Eigenvalue placement for regular matrix pencils with rank one perturbations

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Model of an electrical network

A model for a LC-circuit with simplified transistor behavior is given by the differential-algebraic equation

$$\frac{d}{dt}\underbrace{\begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & c_p & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & L_p \end{pmatrix}}_{=:E} x(t) = \underbrace{\begin{pmatrix} 0 & 0 & -1 & 0 & 0 \\ -g_m & 0 & 0 & -1 & -1 \\ -1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -R_p & 0 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix}}_{=:A} x(t).$$

- the solutions can be described with the eigenvectors and eigenvalues of a matrix pencil

$$\mathcal{A}(s) = sE - A, \quad s \in \mathbb{C}.$$

- Problems: E is singular, A non-Hermitian

Problem statement

- Given: Network described by the pencil $sE - A$, $E, A \in \mathbb{C}^{n \times n}$
- Task: Improve the properties of the network by moving the eigenvalues into a certain region $G \subset \overline{\mathbb{C}}$
- How to move: Add step-by-step new elements to network.
- Example element: A capacity can be described by the pencil $\mathcal{C}_{ij}(s) := sc_{ij}(e_i - e_j)(e_i - e_j)^T, \quad c_{ij} > 0.$
- New circuit: $s(E + c_{ij}(e_i - e_j)(e_i - e_j)^T) - A$
- Difficulty: Number of elements types is restricted, therefore the class of perturbations is restricted.
- Today: Which eigenvalue sets can be obtained under arbitrary rank one perturbations?

Eigenvalues and regularity

- For fixed $\lambda \in \mathbb{C}$ observe that $\mathcal{A}(\lambda) = \lambda E - A$ is a matrix, so we define the spectrum of $\mathcal{A}$ as

$$\sigma(\mathcal{A}) := \{\lambda \in \mathbb{C} \mid 0 \text{ is eig.val. of } \lambda E - A\}, \quad \text{for } E \text{ invertible and}$$
$$\sigma(\mathcal{A}) := \{\lambda \in \mathbb{C} \mid 0 \text{ is eig.val. of } \lambda E - A\} \cup \{\infty\}, \quad \text{for } E \text{ singular.}$$

- $\mathcal{A}(s) = sE - A$ regular $:\Longleftrightarrow \det(sE - A) \neq 0$
- For $\mathcal{A}(s) = sE - A$ regular the set $\sigma(\mathcal{A}) \setminus \{\infty\}$ is the zero set of $\det(sE - A)$.
- For $\mathcal{A}(s) = sE - A$ singular we have $\sigma(\mathcal{A}) = \overline{\mathbb{C}}$

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The rank of matrix pencils

- The rank of $\mathcal{A}(s) = sE - A$ is $\operatorname{rk}\mathcal{A} := \max_{\lambda\in\mathbb{C}} \operatorname{rk}(\lambda E - A)$.
- $\mathcal{A}$ is regular if and only if $\operatorname{rk}\mathcal{A} = n$

Proposition

The pencil $\mathcal{P}(s) = sF - G$ with $F, G \in \mathbb{C}^{n\times n}$ has rank one if and only if there exists $u, v, w \in \mathbb{C}^n$ with $w \neq 0$ and $u \neq 0$ or $v \neq 0$ such that

$$\mathcal{P}(s) = (su + v)w^T \quad \text{or} \quad \mathcal{P}(s) = w(su^T + v^T).$$

Jordan chains and root subspaces

- For $\lambda \in \sigma(\mathcal{A}) \setminus \{\infty\}$ we call $\{g_0, g_1, \dots, g_{m-1}\} \subset \mathbb{C}^n$ a Jordan chain of length m at λ iff $g_0 \neq 0$ and

$$(A - \lambda E)g_0 = 0, (A - \lambda E)g_1 = Eg_0, \dots, (A - \lambda E)g_{m-1} = Eg_{m-2}.$$

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- Define the (kth) root subspace of $\mathcal{A}$ at λ as

$$\mathcal{L}_\lambda^k(\mathcal{A}) := \{g_i \in \mathbb{C}^n \mid g_i \in \{g_0, \dots, g_{k-1}\} \text{ is JC of } \mathcal{A} \text{ at } \lambda\}$$
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Rank one perturbation of root subspaces

Lemma 2.1 in Dopico, Moro, De Terán '08 implies the following perturbation result.

