A new method for network redesign via rank one updates

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We present a method to place the eigenvalues of an electrical network towards a prescribed set of complex numbers by inserting an additional capacitance into the network. For the proof, we use recent results on rank one perturbations of regular matrix pencils.

Introduction. We consider matrix pencils $\mathcal{A}(s) = sE - A$ for $s \in \mathbb{C}$ with $E, A \in \mathbb{C}^{n\times n}$. Here the pencil $\mathcal{A}(s)$ is assumed to be *regular*, which means that $\det(sE - A)$ is not the zero polynomial. For $\mathcal{A}(s)$ regular the finite eigenvalues are given by the zeros of the *characteristic polynomial* $\det(sE - A)$ and ∞ is said to be an eigenvalues of $\mathcal{A}(s)$ if E is not invertible. The set of all eigenvalues of $\mathcal{A}(s)$ is denoted by $\sigma(\mathcal{A})$. It is our aim to investigate the eigenvalues under Hermitian rank one perturbations of the form

$$\mathcal{P}(s) := (\alpha s - \beta)uu^T, \quad \alpha, \beta \in \mathbb{R},\ u \in \mathbb{R}^n, \tag{1}$$

Structured rank one perturbation of the form (1) arise in network redesign considered in [6] and they are given in terms of the canonical unit vectors e_i in $\mathbb{C}^n$ by

$$\mathcal{C}_{ij}(s) := s d_{ij} d_{ij}^T, \quad d_{ij} := \sqrt{c_{ij}}(e_i - e_j) \quad c_{ij} > 0,\ i, j = 1, \dots, n,\ i < j. \tag{2}$$

In this setting, the pencil $(\mathcal{A} + \mathcal{P})(s)$ describes the network with an additional capacitor between the nodes i and j in the network. Note that, various low-rank perturbation results for matrix pencils were obtained in [1, 3–5].

We will study the following problem: Given a set $\{\mu_1, \dots, \mu_{r+1}\}$, $r \in \mathbb{N}$, of pairwise distinct complex numbers and $\alpha, \beta \in \mathbb{R}$, does there exist a rank one pencil of the form (1) such that

$$\sigma(\mathcal{A} + \mathcal{P}) \setminus \{\infty\} = \{\mu_1, \dots, \mu_{r+1}\} \setminus \{\infty\} \tag{3}$$

holds and, in case of existence, is there a way to compute $\mathcal{P}(s)$? We are also interested in finding a perturbation $\mathcal{P}(s)$ that is close to the prescribed structured perturbation class (2) considered in [6].

Preliminaries. For regular $\mathcal{A}(s)$ and $\lambda \in \mathbb{C}$ we denote the *geometric multiplicity of* λ by $\mathrm{gm}_{\mathcal{A}}(\lambda) := \dim\ker(\lambda E - A)$ and the *algebraic multiplicity* $\mathrm{am}_{\mathcal{A}}(\lambda)$ is the zero order of $\det \mathcal{A}(s)$ at λ. For $\lambda = \infty$ one defines $\mathrm{gm}_{\mathcal{A}}(\infty) := \dim\ker E$ and $\mathrm{am}_{\mathcal{A}}(\infty) := n - \deg\det\mathcal{A}(s)$.

In the most models for electrical circuits we have simple eigenvalues, i.e. $\mathrm{am}_{\mathcal{A}}(\lambda) = \mathrm{gm}_{\mathcal{A}}(\lambda) = 1$ at all finite eigenvalues $\lambda \in \sigma(\mathcal{A}) \setminus \{\infty\}$ and a semi-simple eigenvalue at ∞, i.e. $\mathrm{am}_{\mathcal{A}}(\infty) = \mathrm{gm}_{\mathcal{A}}(\infty)$. Hence, we will assume that $\mathcal{A}(s) = sE - A$ is diagonalizable, i.e. there exist invertible matrices $S, T \in \mathbb{C}^{n\times n}$ and $r \in \mathbb{N}$ such that

$$S(sE - A)T = \begin{pmatrix} sI_r - \Lambda & 0 \\ 0 & -I_{n-r} \end{pmatrix}, \quad \Lambda \in \mathbb{C}^{r\times r}, \tag{4}$$

where $\Lambda = \mathrm{diag}(\lambda_i)_{i=1}^r$ is a diagonal matrix with pairwise distinct elements of $\sigma(\mathcal{A}) \setminus \{\infty\}$ on its diagonal.

