On network redesign using rank-one updates with an application to the Miller amplifier

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***Abstract*—We present an approach to the redesign of electrical networks which is based on successive insertion of capacitances to remove peaks in the transfer function while maintaining the bandwidth. We model this as a rank one perturbation problem for matrix pencils and use the Weierstraß canonical form to obtain the size and position of the inserted capacitances. The method is then applied to an operational amplifier 741.**

***Index Terms*—electrical networks, matrix pencils, rank one perturbation, spectral optimization**

I. Introduction

It is well known that electrical networks can be described as differential-algebraic systems which are of the form

$$\tfrac{\mathrm{d}}{\mathrm{d}t}Ex(t) = Ax(t) + Bu(t), \quad y(t) = Cx(t) \tag{1}$$

with coefficient matrices $E, A \in \mathbb{R}^{n\times n}$, input u, output y. Kirchhoff current and voltage laws imply purely algebraic equations and, hence, the matrix E is not invertible.

Important properties for circuit applications like bandwidth or ringing of the network are encoded in the so called *transfer function* of the system. It maps an input to the corresponding output of the system and is given by

$$i\omega \mapsto C(i\omega E - A)^{-1}B, \quad \omega \in \mathbb{R},$$

where we assume that the above inverse exists.

Quite often good properties of the network are obtained if the poles of the transfer function in a specific sector of the left complex half-plane. This sector is given by

$$S_\varphi := \{z \in \mathbb{C} : -\tfrac{3\pi}{2} \leq \arg z \leq \tfrac{\pi}{2}\}, \quad \varphi \in [0, pi/2], \tag{2}$$

where one typically chooses $\varphi = \frac{\pi}{4}$ or larger. Recall that the argument of a complex number $z = |z|e^{i\phi}$ with $\phi \in [0, 2\pi)$ is defined as $\arg z = \phi$.

Here we address the following question: Given an electrical network with peaks in the transfer function, see Figure 2, can one redesign the network, by including new network elements and move the poles of $(sE-A)^{-1}$ into a given sector (2). Poles of $(sE-A)^{-1}$, are the eigenvalues of the matrix pencil $sE-A$, cf. Section II. We focus on the insertion of capacitances only, but our methods can be extended to the case of resistors or inductances, or even RC-combinations. The main idea is to view the insertion of a single capacitance as a structured rank one perturbation of the matrix pencil and to use recent results [1], [2] for these perturbations.

Previously a similar method was worked out in [3]. But here we present a different selection strategy of the capacitances which is based on the Weierstraß canonical form of the underlying matrix pencils $sE-A$. This new method has some advantages like computation time, explicit error bounds, which will be discussed in detail in Section II.

A related problem was considered recently for matrices, where a desired sector for eigenvalues was prescribed in [4], and for matrix pencils in [5], but the results cannot be applied directly to our problem, since our allowed perturbations have a special structure.

We would like to remark that the strong structural assumptions on the perturbations lead to a difficult problem to decide whether it is possible to place the eigenvalues inside a given sector or not. This is different from the related classic feedback pole-placement problem for linear time-invariant differential-algebraic systems. Another issue is the combinatorial nature of the problem. Since a network with n nodes allows $n(n-1)/2$ possible positions for capacitances, one has to obtain exclusion criteria and criteria to determine relevant positions for capacitances.

II. Matrix pencils and Weierstrass form

Matrix pencils are linear matrix polynomials $sE - A$ with coefficients $E, A \in \mathbb{C}^{n\times n}$. If the matrix pencil $sE - A$ is assumed to be regular, i.e. there exists $\lambda \in \mathbb{C}$ such that $\lambda E - A$ is invertible, which is the case for most electrical networks, then the pencil $sE - A$ can be simplified with a transformation to *Weierstraß canonical form*. This means that

there exist invertible matrices $S, T \in \mathbb{C}^{n\times n}$ and $r \in \mathbb{N}$ such that

$$S(sE - A)T = \begin{bmatrix} sI_r - J & 0 \\ 0 & sN - I_{n-r} \end{bmatrix} \tag{3}$$

where J and N are in Jordan canonical form and N is nilpotent, see e.g. [6], [7]. We assume in the following, that

$$N = 0, \quad \text{and} \quad J = \operatorname{diag}(\lambda_1, \ldots, \lambda_r),\ \lambda_i \neq \lambda_j,\ i \neq j,$$

or in other words, that the DAE has index one and that all finite eigenvalues are geometrically simple, respectively. This is also an assumption satisfied by most electrical networks, see e.g. [8]. The lower diagonal block is usually interpreted as eigenvalues at ∞ and in this sense, we have $n-r$ eigenvalues at ∞. The finite eigenvalues are given as the roots of the characteristic polynomial $\det(sE - A)$ and in particular (3) implies

$$\det(sE - A) = (-1)^{n-r} \det(ST)^{-1} \prod_{i=1}^{r} (s - \lambda_i). \tag{4}$$