Theorem (G., Trunk '16; Dopico, Moro, De Terán '08)

For $\mathcal{A}$ regular and $\mathcal{P}$ rank one such that $\mathcal{A}+\mathcal{P}$ is regular, we have for all $\lambda \in \overline{\mathbb{C}}$ and all $k \in \mathbb{N} \setminus \{0\}$

$$\left| \dim \ker(\mathcal{A}+\mathcal{P})(\lambda) - \dim \ker \mathcal{A}(\lambda) \right| \leq 1,$$

$$\left| \dim \frac{\mathcal{L}_\lambda^{k+1}(\mathcal{A}+\mathcal{P})}{\mathcal{L}_\lambda^{k}(\mathcal{A}+\mathcal{P})} - \dim \frac{\mathcal{L}_\lambda^{k+1}(\mathcal{A})}{\mathcal{L}_\lambda^{k}(\mathcal{A})} \right| \leq 1,$$

$$\left| \dim \mathcal{L}_\lambda^{k}(\mathcal{A}+\mathcal{P}) - \dim \mathcal{L}_\lambda^{k}(\mathcal{A}) \right| \leq k.$$

- Similar results by Savchenko '03 for matrices and Behrndt, Leben, Martinez Peria, Trunk '15 for operators

Rank one perturbations of root subspaces

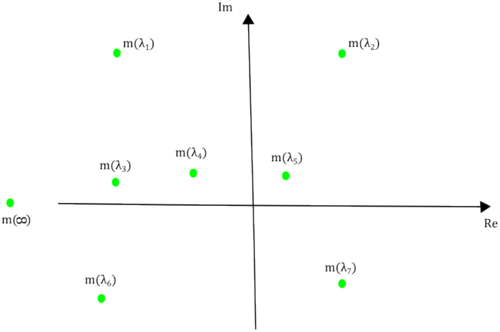
- Denote by $m(\lambda)$ the length of the longest JC at $\lambda \in \sigma(\mathcal{A})$ and define
$$\mathcal{M}(\mathcal{A}) := \sum_{\lambda \in \sigma(\mathcal{A})} m(\lambda)$$

Proposition (G, Trunk '16)

Let $\mathcal{A}$ be regular and $\mathcal{P}$ be of rank one such that $\mathcal{A}+\mathcal{P}$ is regular, then the following estimates hold:

$$\dim \mathcal{L}_\lambda(\mathcal{A}) - m(\lambda) \leq \dim \mathcal{L}_\lambda(\mathcal{A}+\mathcal{P}), \quad \text{for } \lambda \in \sigma(\mathcal{A}),$$
$$\dim \mathcal{L}_\lambda(\mathcal{A}+\mathcal{P}) \leq \dim \mathcal{L}_\lambda(\mathcal{A}) + \mathcal{M}(\mathcal{A}) - m(\lambda), \quad \text{for } \lambda \in \sigma(\mathcal{A}),$$
$$0 \leq \sum_{\mu \in \overline{\mathbb{C}} \setminus \sigma(\mathcal{A})} \dim \mathcal{L}_\mu(\mathcal{A}+\mathcal{P}) \leq \mathcal{M}(\mathcal{A}),$$

Eigenvalue placement for regular pencils



- Given regular $\mathcal{A}(s) = sE - A$ with $\sigma(\mathcal{A}) = \{\lambda_1, \ldots, \lambda_m\}$ and multiplicities $m(\lambda_i)$
- Given pairwise distinct $\mu_1, \ldots, \mu_l \in \overline{\mathbb{C}}$ with multiplicities $m_i > 0$
- Construct rank one $\mathcal{P}$ with $\sigma(\mathcal{A} + \mathcal{P}) = \{\mu_1, \ldots, \mu_l\}$ with multiplicities m_i at μ_i?

Previous Results: Golub '73 for $E = I_n$, A symmetric,
Krupnik '92 for $E = I_n$ (only inverse problem),
Elhay, Golub, Ram '03 for $E > 0$ and A symmetric

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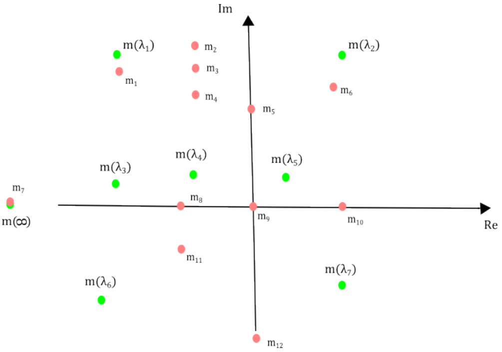

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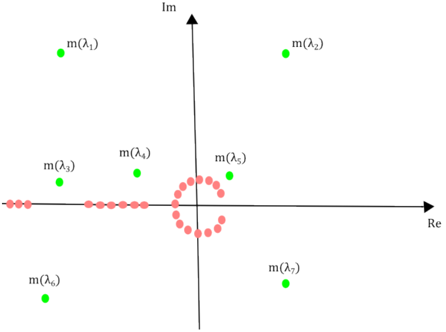

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Eigenvalue placement for regular pencils

Theorem (G, Trunk '16)

