

# EIGENVALUE PLACEMENT FOR REGULAR MATRIX PENCILS WITH RANK ONE PERTURBATIONS

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**Abstract.** A regular matrix pencil  $sE - A$  and its rank one perturbations are considered. We determine the sets in  $\mathbb{C} \cup \{\infty\}$  which are the eigenvalues of the perturbed pencil. We show that the largest Jordan chains at each eigenvalue of  $sE - A$  may disappear and the sum of the length of all destroyed Jordan chains is the number of eigenvalues (counted with multiplicities) which can be placed arbitrarily in  $\mathbb{C} \cup \{\infty\}$ . We prove sharp upper and lower bounds of the change of the algebraic and geometric multiplicity of an eigenvalue under rank one perturbations. Finally we apply our results to a pole placement problem for a single-input differential algebraic equation with feedback.

**Key words.** regular matrix pencils, rank one perturbations, matrix spectral perturbation theory

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**1. Introduction.** For square matrices  $E$  and  $A$  in  $\mathbb{C}^{n \times n}$  we consider the matrix pencil

$$(1) \quad \mathcal{A}(s) := sE - A$$

and study its set of eigenvalues  $\sigma(\mathcal{A})$ , which is called the *spectrum* of the matrix pencil. Here  $\lambda \in \mathbb{C}$  is said to be an eigenvalue if  $0$  is an eigenvalue of the matrix  $\lambda E - A$  and we say that  $\infty$  is an eigenvalue of  $\mathcal{A}(s)$  if  $E$  is not invertible.

The spectral theory of matrix pencils is a generalization of the eigenvalue problem for matrices [17, 26, 31, 35]. Recently, there is a growing interest in the spectral behavior under low rank perturbations of matrices [2, 12, 27, 30, 33, 34] and of matrix pencils [1, 13, 14, 15, 29]. For matrices it was shown in [12] that under generic rank one perturbations only the largest Jordan chain at each eigenvalue might be destroyed. Here a generic set of perturbations is a subset of  $\mathbb{C}^{n \times n}$  which complement is a proper algebraic submanifold.

We consider only *regular* matrix pencils  $\mathcal{A}(s) = sE - A$  which means that the *characteristic polynomial*  $\det(sE - A)$  is not zero. Otherwise we call  $\mathcal{A}(s)$  *singular*. Jordan chains for regular matrix pencils at  $\lambda$  correspond to Jordan chains of the matrix  $J$  at  $\lambda$ , for  $\lambda \neq \infty$ , or to Jordan chains of  $N$  at  $0$ , for  $\lambda = \infty$ , in the Weierstraß canonical form [16],

$$S(sE - A)T = s \begin{pmatrix} I_r & 0 \\ 0 & N \end{pmatrix} - \begin{pmatrix} J & 0 \\ 0 & I_{n-r} \end{pmatrix}, \quad r \in \{0, 1, \dots, n\}$$

with  $J \in \mathbb{C}^{r \times r}$  and  $N \in \mathbb{C}^{(n-r) \times (n-r)}$  in Jordan canonical form,  $N$  nilpotent and invertible  $S, T \in \mathbb{C}^{n \times n}$ . In [14] it was shown that under generic rank one perturbations only the largest chain at  $\lambda$  is destroyed.

In this paper we investigate the behaviour of the spectrum of a regular matrix pencil  $\mathcal{A}(s)$  under a rank one perturbation  $\mathcal{P}(s) := sF - G$ . As we will see in Section 3, the rank one condition allows us to write  $\mathcal{P}(s)$  in the form

$$(2) \quad \mathcal{P}(s) = (su + v)w^* \quad \text{or} \quad \mathcal{P}(s) = w(su^* + v^*)$$

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with non-zero vector  $w$  and vectors  $u$  and  $v$  such that at least one of the two is not zero. Rank one perturbations of the above form are considered in design problems for electrical circuits where the entries of  $E$  are determined by the capacitances of the circuit. The aim is to improve the frequency behavior by adding additional capacitances between certain nodes. This corresponds, within the model, to a (structured) rank one perturbation of the matrix  $E$ , see [4, 7, 19]. Here we follow a more general approach and obtain the following results:

- (i) We find for unstructured rank one perturbations of the form (2) sharp lower and upper bounds for the dimension of the root subspace  $\mathcal{L}_\lambda(\mathcal{A} + \mathcal{P})$  of the perturbed pencil at  $\lambda$  in terms of the dimension of  $\mathcal{L}_\lambda(\mathcal{A})$ , where  $\mathcal{L}_\lambda(\mathcal{A})$  denotes the subspace of all Jordan chains of the matrix pencil  $\mathcal{A}(s)$  at the eigenvalue  $\lambda$ . More precisely, if  $m_1(\lambda)$  denotes the length of the longest chain of  $\mathcal{A}(s)$  at  $\lambda$  and let  $M(\mathcal{A})$  be the sum of all  $m_1(\lambda)$  over all eigenvalues of  $\mathcal{A}(s)$ , that is,  $M(\mathcal{A}) = \sum_{\lambda \in \sigma(\mathcal{A})} m_1(\lambda)$ . Then the bounds are

$$(3) \quad \dim \mathcal{L}_\lambda(\mathcal{A}) - m_1(\lambda) \leq \dim \mathcal{L}_\lambda(\mathcal{A} + \mathcal{P}) \leq \dim \mathcal{L}_\lambda(\mathcal{A}) + M(\mathcal{A}) - m_1(\lambda).$$

- (ii) We show a statement on the eigenvalue placement for regular matrix pencils which is our main result:  $M(\mathcal{A})$  eigenvalues (counted with multiplicities) can be placed arbitrarily in the complex plane. Either by creating new eigenvalues with one chain only or by adding one new (or extending an old) chain at existing eigenvalues of  $\mathcal{A}(s)$ . Here the term new eigenvalue is understood in the sense that this value is an eigenvalue of  $(\mathcal{A} + \mathcal{P})(s)$  but not of  $\mathcal{A}(s)$ . In addition we obtain the same result for real matrices and real rank one perturbations.

Roughly speaking, the behaviour of the spectrum under rank one perturbations described in (i) and (ii) can be summarized in the following way: At each eigenvalue of  $\mathcal{A}(s)$  the longest chain (or parts of it) may disappear but the remaining chains at that eigenvalue are then Jordan chains of  $(\mathcal{A} + \mathcal{P})(s)$ . Moreover, the sum  $M(\mathcal{A})$  of the length of all the longest chains is then the upper bound for the placement of new chains, either at existing eigenvalues or at new eigenvalues. Those new chains have to satisfy two rules. At new eigenvalues there is only one chain with maximal length  $M(\mathcal{A})$  and at existing eigenvalues at most one new chain may appear, again with maximal length  $M(\mathcal{A})$ . For a precise description of this placement result we refer to Theorem 9 below.

The left hand side of (3) is well-known [1, 14]. In the case of matrices, i.e.  $E = I_n$  in (1) and  $u = 0$  in (2), we refer to [33, 34] and to [3, 20] for operators. Results similar to (i) are known in the literature for generic low-rank perturbations [1, 12, 14, 27, 33]. In the generic case it was shown in [14] that only the largest chain at each eigenvalue is eventually destroyed. In Proposition 7 we show a non generic result that gives a bound on the change of the Jordan chains of length  $k$  for all  $k \in \mathbb{N} \setminus \{0\}$ . Similar bounds were previously obtained for matrices in [33] and for operators in [3].

In special cases the placement problem in (ii) is considered in the literature. For  $E$  positive definite and  $A$  symmetric the placement problem was studied in [15]. In the matrix case, i.e.  $E = I_n$  and  $u = 0$ , the placement problem was solved for symmetric  $A$  in [18]. In [22] a related inverse problem was studied: For two given subsets of the complex plane, two matrices were constructed whose set of eigenvalues equal these sets and the matrices differ by rank one. All these eigenvalue placement settings above are special cases of our result in Section 5 below.

In Section 6 we investigate the eigenvalue placement under parameter restrictions in the perturbation, i.e. in the representation (2) we fix  $u, v \in \mathbb{C}^n$ . This allows us to

derive a sharper bound as in (3). For these restricted placement problems, we obtain simple conditions on the number of eigenvalues that can be assigned arbitrarily.

In the final section we present an application. We consider the pole assignment problem under state feedback for single input differential-algebraic equations. This problem is well studied in the literature [8, 23, 24, 25, 28, 32] even for singular matrix pencils, see [11] and the references therein. However, for single input systems we can view this problem as a parameter restricted rank one perturbation problem from Section 6.

**2. Eigenvalues and Jordan Chains of Matrix Pencils.** In this section the notion of eigenvalues and Jordan chains for matrix pencils  $\mathcal{A}(s) = sE - A$  with  $E, A \in \mathbb{C}^{n \times n}$  is recalled. Furthermore we summarize some basic spectral properties which are implied by the well known Weierstraß canonical form [16].

For fixed  $\lambda \in \mathbb{C}$  observe that  $\mathcal{A}(\lambda)$  is a matrix over  $\mathbb{C}$ . Hence the *spectrum* of the matrix pencil  $\mathcal{A}(s) = sE - A$  is defined as

$$\sigma(\mathcal{A}) := \{\lambda \in \mathbb{C} \mid 0 \text{ is an eigenvalue of } \mathcal{A}(\lambda)\}, \quad \text{if } E \text{ is invertible,}$$

and

$$\sigma(\mathcal{A}) := \{\lambda \in \mathbb{C} \mid 0 \text{ is an eigenvalue of } \mathcal{A}(\lambda)\} \cup \{\infty\}, \quad \text{if } E \text{ is singular.}$$

Obviously the spectrum of a matrix pencil is a subset of the extended complex plane  $\overline{\mathbb{C}} := \mathbb{C} \cup \{\infty\}$  and the roots of the characteristic polynomial  $\det(sE - A)$  are exactly the elements of  $\sigma(\mathcal{A}) \setminus \{\infty\}$ . Hence the spectrum of regular matrix pencils consists of finitely many points. For  $\mathcal{A}(s)$  singular one always has  $\sigma(\mathcal{A}) = \overline{\mathbb{C}}$ .