The set of eigenvalues of the pencil $sE - A$ is denoted by

$$\sigma(E, A) := \{\lambda \in \mathbb{C} : \det(\lambda E - A) = 0\} \cup \{\infty\}$$

if E is singular and $\sigma(E, A) := \{\lambda \in \mathbb{C} : \det(\lambda E - A) = 0\}$, if E is invertible.

III. Selection of relevant capacitances

Typically $sE - A$ can be obtained with modified nodal analysis, then (a subset of) the indices $1, \ldots, n$ correspond to nodes in the network and the network with the new capacitance of size $c_{ij} > 0$ between the nodes i, j is given by

$$s(E + c_{ij}(e_i - e_j)(e_i - e_j)^\top) - A \tag{5}$$

where e_i is the i-th canonical unit vector in $\mathbb{R}^n$.

A first approach to select capacitances is via the so called *eigenvalue sensitivity* with respect to the vector $e_i - e_j$ as

$$\operatorname{sens}_{ij}(\lambda_k) = |(e_i - e_j)^\top T^\top e_k e_k^\top S(e_i - e_j)|. \tag{6}$$

This is justified by the Taylor expansion at $c = 0$ of the eigenvalue curve $c \mapsto \lambda(c)$. This curve $\lambda(\cdot)$ exists locally by the implicit function theorem and is an eigenvalue of $s(E + c(e_i - e_j)(e_i - e_j)^\top) - A$ given by

$$\lambda_k(c) = \lambda_k - c\lambda_k \operatorname{sens}_{ij}(\lambda_k) + O(c^2).$$

The eigenvalue sensitivities were applied to electrical networks in [3] to select capacitances which were then used in a gradient based approach to move the eigenvalues towards the sector $S_{\frac{\pi}{4}}$.However, the problem with sensitivities is that they only provide local information on the behavior of the eigenvalues and it was observed in [3] that it might happen that eigenvalue curves as functions of c might suddenly change directions. Another problem is connected with the fact that eigenvalues can pass the eigenvalue infinite on the real line and become unstable.

We use a different approach. The perturbation (5) is of rank one, which enables us to apply recent results from [2] and [1] on rank one perturbations.

In [2] it was shown that for an arbitrary set $\mu_1, \ldots, \mu_r$ there exists a rank one perturbation $uv^\top$, $u, v \in \mathbb{R}^n$ of the matrix E such that $\sigma(E + uv^\top, A) = \{\mu_1, \ldots, \mu_r\}$ if $0 \notin \sigma(E, A)$. If $0 \in \sigma(E, A)$ then $0 \in \sigma(E + uv^T, A)$ for all perturbations of and therefore, we have to fix $\mu_1 = 0$ in the desired eigenvalue set. Unfortunately, [2] contains no formula, how to obtain the vectors u and v, but in general they will not be of the form as in (5).

An explicit formula for the vectors u, v was obtained in [1]. For any set $\{\mu_1, \ldots, \mu_r\} \subseteq \mathbb{C}$ the vectors u, v can be computed explicitly as follows. First, decompose $(u_1, \ldots, u_r, u_\infty^\top)^\top = Su$ and $(v_1, \ldots, v_r, v_\infty^T)^T = T^T v$ according to the block structure of the Weierstraß form (3) and then the entries are given by the equations

$$u_i v_i = \frac{\lambda_i - \mu_i}{\mu_i} \prod_{j=1, j\neq i}^{r} \frac{\lambda_j(\lambda_i - \mu_j)}{\mu_j(\lambda_i - \lambda_j)},\ i = 1, \ldots, r \tag{7}$$

$$u_\infty^\top v_\infty = 0. \tag{8}$$

The second equation in (7) means that the $n - r$ eigenvalues at ∞ are invariant under the particular rank one perturbation.

