On the parametric eigenvalue behavior of matrix pencils under rank one perturbations

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Abstract

We study the eigenvalues of rank one perturbations of regular matrix pencils depending linearly on a complex parameter. We prove properties of the corresponding eigenvalue sets including a convergence result as the parameter tends to infinity and an eigenvalue interlacing property for real valued pencils having real eigenvalues only.

Keywords: matrix pencil, perturbation theory, rank one perturbations

1 Preliminaries

We consider matrix pencils $\mathcal{A}(s) = sE - A$ for $s \in \mathbb{C}$ with $E, A \in \mathbb{C}^{n\times n}$. Here the pencil $\mathcal{A}$ is assumed to be regular, which means that $\det(sE - A)$ is not the zero polynomial. Otherwise we say that $\mathcal{A}$ is singular. For $\mathcal{A}$ regular the finite eigenvalues are given by the zeros of the characteristic polynomial $\det(sE - A)$ and ∞ is said to be an eigenvalues of $\mathcal{A}$ if E is not invertible. The set of all eigenvalues of $\mathcal{A}$ is denoted by $\sigma(\mathcal{A})$. For regular $\mathcal{A}$ and $\lambda \in \mathbb{C}$ we denote the geometric multiplicity of λ by $\mathrm{gm}_{\mathcal{A}}(\lambda) := \dim\ker(\lambda E - A)$ and the algebraic multiplicity $\mathrm{am}_{\mathcal{A}}(\lambda)$ is the zero order of $\det \mathcal{A}(s)$ at λ. For $\lambda = \infty$ one defines $\mathrm{gm}_{\mathcal{A}}(\infty) := \dim\ker E$ and $\mathrm{am}_{\mathcal{A}}(\infty) := n - \deg\det \mathcal{A}(s)$. It is our aim to investigate the eigenvalues under parameter dependent rank one perturbations of the form

$$\tau\mathcal{P}(s) := \tau(\alpha s - \beta)uv^T, \quad \alpha, \beta \in \mathbb{C},\ u, v \in \mathbb{C}^n, \tag{1.1}$$

where τ is a parameter varying over $\mathbb{C}$. For an overview of the theory of regular matrix pencils and general perturbation results see [3, 5, 8]. For previous results on low-rank perturbations of regular matrix pencils we refer to [1, 2, 6, 9].

In this note, we follow the approach from [7], where generic results on the parametric eigenvalue behavior of matrices, i.e. $E = I_n$ were obtained. For this we fix a perturbation $\mathcal{P}$ of the form (1.1) and derive properties of the eigenvalues of $\mathcal{A} + \tau\mathcal{P}$ in dependence of τ, i.e. our results are non-generic.

For $\mathcal{A}(s) = sE - A$ regular there exists $S, T \in \mathbb{C}^{n\times n}$ and $r \in \mathbb{N}$ such that $S\mathcal{A}(s)T$ is in Weierstraß canonical form (cf. [3]), i.e.

$$S(sE-A)T = s\begin{pmatrix} I_r & 0 \\ 0 & N \end{pmatrix} - \begin{pmatrix} J & 0 \\ 0 & I_{n-r} \end{pmatrix}, J \in \mathbb{C}^{r\times r}, N \in \mathbb{C}^{(n-r)\times(n-r)} \quad (1.2)$$

with J and N in Jordan canonical form and N nilpotent. Here, the matrix J contains all the Jordan chains at the finite eigenvalues of $\mathcal{A}$ and the matrix N contains all the Jordan chains of $\mathcal{A}$ at ∞. In particular we have $\mathrm{gm}_{\mathcal{A}}(\infty) = \dim\ker N$ and $\mathrm{am}_{\mathcal{A}}(\infty) = n - r$.

Lemma 1.1. *Let $\mathcal{A}(s) = sE - A$ be regular then the function $s \mapsto (\alpha s - \beta)v^T(sE - A)^{-1}u$ is holomorphic on $\overline{\mathbb{C}} \setminus \sigma(\mathcal{A})$ and the following numbers are equal.*

(a) *The order of the pole of $s \mapsto (\alpha - \beta s)v^T(-sA + E)^{-1}u$ at 0.*

(b) *The order of the pole of $s \mapsto (\alpha s - \beta)v^T(sE - A)^{-1}u$ at ∞, i.e. the order of the pole of $s \mapsto (\alpha 1/s - \beta)v^T(1/sE - A)^{-1}u$ at 0.*

(c) *The smallest $N \in \mathbb{N}$ such that $\lim_{s\to\infty} s^{-N}(\alpha s - \beta)v^T(sE - A)^{-1}u$ exists.*