We recall the notion for Jordan chains and root subspaces [17, Section 1.4], [26, §11.2]). The set  $\{g_0, \dots, g_{m-1}\} \subset \mathbb{C}^n$  is a *Jordan chain of length  $m$*  at  $\lambda \in \mathbb{C}$  if  $g_0 \neq 0$  and

$$(A - \lambda E)g_0 = 0, \quad (A - \lambda E)g_1 = Eg_0, \quad \dots, \quad (A - \lambda E)g_{m-1} = Eg_{m-2}$$

and we call  $\{g_0, \dots, g_{m-1}\} \subset \mathbb{C}^n$  a Jordan chain of length  $m$  at  $\infty$  if

$$g_0 \neq 0, \quad Eg_0 = 0, \quad Eg_1 = Ag_0, \quad \dots, \quad Eg_{m-1} = Ag_{m-2}.$$

Two Jordan chains  $\{g_0, \dots, g_k\}$  and  $\{h_0, \dots, h_l\}$  at  $\lambda \in \overline{\mathbb{C}}$  are called *linearly independent*, if the vectors  $g_0, \dots, g_k, h_0, \dots, h_l$  are linearly independent. Furthermore, we say that  $\mathcal{A}(s)$  has  $k$  Jordan chains of length  $m$  if there exist  $k$  linearly independent Jordan chains of length  $m$  at  $\lambda \in \overline{\mathbb{C}}$ . We denote for  $\lambda \in \overline{\mathbb{C}}$  and  $l \in \mathbb{N} \setminus \{0\}$  the subspace of all elements of all Jordan chains up to the length  $l$  at  $\lambda$  by

$$\mathcal{L}_\lambda^l(\mathcal{A}) := \left\{ g_j \in \mathbb{C}^n \mid 0 \leq j \leq l-1, \{g_0, \dots, g_j\} \text{ is a Jordan chain at } \lambda \right\}$$

and the root subspace which consists of all elements of all Jordan chains at  $\lambda$ ,

$$\mathcal{L}_\lambda(\mathcal{A}) := \bigcup_{l=1}^{\infty} \mathcal{L}_\lambda^l(\mathcal{A}).$$

It is well known that regular pencils  $\mathcal{A}(s) = sE - A$  can be transformed into the Weierstraß canonical form [16, Chapter XII, §2], i.e. there exist invertible matrices  $S, T \in \mathbb{C}^{n \times n}$  and  $r \in \{0, 1, \dots, n\}$  such that

$$(4) \quad S(sE - A)T = s \begin{pmatrix} I_r & 0 \\ 0 & N \end{pmatrix} - \begin{pmatrix} J & 0 \\ 0 & I_{n-r} \end{pmatrix},$$

with  $J \in \mathbb{C}^{r \times r}$  and  $N \in \mathbb{C}^{(n-r) \times (n-r)}$  in Jordan canonical form and  $N$  nilpotent. From the Weierstraß canonical form, we have some well known properties [7, 16].

PROPOSITION 1. *For a regular matrix pencil  $\mathcal{A}(s) = sE - A$  with Weierstraß canonical form (4) the following holds.*

- (a) *A Jordan chain  $\{g_0, \dots, g_{m-1}\}$  of the matrix pencil at  $\lambda \in \mathbb{C}$  of length  $m$  corresponds to a Jordan chain  $\{\pi_r T^{-1} g_0, \dots, \pi_r T^{-1} g_{m-1}\} \subset \mathbb{C}^r$  of  $J$  at  $\lambda$  of length  $m$ . Here  $\pi_r$  denotes the projection of  $x \in \mathbb{C}^n$  onto the first  $r$  entries. Vice versa a Jordan chain  $\{h_0, \dots, h_{m-1}\}$  of  $J$  at  $\lambda$  corresponds to a Jordan chain  $\left\{ T \begin{pmatrix} h_0 \\ 0 \end{pmatrix}, \dots, T \begin{pmatrix} h_{m-1} \\ 0 \end{pmatrix} \right\}$  of the matrix pencil at  $\lambda$ .*
- (b) *A Jordan chain  $\{g_0, \dots, g_{m-1}\}$  of the matrix pencil at  $\infty$  of length  $m$  corresponds to a Jordan chain  $\{\pi_{n-r} T^{-1} g_0, \dots, \pi_{n-r} T^{-1} g_{m-1}\} \subset \mathbb{C}^{n-r}$  of  $N$  at  $0$  of length  $m$ . Here  $\pi_{n-r}$  denotes the projection of  $x \in \mathbb{C}^n$  onto the last  $n-r$  entries. Vice versa a Jordan chain  $\{h_0, \dots, h_{m-1}\}$  of  $N$  at  $0$  corresponds to a Jordan chain  $\left\{ T \begin{pmatrix} 0 \\ h_0 \end{pmatrix}, \dots, T \begin{pmatrix} 0 \\ h_{m-1} \end{pmatrix} \right\}$  of the matrix pencil at  $\infty$ .*
- (c) *A Jordan chain  $\{g_0, \dots, g_{m-1}\}$  of the matrix pencil at  $\infty$  of length  $m$  corresponds to a Jordan chain  $\{h_0, \dots, h_{m-1}\}$  of the dual pencil  $\mathcal{A}'(s) = -sA + E$  at  $0$ .*
- (d) *If  $r \geq 1$  we have  $\sigma(\mathcal{A}) \setminus \{\infty\} = \sigma(J)$  and the characteristic polynomial of  $sE - A$  is divisible by the minimal polynomial  $m_J(s)$  of  $J$  with*

$$(5) \quad \det(sE - A) = (-1)^{n-r} \det(ST)^{-1} m_J(s) q(s),$$

where  $q(s)$  is a monic polynomial of degree  $r - \deg m_J$ . The value  $\lambda \in \sigma(\mathcal{A}) \setminus \{\infty\}$  is a root of  $q(s)$  if and only if  $\dim \ker \mathcal{A}(\lambda) \geq 2$ . Moreover the multiplicity of a root  $\lambda$  of  $\det(sE - A)$  is equal to  $\dim \mathcal{L}_\lambda(\mathcal{A})$  and we have

$$(6) \quad \sum_{\lambda \in \sigma(\mathcal{A})} \dim \mathcal{L}_\lambda(\mathcal{A}) = n.$$

There are  $\dim \ker \mathcal{A}(\lambda)$  linearly independent Jordan chains at  $\lambda$  and, by Proposition 1, this corresponds to the number of linearly independent Jordan chains of  $J$  at  $\lambda \neq \infty$  or  $N$  at  $0$  for  $\lambda = \infty$ . Each of these  $\dim \ker \mathcal{A}(\lambda)$  different Jordan chains has a length which we denote by  $m_j(\lambda)$ ,  $1 \leq j \leq \dim \ker \mathcal{A}(\lambda)$ . These numbers  $m_j(\lambda)$  are not uniquely determined, more precisely, they depend on the chosen Weierstraß canonical form (4) but they are unique up to permutations. In the following, we will choose those numbers in a specific way and we fix this in the following assumption.

ASSUMPTION 2. *Given a regular pencil  $\mathcal{A}(s) = sE - A$  which has Weierstraß canonical form (4) with  $r \in \{0, 1, \dots, n\}$  and matrices  $J \in \mathbb{C}^{r \times r}$  and  $N \in \mathbb{C}^{(n-r) \times (n-r)}$ . Then we assume that for  $\lambda \in \sigma(\mathcal{A})$  the numbers  $m_j(\lambda)$ ,  $1 \leq j \leq \dim \ker \mathcal{A}(\lambda)$ , are sorted in a non-decreasing order*

$$(7) \quad m_1(\lambda) \geq \dots \geq m_{\dim \ker \mathcal{A}(\lambda)}(\lambda).$$

Observe that Assumption 2 is no restriction for regular pencils. This means that for every regular pencil the matrices  $S$  and  $T$  in (4) can be chosen in such a way that the Jordan blocks of  $J$  satisfy the condition (7), see [16]. Therefore with Assumption 2 the minimal polynomial  $m_J(s)$  of  $J$  can be written as

$$m_J(s) = \prod_{\lambda \in \sigma(J)} (s - \lambda)^{m_1(\lambda)}$$

and we introduce

$$(8) \quad m_{\mathcal{A}}(s) := \begin{cases} \prod_{\lambda \in \sigma(J)} (s - \lambda)^{m_1(\lambda)}, & \text{for } r \geq 1, \\ 1, & \text{for } r = 0. \end{cases}$$

Note that with this definition the equation (5) also holds for  $r = 0$  after replacing  $m_J(s)$  by  $m_{\mathcal{A}}(s)$ .

**3. The structure of rank one pencils.** In this section we study pencils of rank one. Recall that the rank of a pencil  $\mathcal{A}(s)$  is the largest  $r \in \mathbb{N}$  such that  $\mathcal{A}(s)$ , viewed as a matrix with polynomial entries, has minors of size  $r$  that are not the zero polynomial [14, 16]. This implies that  $\mathcal{A}(s)$  has rank equal to  $n$  if and only if  $\mathcal{A}(s)$  is regular. Hence, pencils of rank one are not regular for  $n \geq 2$ , meaning that they cannot be transformed to Weierstraß canonical form. Nevertheless, there is a simple representation given in the following proposition.

**PROPOSITION 3.** *The pencil  $\mathcal{P}(s) = sF - G$  with  $F, G \in \mathbb{C}^{n \times n}$  has rank one if and only if there exists  $u, v, w \in \mathbb{C}^n$  with  $w \neq 0$  and ( $u \neq 0$  or  $v \neq 0$ ) such that*

$$(9) \quad \mathcal{P}(s) = (su + v)w^* \quad \text{or} \quad \mathcal{P}(s) = w(su^* + v^*).$$

*If  $F, G$  are real matrices, then  $u, v, w$  can be chosen to have real-valued entries.*

*Proof.* Given that  $\mathcal{P}(s)$  has rank one, then all minors of  $\mathcal{P}(s)$  of size strictly larger than one are the zero polynomial. Since

$$(10) \quad \text{rank}(\mathcal{P}(\lambda)) = \text{rank}(\lambda F - G) \leq 1 \quad \text{for all } \lambda \in \mathbb{C},$$

for  $\lambda = 0$  we have  $\text{rank}(G) \leq 1$ . Then there exist  $u, v \in \mathbb{C}^n$  with  $G = uv^*$ . For  $\lambda = 1$  we have  $\text{rank}(F - G) \leq 1$ , so there exists  $w, z \in \mathbb{C}^n$  with  $F - G = wz^*$ . Using the representations above we see

$$2F - G = 2(F - G) + G = 2wz^* + uv^*.$$

From (10), for  $\lambda = 2$  we have that  $\text{rank}(2F - G) \leq 1$ . If  $u$  and  $w$  are linearly independent then  $z = \alpha v$  or  $v = \alpha z$  for some  $\alpha \in \mathbb{C}$ . Let  $z = \alpha v$  (the case  $v = \alpha z$  can be proven similarly). Then

$$sF - G = s(uv^* + wz^*) - uv^* = (s(u + \alpha w) - u)v^*,$$

therefore  $\mathcal{P}(s)$  admits a representation as in (9). Now, assume that  $u$  and  $w$  are linearly dependent. Let  $u = \beta w$  for some  $\beta \in \mathbb{C}$  (the case  $w = \beta u$  can be proven similarly), then

$$sF - G = s(uv^* + wz^*) - uv^* = w(s(\beta v^* + z^*) - \beta v^*)$$

holds, hence (9) is proven. The converse statement is obvious. For  $F, G \in \mathbb{R}^{n \times n}$  the arguments above remain valid after replacing  $\mathbb{C}$  by  $\mathbb{R}$  and  $u^*, v^*, z^*$  and  $w^*$  by  $u^T, v^T, z^T$  and  $w^T$ .  $\square$

The following example illustrates that both representations in (9) are necessary.

**EXAMPLE 4.** *A short computation shows that the matrix pencils*

$$\begin{aligned} \mathcal{P}_1(s) &:= \begin{pmatrix} s+1 & s+1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \end{pmatrix} (1, 1), \\ \mathcal{P}_2(s) &:= \begin{pmatrix} s+1 & 1 \\ s+1 & 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} (s(1, 0) + (1, 1)) \end{aligned}$$

*admit only one of the representations given in Proposition 3.*

If in (9) the elements  $u, v \in \mathbb{C}^n$  are linearly dependent both representations in (9) coincide and without restriction we can write for non-zero  $(\alpha, \beta) \in \mathbb{C}^2$

$$(11) \quad \mathcal{P}(s) = (\alpha s - \beta)uw^*.$$

The next lemma provides a simple criterion for  $(\mathcal{A} + \mathcal{P})(s)$  to be regular when  $\mathcal{P}(s)$  is of the form (11).