We use the main construction from [4], to construct the pencil $\mathcal{P}(s)$ such that (3) holds. Consider the representation (4) and decompose according to the block structure $(u_1, \dots, u_r, u_\infty^T)^T := Su$, $(v_1, \dots, v_r, v_\infty^T)^T := T^T u$ we can write

$$\begin{aligned} &\det(sE - A + (\alpha s - \beta)uu^T) \\ &= \det(sE - A)\left(1 + (\alpha s - \beta)\left(\sum_{\lambda_i \in \sigma(\mathcal{A})\setminus\{\infty\}} \frac{v_i u_i}{s - \lambda_i} - v_\infty^T u_\infty\right)\right) \\ &= (-1)^{n-r}\det(ST)^{-1}\prod_{i=1}^r (s - \lambda_i)\left(1 + (\alpha s - \beta)\left(\sum_{\lambda_i \in \sigma(\mathcal{A})\setminus\{\infty\}} \frac{v_i u_i}{s - \lambda_i} - v_\infty^T u_\infty\right)\right). \end{aligned} \tag{5}$$

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In the last step we used [4, Proposition 2.1 (d)] where the identity $\det(sE-A) = (-1)^{n-r}\det(ST)^{-1}\prod_{i=1}^{r}(s-\lambda_i)$ was shown. For a given set of pairwise distinct numbers $\mu_1,\dots,\mu_{r+1} \in \mathbb{C}\setminus\{\beta/\alpha\}$, $\alpha \neq 0$ we define the polynomials

$$q(s) := \det(\beta/\alpha E - A)\prod_{i=1}^{r+1}\frac{s-\mu_i}{\beta/\alpha-\mu_i}, \quad p_\lambda(s) := \begin{cases} \frac{\det(sE-A)}{s-\lambda} & \text{for } \lambda \in \sigma(\mathcal{A})\setminus\{\infty\}, \\ -\det(sE-A) & \text{for } \lambda = \infty. \end{cases} \tag{6}$$

Observe that (3) holds if and only if $q(s) = \det(sE-A+(\alpha s-\beta)uu^T)$ and therefore from (5) we obtain that (3) is equivalent to

$$\frac{q(s)-\det(sE-A)}{\alpha s-\beta} = \sum_{\lambda_i\in\sigma(\mathcal{A})\setminus\{\infty\}} v_i u_i p_{\lambda_i}(s) + v_\infty^T u_\infty p_\infty(s) \tag{7}$$

where the left hand side of (7) is a polynomial of degree at most r.

Main result. From equation (7) we obtain the following main result.

Proposition 1 *Let $\mathcal{A}(s) = sE - A$ be a regular matrix pencil satisfying* (4), $\alpha \neq 0$, $\beta/\alpha \notin \sigma(\mathcal{A})$. *Given pairwise distinct $\mu_1,\dots,\mu_{r+1} \in \mathbb{C}\setminus\{\beta/\alpha\}$ then there exists a perturbation $\mathcal{P}(s) = (\alpha s-\beta)uu^T$ satisfying $\sigma(\mathcal{A}+\mathcal{P})\setminus\{\infty\} = \{\mu_1,\dots,\mu_{r+1}\}$ if and only if there is a solution of*

$$v_i u_i = \frac{q(\lambda_i)}{(\alpha\lambda_i-\beta)p_{\lambda_i}(\lambda_i)},\ \lambda_i \in \sigma(\mathcal{A})\setminus\{\infty\}, \quad v_\infty^T u_\infty = -\frac{\prod_{i=1}^{r}(\frac{\beta}{\alpha}-\lambda_i)}{\alpha\prod_{i=1}^{r+1}(\frac{\beta}{\alpha}-\mu_i)}. \tag{8}$$

