On the essential spectrum of operator pencils

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joint work with

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Motivation: Spectrum of operator pencils

Abstract differential algebraic equations are of the form

$$\frac{d}{dt}Ex(t) = Ax(t),\ t \geq 0, \quad (Ex)(0) = Ex_0$$

with E and A are (unbounded) operators between Hilbert spaces X, Z can be used to model

- coupled systems of PDEs, ODEs and linear equations
- e.g. electric circuits with transmission lines, see **Reis '06**
- Problem: The operator $E : X \to Z$ is in general not invertible!

Estimating the spectrum leads e.g. to stability results for the ADAE.

Problem formulation

- Let X, Z be Hilbert spaces and $E, A : X \to Z$ bounded linear operators the spectrum of $sE - A$ is given by

$$\sigma(E, A) := \{\lambda \in \mathbb{C} : 0 \in \sigma(\lambda E - A)\}.$$

- The essential spectrum is given by

$$\sigma_{\text{ess}}(E, A) := \{\lambda \in \mathbb{C} : \lambda E - A \text{ is not Fredholm}\}.$$

  Recall that T is Fredholm iff $\operatorname{codim} \operatorname{ran} T < \infty$ and $\dim \ker T < \infty$.

- Aim: Given two operator pencils $s\hat{E} - \hat{A}$ and $sE - A$, then find conditions on the coefficients such that

$$\sigma_{\text{ess}}(E, A) = \sigma_{\text{ess}}(\hat{E}, \hat{A}).$$

Some examples for $\sigma_{\text{ess}}(E, A) = \sigma_{\text{ess}}(\hat{E}, \hat{A})$

Expl 1: Assume that $\hat{E} - E$ and $\hat{A} - A$ are both compact then $\sigma_{\text{ess}}(E, A) = \sigma_{\text{ess}}(\hat{E}, \hat{A})$, because compact perturbation of a Fredholm operator is still Fredholm

Expl 2: We consider for $M, M^{-1} \in \mathcal{L}(Z)$, $T \in \mathcal{L}(X, Z)$ the operator pencils

$$sE - A = sI - T, \quad s\hat{E} - \hat{A} = sM - TM$$

then obviously $\sigma_{\text{ess}}(E, A) = \sigma_{\text{ess}}(\hat{E}, \hat{A})$ but the difference of the coefficients can be arbitrarily large and non-compact.

Main Idea: Use linear relations

We associate with the operator pencil $sE - A$ the linear subspace AE^{-1} as follows:

- 1. We identify the operators with their graphs which gives a linear subspace.
- 2. We can invert the graph of E:

$$E^{-1} := \{(Ex, x) : x \in X\} \subset Z \times X.$$

- 3. We can multiply the (graph of) A and E^{-1}

$$AE^{-1} := \{(x, y) \in Z \times Z : (x, z) \in E^{-1},\ (z, y) \in A\} = \operatorname{ran} \begin{bmatrix} E \\ A \end{bmatrix}.$$

In the second example from before we see

$$\operatorname{ran} \begin{bmatrix} \hat{E} \\ \hat{A} \end{bmatrix} = \operatorname{ran} \begin{bmatrix} M \\ TM \end{bmatrix} = \operatorname{ran} \begin{bmatrix} I \\ T \end{bmatrix} = \operatorname{ran} \begin{bmatrix} E \\ A \end{bmatrix}.$$

Spectrum of linear relations

The resolvent set of a linear relation Θ is given by

$$\rho(\Theta) := \{\lambda \in \mathbb{C} |\ (\Theta - \lambda)^{-1} \text{ is the graph of a bounded operator}\}.$$

and the spectrum of Θ is given by $\sigma(\Theta) := \mathbb{C} \setminus \rho(\Theta)$. With

$$\ker \Theta := \{x \in X : (x, 0) \in \Theta\}, \quad \operatorname{ran} \Theta := \{y \in Z : (x, y) \in \Theta\}$$

the essential spectrum of linear relation $\Theta \subseteq X \times Z$ is given by

$$\sigma_{\text{ess}}(\Theta) := \mathbb{C} \setminus \{\lambda \in \mathbb{C} : \dim \ker(\Theta - \lambda), \operatorname{codim} \operatorname{ran}(\Theta - \lambda) < \infty\}$$

Perturbations of essential spectrum of linear relations was also investigated by **Cross '98, Wilcox '14, Amar, Jeribi,Álvarez**

Spectral correspondence between $sE - A$ and AE^{-1}

We need to understand how kernels and ranges are related:

$\ker(AE^{-1} - \lambda) = E\ker(\lambda E - A), \quad \mathrm{ran}\,(AE^{-1} - \lambda) = \mathrm{ran}\,(\lambda E - A)$.