LEMMA 5. *Let  $\mathcal{A}(s) = sE - A$  be regular. Choose  $(\alpha, \beta) \in \mathbb{C}^2$  non-zero and let  $\mathcal{P}(s)$  be given by (11). Then the following holds.*

- (a) *Assume  $\alpha \neq 0$ . If  $\beta/\alpha \in \sigma(\mathcal{A})$  then  $\beta/\alpha \in \sigma(\mathcal{A} + \mathcal{P})$ . If  $\beta/\alpha \notin \sigma(\mathcal{A})$  then  $(\mathcal{A} + \mathcal{P})(s)$  is regular.*
- (b) *Assume  $\alpha = 0$ . If  $\infty \in \sigma(\mathcal{A})$  then  $\infty \in \sigma(\mathcal{A} + \mathcal{P})$ . If  $\infty \notin \sigma(\mathcal{A})$  then  $(\mathcal{A} + \mathcal{P})(s)$  is regular.*

*Proof.* For  $\alpha \neq 0$  we have

$$\det(\mathcal{A} + \mathcal{P})\left(\frac{\beta}{\alpha}\right) = \det\left(\frac{\beta}{\alpha}E - A + (\alpha\frac{\beta}{\alpha} - \beta)uw^*\right) = \det\left(\frac{\beta}{\alpha}E - A\right) = \det\mathcal{A}\left(\frac{\beta}{\alpha}\right)$$

and (a) follows. For  $\alpha = 0$  we use that  $\infty \in \sigma(\mathcal{A})$  if and only if the leading coefficient  $E$  is singular. Since  $\alpha = 0$ , the leading coefficient of  $(\mathcal{A} + \mathcal{P})(s)$  is  $E$  for all  $u, w \in \mathbb{C}^n$ , hence (b) is proved.  $\square$

**4. Change of the root subspaces under rank one perturbations.** In this section, we obtain bounds on the number of eigenvalues which can be changed by a rank one perturbation. The following lemma is a special case ( $r = 1$ ) of [14, Lemma 2.1].

LEMMA 6. *Let  $\mathcal{A}(s)$  be a regular matrix pencil satisfying Assumption 2 and let  $\mathcal{P}(s)$  be of rank one. Assume that  $(\mathcal{A} + \mathcal{P})(s)$  is regular and let  $\lambda \in \overline{\mathbb{C}}$  be an eigenvalue of  $\mathcal{A}(s)$ . Then  $(\mathcal{A} + \mathcal{P})(s)$  has at least  $\dim \ker \mathcal{A}(\lambda) - 1$  linearly independent Jordan chains at  $\lambda$ . For  $\dim \ker \mathcal{A}(\lambda) \geq 2$  these chains can be sorted in such a way that for the length of the  $i$ th chain  $\tilde{m}_i(\lambda)$  the following holds*

$$\tilde{m}_2(\lambda) \geq \dots \geq \tilde{m}_{\dim \ker \mathcal{A}(\lambda)}(\lambda) \quad \text{and} \quad \tilde{m}_i(\lambda) \geq m_i(\lambda), \quad 2 \leq i \leq \dim \ker \mathcal{A}(\lambda).$$

The following result describes the maximal change of the root subspace dimension under rank one perturbations. For matrices, that is, if  $E = I_n$ , this result was obtained in [33], see also [3, 37].

PROPOSITION 7. *Let  $\mathcal{A}(s)$  be a regular matrix pencil satisfying Assumption 2, then for any rank one pencil  $\mathcal{P}(s)$  such that  $(\mathcal{A} + \mathcal{P})(s)$  is regular we have for all  $\lambda \in \overline{\mathbb{C}}$  and  $k \in \mathbb{N} \setminus \{0\}$*

$$(12) \quad \left| \dim \frac{\mathcal{L}_\lambda^{k+1}(\mathcal{A} + \mathcal{P})}{\mathcal{L}_\lambda^k(\mathcal{A} + \mathcal{P})} - \dim \frac{\mathcal{L}_\lambda^{k+1}(\mathcal{A})}{\mathcal{L}_\lambda^k(\mathcal{A})} \right| \leq 1,$$

$$(13) \quad |\dim \mathcal{L}_\lambda^k(\mathcal{A} + \mathcal{P}) - \dim \mathcal{L}_\lambda^k(\mathcal{A})| \leq k.$$

*Proof.* We prove the inequality (12). Assume  $\lambda \neq \infty$  and that for  $k, l \in \mathbb{N} \setminus \{0\}$  we have

$$(14) \quad \dim \frac{\mathcal{L}_\lambda^{k+1}(\mathcal{A})}{\mathcal{L}_\lambda^k(\mathcal{A})} = l \geq 2.$$

This is equivalent to the fact  $\mathcal{A}(s)$  has  $l$  linearly independent Jordan chains at  $\lambda$  with length at least  $k + 1$  which means

$$m_1(\lambda) \geq m_2(\lambda) \dots \geq m_l(\lambda) \geq k + 1.$$

It follows from Lemma 6 that  $(\mathcal{A} + \mathcal{P})(s)$  has at least  $l - 1$  linearly independent Jordan chains with lengths

$$\tilde{m}_2(\lambda) \geq \dots \geq \tilde{m}_l(\lambda) \geq m_l(\lambda) \geq k + 1$$

which leads to

$$(15) \quad \dim \frac{\mathcal{L}_\lambda^{k+1}(\mathcal{A} + \mathcal{P})}{\mathcal{L}_\lambda^k(\mathcal{A} + \mathcal{P})} \geq l - 1.$$

It remains to show that the expression in (15) is less or equal to  $l + 1$ . Indeed, assume

$$\dim \frac{\mathcal{L}_\lambda^{k+1}(\mathcal{A} + \mathcal{P})}{\mathcal{L}_\lambda^k(\mathcal{A} + \mathcal{P})} \geq l + 2.$$

If we consider the regular pencil  $(\mathcal{A} + \mathcal{P})(s)$  and the rank one pencil  $-\mathcal{P}(s)$  and apply the above arguments, we have that

$$\dim \frac{\mathcal{L}_\lambda^{k+1}(\mathcal{A})}{\mathcal{L}_\lambda^k(\mathcal{A})} \geq l + 1,$$

which is a contradiction to (14). Hence, (12) is shown for  $l \geq 2$ . If  $l = 1$ , i.e.

$$\dim \frac{\mathcal{L}_\lambda^{k+1}(\mathcal{A})}{\mathcal{L}_\lambda^k(\mathcal{A})} = 1$$

and assume that  $\dim \frac{\mathcal{L}_\lambda^{k+1}(\mathcal{A} + \mathcal{P})}{\mathcal{L}_\lambda^k(\mathcal{A} + \mathcal{P})} \geq 3$ . Lemma 6 applied to the regular matrix pencil  $(\mathcal{A} + \mathcal{P})(s)$  shows  $\dim \frac{\mathcal{L}_\lambda^{k+1}(\mathcal{A})}{\mathcal{L}_\lambda^k(\mathcal{A})} \geq 2$ , a contradiction and (12) follows.

Now we show (13). For  $k = 1$  the definition of  $\mathcal{L}_\lambda^i(\mathcal{A})$  implies  $\mathcal{L}_\lambda^1(\mathcal{A}) = \ker \mathcal{A}(\lambda)$ . Since  $\mathcal{A}(\lambda)$  and  $(\mathcal{A} + \mathcal{P})(\lambda)$  are matrices and  $\mathcal{P}(\lambda)$  is a matrix of rank at most one (see Proposition 3), the estimates

$$\begin{aligned} \text{rank}(\mathcal{A}(\lambda)) &= \text{rank}((\mathcal{A} + \mathcal{P})(\lambda) - \mathcal{P}(\lambda)) \leq \text{rank}((\mathcal{A} + \mathcal{P})(\lambda)) + \text{rank}(\mathcal{P}(\lambda)), \\ \text{rank}((\mathcal{A} + \mathcal{P})(\lambda)) &\leq \text{rank}(\mathcal{A}(\lambda)) + 1 \end{aligned}$$

imply  $|\text{rank}((\mathcal{A} + \mathcal{P})(\lambda)) - \text{rank}(\mathcal{A}(\lambda))| \leq 1$  and together with the dimension formula  $\dim \ker \mathcal{A}(\lambda) + \text{rank}(\mathcal{A}(\lambda)) = n$  this leads to

$$(16) \quad \begin{aligned} &|\dim \ker \mathcal{A}(\lambda) - \dim \ker(\mathcal{A} + \mathcal{P})(\lambda)| \\ &= |n - \text{rank}(\mathcal{A}(\lambda)) - (n - \text{rank}((\mathcal{A} + \mathcal{P})(\lambda)))| \leq 1. \end{aligned}$$

Therefore (13) holds for  $k = 1$ . For  $k \geq 2$  we have the identity

$$\dim \mathcal{L}_\lambda^k(\mathcal{A}) = \dim \ker \mathcal{A}(\lambda) + \sum_{m=1}^{k-1} \dim \frac{\mathcal{L}_\lambda^{m+1}(\mathcal{A})}{\mathcal{L}_\lambda^m(\mathcal{A})}$$

which leads to

$$\begin{aligned} & |\dim \mathcal{L}_\lambda^k(\mathcal{A}) - \dim \mathcal{L}_\lambda^k(\mathcal{A} + \mathcal{P})| \\ & \leq |\dim \ker \mathcal{A}(\lambda) - \dim \ker(\mathcal{A} + \mathcal{P})(\lambda)| + \sum_{m=1}^{k-1} \left| \dim \frac{\mathcal{L}_\lambda^{m+1}(\mathcal{A})}{\mathcal{L}_\lambda^m(\mathcal{A})} - \dim \frac{\mathcal{L}_\lambda^{m+1}(\mathcal{A} + \mathcal{P})}{\mathcal{L}_\lambda^m(\mathcal{A} + \mathcal{P})} \right| \end{aligned}$$

and (16) together with (12) imply (13).

For  $\lambda = \infty$  we consider the dual pencil  $\mathcal{A}'(s) = -As + E$  at  $\lambda = 0$ . Obviously  $\mathcal{A}'(s)$  and the dual pencil  $(\mathcal{A} + \mathcal{P})'(s)$  of  $(\mathcal{A} + \mathcal{P})(s)$  are regular. By Proposition 1 (c), it remains to apply (12) and (13) for  $\lambda = 0$  to  $\mathcal{A}'(s)$  and  $(\mathcal{A} + \mathcal{P})'(s)$  to see that (12) and (13) hold for  $\mathcal{A}(s)$  and  $(\mathcal{A} + \mathcal{P})(s)$  at  $\lambda = \infty$ .  $\square$

For  $k = 1$  the inequality (13) leads to the following statement.