Proof. From the Weiertraß canonical form (1.2) with $S, T \in \mathbb{C}^{n\times n}$ it can be seen that

$$v^T(sE-A)^{-1}u = v^T T \begin{pmatrix} (sI_r - J)^{-1} & \\ & (N - I_{n-r})^{-1} \end{pmatrix} Su. \quad (1.3)$$

Thus $s \mapsto (\alpha s - \beta)v^T(sE - A)^{-1}u$ is just the sum of rational functions with poles only in a subset of $\sigma(\mathcal{A})$, hence it is holomorphic on $\overline{\mathbb{C}} \setminus \sigma(\mathcal{A})$. By definition, the order of the pole of $s \mapsto (\alpha 1/s - \beta)v^T(1/sE - A)^{-1}u$ at 0 is the smallest $N \in \mathbb{N}$ such that

$$\lim_{s\to 0} s^N(\alpha 1/s - \beta)v^T(1/sE - A)^{-1}u = \lim_{s\to 0} s^N(\alpha - \beta s)v^T(E - sA)^{-1}u$$

exists. This is the order of the pole of $s \mapsto (\alpha - \beta s)v^T(-sA + E)^{-1}u$ at 0. □

For $\lambda \in \sigma(\mathcal{A}) \setminus \{\infty\}$ we denote by $m_{uv}(\lambda)$ the order of the pole of $s \mapsto (\alpha s - \beta)v^T(sE - A)^{-1}u$ at λ and set $m_{uv}(\lambda) := 0$ if there is no pole. For $\infty \in \sigma(\mathcal{A})$ the number $m_{uv}(\infty)$ is the order of the pole of $s \mapsto (\alpha - \beta s)v^T(-sA + E)^{-1}u$ at 0.

In the next lemma we provide a common method to characterize the eigenvalue at ∞.

Lemma 1.2. *Let $\mathcal{A}(s) = sE - A$ be regular then we define the dual pencil $\mathcal{A}'(s) = -sA + E$ then the following holds.*

(a) *We have $\lambda \in \sigma(\mathcal{A})\setminus\{0,\infty\}$ if and only if $\lambda^{-1} \in \sigma(\mathcal{A}')\setminus\{0,\infty\}$ with $\mathrm{gm}_{\mathcal{A}}(\lambda) = \mathrm{gm}_{\mathcal{A}'}(\lambda^{-1})$ and $\mathrm{am}_{\mathcal{A}}(\lambda) = \mathrm{am}_{\mathcal{A}'}(\lambda^{-1})$.*

(b) *We have $0 \in \sigma(\mathcal{A})$ if and only if $\infty \in \sigma(\mathcal{A}')$ with $\mathrm{gm}_{\mathcal{A}}(0) = \mathrm{gm}_{\mathcal{A}'}(\infty)$ and $\mathrm{am}_{\mathcal{A}}(0) = \mathrm{am}_{\mathcal{A}'}(\infty)$.*

(c) *We have $\infty \in \sigma(\mathcal{A})$ if and only if $0 \in \sigma(\mathcal{A}')$ with $\mathrm{gm}_{\mathcal{A}}(\infty) = \mathrm{gm}_{\mathcal{A}'}(0)$ and $\mathrm{am}_{\mathcal{A}}(\infty) = \mathrm{am}_{\mathcal{A}'}(0)$.*

Proof. We use the transformation matrices $S, T \in \mathbb{C}^{n\times n}$ from (1.2) and obtain

$$S(-sA-E)T = -s\begin{pmatrix} J & 0 \\ 0 & I_{n-r}\end{pmatrix} + \begin{pmatrix} I_r & 0 \\ 0 & N\end{pmatrix}, \quad J \in \mathbb{C}^{r\times r}, N \in \mathbb{C}^{(n-r)\times(n-r)}$$

which already in block diagonal form. It remains to transform it into Weierstraß canonical from. Now let us consider a Jordan block $\mathcal{J}_m(\lambda)$ of J at $\lambda \in \sigma(\mathcal{A}) \setminus \{0,\infty\}$ of size $m \in \mathbb{N}\setminus\{0\}$. Then a computation of $\mathcal{J}_m(\lambda)^{-1}$ shows that there exists $U \in \mathbb{C}^{m\times m}$ with $\mathcal{J}_m(\lambda)^{-1} = U^{-1}\mathcal{J}_m(\lambda^{-1})U$. With the transformations $\tilde{S} = U^{-1}$ and $\tilde{T} = \mathcal{J}_m(\lambda)^{-1}U$ we obtain

$$\tilde{S}\mathcal{J}_m(\lambda)\tilde{T} = U^{-1}\mathcal{J}_m(\lambda)\mathcal{J}_m(\lambda)^{-1}U = I_m, \quad \tilde{S}I_m\tilde{T} = \mathcal{J}_m(\lambda^{-1}).$$