Since for large systems, the computation of the products in (7) might be expensive and has also the disadvantage, that one must prescribe all of the values of new eigenvalues μ_i explicitly, we prescribe only a typically compared to n small fixed number, say $\mu_1, \ldots, \mu_f$ of new eigenvalues which lie outside a given sector and set $\mu_i = \lambda_i$ for $f < i \leq r$ for the remaining eigenvalues. This results in the following simplified system

$$\begin{aligned} &u_i v_i = \frac{\lambda_i - \mu_i}{\mu_i} \prod_{j=1, j\neq i}^{f} \frac{\lambda_j(\lambda_i - \mu_j)}{\mu_j(\lambda_i - \lambda_j)},\ 1 \leq i \leq f, \\ &u_i v_i = 0,\ f < i \leq r, \quad u_\infty^\top v_\infty = 0. \end{aligned} \tag{9}$$

Later in the application, we will have $f = 1$ or $f = 2$ and assign only a small amount of complex pole pairs.

However these results cannot directly be applied to this class of perturbations (5), since we have additional restrictions on the structure of u, v. Therefore we select the capacitances which fulfill (7) with smallest possible error. In the following, we will fix a pair of indices $i, j \in \mathbb{N}$ and consider $b_{ij} = \sqrt{c}(e_i - e_j)$ for some parameter $c > 0$ and vectorize the system (9) with $X = (X_k)_{k=1}^{r+1}$ and $g = (g_k)_{k=1}^{r+1}$ given by

$$X_k := \begin{cases} (e_i - e_j)^\top T^\top e_k e_k^\top S(e_i - e_j), & \text{if } 1 \leq k \leq r, \\ (e_i - e_j)^\top T_\infty^\top S_\infty (e_i - e_j), & \text{if } k = r + 1 \end{cases},$$

$$g_k := \begin{cases} \frac{\lambda_k - \mu_k}{\mu_k} \prod_{j=1, j\neq k}^{f} \frac{\lambda_j(\lambda_k - \mu_j)}{\mu_j(\lambda_k - \lambda_j)} & \text{if } k \leq 1 \leq f, \\ 0, & \text{if } f < k \leq r + 1, \end{cases}$$

with $T_\infty = [e_{r+1}, \ldots, e_n]^\top T$ and $S_\infty := [e_{r+1}, \ldots, e_n]^\top S$. The idea is to choose now the parameter c in such a way that (7) is fulfilled approximately, i.e. $cX \approx g$ with smallest

possible error. Since the right-hand side is fixed and the left-hand side depends on the parameter $c > 0$, the optimal value of c_* which minimizes the approximation error is given by the orthogonal projection formula as

$$c_* = \frac{X^* g}{\|g\|^2} \tag{10}$$

with minimized error

$$\|c_* X - g\| = \min_{c \in \mathbb{C}} \|cX - g\|. \tag{11}$$

Hence we also obtain an explicit value for the capacity c_{ij}, by setting $c_{ij} := \operatorname{Re} c_*$.

Classical eigenvalue bounds from [7, Theorem VI.2.6] in the chordal distance can be used to show that a sufficiently small error in (11) implies that the eigenvalues of the pencils perturbed with $u = v$ and with $e_i - e_j$ lie in the same sector for a suitable choice of the sector. This guarantees two things, first that the eigenvalues will actually move into this sector after inclusion of a capacitance and second that there are no eigenvalues of the unperturbed pencil that move outside the sector after the perturbation.

IV. New Algorithm for network redesign

We propose the following algorithm for redesigning the network by inserting step-by-step capacitances such that the eigenvalues eventually move into a prescribed sector S_{φ_*} for given φ_*. For the sake of simplicity, we restrict to the case that there is only one pair of eigenvalues outside the prescribed sector, i.e. $f = 1$ in (9), but the algorithm can easily be extended to the case $f > 1$.