COROLLARY 8. *Let  $\mathcal{A}(s)$  be a regular matrix pencil, then for any rank one pencil  $\mathcal{P}(s)$  such that  $(\mathcal{A} + \mathcal{P})(s)$  is regular we have*

$$\left\{ \lambda \in \sigma(\mathcal{A}) \mid \dim \ker \mathcal{A}(\lambda) \geq 2 \right\} \subseteq \sigma(\mathcal{A} + \mathcal{P})$$

and for every  $\mu \in \sigma(\mathcal{A} + \mathcal{P}) \setminus \sigma(\mathcal{A})$

$$\dim \ker(\mathcal{A} + \mathcal{P})(\mu) = 1,$$

i.e., in this case, there is only one Jordan chain of length  $\dim \mathcal{L}_\mu(\mathcal{A} + \mathcal{P})$ .

Proposition 7 states, roughly speaking, that the largest possible change in the dimensions of  $\mathcal{L}_\lambda(\mathcal{A} + \mathcal{P})$  compared with  $\mathcal{L}_\lambda(\mathcal{A})$  is bounded by the length of the largest Jordan chain of  $\mathcal{A}(s)$  and  $(\mathcal{A} + \mathcal{P})(s)$ . However, the aim of the following Theorem 9 is to give bounds for the change of dimension of  $\mathcal{L}_\lambda(\mathcal{A} + \mathcal{P})$  only in terms of the unperturbed pencil  $\mathcal{A}(s)$ . For this, we use the number  $m_1(\lambda)$  which is according to Assumption 2 the length of the largest Jordan chain of  $\mathcal{A}(s)$  at  $\lambda$  and the number

$$M(\mathcal{A}) := \sum_{\mu \in \sigma(\mathcal{A})} m_1(\mu).$$

THEOREM 9. *Let  $\mathcal{A}(s)$  be a regular matrix pencil satisfying Assumption 2. Then for any rank one pencil  $\mathcal{P}(s)$  such that  $(\mathcal{A} + \mathcal{P})(s)$  is regular we have for  $\lambda \in \sigma(\mathcal{A})$*

$$(17) \quad \dim \mathcal{L}_\lambda(\mathcal{A}) - m_1(\lambda) \leq \dim \mathcal{L}_\lambda(\mathcal{A} + \mathcal{P}) \leq \dim \mathcal{L}_\lambda(\mathcal{A}) + M(\mathcal{A}) - m_1(\lambda),$$

whereas the change in the dimension for  $\lambda \in \overline{\mathbb{C}} \setminus \sigma(\mathcal{A})$  is bounded by

$$(18) \quad 0 \leq \dim \mathcal{L}_\lambda(\mathcal{A} + \mathcal{P}) \leq M(\mathcal{A}).$$

Summing up, we obtain the following bounds

$$(19) \quad \begin{aligned} & \sum_{\lambda \in \sigma(\mathcal{A})} \dim \mathcal{L}_\lambda(\mathcal{A} + \mathcal{P}) \geq n - M(\mathcal{A}), \\ & \sum_{\lambda \in \sigma(\mathcal{A} + \mathcal{P}) \setminus \sigma(\mathcal{A})} \dim \mathcal{L}_\lambda(\mathcal{A} + \mathcal{P}) \leq M(\mathcal{A}). \end{aligned}$$



*Proof.* By Assumption 2, we have  $\mathcal{L}_\lambda(\mathcal{A}) = \mathcal{L}_\lambda^{m_1(\lambda)}(\mathcal{A})$ . Then (13) implies for  $\lambda \in \sigma(\mathcal{A})$

$$(20) \quad \begin{aligned} \dim \mathcal{L}_\lambda(\mathcal{A}) - m_1(\lambda) &= \dim \mathcal{L}_\lambda^{m_1(\lambda)}(\mathcal{A}) - m_1(\lambda) \\ &\leq \dim \mathcal{L}_\lambda^{m_1(\lambda)}(\mathcal{A} + \mathcal{P}) \leq \dim \mathcal{L}_\lambda(\mathcal{A} + \mathcal{P}). \end{aligned}$$

This is the lower bound in (17). Since  $(\mathcal{A} + \mathcal{P})(s)$  is regular we can apply (6), (20) and the upper bound for  $\lambda \in \sigma(\mathcal{A})$  follows from

$$\begin{aligned} \dim \mathcal{L}_\lambda(\mathcal{A} + \mathcal{P}) &= n - \sum_{\mu \in \sigma(\mathcal{A} + \mathcal{P}) \setminus \{\lambda\}} \dim \mathcal{L}_\mu(\mathcal{A} + \mathcal{P}) \\ &\leq \sum_{\mu \in \sigma(\mathcal{A})} \dim \mathcal{L}_\mu(\mathcal{A}) - \sum_{\mu \in \sigma(\mathcal{A}) \setminus \{\lambda\}} \dim \mathcal{L}_\mu(\mathcal{A} + \mathcal{P}) \\ &= \dim \mathcal{L}_\lambda(\mathcal{A}) + \sum_{\mu \in \sigma(\mathcal{A}) \setminus \{\lambda\}} \dim \mathcal{L}_\mu(\mathcal{A}) - \sum_{\mu \in \sigma(\mathcal{A}) \setminus \{\lambda\}} \dim \mathcal{L}_\mu(\mathcal{A} + \mathcal{P}) \\ &\leq \dim \mathcal{L}_\lambda(\mathcal{A}) + \sum_{\mu \in \sigma(\mathcal{A}) \setminus \{\lambda\}} m_1(\mu) = \dim \mathcal{L}_\lambda(\mathcal{A}) + M(\mathcal{A}) - m_1(\lambda). \end{aligned}$$

Hence (17) is proved and applying the same estimates for  $\lambda \in \overline{\mathbb{C}} \setminus \sigma(\mathcal{A})$  proves (18). We continue with the proof of (19). Relation (20) implies

$$\sum_{\lambda \in \sigma(\mathcal{A})} \dim \mathcal{L}_\lambda(\mathcal{A} + \mathcal{P}) \geq \sum_{\lambda \in \sigma(\mathcal{A})} (\dim \mathcal{L}_\lambda(\mathcal{A}) - m_1(\lambda)) = n - \sum_{\lambda \in \sigma(\mathcal{A})} m_1(\lambda) = n - M(\mathcal{A})$$

and this yields

$$\sum_{\lambda \in \sigma(\mathcal{A} + \mathcal{P}) \setminus \sigma(\mathcal{A})} \dim \mathcal{L}_\lambda(\mathcal{A} + \mathcal{P}) = n - \sum_{\lambda \in \sigma(\mathcal{A})} \dim \mathcal{L}_\lambda(\mathcal{A} + \mathcal{P}) \leq M(\mathcal{A}). \quad \square$$

From the inequality (19) we see that the number of changeable eigenvalues under a rank one perturbation is bounded by  $M(\mathcal{A})$ .

**5. Eigenvalue placement with rank one perturbations.** In this section we study which sets of eigenvalues can be obtained by rank one perturbations. The following theorem is the main result. It states that for a given set of complex numbers there exists a rank one perturbation  $\mathcal{P}(s)$  such that the set is included in  $\sigma(\mathcal{A} + \mathcal{P})$ , provided the given set has not more than  $M(\mathcal{A})$  elements.

**THEOREM 10.** *Let  $\mathcal{A}(s)$  be a regular matrix pencil satisfying Assumption 2 and choose pairwise distinct numbers  $\mu_1, \dots, \mu_l \in \overline{\mathbb{C}}$  with  $l \leq M(\mathcal{A})$ . Choose multiplicities  $m_1, \dots, m_l \in \mathbb{N} \setminus \{0\}$  with  $\sum_{i=1}^l m_i = M(\mathcal{A})$ . Then the following statements hold true.*

- (a) *There exists a rank one pencil  $\mathcal{P}(s) = (\alpha s - \beta)uv^*$  with  $\alpha \in \mathbb{C} \setminus \{0\}$ ,  $\beta \in \mathbb{C}$  and  $u, v \in \mathbb{C}^n$  such that  $(\mathcal{A} + \mathcal{P})(s)$  is regular, and*

$$(21) \quad \sigma(\mathcal{A} + \mathcal{P}) = \{\mu_1, \dots, \mu_l\} \cup \{\lambda \in \sigma(\mathcal{A}) \mid \dim \ker \mathcal{A}(\lambda) \geq 2\}$$

with multiplicities

$$(22) \quad \dim \mathcal{L}_\lambda(\mathcal{A} + \mathcal{P}) = \begin{cases} \dim \mathcal{L}_\lambda(\mathcal{A}) - m_1(\lambda) + m_i, & \text{for } \lambda = \mu_i \in \sigma(\mathcal{A}), \\ \dim \mathcal{L}_\lambda(\mathcal{A}) - m_1(\lambda), & \text{for } \lambda \in \sigma(\mathcal{A}) \setminus \{\mu_1, \dots, \mu_l\}, \\ m_i, & \text{for } \lambda = \mu_i \notin \sigma(\mathcal{A}), \\ 0, & \text{for } \lambda \notin \sigma(\mathcal{A}) \cup \{\mu_1, \dots, \mu_l\}. \end{cases}$$

- (b) If  $E, A$  are real matrices and  $\{\mu_1, \dots, \mu_l\}$  is symmetric with respect to the real line with  $m_i = m_j$  if  $\mu_j = \bar{\mu}_i$  and all  $i, j = 1, \dots, l$ , there exists  $\alpha, \beta \in \mathbb{R}$  and  $u, v \in \mathbb{R}^n$  such that  $\mathcal{P}(s) = (\alpha s - \beta)uv^T$  satisfies (21) and (22).

We formulate a special case of Theorem 10.

COROLLARY 11. In addition to the assumptions of Theorem 10, assume that  $\dim \ker \mathcal{A}(\lambda) = 1$  for all  $\lambda \in \sigma(\mathcal{A})$ . Hence  $m_1(\lambda) = \dim \mathcal{L}_\lambda(\mathcal{A})$  holds for all  $\lambda \in \sigma(\mathcal{A})$  and  $M(\mathcal{A}) = n$ . Then there exists a rank one pencil  $\mathcal{P}(s) = (\alpha s - \beta)w^*$  such that  $(\mathcal{A} + \mathcal{P})(s)$  is regular and the equations (21) and (22) take the following form

$$\sigma(\mathcal{A} + \mathcal{P}) = \{\mu_1, \dots, \mu_l\} \quad \text{and} \quad \dim \mathcal{L}_\lambda(\mathcal{A} + \mathcal{P}) = \begin{cases} m_i, & \text{for } \lambda = \mu_i, \\ 0, & \text{for } \lambda \notin \{\mu_1, \dots, \mu_l\}. \end{cases}$$

Therefore, for each  $\mu_i \in \sigma(\mathcal{A} + \mathcal{P})$  there is only one Jordan chain of  $(\mathcal{A} + \mathcal{P})(s)$  of length  $m_i$ .

Combining Theorem 9, Theorem 10 and Corollary 11 we get the following result which solves an inverse problem which was investigated for matrices in [22].