From the block decomposition we see that there exist $\tilde{S}, \tilde{T} \in \mathbb{C}^{n\times n}$ such that

$$\tilde{S}(-sA-E)\tilde{T} = s\begin{pmatrix} I & 0 \\ 0 & \tilde{N}\end{pmatrix} - \begin{pmatrix} \tilde{J} & 0 \\ 0 & I\end{pmatrix}$$

where for each Jordan block $\mathcal{J}_m(\lambda)$ in J there is a Jordan block $\mathcal{J}_m(\lambda^{-1})$ in $\tilde{J}$. Furthermore $\tilde{J}$ contains the block N which contains now the Jordan chains of $\mathcal{A}'$ at 0 and $\tilde{N}$ is nilpotent and consist of the Jordan blocks of J at 0. This proves the claims (a) - (c). ☐

Lemma 1.3. *For $\mathcal{A}$ regular and $\mathcal{P}$ of the form (1.1) we introduce the polynomials $m(s) := \prod_{\lambda\in\sigma(\mathcal{A})\setminus\{\infty\}}(s-\lambda)^{m_{uv}(\lambda)}$ and $p(s) := (\alpha s-\beta)v^T m(s)(sE-A)^{-1}u$. Then the following holds.*

(a) *For $m_{uv}(\infty) > 0$ we have $\deg p = \deg m + m_{uv}(\infty)$.*

(b) *We have the polynomial factorization $\det(\mathcal{A}+\tau\mathcal{P})(s) = \frac{\det\mathcal{A}(s)}{m(s)}(m(s)+\tau p(s))$.*

(c) *We have the dual factorization $\det(\mathcal{A}'+\tau\mathcal{P}')(s) = \frac{\det\mathcal{A}'(s)}{m^\sharp(s)}(m^\sharp(s)+\tau p^\sharp(s))$ with*

$$m^\sharp(s) := s^{m_{uv}(\infty)} \prod_{\lambda\in\sigma(\mathcal{A})\setminus\{0,\infty\}} (s-\lambda^{-1})^{m_{uv}(\lambda)},$$
$$p^\sharp(s) := (\alpha-\beta s)v^T m^\sharp(s)(-As+E)^{-1}u$$

Proof. (a) An evaluation of $(\alpha s-\beta)v^T(sE-A)^{-1}u$ with the help of Weierstraß canonical form shows that this function is the sum of rational functions $\frac{r(s)}{q(s)}$ with $q(s) = (s-\lambda)^k$ for $\lambda \in \sigma(\mathcal{A})\setminus\{\infty\}$ and r is constant or r is a polynomial of degree at most $m_{uv}(\infty)$ and q is constant. Therefore the degree of p is $\deg m + m_{uv}(\infty)$.

(b) We use Sylvester's determinant identity and obtain

$$\begin{aligned}\det(\mathcal{A}+\tau\mathcal{P})(s) &= \det(sE-A)\det(I_n + (sE-A)^{-1}\tau(\alpha s-\beta)uv^T) \\ &= \det(sE-A)(1+v^T(sE-A)^{-1}\tau(\alpha s-\beta)u) \\ &= \frac{\det\mathcal{A}(s)}{m(s)}(m(s)+\tau p(s)).\end{aligned}$$

Note that $\det \mathcal{A}(s)$ is divisible by $m(s)$ since $m_{uv}(\lambda) \leq \mathrm{am}_{\mathcal{A}}(\lambda)$ holds.

(c) This can be proven in the same way as (b) with an application of Lemma 1.2 and Lemma 1.1 to rewrite $m^\sharp$.

□

2 The algebraic multiplicity and eigenvalue convergence

In the next proposition we describe the regularity of $\mathcal{A} + \tau\mathcal{P}$ and the change of the algebraic multiplicities in dependence of τ.