Algorithm 1: Finding an optimal structured perturbation

1 **Input:** Matrix pencil $sE - A$, desired sector S_{φ_*}, critical sensitivity σ, capacity threshold $c_{\max}$, error bound $\epsilon_{\max}$, stepsize τ, relevant capacitances $\mathcal{S} := \emptyset$;
2 **while** $\varphi < \varphi_*$ **do**
3 Compute $S, T \in \mathbb{C}^{n \times n}$ for Weierstraß form (3) with eigenvalues $\lambda_1 = \overline{\lambda_2} \notin S_{\varphi_*}$;
4 **for** $i, j = 1$ *to* n **do**
5 **if** $\operatorname{sens}_{ij}(\lambda_1) > \sigma$ **then**
6 **for** $l = 1$ *to* K **do**
7 Set $\mu_1 = \tau e^{+i(\frac{\pi}{2}+\varphi)} l = \overline{\mu_2}$ and $\mu_i = \lambda_i$ for $i = 3 \ldots, r$;
8 Compute optimal $c_{ij} = c_*$ via (10);
9 $\mathcal{S} := \mathcal{S} \cup \{(i,j)\}$ if $\operatorname{Re} c_{ij} \le c_{max}$, $\operatorname{Re} c_{ij} > 0$ and $\|c_{ij} X - g\| < \varepsilon_{\max}$;
10 Choose $(i_o, j_o) \in \mathcal{S}$ as a minimizer of $\min_{(i,j)\in\mathcal{S}} \frac{\operatorname{Im} \hat{\lambda}_1}{\operatorname{Re} \hat{\lambda}_1}$ where $\hat{\lambda}_1 = \overline{\hat{\lambda}_2}$ are the eigenvalues of $s(E + c_{ij}(e_i - e_j)(e_i - e_j)^\top) - A$ not in S_{φ_*};
11 Set $E := E + \operatorname{Re} c_{ij}(e_{i_o} - e_{j_o})(e_{i_o} - e_{j_o})^\top$ and increase the sector angle φ

Note that the columns of T in Line 3 of Algorithm 1 are the right eigenvectors of the matrix pencil, therefore one can rely on the built in MATLAB function `eig`(A,E), to find the columns of T. Similarly, the rows of S are given by `eig`(A^*,E^*) for the adjoint pencil, where one has to switch and conjugate the eigenvectors that correspond to a pair of complex eigenvalues. To obtain the diagonal entry I_{n-r} in (3) for the eigenvalue at ∞ one has to apply the singular value decomposition.

We would like to mention that the algorithm can be modified in such a way that we consider the system of equations (9) only for the eigenvalues that are dominant with respect to a given capacitance. Also, the Algorithm is formulated for sectors, but not restricted to this type of subsets of the complex plane.

V. Application to an operational amplifier 741

In this section, we show how Algorithm 1 can be applied to an operational amplifier shown in Figure 1 and discuss the results. The underlying DAE is given by matrices $E, A \in \mathbb{R}^{59\times 59}$. We have 49 eigenvalues between $-4.1 \cdot 10^{11}$ and $-2 \cdot 10^8$ with 7 pairs of complex eigenvalues from which the two pairs

$$\lambda_{1,2} = (-6.982 \pm 82.286i) \cdot 10^7, \lambda_{3,4} = (-0.517 \pm 1.844i) \cdot 10^8$$

lie outside the sector $S_{\frac{\pi}{4}}$. Furthermore, there are 10 eigenvalues at ∞. The network consists of 27 nodes which are available for inserting capacitances in the network. The indices from 28 to 59 correspond to interior nodes inside the transistors or to current and voltage sources.

As an input for the algorithm we use the sector angle $\varphi_* = \frac{\pi}{4}$, the sensitivity $\sigma = 10^{10}$, maximal capacity $c_{\max} = 2.5\,\mathrm{pF}$. The Algorithm 1 moves the eigenvalues in the sector $S_{\frac{\pi}{4}}$ in four Steps shown in Table I with a total capacity of $7.01\,\mathrm{pF}$. The computation time is less then a second for each step. Furthermore, Algorithm 1 recovers the typically used Miller compensation between the nodes $i = 17$ and $j = 18$ in the first step.

TABLE I
The steps of Algorithm 1 until all eigenvalues are contained in $S_{\frac{\pi}{4}}$.

Step	$(\mathbf{i}, \mathbf{j})$	$\mathbf{c_{ij}}$	$\|c_* X - g\|$	$\max_{i=1,2} \frac{\mathbf{Im}(\mu_i)}{\mathbf{Re}(\mu_i)}$	φ
1	$(17, 18)$	2.36 pF	0.575	3.20	$\frac{\pi}{8}$
2	$(3, 19)$	2.07 pF	0.173	1.99	$\frac{\pi}{6}$
3	$(3, 12)$	2.03 pF	10^{-4}	1.20	$\frac{\pi}{4}$
4	$(7, 22)$	0.55 pF	0.058	0.97	$\frac{\pi}{3}$

The input is the voltage V_{id} at node 4 and the output is at node 26. The original transfer function $\omega \mapsto c(i\omega E - A)^{-1} b$ and the transfer functions of the redesigned networks are shown in Figure 2. We also considered the transfer function of the network with standard Miller compensation $C_1 = 7\,\mathrm{pF}$ and $C_1 = 2\,\mathrm{pF}$. It can be seen that the network with $C_1 = 2\,\mathrm{pF}$ still has a small peak and the network with $C_1 = 7\,\mathrm{pF}$ has a