THEOREM 12. Given pairwise distinct numbers  $\lambda_1, \dots, \lambda_k \in \bar{\mathbb{C}}$  and  $\mu_1, \dots, \mu_l \in \bar{\mathbb{C}}$  with  $k \leq n$ ,  $l \leq n$  and multiplicities  $m(\lambda_1), \dots, m(\lambda_k)$ ,  $m(\mu_1), \dots, m(\mu_l) \in \mathbb{N} \setminus \{0\}$  such that

$$\sum_{i=1}^k m(\lambda_i) = \sum_{i=1}^l m(\mu_i) = n.$$

Then, there exists a regular matrix pencil  $\mathcal{A}(s) \in \mathbb{C}[s]^{n \times n}$  and a rank one pencil  $\mathcal{P}(s) \in \mathbb{C}[s]^{n \times n}$  such that

$$\begin{aligned} \sigma(\mathcal{A}) &= \{\lambda_1, \dots, \lambda_k\}, & \dim \mathcal{L}_{\lambda_i}(\mathcal{A}) &= m(\lambda_i), \quad i = 1, \dots, k, \\ \sigma(\mathcal{A} + \mathcal{P}) &= \{\mu_1, \dots, \mu_l\}, & \dim \mathcal{L}_{\mu_i}(\mathcal{A} + \mathcal{P}) &= m(\mu_i), \quad i = 1, \dots, l. \end{aligned}$$

LEMMA 13. Let  $\mathcal{A}(s)$  be a regular matrix pencil satisfying Assumption 2. Then for the polynomial  $m_{\mathcal{A}}(s)$  defined in (8) and for every  $p(s) \in \mathbb{C}[s]$  with

$$\deg p \leq M(\mathcal{A}) - 1$$

there exist  $u, v \in \mathbb{C}^n$  such that

$$(23) \quad p(s) = v^* m_{\mathcal{A}}(s)(sE - A)^{-1} u.$$

If  $E$  and  $A$  are real matrices and  $p(s) \in \mathbb{R}[s]$ , then there exists  $u, v \in \mathbb{R}^n$  satisfying (23).

*Proof.* We introduce

$$\Theta_{\mathcal{A}} : \mathbb{C}^n \times \mathbb{C}^n \rightarrow \{p(s) \in \mathbb{C}[s] \mid \deg p \leq M(\mathcal{A}) - 1\}, \quad (u, v) \mapsto v^* m_{\mathcal{A}}(s)(sE - A)^{-1}u$$

and show the surjectivity of this map. Since the surjectivity of  $\Theta_{\mathcal{A}}$  is invariant under equivalence transformations of the form of  $sE - A$  to  $S(sE - A)T$  with invertible  $S, T \in \mathbb{C}^{n \times n}$ , we can assume that  $\mathcal{A}(s)$  is given in Weierstraß canonical form (4) with matrices  $J$  and  $N$ . If  $\sigma(J) = \{\lambda_1, \dots, \lambda_m\}$  for some complex numbers  $\lambda_1, \dots, \lambda_m$ , then  $J$  and  $N$  are given by

$$(24) \quad J = \bigoplus_{i=1}^m \bigoplus_{j=1}^{\dim \ker \mathcal{A}(\lambda_i)} J_{m_j(\lambda_i)}(\lambda_i), \quad N = \bigoplus_{j=1}^{\dim \ker \mathcal{A}(\infty)} J_{m_j(\infty)}(0)$$

with Jordan blocks  $J_k(\lambda)$  of size  $k$  at  $\lambda \in \mathbb{C}$  given by

$$J_k(\lambda) = \begin{pmatrix} \lambda & 1 & & \\ & \cdot & \cdot & \\ & & \cdot & \cdot \\ & & & \cdot & 1 \\ & & & & \lambda \end{pmatrix} \in \mathbb{C}^{k \times k}.$$

This allows us to simplify the resolvent representation with  $u = (u_0^*, u_1^*)^*$ ,  $v = (v_0^*, v_1^*)^*$ ,  $u_0, v_0 \in \mathbb{C}^r$  and  $u_1, v_1 \in \mathbb{C}^{n-r}$  to

$$(25) \quad \begin{aligned} & v^* m_{\mathcal{A}}(s)(sE - A)^{-1}u = v_0^* m_{\mathcal{A}}(s)(sI_r - J)^{-1}u_0 + v_1^* m_{\mathcal{A}}(s)(sN - I_{n-r})^{-1}u_1 \\ & = v_0^* m_{\mathcal{A}}(s) \bigoplus_{\substack{i=1, \dots, m, \\ j=1, \dots, \dim \ker \mathcal{A}(\lambda_i)}} \begin{pmatrix} (s - \lambda_i)^{-1} & \dots & (-1)^{-m_j(\lambda_i)+1} (s - \lambda_i)^{-m_j(\lambda_i)} \\ & \ddots & \vdots \\ & & (s - \lambda_i)^{-1} \end{pmatrix} u_0 \\ & + v_1^* m_{\mathcal{A}}(s) \bigoplus_{j=1, \dots, \dim \ker \mathcal{A}(\infty)} \begin{pmatrix} -1 & -s & \dots & -s^{m_j(\infty)-1} \\ & -1 & \dots & -s^{m_j(\infty)-2} \\ & & \ddots & \vdots \\ & & & -1 \end{pmatrix} u_1. \end{aligned}$$

Observe  $M(\mathcal{A}) = \deg m_{\mathcal{A}} + m_1(\infty)$ . From (7) and (8) we see that  $\Theta_{\mathcal{A}}$  maps into the set

$$\{p(s) \in \mathbb{C}[s] \mid \deg p \leq M(\mathcal{A}) - 1\}.$$

Obviously, the right hand side of (25) consists of the sum of products involving two block matrices. Consider the first summand, then the entries of the first row of the blocks of the block matrix for  $j = 1$  and  $i = 1, \dots, m$  are linearly independent as they are functions with a pole in  $\lambda_i$  of order from one up to  $m_1(\lambda_i)$ . Choosing suitable contours and applying the residue theorem, one sees that

$$\{(s - \lambda_i)^{-r} \mid i = 1, \dots, m, r = 1, \dots, m_1(\lambda_i)\}.$$

is also linearly independent. After multiplication with  $m_{\mathcal{A}}(s)$  this set of functions remains linearly independent. Therefore the set

$$P_1 := \{m_{\mathcal{A}}(s)(s - \lambda_i)^{-r} \mid i = 1, \dots, m, r = 1, \dots, m_1(\lambda_i)\}$$

is linearly independent and it contains  $\sum_{i=1}^m m_1(\lambda_i) = \deg m_{\mathcal{A}}$  elements, each of degree less or equal to  $\deg m_{\mathcal{A}} - 1$ . Moreover, for  $j = 1$ , the entries of the first row of the blocks in the block matrix of the second summand on the right hand side of (25) are linearly independent and form the linearly independent set of polynomials

$$P_2 := \{m_{\mathcal{A}}(s)s^r \mid r = 0, \dots, m_1(\infty) - 1\}$$

which contains  $m_1(\infty)$  elements of degree between  $\deg m_{\mathcal{A}}$  and  $\deg m_{\mathcal{A}} + m_1(\infty) - 1 = M(\mathcal{A}) - 1$ . Hence  $P_1 \cup P_2$  consists of

$$(26) \quad \deg m_{\mathcal{A}} + m_1(\infty) = M(\mathcal{A})$$

linearly independent elements. Furthermore, one can choose certain entries of  $v = (v_0^*, v_1^*)$  as one and all others as zero such that the multiplication with  $v_0^*$  and  $v_1^*$  in (25) picks exactly the first row of each block with  $j = 1$  for all  $i = 1, \dots, m$ . By choosing one entry of  $u$  as one and all others as zero we see

$$P_1 \cup P_2 \subset \text{ran } \Theta_{\mathcal{A}}$$

and from the linearity of the map  $u \mapsto \Theta_{\mathcal{A}}(u, v)$  with (26) the lemma is proved for matrices  $E, A \in \mathbb{C}^{n \times n}$ .

We consider the case where  $E, A$  are real matrices. Here we use the Weierstraß canonical form over  $\mathbb{R}$ , obtained in [16], with transformation matrices  $S, T \in \mathbb{R}^{n \times n}$ . The matrix  $N$  is the same as in (24) and  $J$  is in real Jordan canonical form, see [21, Section 3.4.1],

$$J = \bigoplus_{\substack{\lambda \in \sigma(J), \\ \text{Im } \lambda > 0}} \bigoplus_{j=1}^{\dim \ker \mathcal{A}(\lambda)} J_{m_j(\lambda)}^{\mathbb{R}}(\lambda) \oplus \bigoplus_{\lambda \in \sigma(J) \cap \mathbb{R}} \bigoplus_{j=1}^{\dim \ker \mathcal{A}(\lambda)} J_{m_j(\lambda)}(\lambda),$$

where  $J_{m_j(\lambda)}(\lambda)$ ,  $\lambda \in \sigma(\mathcal{A}) \cap \mathbb{R}$ , are Jordan blocks of size  $m_j(\lambda)$  and  $J_l^{\mathbb{R}}(\lambda) \in \mathbb{R}^{2l \times 2l}$  for some  $l \in \mathbb{N} \setminus \{0\}$  is a real Jordan block at  $\lambda = a + ib$  with  $a \in \mathbb{R}$ ,  $b > 0$ , given by

$$J_l^{\mathbb{R}}(\lambda) := \begin{pmatrix} C(a, b) & I_2 & & & \\ & \cdot & \cdot & & \\ & & \cdot & \cdot & \\ & & & \cdot & I_2 \\ & & & & C(a, b) \end{pmatrix} \in \mathbb{R}^{2l \times 2l}, \quad C(a, b) := \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \in \mathbb{R}^{2 \times 2}.$$

Therefore the resolvent of  $J_l^{\mathbb{R}}(a, b)$  is given by

$$(27) \quad (sI_{2l} - J_l^{\mathbb{R}}(a, b))^{-1} = \begin{pmatrix} (s - C(a, b))^{-1} & \dots & (-1)^{l-1}(s - C(a, b))^{-l} \\ & \ddots & \vdots \\ & & (s - C(a, b))^{-1} \end{pmatrix}$$

where the entries are given by

$$(s - C(a, b))^{-k} = ((s - C(a, b))^{-1})^k = \left( \frac{1}{(s - a)^2 + b^2} \begin{pmatrix} s - a & b \\ -b & s - a \end{pmatrix} \right)^k, \quad k \in \mathbb{N} \setminus \{0\}.$$

Using the expression (27) instead of the blocks occurring in the block matrix in the first summand of the right hand side of (25) for the non-real eigenvalues, one can

define again a linearly independent set of polynomials  $P_1$  by picking all first row entries. This set consists again of polynomials all of distinct degree, because the factor  $((s-a)^2 + b^2)^{m_1(\lambda)}$  occurs in the minimal polynomial  $m_{\mathcal{A}}(s)$ . The set  $P_2$  remains the same as in the complex valued case. Therefore the same arguments imply the surjectivity of  $\Theta_{\mathcal{A}}$  in this case.  $\square$

*Proof of Theorem 10.* Choose  $\alpha, \beta \in \mathbb{C}$  such that  $\alpha \neq 0$  and

$$\frac{\beta}{\alpha} \notin \{\mu_1, \dots, \mu_l\} \cup \sigma(\mathcal{A})$$

holds. We set

$$\gamma := m_{\mathcal{A}}(\beta/\alpha) \cdot \prod_{\mu_i \in \{\mu_1, \dots, \mu_l\} \setminus \{\infty\}} (\beta/\alpha - \mu_i)^{-m_i}.$$

The condition  $\beta/\alpha \notin \{\mu_1, \dots, \mu_l\} \cup \sigma(\mathcal{A})$  implies  $m_{\mathcal{A}}(\beta/\alpha) \neq 0$ , hence  $\gamma \neq 0$ . We consider

$$q_{\gamma}(s) := \gamma \prod_{\mu_i \in \{\mu_1, \dots, \mu_l\} \setminus \{\infty\}} (s - \mu_i)^{m_i}.$$