Proposition 2.1. *Let $\mathcal{A}(s) = sE - A$ be regular, let $\mathcal{P}$ be of the form (1.1).*

(a) *Assume that there exists $\lambda \in \sigma(\mathcal{A})$ with $m_{uv}(\lambda) > 0$. Then $\mathcal{A} + \tau\mathcal{P}$ is regular for all $\tau \in \mathbb{C}$ and the following holds.*

  (i) *For all $\mu \in \sigma(\mathcal{A})$ with $m_{uv}(\mu) > 0$ we have $\mathrm{am}_{\mathcal{A}+\tau\mathcal{P}}(\mu) = \mathrm{am}_{\mathcal{A}}(\mu) - m_{uv}(\lambda)$ for all $\tau \in \mathbb{C} \setminus \{0\}$.*

  (ii) *For $\mu \in \sigma(\mathcal{A})\setminus\{\infty\}$ with $m_{uv}(\mu) = 0$ and $p(\mu) = 0$ we have $\mathrm{am}_{\mathcal{A}+\tau\mathcal{P}}(\mu) = \mathrm{am}_{\mathcal{A}}(\mu)$ for all $\tau \in \mathbb{C}$.*

  (iii) *For $\mu \in \sigma(\mathcal{A})\setminus\{\infty\}$ with $m_{uv}(\mu) = 0$ and $p(\mu) \neq 0$ we have $\mathrm{am}_{\mathcal{A}+\tau\mathcal{P}}(\mu) = \mathrm{am}_{\mathcal{A}}(\mu)$ for all $\tau \in \mathbb{C} \setminus \{-\frac{m(\mu)}{p(\mu)}\}$.*

  (iv) *For $\mu = \infty \in \sigma(\mathcal{A})$ with $m_{uv}(\infty) = 0$ and $p^\sharp(0) = 0$ we have $\mathrm{am}_{\mathcal{A}+\tau\mathcal{P}}(\mu) = \mathrm{am}_{\mathcal{A}}(\mu)$ for all $\tau \in \mathbb{C}$.*

  (v) *For $\mu = \infty \in \sigma(\mathcal{A})$ with $m_{uv}(\infty) = 0$ and $p^\sharp(0) \neq 0$ we have $\mathrm{am}_{\mathcal{A}+\tau\mathcal{P}}(\mu) = \mathrm{am}_{\mathcal{A}}(\mu)$ for all $\tau \in \mathbb{C} \setminus \{-\frac{m^\sharp(0)}{p^\sharp(0)}\}$.*

(b) *Assume that $m_{uv}(\lambda) = 0$ for all $\lambda \in \sigma(\mathcal{A})$. Then $m \equiv 1$, the polynomial p is constant and the following holds.*

  (i) *If $p \equiv 0$ then $\det(\mathcal{A}+\tau\mathcal{P})(s) = \det \mathcal{A}(s)$ for all $\tau \in \mathbb{C}$. Hence $\mathrm{am}_{\mathcal{A}+\tau\mathcal{P}}(\lambda) = \mathrm{am}_{\mathcal{A}}(\lambda)$ for all $\tau \in \mathbb{C}$ and all $\lambda \in \sigma(\mathcal{A})$.*

  (ii) *If $p \equiv c$ for some $c \in \mathbb{C} \setminus \{0\}$ then $\mathrm{am}_{\mathcal{A}+\tau\mathcal{P}}(\lambda) = \mathrm{am}_{\mathcal{A}}(\lambda)$ for all $\lambda \in \sigma(\mathcal{A})$ and $\tau \in \mathbb{C} \setminus \{-1/c\}$ and $\mathcal{A} - 1/c\mathcal{P}$ is singular.*

Proof. (a) For $\lambda \in \sigma(\mathcal{A})$ with $\lambda \in \sigma(\mathcal{A}) \setminus \{\infty\}$ we have $m(\lambda) = 0$ and $p(\lambda) \neq 0$. Otherwise, for $p(\lambda) = 0$, we could divide by $s - \lambda$ which is a contradiction to the definition of $m_{uv}(\lambda)$ as the pole order at λ. This implies that (i) holds and that $m + \tau p$ is not the zero polynomial. Hence Lemma 1.3 (b) implies that $\mathcal{A} + \tau\mathcal{P}$ is regular for all $\tau \in \mathbb{C}$. For $\lambda = \infty$ and $m_{uv}(\infty) > 0$ we see from Lemma 1.3 (a) that $\deg m < \deg p$. Therefore $m + \tau p$ is unequal to the zero polynomial for all $\tau \in \mathbb{C}$ and (i) holds in this case. The statements (ii) and (iii) can be verified easily. The dual factorization from Lemma 1.3 (c) implies (iv) and (v).

