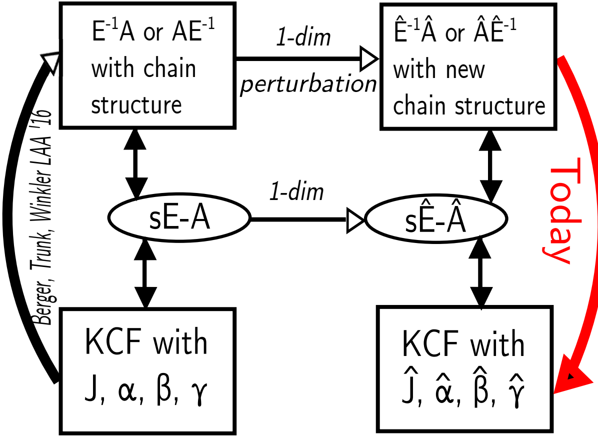
$$(su+v)w^\top, \quad w(su+v)^\top, \quad u,v,w\in\mathbb{C}^n?$$

Proposition (rk 1 ⇒ 1-dim)

- AE^{-1} and $(A+vw^\top)(E+uw^\top)^{-1}$ are 1-dim perturbations of each other.
- $E^{-1}A$ and $(E+wu^\top)^{-1}(A+wv^\top)$ are 1-dim perturbations of each other.

This is why we need both representations.

An overview



E⁻¹A or AE⁻¹ with chain structure
1-dim perturbation
Ê⁻¹Â or ÂÊ⁻¹ with new chain structure
sE-A
1-dim
sÊ-Â
KCF with J, α, β, γ
KCF with Ĵ, α̂, β̂, γ̂
Berger, Trunk, Winkler LAA '16
Today

Correspondence of KCF and chains in linear relations

To obtain the Jordan part of the KCF we have to factor out the minimal reducing subspace from Henrik's talk

$$\mathcal{R}_c(L) = \mathcal{R}_0(L) \cap \mathcal{R}_\infty(L)$$

and follow a chain definition **Berger, Trunk, de Snoo, Winkler**. We introduce the Weyr characteristics of a subspace L and $k \geq 1$ by

$$j_k^L(\lambda) := \dim \frac{N((L-\lambda)^k) + \mathcal{R}_c(L)}{N((L-\lambda)^{k-1}) + \mathcal{R}_c(L)}, \quad j_k^L(\infty) := \dim \frac{M(L^k) + \mathcal{R}_c(L)}{M(L^{k-1}) + \mathcal{R}_c(L)},$$

$$s_k^L := \dim \frac{M(L^k) \cap N(L)}{M(L^{k-1}) \cap N(L)}, \quad m_k^L := \dim \frac{R(L^k) + \sum_{\lambda \in \sigma(L)} \mathcal{R}_\lambda(L)}{R(L^{k+1}) + \sum_{\lambda \in \sigma(L)} \mathcal{R}_\lambda(L)},$$

Invisible blocks of the Kronecker form

For the subspace $E^{-1}A$ we consider two examples:

$$sE - A = \begin{pmatrix} s & 0 \\ 1 & 0 \end{pmatrix} \longrightarrow E^{-1}A = \ker[A, -E] = \operatorname{span}\left\{ \begin{pmatrix} 0 \\ e_2 \end{pmatrix}, \begin{pmatrix} e_2 \\ 0 \end{pmatrix} \right\}.$$

In the example we have $\gamma = 2$ and $\beta = 1$ but we don't see a multishift chain in the linear relation. Increasing the size of the singular block to $\gamma = 3$ we observe that

$$sE - A = \begin{pmatrix} s & 0 & 0 \\ 1 & s & 0 \\ 0 & 1 & 0 \end{pmatrix} \longrightarrow E^{-1}A = \operatorname{span}\left\{ \begin{pmatrix} e_1 \\ e_2 \end{pmatrix}, \begin{pmatrix} 0 \\ e_3 \end{pmatrix}, \begin{pmatrix} e_3 \\ 0 \end{pmatrix} \right\}.$$

Now the multishift appears but the chain is shorter then the size of the block. Consequence: The matrices associated with the subspace could have a different Kronecker canonical form.

Main theorem: Weyr characteristics $\Rightarrow$ Kronecker CF

Theorem

Given the subspace $E^{-1}A$ or AE^{-1} with $E, A \in \mathbb{C}^{n\times m}$. If n and m are known then the Weyr characteristic of $E^{-1}A$ (resp. AE^{-1}) uniquely determines the Kronecker canonical form of $sE - A$ with

- $j_k(\lambda) - j_{k+1}(\lambda) =$ number of Jordan blocks at λ of length k,
- $s_k - s_{k+1} =$ number of blocks with $\beta_i = k$ and so on

Idea of the proof:

- Weyr characteristics is invariant under transformations of the form $AE^{-1} \mapsto TAE^{-1}T^{-1}$ for invertible $T \in \mathbb{C}^{n\times n}$.
- Therefore we can assume that $sE - A$ is already given in KCF and determine the Weyr characteristics.
- Knowing n and m we can count the invisible blocks.

Perturbation results for chains

Theorem

Let L and M be subspaces that are one-dimensional perturbations of each other then for all $n \geq 1$ and all $\lambda \in \mathbb{C} \cup \{\infty\}$ we have

$$|j_n^L(\lambda) - j_n^M(\lambda)| \leq 2.$$

If $\mathcal{R}_c(L) = \mathcal{R}_c(M) = \{0\}$ or $L \subseteq M$ then

$$|j_n^L(\lambda) - j_n^M(\lambda)| \leq 1.$$

- The bound is sharp as we will see from a later example.
- From our correspondence we immediately get the same perturbations result for the blocks in the KCF!

Bounds for the change of singular chains?

- This result can be used to understand the change of the singular blocks
- We know from **Leben, Martínez Pería, Philipp, Trunk, Winkler** that the total change of the root subspace at 0 is

$$\left| \frac{N(L^{n+1})}{N(L^n)} - \frac{N(M^{n+1})}{N(M^n)} \right| \leq n+1.$$

  and here we bound the change of both: Jordan chains of length $n+1$ and singular chains of length at least $n+2$.
- We recall the example where the bound above is sharp for $n=2$ for the general case we refer to the above paper.
- This also gives the sharpness of the upper bound 2 for the change of Jordan chains!

Sharpness example from LMPPTW

- We consider for linearly independent $x_i, y_i, u, v, w,\ i = 1, 2, 3$

$$L = \mathrm{span}\{(y_2, y_1), (y_1, y_0), (y_0, 0),$$
$$(w, x_2 - v), (x_2 - v, u), (u, 0)$$
$$(0, y_1 - v), (y_1 - v, y_0 - x_1 + u), (y_0 - x_1 + u, -x_0), (-x_0, 0)\}$$
$$.$$
$$M = \mathrm{span}\{(0, x_1 - u), (x_1 - u, x_0), (x_0, 0)$$
$$(0, y_1 - v), (y_1 - v, y_0), (y_0, 0)$$
$$(y_2, v), (v, 0), \quad (x_2, u), (u, 0)\}.$$

- Exchange only the element $(w, x_2 - v) \in L$ by $(v, 0) \in M$ to see that M and L are 1-dim perturbations.
- We can easily obtain a pencil, by using the subspace $AE^{-1} = \mathrm{ran} \begin{bmatrix} E \\ A \end{bmatrix}$.

Future work

- Perturbation results for structured pencils like hermitian pencils.
- Investigate the interplay between the chains under perturbations to obtain more detailed perturbation results for the change of β and γ.
- Compare our results with generic perturbation results obtained by De Téran, Dopico '07.