As  $\sum_{i=1}^l m_i = M(\mathcal{A})$ , the polynomial  $q_{\gamma}(s)$  satisfies  $\deg q_{\gamma} \leq M(\mathcal{A})$ . The degree of  $m_{\mathcal{A}}(s)$  is  $M(\mathcal{A}) - m_1(\infty)$  which is smaller or equal to  $M(\mathcal{A})$ . From the choice of  $\gamma$  we see that  $(q_{\gamma} - m_{\mathcal{A}})(\beta/\alpha) = 0$  holds and therefore

$$\frac{q_{\gamma}(s) - m_{\mathcal{A}}(s)}{\alpha s - \beta}$$

is a polynomial of degree less or equal to  $M(\mathcal{A}) - 1$ . By Lemma 13 there exist  $u, v \in \mathbb{C}^n$  with

$$\frac{q_{\gamma}(s) - m_{\mathcal{A}}(s)}{\alpha s - \beta} = v^* m_{\mathcal{A}}(s) (sE - A)^{-1} u.$$

This equation combined with Sylvester's determinant identity leads to

$$(28) \quad \frac{q_{\gamma}(s)}{m_{\mathcal{A}}(s)} = 1 + (\alpha s - \beta) v^* (sE - A)^{-1} u = \det(I_n + (sE - A)^{-1} (\alpha s - \beta) uv^*).$$

Now, set  $\mathcal{P}(s) = (\alpha s - \beta) uv^*$ . Then by Lemma 5  $(\mathcal{A} + \mathcal{P})(s)$  is regular and from (28) we obtain

$$\begin{aligned} \det(\mathcal{A} + \mathcal{P})(s) &= \det(sE - A + (\alpha s - \beta) uv^*) \\ &= \det(sE - A) \det(I_n + (sE - A)^{-1} (\alpha s - \beta) uv^*) \\ &= \det(sE - A) \frac{q_{\gamma}(s)}{m_{\mathcal{A}}(s)}. \end{aligned}$$

Since  $\det(sE - A)$  is by Proposition 1 (d) divisible by  $m_{\mathcal{A}}(s)$ , (21) follows. The equation (22) follows from Proposition 1 (d) and Theorem 10 (a) is proved. The statements in (b) follow by the same construction as above and by Lemma 13 for real matrices.  $\square$

**6. Eigenvalue placement under parameter restrictions.** In this section we study the eigenvalue placement under perturbations of the form

$$(29) \quad \mathcal{P}(s) = (su + v)w^*, \quad w \in \mathbb{C}^n$$

for fixed  $u$  and  $v$  in  $\mathbb{C}^n$  (cf. Proposition 3). Special cases of this parameter restricted placement are the pole assignment problem for linear systems under state feedback (cf. Section 7) and also the eigenvalue placement problem for matrices under rank one matrices. Obviously, for the perturbation (29) the bounds on the multiplicities from Theorem 9 still hold. But since  $u$  and  $v$  are now fixed, we obtain tighter bounds which are given below. In the formulation of these bounds we introduce the number  $m_{u,v}(\lambda)$  which basically replaces  $m_1(\lambda)$  in the bounds obtained in Theorem 9. Assume that the vector-valued function

$$(30) \quad s \mapsto (sE - A)^{-1}(su + v)$$

has a pole at  $\lambda \in \sigma(\mathcal{A}) \setminus \{\infty\}$ , i.e. one of the entries has a pole at  $\lambda$ . Then, we denote by  $m_{u,v}(\lambda)$  the order of  $\lambda$  as a pole of (30), which is the maximal order of  $\lambda$  as a pole of one of the entries. For  $\lambda = \infty$  we define  $m_{u,v}(\infty)$  as the order of the pole of

$$s \mapsto (-sA + E)^{-1}(sv + u)$$

at  $s = 0$ . In the case where no pole occurs we set  $m_{u,v}(\lambda) = 0$ . Hence  $m_{u,v}(\lambda)$  is defined for all  $\lambda \in \sigma(\mathcal{A})$ . Note that another way of introducing poles and their order is given by the use of the Smith-McMillan form [9], [16, Ch. VI], [17, Ch. S1].

**PROPOSITION 14.** *Let  $\mathcal{A}(s) = sE - A$  be regular and let  $\mathcal{P}(s)$  be given by (29) with  $u, v \in \mathbb{C}^n$  fixed such that  $(\mathcal{A} + \mathcal{P})(s)$  is regular. For*

$$M(\mathcal{A}, u, v) := \sum_{\mu \in \sigma(\mathcal{A})} m_{u,v}(\mu)$$

and  $\lambda \in \sigma(\mathcal{A})$  we have

$$(31) \quad \dim \mathcal{L}_\lambda(\mathcal{A}) - m_{u,v}(\lambda) \leq \dim \mathcal{L}_\lambda(\mathcal{A} + \mathcal{P}) \leq \dim \mathcal{L}_\lambda(\mathcal{A}) + M(\mathcal{A}, u, v) - m_{u,v}(\lambda).$$

For  $\lambda \in \overline{\mathbb{C}} \setminus \sigma(\mathcal{A})$  we have

$$(32) \quad 0 \leq \dim \mathcal{L}_\lambda(\mathcal{A} + \mathcal{P}) \leq M(\mathcal{A}, u, v).$$

From this we obtain

$$\begin{aligned} \sum_{\lambda \in \sigma(\mathcal{A})} \dim \mathcal{L}_\lambda(\mathcal{A} + \mathcal{P}) &\geq n - \mathcal{M}(\mathcal{A}, u, v), \\ \sum_{\lambda \in \sigma(\mathcal{A} + \mathcal{P}) \setminus \sigma(\mathcal{A})} \dim \mathcal{L}_\lambda(\mathcal{A} + \mathcal{P}) &\leq \mathcal{M}(\mathcal{A}, u, v). \end{aligned}$$

*Proof.* From Sylvester's formula we conclude

$$\det(\mathcal{A} + \mathcal{P})(s) = \det \mathcal{A}(s)(1 + w^*(sE - A)^{-1}(su + v)).$$

If  $\lambda \in \sigma(\mathcal{A}) \setminus \{\infty\}$  then the characteristic polynomial of  $\mathcal{A}(s)$  has at  $\lambda$  a zero with multiplicity  $\dim \mathcal{L}_\lambda(\mathcal{A})$ . Since the order of  $\lambda$  as a pole of  $s \mapsto 1 + w^*(sE - A)^{-1}(su + v)$

is at most  $m_{u,v}(\lambda)$ , the multiplicity of  $\lambda$  as a zero of  $\det(\mathcal{A} + \mathcal{P})(s)$ , hence the dimension of  $\mathcal{L}_\lambda(\mathcal{A} + \mathcal{P})$ , can be bounded by

$$\dim \mathcal{L}_\lambda(\mathcal{A} + \mathcal{P}) \geq \dim \mathcal{L}_\lambda(\mathcal{A}) - m_{u,v}(\lambda).$$

This shows the lower bound in (31). The upper bound in (31) for  $\lambda \in \sigma(\mathcal{A}) \setminus \{\infty\}$  and (32) for  $\lambda \in \mathbb{C} \setminus \sigma(\mathcal{A})$  are obtained in the same way as in the proof of Theorem 9. The estimates (31) and (32) hold also for the dual pencil  $\mathcal{A}'(s) = -sA + E$  at  $\lambda = 0$  and, hence, by Proposition 1 (c) the estimates (31) and (32) also hold for  $\mathcal{A}(s)$  at  $\lambda = \infty$ . Now the remaining assertions follow in the same way as in Theorem 9.  $\square$

Within these bounds we investigate the placement of the eigenvalues. We start with a result which can be seen as a generalization of Lemma 5.

LEMMA 15. *Let  $\mathcal{A}(s) = sE - A$  be a regular matrix pencil satisfying Assumption 2 and let  $u, v, w \in \mathbb{C}^n$  and  $\mathcal{P}(s) = (su + v)w^*$ , such that  $(\mathcal{A} + \mathcal{P})(s)$  is regular. Then we have*

$$\{\lambda \in \sigma(\mathcal{A}) \mid \dim \ker \mathcal{A}(\lambda) \geq 2 \text{ or } m_{u,v}(\lambda) < m_1(\lambda)\} \subseteq \sigma(\mathcal{A} + \mathcal{P}).$$

*Proof.* From Corollary 8 we deduce

$$\{\lambda \in \sigma(\mathcal{A}) \mid \dim \ker \mathcal{A}(\lambda) \geq 2\} \subseteq \sigma(\mathcal{A} + \mathcal{P}).$$

First, we consider the case  $\lambda \in \sigma(\mathcal{A}) \setminus \{\infty\}$ . The resolvent representation (25) implies that  $s \mapsto (sE - A)^{-1}$  has an entry with a pole at  $\lambda$  of order  $m_1(\lambda)$ . Hence  $m_{u,v}(\lambda) \leq m_1(\lambda)$ . The condition  $m_{u,v}(\lambda) < m_1(\lambda)$ , Sylvester's determinant formula, and the representation from Proposition 1 (d) lead to

$$(33) \quad \begin{aligned} \det(\mathcal{A} + \mathcal{P})(s) &= \det(sE - A)(1 + w^*(sE - A)^{-1}(su + v)) \\ &= (-1)^{n-r} \det(ST)^{-1} m_{\mathcal{A}}(s) q(s) (1 + w^*(sE - A)^{-1}(su + v)). \end{aligned}$$

Hence  $\det(\mathcal{A} + \mathcal{P})(s)$  contains the factor  $(s - \lambda)$ , therefore  $\lambda \in \sigma(\mathcal{A} + \mathcal{P})$  and the set inclusion is proven for  $\lambda \in \sigma(\mathcal{A}) \setminus \{\infty\}$ . In the case  $\lambda = \infty \in \sigma(\mathcal{A})$ , one has to apply the above arguments to the dual pencil  $\mathcal{A}'(s) = -sA + E$  and  $(\mathcal{A} + \mathcal{P})'(s) = -sA + E + (sv + u)w^*$  at  $\lambda = 0$ .  $\square$

THEOREM 16. *Let  $\mathcal{A}(s) = sE - A$  be a regular matrix pencil satisfying Assumption 2 and let  $u, v \in \mathbb{C}^n$  be fixed. Choose  $\mu_1, \dots, \mu_l \in \overline{\mathbb{C}}$  with  $l \leq M(\mathcal{A}, u, v)$  and multiplicities  $m_i \in \mathbb{N} \setminus \{0\}$  satisfying  $\sum_{i=1}^l m_i = M(\mathcal{A}, u, v)$ . If  $u, v$  are linearly dependent we make the following further assumptions.*

- (i) *If  $u \neq 0$  and  $v = -\mu u$ . Then  $\mu \notin \sigma(\mathcal{A})$  implies  $\mu \notin \{\mu_1, \dots, \mu_l\}$ .*
- (ii) *If  $u = 0$  and  $v \neq 0$ . Then  $\infty \notin \sigma(\mathcal{A})$  implies  $\infty \notin \{\mu_1, \dots, \mu_l\}$ .*

*Then there exists  $w \in \mathbb{C}^n$  such that for  $\mathcal{P}(s) = (su + v)w^*$  the pencil  $(\mathcal{A} + \mathcal{P})(s)$  is regular and satisfies*

$$(34) \quad \sigma(\mathcal{A} + \mathcal{P}) = \{\mu_1, \dots, \mu_l\} \cup \{\lambda \in \sigma(\mathcal{A}) \mid \dim \ker \mathcal{A}(\lambda) \geq 2 \text{ or } m_{u,v}(\lambda) < m_1(\lambda)\}$$

*and the dimensions of the root subspaces are*

$$(35) \quad \dim \mathcal{L}_\lambda(\mathcal{A} + \mathcal{P}) = \begin{cases} \dim \mathcal{L}_\lambda(\mathcal{A}) - m_{u,v}(\lambda) + m_i, & \text{for } \lambda = \mu_i \in \sigma(\mathcal{A}), \\ \dim \mathcal{L}_\lambda(\mathcal{A}) - m_{u,v}(\lambda), & \text{for } \lambda \in \sigma(\mathcal{A}) \setminus \{\mu_1, \dots, \mu_l\}, \\ m_i, & \text{for } \lambda = \mu_i \notin \sigma(\mathcal{A}), \\ 0, & \text{for } \lambda \notin \sigma(\mathcal{A}) \cup \{\mu_1, \dots, \mu_l\}. \end{cases}$$

If  $E, A$  are real matrices and  $u, v \in \mathbb{R}^n$  then we can find  $w \in \mathbb{R}^n$  such that (34) and (35) hold.