(b) Assume $m_{uv}(\lambda) = 0$ for all $\lambda \in \sigma(\mathcal{A})$, then the definition of m implies $m \equiv 1$. Furthermore p we see from the Weierstraß canonical form (1.2) that p is a polynomial. Since $m_{uv}(\infty) = 0$ implies by Lemma 1.1 (c) that p is bounded on $\mathbb{C}$ we infer that p is constant. Now (i) and (ii) are easy to see.

□

In the next theorem we describe the eigenvalue behavior as $\tau \to \infty$.

Theorem 2.2. *Let $\mathcal{A}$ be regular and let $\mathcal{P}$ be of the form (1.1) such that there exists $\lambda \in \sigma(\mathcal{A})$ with $m_{uv}(\lambda) > 0$.*

(a) *For $\mu \in \mathbb{C} \setminus \sigma(\mathcal{A})$ with $p(\mu) \neq 0$ then $\tau = -\frac{m(\mu)}{p(\mu)}$ is the only value such that $\mu \in \sigma(\mathcal{A} + \tau\mathcal{P})$.*

(b) *There exists $\tau_0 \in \mathbb{C}$ such that for all $\tau \geq \tau_0$*

$$\sum_{\mu \in \sigma(\mathcal{A}+\tau\mathcal{P}) \setminus \sigma(\mathcal{A})} \mathrm{am}_{\mathcal{A}+\tau\mathcal{P}}(\mu) = \sum_{\lambda \in \sigma(\mathcal{A})} m_{uv}(\lambda) = \max\{\deg p, \deg m\}.$$

(c) *There are $\deg p$ eigenvalues, counting with multiplicity, that converge for $\tau \to \infty$ to the zero set of p and $\max\{0, \deg m - \deg p\}$ eigenvalues converge to ∞.*

Proof. (a) Consider the factorization from Lemma 1.3 (a) for $s = \mu$. From $\mu \in \mathbb{C} \setminus \sigma(\mathcal{A})$ we conclude $m(\mu) \neq 0$ and $\det(\mu E - A) \neq 0$. Hence $\mu \in \sigma(\mathcal{A} + \tau\mathcal{P})$ if and only if $m(\mu) + \tau p(\mu) = 0$, but this is a linear equation in τ with the unique solution $\tau = -\frac{m(\mu)}{p(\mu)}$.

(b) From the Proposition 2.1 (a) we see that there is a $\tau_0 \in \mathbb{C}$ such that for all $\tau \geq \tau_0$ the following equations hold

$$\begin{aligned} n &= \sum_{\mu \in \sigma(\mathcal{A}+\tau\mathcal{P})} \mathrm{am}_{\mathcal{A}+\tau\mathcal{P}}(\mu) \\ &= \sum_{\mu \in \sigma(\mathcal{A}+\tau\mathcal{P}) \cap \sigma(\mathcal{A})} \mathrm{am}_{\mathcal{A}+\tau\mathcal{P}}(\mu) + \sum_{\mu \in \sigma(\mathcal{A}+\tau\mathcal{P}) \setminus \sigma(\mathcal{A})} \mathrm{am}_{\mathcal{A}+\tau\mathcal{P}}(\mu) \\ &= \sum_{\mu \in \sigma(\mathcal{A})} \mathrm{am}_{\mathcal{A}}(\mu) - \sum_{\lambda \in \sigma(\mathcal{A})} m_{uv}(\lambda) + \sum_{\mu \in \sigma(\mathcal{A}+\tau\mathcal{P}) \setminus \sigma(\mathcal{A})} \mathrm{am}_{\mathcal{A}+\tau\mathcal{P}}(\mu) \end{aligned}$$

Now $\sum_{\mu \in \sigma(\mathcal{A})} \mathrm{am}_{\mathcal{A}}(\mu) = n$ implies

$$\sum_{\mu \in \sigma(\mathcal{A}+\tau\mathcal{P}) \setminus \sigma(\mathcal{A})} \mathrm{am}_{\mathcal{A}+\tau\mathcal{P}}(\mu) = \sum_{\lambda \in \sigma(\mathcal{A})} m_{uv}(\lambda) = \deg m + m_{uv}(\infty).$$

For $m_{uv}(\infty) > 0$ this is by Lemma 1.3 (c) equal to $\deg p$. For $m_{uv}(\infty) = 0$ we have $\deg p \leq \deg m$. This proves the claim