Proof. It was already observed that there is a solution to (3) if and only if (7) holds. By plugging in the eigenvalues $\lambda \in \sigma(\mathcal{A})$ in (7) and comparing the coefficients on both sides we obtain the equation on the left hand side of (8) for all $\lambda_i \in \sigma(\mathcal{A})\setminus\{\infty\}$. Furthermore, comparing the leading coefficients leads to the second equation in (8). Assume conversely that (8) holds. Then the leading coefficients of the polynomials on the right and left hand side of (7) coincide. Cancelling the leading term on both sides, we obtain polynomials of degree less or equal to r. Again (8) implies that (7) holds for all $\lambda_i \in \sigma(\mathcal{A})$, $i = 1,\dots,r$ and therefore the polynomial on the left hand side in (8) is the same as the polynomial on the right hand side of (8). Thus we have proven (8). □

As a special case of Proposition 1 we consider the case where $E = E^*$ and $A = A^*$ and $\sigma(\mathcal{A}) \subset \mathbb{R}\cup\{\infty\}$. Then [8, Theorem 1] implies that we can choose $T = S^*\Sigma$ in (4), where $\Sigma = \operatorname{diag}(\varepsilon_i)_{i=1}^n$ is a matrix of characteristic signs $\varepsilon_i = \pm 1$.

Corollary 2 *Additionally to the assumptions of Proposition 1 assume that $E = E^*$, $A = A^*$ and $\sigma(\mathcal{A}) \subset \mathbb{R}\cup\{\infty\}$ with the characteristic signs $\Sigma = \operatorname{diag}(\varepsilon_i)_{i=1}^n$ then* (8) *has a solution if and only if*

$$\varepsilon_i = \operatorname{sgn}\frac{q(\lambda_i)}{(\alpha\lambda_i-\beta)p_{\lambda_i}(\lambda_i)}, \quad -\operatorname{sgn}\frac{\prod_{i=1}^{r}(\frac{\beta}{\alpha}-\lambda_i)}{\alpha\prod_{i=1}^{r+1}(\frac{\beta}{\alpha}-\mu_i)} \in \{\varepsilon_{r+1},\dots,\varepsilon_n\}.$$

Proof. The relation $T = S^*\Sigma$ from [8, Theorem 1] implies that $(v_1,\dots,v_r,v_\infty^T) = \Sigma\overline{S}u$ and $(u_1,\dots,u_r,u_\infty^T) = Su$. Since $u \in \mathbb{R}$ we have $v_i = \varepsilon_i\overline{u_i}$ for all $i = 1,\dots,r$ and $v_\infty = \operatorname{diag}(\varepsilon_{r+1},\dots,\varepsilon_n)\overline{u_\infty}$. Plugging this into the conditions (8) proves the corollary. □

Outlook. We apply the results to the setting of a more restricted perturbation class given by (2). In this case we have $\alpha = 1$ and $\beta = 0$. The basic idea is to select the element $\mathcal{C}_{ij}(s) = sd_{ij}d_{ij}^T$ of the form (2) with $(x_1,\dots,x_r,x_\infty^T)^T := S(e_i-e_j)$, $(y_1,\dots,y_k,y_\infty^T)^T := T^T(e_i-e_j)$ that minimizes the total error in (8):

$$\min_{c_{ij}>0} \qquad \sum_{k=1}^{r}\left|c_{ij}x_k y_k - \frac{q(\lambda_k)}{\lambda_k p_{\lambda_k}(\lambda_k)}\right|^2 + \left|c_{ij}y_\infty^T x_\infty - \frac{\prod_{i=1}^{r}\lambda_i}{\prod_{i=1}^{r+1}\mu_i}\right|^2. \tag{9}$$

The capacitance $\mathcal{C}_{ij}(s)$ minimizing (9) can be found by performing a simple line search over the parameter c_{ij}.

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