*Proof.* For the proof, we essentially repeat the arguments from the proof of Lemma 13. First, let us assume that  $u$  and  $v$  are linearly dependent with  $u \neq 0$  and, hence,  $v = -\mu u$  and  $\mathcal{P}(s) = (s - \mu)uw^*$ . For  $S, T \in \mathbb{C}^{n \times n}$  such that  $S\mathcal{A}(s)T$  is in Weierstraß canonical form we write in (33)

$$(1 + (s - \mu)w^*(sE - A)^{-1}u) = (1 + (s - \mu)w^*T(sSET - SAT)^{-1}Su).$$

If  $\mathcal{A}(s) = sE - A$  has only one eigenvalue  $\lambda \neq \infty$  with one Jordan chain at  $\lambda$  of length  $m_1(\lambda)$  and  $m_{u,v}(\lambda) \geq 1$ , one can easily write down the result of the multiplication of  $(sSET - SAT)^{-1}$  with  $Su = ((Su)_j)_{j=1}^{m_1(\lambda)}$ , cf. (25),

$$(36) \quad (sSET - SAT)^{-1}Su = \left( \sum_{j=k}^{m_1(\lambda)} (-1)^{j-k} (s - \lambda)^{-j+k-1} (Su)_j \right)_{k=1}^{m_1(\lambda)}.$$

Denote by  $m \in \mathbb{N}$  with  $1 \leq m \leq m_1(\lambda)$  the largest index such that  $(Su)_m \neq 0$ . Such an index exists because of  $m_{u,v}(\lambda) \geq 1$  and then  $(Su)_k = 0$  for all  $m < k \leq m_1(\lambda)$ . Let us now assume that  $\mu \notin \sigma(\mathcal{A})$ . The assumption  $\mu \notin \sigma(\mathcal{A})$  implies that  $m_{u,v}(\lambda)$  is not only the order of  $\lambda$  as a pole of  $s \mapsto (s - \mu)(sE - A)^{-1}u$  but also the order of  $\lambda$  as a pole of  $s \mapsto (sE - A)^{-1}u$ . Therefore  $m = m_{u,v}(\lambda)$  hence  $(Su)_{m_{u,v}(\lambda)} \neq 0$  and we consider the following functions given by the right hand side of (36)

$$(37) \quad p_{\lambda,k}(s) := \sum_{j=k}^{m_{u,v}(\lambda)} (-1)^{j-k} (s - \lambda)^{-j+k-1} (Su)_j$$

for  $1 \leq k \leq m_{u,v}(\lambda)$ . The summand in each  $p_{\lambda,k}(s)$  with the pole of the highest order has, as  $(Su)_{m_{u,v}(\lambda)} \neq 0$ , a non-zero coefficient so that (37) defines a linearly independent set of functions. If  $\mathcal{A}(s) = sE - A$  has only the eigenvalue at  $\infty$  with only one Jordan chain of length  $m_1(\infty)$  and  $m_{u,v}(\infty) \geq 1$  then the largest index  $m \in \mathbb{N}$  such that  $(Su)_m \neq 0$  holds is  $m = m_{u,v}(\infty)$  and the right hand side of (36) consists of the functions

$$p_{\infty,k}(s) := - \sum_{j=k}^{m_{u,v}(\infty)-1} s^{j-k} (Su)_{j+1}$$

for  $0 \leq k \leq m_{u,v}(\infty) - 1$ .

Since  $(sSET - SAT)^{-1}$  has block diagonal structure (25), similar expressions as in (36) occur when  $\mathcal{A}(s)$  has more than one Jordan chain at  $\lambda \in \sigma(\mathcal{A})$  or more than one eigenvalue. Here there exists for all  $\lambda \in \sigma(\mathcal{A})$  with  $m_{u,v}(\lambda) \geq 1$  some block on the right hand side of (25) such that the above construction of the functions  $p_{\lambda,k}(s)$  can be carried out. As in the proof of Lemma 13 we multiply with a polynomial

$$\tilde{m}(s) := \prod_{\lambda \in \sigma(\mathcal{A}) \setminus \{\infty\}} (s - \lambda)^{m_{u,v}(\lambda)}$$

and conclude that the set

$$(38) \quad \{p_{\lambda,k}(s)\tilde{m}(s) \mid \lambda \in \sigma(\mathcal{A}) \setminus \{\infty\}, 1 \leq k \leq m_{u,v}(\lambda)\} \\ \cup \{p_{\infty,k}(s)\tilde{m}(s) \mid \infty \in \sigma(\mathcal{A}), 0 \leq k \leq m_{u,v}(\infty) - 1\}$$



is linearly independent and consists of  $\sum_{\lambda \in \sigma(\mathcal{A})} m_{u,v}(\lambda) = M(\mathcal{A}, u, v)$  polynomials of degree at most  $M(\mathcal{A}, u, v) - 1$  since we consider only those  $\lambda \in \sigma(\mathcal{A})$  with  $m_{u,v}(\lambda) \geq 1$ . Therefore it is a basis in the space of polynomials of degree less or equal to  $M(\mathcal{A}, u, v) - 1$ . By (i) we have  $\mu \notin \{\mu_1, \dots, \mu_l\}$  and we consider

$$\gamma := \tilde{m}(\mu) \prod_{\mu_i \in \{\mu_1, \dots, \mu_l\} \setminus \{\infty\}} (\mu - \mu_i)^{-m_i}$$

and the polynomial

$$\tilde{q}_\gamma(s) := \gamma \prod_{\mu_i \in \{\mu_1, \dots, \mu_l\} \setminus \{\infty\}} (s - \mu_i)^{m_i}.$$

Since  $T$  is invertible and from the linear independence of the polynomials there exists  $w \in \mathbb{C}^n$  such that

$$(39) \quad \frac{\tilde{q}_\gamma(s) - \tilde{m}(s)}{s - \mu} = w^* T \tilde{m}(s) (sSET - SAT)^{-1} Su$$

holds. Plugging this into (33) proves (34) and (35) in the case  $\mu \notin \sigma(\mathcal{A})$ .

Let us now assume that  $\mu \in \sigma(\mathcal{A})$  and we consider again the case that  $\mathcal{A}(s) = sE - A$  has only one Jordan chain at  $\lambda$  of length  $m_1(\lambda)$  and  $m_{u,v}(\lambda) \geq 1$ . This means that  $\mu = \lambda$  and the definition of  $m_{u,v}(\lambda)$  implies  $m = m_{u,v}(\lambda) + 1$  and we consider the following functions given by the right hand side of (36)

$$(40) \quad p_{\lambda,k}(s) := \sum_{j=k}^{m_{u,v}(\lambda)+1} (-1)^{j-k} (s - \lambda)^{-j+k-1} (Su)_j$$

for  $1 \leq k \leq m_{u,v}(\lambda) + 1$ . These functions define a linearly independent set. As in the previous sub case, we are looking for a solution  $w \in \mathbb{C}^n$  with  $n = m_{u,v}(\lambda) + 1$  and  $(w_k)_{k=1}^n := w^* T$  of the equation

$$(41) \quad \prod_{\mu_i \in \{\mu_1, \dots, \mu_l\} \setminus \{\infty\}} (s - \mu_i)^{m_i} - \tilde{m}(s) = w^* T \tilde{m}(s) (s - \mu) (sSET - SAT)^{-1} Su \\ = \sum_{k=1}^{m_{u,v}(\lambda)+1} w_k \tilde{m}(s) (s - \mu) p_{\lambda,k}(s).$$

If we set  $w_1 = \dots = w_{m_{u,v}(\lambda)} = 0$  and  $w_{m_{u,v}(\lambda)+1} = -((Su)_{m_{u,v}(\lambda)+1})^{-1}$  the right hand side of (41) is equal to  $-\tilde{m}(s)$ . On the other hand, by the linear independence of  $\{\tilde{m}(s) p_{\lambda,k}(s)\}_{k=1}^{m_{u,v}(\lambda)+1}$  in the space of polynomials of degree at most  $m_{u,v}(\lambda) = M(\mathcal{A}, u, v)$ , there is a solution  $w \in \mathbb{C}^n$  of the equation

$$\prod_{\mu_i \in \{\mu_1, \dots, \mu_l\} \setminus \{\infty\}} (s - \mu_i)^{m_i} = \sum_{k=1}^{m_{u,v}(\lambda)+1} w_k \tilde{m}(s) p_{\lambda,k}(s)$$

This, together with the linearity of equation (41) in  $w$  implies the existence of  $w \in \mathbb{C}^n$  such that (41) holds which proves the assertions in this sub case, that there is only one eigenvalue  $\lambda \neq \infty$ . The block diagonal structure of  $(sSET - SAT)^{-1}$  as in (25) implies the existence of  $w \in \mathbb{C}^n$  such that (34) and (35) hold in the case when  $\mathcal{A}(s)$

has more than one Jordan chain at  $\lambda \in \sigma(\mathcal{A})$  or more than one eigenvalue. One only has to replace the functions  $p_{\lambda,k}(s)$  for  $\lambda = \mu$  with those defined in (40).