(c) We consider the factorization from Lemma 1.3 (b) and write the last factor for $\tau \in \mathbb{C} \setminus \{0\}$ as $\tau^{-1}m + p$ and consider this polynomial in a neighborhood of the zeros of p given by $C_j(\varepsilon) := \{\lambda \in \mathbb{C} \mid |\lambda - \mu_j| < \varepsilon\}$ with $j = 1, \ldots, k$. Here $\varepsilon > 0$ is chosen such that these discs are pairwise disjoint. As $\tau \to \infty$ the polynomial converges on $\cup_{j=1}^{k} C_j(\varepsilon)$ uniformly to p. By Rouche's theorem the number of zeros of p and $\tau^{-1}m+p$ inside the discs coincide, they are eigenvalues of $\mathcal{A} + \tau\mathcal{P}$ by Lemma 1.3 (b), and they converge to the zeros of p. For $\deg p < \deg m$ the convergence to ∞ follows from considering the dual pencil $\mathcal{A}' + \tau\mathcal{P}'$ and applying the above argument to the dual factorization from Lemma 1.3 (c) and the expression $\tau^1 m^\sharp + p^\sharp$ on a disc around 0.

□

3 Eigenvalue interlacing for real valued matrix pencils

In the following we consider matrix pencils with $\sigma(\mathcal{A}) \subseteq \mathbb{R} \cup \{\infty\}$ and semi-simple eigenvalues, i.e. $\mathrm{am}_{\mathcal{A}}(\lambda) = \mathrm{gm}_{\mathcal{A}}(\lambda)$ for all $\lambda \in \sigma(\mathcal{A})$. Under the assumption that all eigenvalues move in the same direction, we show in the next theorem that the eigenvalues interlace, i.e. roughly speaking that there is only one eigenvalue of $\mathcal{A} + \tau\mathcal{P}$ between two consecutive eigenvalues of $\mathcal{A}$.

Theorem 3.1. *Let $\mathcal{A}(s) = sE - A$ be regular with $E, A \in \mathbb{R}^{n\times n}$ and only real semi-simple eigenvalues $\lambda_0 < \ldots < \lambda_m < \infty$ and let $\mathcal{P}(s) = (\alpha s - \beta)uv^T$ with $u, v \in \mathbb{R}^n$, $\alpha \in \mathbb{R} \setminus \{0\}$, $\beta \in \mathbb{R}$ and $\beta/\alpha \notin \sigma(\mathcal{A})$. Then there exist transformation matrices $S, T \in \mathbb{R}^{n\times n}$ in (1.2) such that J is diagonal and $N = 0$ we decompose according to the eigenspaces $Su = (u_0^T, \ldots, u_m^T, u_\infty^T)^T, v^T T = (v_0^T, \ldots, v_m^T, v_\infty^T)$ with $u_i, v_i \in \mathbb{R}^{m_i}$ and $u_\infty, v_\infty \in \mathbb{R}^{n-r}$. Denote by $i_1 < \ldots < i_{m'} < \infty$ all indices with $(\beta - \alpha\lambda_{i_k})v_{i_k}^T u_{i_k} \neq 0$ for $k = 1, \ldots, m'$ and assume that $m' \geq 2$.*

(a) *If $v_i^T u_i \neq 0$ then $\mathrm{am}_{\mathcal{A}+\tau\mathcal{P}}(\lambda_i) = \mathrm{am}_{\mathcal{A}}(\lambda_i) - 1$ for all $\tau \in \mathbb{C} \setminus \{0\}$. If $v_i^T u_i = 0$ then $\mathrm{am}_{\mathcal{A}+\tau\mathcal{P}}(\lambda_i) = \mathrm{am}_{\mathcal{A}}(\lambda_i)$ for all $\tau \in \mathbb{C} \setminus \{-\frac{m(\lambda_i)}{p(\lambda_i)}\}$.*

(b) *Assume $(\beta - \alpha\lambda_{i_k})v_{i_k}^T u_{i_k} > 0$ for all $k = 1, \ldots, m' - 1$.*

  (i) *If $\lambda_{i_{m'}} < \infty$ and $(\beta - \alpha\lambda_{i_{m'}})v_{i_{m'}}^T u_{i_{m'}} > 0$ then for all $\tau \in (0, \infty)$ there exists $\lambda_{i_k}(\tau) \in \overline{\mathbb{C}}$ with $\sigma(\mathcal{A} + \tau\mathcal{P}) \cap (\lambda_{i_k}, \lambda_{i_{k+1}}) = \{\lambda_{i_k}(\tau)\} \cup (\sigma(\mathcal{A}) \cap (\lambda_{i_k}, \lambda_{i_{k+1}}))$ for all $k = 1, \ldots, m' - 1$ and $\sigma(\mathcal{A} + \tau\mathcal{P}) \cap ((\lambda_{m'}, \infty] \cup (-\infty, \lambda_{i_1})) = \{\lambda_{i_{m'}}(\tau)\} \cup (\sigma(\mathcal{A}) \cap ((\lambda_{m'}, \infty] \cup (-\infty, \lambda_{i_1})))$.*