Moreover one can show that

$$\dim \ker(AE^{-1} - \lambda) = \dim \frac{\ker(\lambda E - A)}{\ker E \cap \ker A}.$$

Proposition

Let $sE - A$ be an operator pencil with E, A bounded then the following holds

- $\sigma(AE^{-1}) \subseteq \sigma(E, A)$
- If $\ker E \cap \ker A = \{0\}$ then $\sigma(AE^{-1}) = \sigma(E, A)$.
- If $\rho(E, A) \neq \emptyset$ and $\dim \ker E \cap \ker A < \infty$

$$\sigma_{\mathrm{ess}}(E, A) = \sigma_{\mathrm{ess}}(AE^{-1}).$$

Some remarks

- For E invertible one has obviously $\sigma(AE^{-1}) = \sigma(E, A)$ and $\sigma_{\mathrm{ess}}(AE^{-1}) = \sigma_{\mathrm{ess}}(E, A)$.
- The correspondence holds also for unbounded operators.
- For $0 \in \sigma(E)$ and E and A self-adjoint **Nakic '16**[1] showed a correspondence of spectra and point spectra of $sE - A$ and $E^{-1}A$.
- **Trostorff, Waurick '17**[2] showed correspondence of spectra for pencils with E and A bounded under some additional assumption (index 1).

[1] I. Nakić, *On the correspondence between spectra of the operator pencil $A - \lambda B$ and the operator $B^{-1}A$*, Glas. Mat. III. Ser. **51** (2016), 197–221.

[2] S. Trostorff, M. Waurick, *On Differential-Algebraic Equations in Infinite Dimensions* , Journal of Differential Equations, 266(1): 526-561, 2019.

Compact perturbations of linear relations

Definition (Azizov, Behrndt, Jonas, Trunk '09[3])

Two subspaces Θ and Θ' in a Hilbert space are said to be compact perturbations if

$$P_\Theta - P_{\Theta'} \quad \text{is compact,}$$

where P_Θ denotes the orthogonal projection onto Θ.

- Different from compactness in **Cross '98, Wilcox '02** where one needs an inclusion assumption on the multivalued parts, i.e. $A \ker E \subseteq \hat{A} \ker \hat{E}$.
- One can show that $P_\Theta - P_{\Theta'}$ compact implies that Θ Fredholm if and only if Θ' Fredholm.

[3] T. Azizov, J. Behrndt, P. Jonas, C. Trunk, *Compact and finite rank perturbations of closed linear operators and relations in Hilbert spaces*, Integral Eq. Oper. Th. 63 (2009), 151-163.

A first result on the invariance of $\sigma_{\text{ess}}(E, A)$

Recall the set of upper semi Fredholm operators given by

$$\Phi_+(X) := \{ T \in \mathcal{L}(X) |\ \dim \ker T < \infty,\ \mathrm{ran}\ T \text{ closed } \}.$$

For $\rho(E, A) \neq \emptyset$ we can rewrite the projection on the range with the pseudo-inverse $P_{AE^{-1}} = \begin{bmatrix} E \\ A \end{bmatrix} \begin{bmatrix} E \\ A \end{bmatrix}^\dagger$ and obtain:

Theorem

Let $E_1, E_2, A_1, A_2 \in \mathcal{L}(X, Y)$. Assume that $(E_i^* E_i + A_i^* A_i) \in \Phi_+(X)$, $i = 1, 2$ then $Z_i := (E_i^* E_i + A_i^* A_i)^\dagger$ exists and if

$$\begin{bmatrix} E_1 Z_1 E_1^* - E_2 Z_2 E_2^* & E_1 Z_1 A_1^* - E_2 Z_2 A_2^* \\ A_1 Z_1 E_1^* - A_2 Z_2 E_2^* & A_1 Z_1 A_1^* - A_2 Z_2 A_2^* \end{bmatrix} \quad \text{is compact}$$

then $\sigma_{\text{ess}}(E_1, A_1) = \sigma_{\text{ess}}(E_2, A_2)$.

A second result on the invariance of $\sigma_{\text{ess}}(E, A)$

Theorem

Assume that $E_1, E_2 \in \mathcal{L}(X, Y)$ are Fredholm and that the following operators are compact for all $i, j = 1, 2$

$$A_i E_j^\dagger (E_i - E_j), \quad (E_i - E_j) E_j^\dagger A_i,$$
$$(A_i - A_j) A_i^\dagger E_j, \quad E_i A_j^\dagger (A_i - A_j)$$

then $\sigma_{\text{ess}}(E_1, A_1) = \sigma_{\text{ess}}(E_2, A_2)$.

Final remarks

- If A is a closed, densely defined linear operator with $\rho(E,A) \neq \emptyset$ then we can rewrite the linear relation

$$\mathrm{ran}\begin{bmatrix} E \\ A \end{bmatrix} = \mathrm{ran}\begin{bmatrix} E(A-\lambda E)^{-1} \\ I + \lambda E(A-\lambda E)^{-1} \end{bmatrix}$$

- If $\lambda \in \rho(E,A) \cap \rho(\hat{E},\hat{A})$ then

$$E(A-\lambda E)^{-1} - \hat{E}(\hat{A}-\lambda\hat{E})^{-1} \text{ compact}$$

  implies $\sigma_{\mathrm{ess}}(E,A) = \sigma_{\mathrm{ess}}(\hat{E},\hat{A})$.

- There are some more detailed results on upper and lower semi-Fredholm spectrum

$$\sigma_{e2}^{+}(E,A) := \{\lambda \in \mathbb{C} : \ker(\lambda E - A) < \infty,\ \mathrm{ran}\,(\lambda E - A) \text{ closed}\},$$
$$\sigma_{e2}^{-}(E,A) := \{\lambda \in \mathbb{C} : \mathrm{codim}\,\mathrm{ran}\,(\lambda E - A) < \infty\}.$$