Next, when  $u$  and  $v$  are linearly dependent with  $u = 0$  then we can further assume that  $v \neq 0$  because  $v = 0$  implies that  $m_{u,v}(\lambda) = 0$  for all  $\lambda \in \sigma(\mathcal{A})$  and then (34) and (35) hold trivially. This case can be treated by considering the dual pencils  $\mathcal{A}'(s)$  and  $(\mathcal{A} + \mathcal{P})'(s)$ , because for the dual pencils we are in the case of (i) with  $\mu = 0$ . Therefore, we have already proven that (34) and (35) hold for the dual pencils under the assumption that  $0 \notin \sigma(\mathcal{A}')$  implies  $0 \notin \{\mu_1, \dots, \mu_l\}$ . By Proposition 1 (c) this condition follows from the assumption (ii) that  $\infty \notin \sigma(\mathcal{A})$  implies  $\infty \notin \{\mu_1, \dots, \mu_l\}$ . Furthermore we see immediately that  $S\mathcal{A}'(s)T$  is block diagonal and that  $\dim \mathcal{L}_\lambda(\mathcal{A})$  for  $\lambda \in \sigma(\mathcal{A}) \setminus \{0\}$  is equal to  $\dim \mathcal{L}_{\lambda^{-1}}(\mathcal{A}')$ . This proves (34) and (35) for  $\mathcal{A}(s)$  and  $(\mathcal{A} + \mathcal{P})(s)$ . This finishes the proof of the theorem given that  $u$  and  $v$  are linearly dependent.

In the case where  $u$  and  $v$  are linearly independent we have in (33)

$$(1 + w^*(sE - A)^{-1}(su + v)) = (1 + w^*T(sSET - SAT)^{-1}(sSu + Sv)).$$

If  $\mathcal{A}(s) = sE - A$  has only one eigenvalue  $\lambda \neq \infty$  with one Jordan chain of length  $m_1(\lambda)$  and  $m_{u,v}(\lambda) \geq 1$  then the product  $(sSET - SAT)^{-1}(sSu + Sv)$  is given by

$$(42) \quad \left( \sum_{j=k}^{m_1(\lambda)} (-1)^{j-k} (s - \lambda)^{-j+k-1} (s(Su)_j + (Sv)_j) \right)_{k=1}^{m_1(\lambda)}.$$

This again allows us to define a linearly independent set of polynomials. For this we consider the largest index  $m \in \mathbb{N}$  with  $1 \leq m \leq m_1(\lambda)$  such that  $s(Su)_m + (Sv)_m \neq 0$ . This implies that  $s(Su)_k + (Sv)_k = 0$  for all  $m < k \leq m_1(\lambda)$  and we obtain the following to cases. If  $(s(Su)_m + (Sv)_m)$  and  $(s - \lambda)$  are linearly independent in  $\mathbb{C}[s]$  then we have  $m = m_{u,v}(\lambda)$  and we consider the following entries of (42)

$$p_{\lambda,k}(s) := \sum_{j=k}^{m_{u,v}(\lambda)} (-1)^{j-k} (s - \lambda)^{-j+k-1} (s(Su)_j + (Sv)_j)$$

for  $1 \leq k \leq m_{u,v}(\lambda)$ . If  $(s(Su)_m + (Sv)_m)$  and  $(s - \lambda)$  are linearly dependent then we have  $m = m_{u,v}(\lambda) + 1$  and define

$$p_{\lambda,k}(s) := \sum_{j=k}^{m_{u,v}(\lambda)+1} (-1)^{j-k} (s - \lambda)^{-j+k-2} (s(Su)_j + (Sv)_j)$$

for  $1 \leq k \leq m_{u,v}(\lambda) + 1$ . In both cases we have from the condition  $(s(Su)_m + (Sv)_m) \neq 0$  that the functions  $p_{\lambda,k}(s) \neq 0$  define a linearly independent set. If  $\mathcal{A}(s)$  has only the eigenvalue  $\infty$  with one Jordan chain of length  $m_1(\infty)$  we consider in the case  $(Su)_{m_{u,v}(\infty)} \neq 0$

$$p_{\infty,k}(s) := - \sum_{j=k}^{m_{u,v}(\infty)-1} s^{j-k-1} (s(Su)_{j+1} + (Sv)_{j+1})$$

for  $0 \leq k \leq m_{u,v}(\infty) - 1$  and if  $(Su)_{m_{u,v}(\infty)} = 0$

$$p_{\infty,k}(s) := - \sum_{j=k}^{m_{u,v}(\infty)-1} s^{j-k} (s(Su)_{j+1} + (Sv)_{j+1})$$

for  $0 \leq k \leq m_{u,v}(\infty) - 1$ . If  $\mathcal{A}(s)$  has more than one eigenvalue and with more than one Jordan chain, then we can use again the block diagonal structure of  $(sSET - SAT)^{-1}$  to define the functions  $p_{\lambda,k}(s)$  for all  $\lambda \in \sigma(\mathcal{A})$ . The linear independence of these functions follows from the linear independence of  $u$  and  $v$ .

Now the same linear independency arguments from above can be applied. This implies the existence of  $w \in \mathbb{C}^n$  satisfying the equation

$$\prod_{\mu_i \in \{\mu_1, \dots, \mu_l\} \setminus \{\infty\}} (s - \mu_i)^{m_i} - \tilde{m}(s) = w^* T \tilde{m}(s) (sSET - SAT)^{-1} S (su + v).$$

For real matrices  $E$  and  $A$  one uses the transformation to the real Weierstraß canonical form with  $S, T \in \mathbb{R}^{n \times n}$  such that for  $\lambda = a + ib \in \sigma(\mathcal{A}) \setminus \mathbb{R}$  with  $b > 0$  we use the blocks given by (27) to define the polynomials  $p_{\lambda,k}(s)$ . These polynomials have to be inserted into (38) which gives a linearly independent set. Since the transformation matrices  $S, T$  are real, this proves the statement in the real case.  $\square$

If  $E = I_n$  then  $\sigma(\mathcal{A})$  equals the set  $\sigma(A)$  of all eigenvalues of  $A$  and  $\infty \notin \sigma(\mathcal{A})$  holds. For the perturbation class given for a fixed  $v \in \mathbb{C}^n$  by

$$\mathcal{P}(s) := vw^*, \quad w \in \mathbb{C}^n,$$

Theorem 16 for  $u = 0$  leads to the following eigenvalue placement result for matrices which is a generalization of the placement results obtained in [10, 18, 22, 36].

**COROLLARY 17.** *Let  $\mathcal{A}(s) = sI_n - A$  be a matrix pencil satisfying Assumption 2 with minimal polynomial  $m_A(s)$  of the matrix  $A \in \mathbb{C}^{n \times n}$  and fixed  $v \in \mathbb{C}^n$ . Then for any given values  $\mu_1, \dots, \mu_l \in \mathbb{C}$  with  $l \leq M(\mathcal{A}, 0, v)$  and multiplicities  $m_i \in \mathbb{N} \setminus \{0\}$  satisfying  $\sum_{i=1}^l m_i = M(\mathcal{A}, 0, v)$  there exists  $w \in \mathbb{C}^n$  such that*

$$\sigma(A + vw^*) = \{\mu_1, \dots, \mu_l\} \cup \{\lambda \in \sigma(A) \mid \dim \ker(\lambda I_n - A) \geq 2, m_{0,v}(\lambda) < m_1(\lambda)\}$$

and (35) holds. If  $v \in \mathbb{C}^n$  is not fixed, then one can always choose  $v$  such that  $m_{0,v}(\lambda) = m_1(\lambda)$  holds for all  $\lambda \in \sigma(A)$  implying that  $M(\mathcal{A}, 0, v) = \deg m_A$ . For this choice of  $v$  there exists  $w \in \mathbb{C}^n$  such that

$$\sigma(A + vw^*) = \{\mu_1, \dots, \mu_l\} \cup \{\lambda \in \sigma(A) \mid \dim \ker(\lambda I_n - A) \geq 2\}.$$

If  $E, A$  are real matrices then  $u, v$  can be chosen in  $\mathbb{R}^n$ .

**7. Application to single input differential-algebraic equations with feedback.** Parameter restricted perturbations of the form (29) occur naturally in the study of differential algebraic equations with a single input given by  $E, A \in \mathbb{C}^{n \times n}$ ,  $b \in \mathbb{C}^n$ ,  $x_0 \in \mathbb{C}^n$  and the equation

$$(43) \quad \frac{d}{dt} Ex(t) = Ax(t) + bu(t), \quad t \in [0, \infty), \quad x(0) = x_0.$$

For this equation we consider a state feedback of the form  $u(t) = f^* x(t)$  with  $f \in \mathbb{C}^n$ . It is well known that the solution of the closed loop-system

$$\frac{d}{dt} Ex(t) = (A + bf^*)x(t), \quad t \in [0, \infty), \quad x(0) = x_0$$

can be expressed with the eigenvalues and Jordan chains of the matrix pencil  $sE - (A + bf^*)$ , see [5]. This pencil can be written as a perturbation of  $sE - A$  with the rank one pencil  $\mathcal{P}(s) = -bf^*$ . By fixing  $u = 0$  and  $v = -b$  in (29) we can write

$$\mathcal{P}(s) = -bf^* = (su + v)f^*.$$

For  $E$  singular we have  $\infty \in \sigma(\mathcal{A})$  and for  $E$  invertible we have  $\infty \notin \sigma(\mathcal{A})$ .

In [6] it was shown that a system given by  $(E, A, b)$  with  $sE - A$  regular is controllable in the behavioural sense if and only if the Hautus condition

$$\text{rank}[\lambda E - A, b] = n, \quad \text{for all } \lambda \in \mathbb{C}$$

holds. A transformation to Weierstraß canonical form (4) reveals that this Hautus condition is equivalent to  $\dim \ker \mathcal{A}(\lambda) = 1$  for all  $\lambda \in \mathbb{C} \cap \sigma(\mathcal{A})$  and

$$m_{0,-b}(\lambda) = m_1(\lambda) = \dim \mathcal{L}_\lambda(\mathcal{A}), \quad \text{for all } \lambda \in \mathbb{C} \cap \sigma(\mathcal{A}).$$

Hence Theorem 16 implies that all finite eigenvalues can be placed arbitrarily in  $\mathbb{C}$ .

The feedback placement problem for regular matrix pencils was studied in [8, 24, 25, 28]. In the following, as in [25], we do not assume controllability. Then Theorem 16 implies the following.

**THEOREM 18.** *Let  $(E, A, b)$  be a system given by (43) such that  $\mathcal{A}(s) = sE - A$  is a regular matrix pencil satisfying Assumption 2. Choose pairwise distinct numbers  $\mu_1, \dots, \mu_l \in \overline{\mathbb{C}}$  with  $l \leq M(\mathcal{A}, 0, -b)$ . Choose multiplicities  $m_1, \dots, m_l \in \mathbb{N} \setminus \{0\}$  with  $\sum_{i=1}^l m_i = M(\mathcal{A}, 0, -b)$ . Then:*

- (a) *For  $E$  singular there exists a feedback vector  $f \in \mathbb{C}^n$ , such that  $sE - (A + bf^*)$  is regular and  $\sigma(sE - (A + bf^*))$  equals to*

$$(44) \quad \{\mu_1, \dots, \mu_l\} \cup \{\lambda \in \sigma(\mathcal{A}) \mid \dim \ker \mathcal{A}(\lambda) \geq 2 \text{ or } m_{0,-b}(\lambda) < m_1(\lambda)\}.$$

- (b) *For  $E$  invertible and  $\infty \notin \{\mu_1, \dots, \mu_l\}$  there exists a feedback vector  $f \in \mathbb{C}^n$  such that  $sE - (A + bf^*)$  is regular and  $\sigma(sE - (A + bf^*))$  equals (44).*

*In both cases (a) and (b) the dimensions of the root subspaces are given by formula (35). If  $E, A$  are real matrices and  $b \in \mathbb{R}^n$  then  $f$  can be chosen in  $\mathbb{R}^n$ .*

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