  (ii) *For $\lambda_{i_{m'}} = \infty$ and $\alpha(-1)^{n-r} v_\infty^T u_\infty < 0$ then for all $\tau \in (0, \infty)$ there exists $\lambda_{i_k}(\tau) \in \overline{\mathbb{C}}$ with $\sigma(\mathcal{A} + \tau\mathcal{P}) \cap (\lambda_{i_k}, \lambda_{i_{k+1}}) = \{\lambda_{i_k}(\tau)\} \cup (\sigma(\mathcal{A}) \cap (\lambda_{i_k}, \lambda_{i_{k+1}}))$ for all $k = 1, \ldots, m' - 1$ and $\sigma(\mathcal{A} + \tau\mathcal{P}) \cap (-\infty, \lambda_{i_1}) = \{\lambda_{m'}(\tau)\}$.*

(c) *Assume $(\beta - \alpha\lambda_{i_k})v_{i_k}^T u_{i_k} < 0$ for all $k = 1, \ldots, m' - 1$.*

(i) *If* $\lambda_{i_{m'}} < \infty$ *and* $(\beta - \alpha\lambda_{i_{m'}})v_{i_{m'}}^T u_{i_{m'}} < 0$ *then for all* $\tau \in (0, \infty)$ *there exists* $\lambda_{i_{k+1}}(\tau) \in \overline{\mathbb{C}}$ *with* $\sigma(\mathcal{A}+\tau\mathcal{P}) \cap (\lambda_{i_k}, \lambda_{i_{k+1}}) = \{\lambda_{i_{k+1}}(\tau)\} \cup (\sigma(\mathcal{A}) \cap (\lambda_{i_k}, \lambda_{i_{k+1}}))$ *for all* $k = 1, \dots, m'-1$ *and* $\sigma(\mathcal{A} + \tau\mathcal{P}) \cap ((\lambda_{m'}, \infty] \cup (-\infty, \lambda_{i_1})) = \{\lambda_{i_1}(\tau)\} \cup (\sigma(\mathcal{A}) \cap ((\lambda_{m'}, \infty] \cup (-\infty, \lambda_{i_1})))$.

(ii) *If* $\lambda_{i_{m'}} = \infty$ *and* $\alpha(-1)^{n-r} v_\infty^T u_\infty > 0$ *then for all* $\tau \in (0, \infty)$ *there exists* $\lambda_{i_{k+1}}(\tau) \in \overline{\mathbb{C}}$ *with* $\sigma(\mathcal{A} + \tau\mathcal{P}) \cap (\lambda_{i_k}, \lambda_{i_{k+1}}) = \{\lambda_{i_{k+1}}(\tau)\} \cup (\sigma(\mathcal{A}) \cap (\lambda_{i_k}, \lambda_{i_{k+1}}))$ *for all* $k = 1, \dots, m'-1$ *and* $\sigma(\mathcal{A} + \tau\mathcal{P}) \cap (\lambda_{m'-1}, \infty) = \{\lambda_{i_{m'}}(\tau)\} \cup (\sigma(\mathcal{A}) \cap (\lambda_{m'-1}, \infty))$.

The assumptions of (b) and (c) imply that $\sigma(\mathcal{A} + \tau\mathcal{P}) \subseteq \mathbb{R} \cup \{\infty\}$ *holds for all* $\tau \in \mathbb{R}$.

Proof. First note that for $E, A \in \mathbb{R}^{n\times n}$ and $\sigma(\mathcal{A}) \subset \mathbb{R} \cup \{\infty\}$ there exist $S, T \in \mathbb{R}^{n\times n}$ such that (1.2) holds with J diagonal and $N = 0$ (cf. [3]). This readily implies $(\beta - \alpha\lambda_i)v_i^T u_i \in \mathbb{R}$ for all $i = 1, \dots m$ and it allows us write

$$v^T(sE - A)^{-1}u = (-1)^{n-r} v_\infty^T u_\infty + \sum_{i=1}^{m} \frac{v_i^T u_i}{s - \lambda_i}, \tag{3.1}$$

$$v^T(-sA + E)^{-1}u = (-1)^{n-r} s^{-1} v_\infty^T u_\infty + \sum_{i=1}^{m} \frac{v_i^T u_i}{1 - \lambda_i s}. \tag{3.2}$$