Given pairwise distinct $\mu_1, \ldots, \mu_l \in \overline{\mathbb{C}}$, with multiplicities $m_i \in \mathbb{N} \setminus \{0\}$ such that $\sum_{i=1}^{l} m_i = \mathcal{M}(\mathcal{A})$, then we find

$$\mathcal{P}(s) = (\alpha s - \beta) u v^T, \quad \alpha, \beta \in \mathbb{C} \quad u, v \in \mathbb{C}^n$$

such that

$$\sigma(\mathcal{A} + \mathcal{P}) = \{\mu_1, \ldots, \mu_l\} \cup \{\lambda \in \sigma(\mathcal{A}) | \dim \ker \mathcal{A}(\lambda) \geq 2\}.$$

and $\dim \mathcal{L}_\lambda(\mathcal{A} + \mathcal{P})$ is equal to

$$\begin{cases} \dim \mathcal{L}_\lambda(\mathcal{A}) - m(\lambda) + m_i, & \text{for } \lambda = \mu_i \in \sigma(\mathcal{A}), \\ \dim \mathcal{L}_\lambda(\mathcal{A}) - m(\lambda), & \text{for } \lambda \in \sigma(\mathcal{A}) \setminus \{\mu_1, \ldots, \mu_l\}, \\ m_i, & \text{for } \lambda = \mu_i \notin \sigma(\mathcal{A}), \\ 0, & \text{for } \lambda \notin \sigma(\mathcal{A}) \cup \{\mu_1, \ldots, \mu_l\}. \end{cases}$$

Remarks on the placement

- There is only one Jordan chain at each $\lambda \in \sigma(\mathcal{A} + \mathcal{P}) \setminus \sigma(\mathcal{A})$
- Similar result for $E, A \in \mathbb{R}^{n \times n}$ and $\{\mu_1, \ldots, \mu_l\}$ symmetric w.r.t. the real line, then we can find $\alpha, \beta \in \mathbb{R}$ and $u, v \in \mathbb{R}^n$.

Corollary

For $A \in \mathbb{C}^{n \times n}$ with $\dim \ker(\lambda - A) \leq 1$ and $\mu_1, \ldots, \mu_l \in \mathbb{C}$ with multiplicities $m_i \in \mathbb{N} \setminus \{0\}$ satisfying $\sum_{i=1}^{l} m_i = n$ there exist $u, v \in \mathbb{C}^n$ such that $\sigma(A + uv^T) = \{\mu_1, \ldots, \mu_l\}$ and

$$\dim \mathcal{L}_\lambda(A + uv^T) = \begin{cases} m_i, & \text{if } \lambda = \mu_i, \\ 0, & \text{if } \lambda \notin \{\mu_1, \ldots, \mu_l\}. \end{cases}$$

Application: DAEs with feedback

- Given a single input DAE with state feedback $u(t) = f^T x(t)$, $f \in \mathbb{C}^n$

$$E\frac{d}{dt}x(t) = Ax(t) + b \cdot f^T x(t), \quad x(0) = x_0.$$

- Solutions are given by the eigenvalues and Jordan chains of

$$(\mathcal{A} + \mathcal{P})(s) := sE - (A + bf^T), \quad \mathcal{P}(s) = -bf^T$$

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- Let's compare the rank one perturbations

$$\mathcal{P}(s) = (\alpha s - \beta)uv^T \quad \text{vs.} \quad \mathcal{P}(s) = -bf^T.$$

Here $b \in \mathbb{C}^n$ is fixed and this affects the placement.

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For simplicity, assume that $\mathcal{A}(s) = sE - A$, $E, A \in \mathbb{C}^{n\times n}$ satisfies the Hautus condition

$$\operatorname{rk}[\lambda E - A, b] = n, \quad \text{for all } \lambda \in \overline{\mathbb{C}}. \tag{1}$$

This implies $\dim\ker(\lambda E - A) \leq 1$ for $\lambda \in \overline{\mathbb{C}}$ hence $\mathcal{M}(\mathcal{A}) = n$.

Theorem (Feedback placeability)

Let $sE - A$ be regular satisfying (1) with E singular such that $sE - (A + bf^T)$ is regular. Given $\mu_1 = \infty$ and $\mu_2, \ldots, \mu_l \in \mathbb{C}$ and multiplicities $m_i \in \mathbb{N} \setminus \{0\}$ with $\sum_{i=1}^{l} m_i = \mathcal{M}(\mathcal{A})$ there exists an $f \in \mathbb{C}^n$ such that $\mathcal{P}(s) = -bf^T$ satisfies

$$\sigma(\mathcal{A} + \mathcal{P}) = \{\infty, \mu_2, \ldots, \mu_l\} \cup \{\lambda \in \sigma(\mathcal{A}) : \dim\ker \mathcal{A}(\lambda) \geq 2\}$$

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