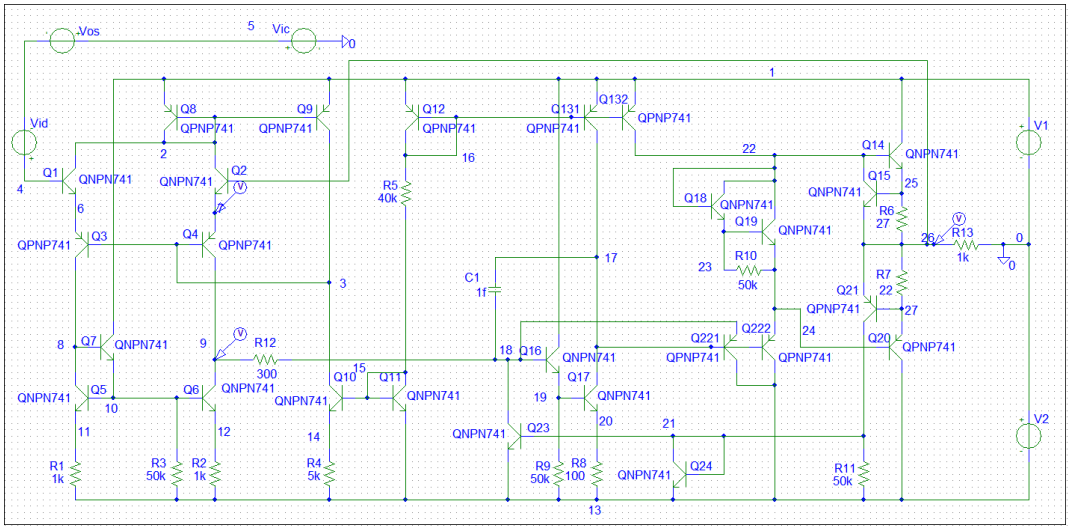

Fig. 1. Circuit diagram of an operational amplifier 741 with the typical Miller precompensation with $C_1 = 1\,\mathrm{fF}$.

significantly smaller bandwith compared to the network from Algorithm 1.

The results in Table II show that the conditions (9) in combination with a sensitivity check, can be used to further reduce the number of relevant capacitances. Here we consider the first step of Algorithm 1 for different values of the sensitivity σ and the error $\varepsilon_{\max}$.

TABLE II
NUMBER OF CONSIDERED CAPACITANCES IN THE FIRST STEP OF ALGORITHM 1 OUT OF POSSIBLE 351.

sensitivity σ	10^8	10^9	10^{10}	10^{11}
# of capacitances	346	302	222	102

$\varepsilon_{\max}$	1	0.5	0.25	0.1
# of capacitances, $\sigma = 10^{10}$	64	54	24	13
# of capacitances, $\sigma = 10^{11}$	40	30	9	7

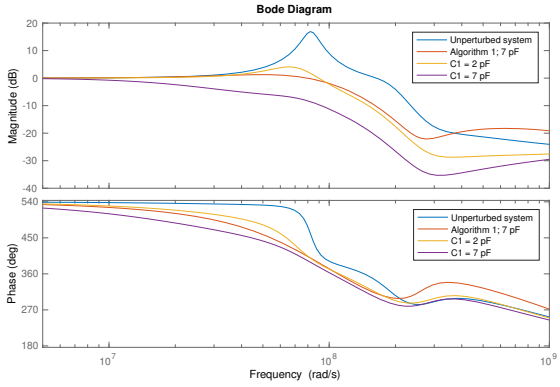

Fig. 2. The Bode diagram of the unperturbed operational amplifier shown in Figure 1, after the application of Algorithm 1 and with the standard Miller compensation with different capacities.

VI. CONCLUSION AND OUTLOOK

We obtained a new condition (7) which allowed us to select capacitances that move the eigenvalues towards a prescribed set. Based on these equations we proposed an algorithm which moves the eigenvalues towards a given sector. This method was applied to improve the transfer function of an operational amplifier. The new capacitances reduces the peaks in the transfer function while keeping the bandwidth with a small total capacity. For an operational amplifier, we have seen that the typically used Miller compensation with the same capacity has a significantly smaller bandwidth.

In comparison with the previous gradient based approach from [3], we have observed several advantages: a lower computation time, no need of a target function for the gradient approach.

Our general approach is for structured perturbations of matrix pencils and therefore not restricted to circuit applications. But also the generalization to other type of network elements, like resistances or inductances is immediate.

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