Therefore $v_i^T u_i \neq 0$ implies $m_{uv}(\lambda_i) = 1$ and $v_i^T u_i = 0$ implies $m_{uv}(\lambda_i) = 0$. Together with Proposition 2.1 this proves (a). We continue with the proof of (b). For given $\tau > 0$ and $\lambda \in \sigma(\mathcal{A} + \tau\mathcal{P}) \setminus \sigma(\mathcal{A})$ with $\lambda \neq \infty$ we obtain from (3.1) the equation $m(\lambda) + \tau p(\lambda) = 0$. We show that there is a solution to this equation $\lambda_{i_k}(\tau) \in (\lambda_{i_k}, \lambda_{i_{k+1}})$. Under the assumption of (i) we have $v_\infty^T u_\infty = 0$ and the equation $m(\lambda) + \tau p(\lambda) = 0$ can be reduced with (3.1) to

$$\tau^{-1} = \sum_{k=1}^{m'} \frac{(\beta - \alpha\lambda) v_{i_k}^T u_{i_k}}{\lambda - \lambda_{i_k}}. \tag{3.3}$$

On the interval $(\lambda_{i_k}, \lambda_{i_{k+1}})$ the right hand side of (3.3) is a continuous function that maps onto $\mathbb{R}$ in the case $\beta/\alpha \notin (\lambda_{i_k}, \lambda_{i_{k+1}})$ and onto $[0, \infty)$ for $\beta/\alpha \in (\lambda_{i_k}, \lambda_{i_{k+1}})$. Hence we conclude that for every $\tau \in (0, \infty)$ there is a solution $\lambda_{i_k}(\tau) \in (\lambda_{i_k}, \lambda_{i_{k+1}})$ for all $k = 1, \dots, m'-1$. For the interval around ∞ given by $(\lambda_{i_{m'}}, \infty] \cup (-\infty, \lambda_{i_1})$ one can see that the right hand side of (3.3) can be extended continuously to $\lambda = \infty$ which leads to the existence of a solution of the equation (3.3). Under the assumption $\lambda_{i_{m'}} = \infty$ of (ii) one has to solve the equation

$$\tau^{-1} = (-1)^{n-r}(\beta - \alpha\lambda) v_\infty^T u_\infty + \sum_{k=1}^{m'-1} \frac{(\beta - \alpha\lambda) v_{i_k}^T u_{i_k}}{\lambda - \lambda_{i_k}}. \tag{3.4}$$

The arguments for the existence of a solution of (3.4) in $(\lambda_{i_k}, \lambda_{i_{k+1}})$ remain the same for $k = 1, \dots, m'-1$. For $k = m'$ one uses the dual pencil $\mathcal{A}' + \tau\mathcal{P}'$ with the equation $m^\sharp(\lambda) + \tau p^\sharp(\lambda)$, to show the existence of a solution in $(-\infty, \lambda_{i_1})$.

In summary we have shown that there are m' eigenvalues in these disjoint intervals. Furthermore (a) implies that $m' = \deg m$ and Proposition 2.1 shows that there are at most $\deg m$ eigenvalues in $\sigma(\mathcal{A}+\tau\mathcal{P})\setminus\sigma(\mathcal{A})$. Hence each of these eigenvalues $\lambda_{i_k}(\tau)$ is simple when there are not an element of $\sigma(\mathcal{A})$ which proves (b) and we also have $\sigma(\mathcal{A}+\tau\mathcal{P}) \subseteq \mathbb{R}\cup\{\infty\}$ for all $\tau \in \mathbb{R}$ under the assumptions of (b). The proof of (c) can be carried out in the same way as the proof of (b). □

Remark 3.2. The above Theorem 3.1 remains true for $m' \leq 1$. For $m' = 0$ we see from Proposition 2.1 that $\mathrm{am}_{\mathcal{A}+\tau\mathcal{P}}(\lambda) = \mathrm{am}_{\mathcal{A}}(\lambda)$ holds for all $\lambda \in \sigma(\mathcal{A})$ and all $\tau \in \mathbb{C}$ and therefore $\sigma(\mathcal{A}+\tau\mathcal{P}) = \sigma(\mathcal{A})$ for all $\tau \in \mathbb{C}$. For $m' = 1$ one considers the open and connected set $\overline{\mathbb{R}} \setminus \{\lambda_{i_1}\}$ instead of the intervals $(\lambda_{i_k}, \lambda_{i_{k+1}})$.

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